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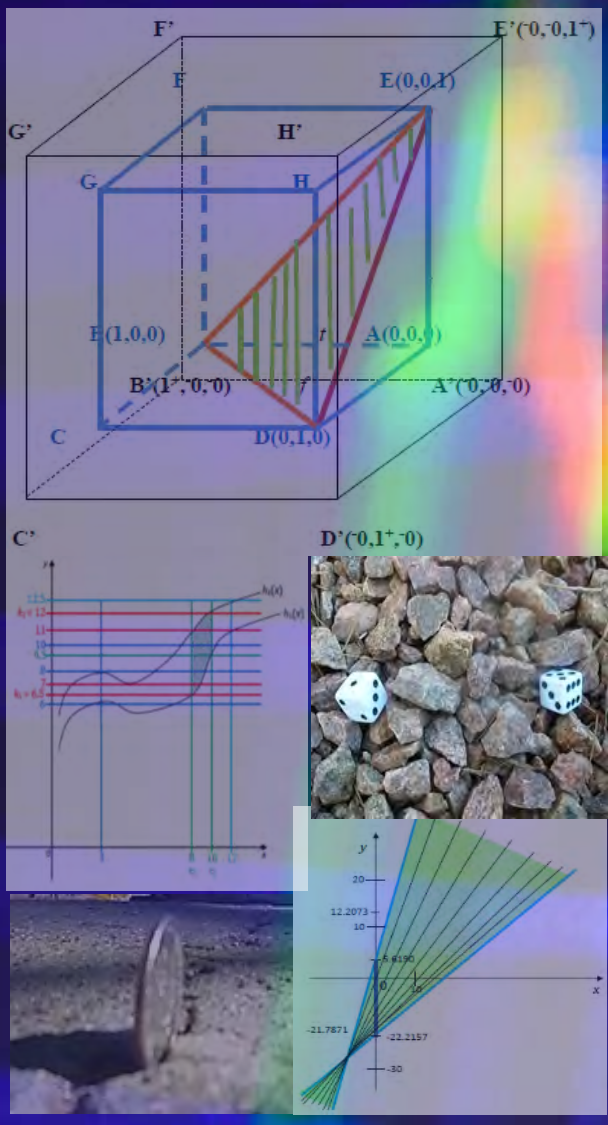

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The submitted papers should be professional, in good English, containing a brief review of a problem and obtained results.

Neutrosophy is a new branch of philosophy that studies the origin, nature, and scope of neutralities, as well as their interactions with different ideational spectra.

This theory considers every notion or idea <A> together with its opposite or negation <antiA> and with their spectrum of neutralities <neutA> in between them (i.e. notions or ideas supporting neither <A> nor <antiA>). The <neutA> and <antiA> ideas together are referred to as <nonA>.

Neutrosophy is a generalization of Hegel's dialectics (the last one is based on <A> and <antiA> only).

According to this theory every idea <A> tends to be neutralized and balanced by <antiA> and <nonA> ideas - as a state of equilibrium.

In a classical way <A>, <neutA>, <antiA> are disjoint two by two. But, since in many cases the borders between notions are vague, imprecise, Sorites, it is possible that <A>, <neutA>, <antiA> (and <nonA> of course) have common parts two by two, or even all three of them as well.

Neutrosophic Set and *Neutrosophic Logic* are generalizations of the fuzzy set and respectively fuzzy logic (especially of intuitionistic fuzzy set and respectively intuitionistic fuzzy logic). In neutrosophic logic a proposition has a degree of truth (*T*), a degree of indeterminacy (*I*), and a degree of falsity (*F*), where *T, I, F* are standard or non-standard subsets of $]^{-}0, 1^{+}[$.

Neutrosophic Probability is a generalization of the classical probability and imprecise probability.

Neutrosophic Statistics is a generalization of the classical statistics.

What distinguishes the neutrosophics from other fields is the <neutA>, which means neither <A> nor <antiA>.

<neutA>, which of course depends on <A>, can be indeterminacy, neutrality, tie game, unknown, contradiction, ignorance, imprecision, etc.

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FOLDOC Dictionary of Computing (**1, 2**), Webster
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Dictionary (**1**), Wiktionary (**2**), YourDictionary (**1, 2**),OneLook Dictionary (**1, 2**), Dictionary / Thesaurus (**1**), Online Medical Dictionary (**1,2**), Encyclopedia (**1, 2**), Chinese Fanyi Baidu Dictionary (**2**), Chinese Youdao Dictionary (**2**) etc. have included these scientific neologisms.

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Contents

Florentin Smarandache, Maikel Y. Leyva Vázquez, From Di-alectics to N-alectics: Indigenous Cultures and Ancestral Philosophies in Latin America……1-11
M.Palanikumar, Nasreen K.,Tonguc C., Neutrosophic normal vague set fitting to trigonometric concept via aggregation operators and its augmentation… ….12-26
Ruijuan Zhang, Type-2 Neutrosophic Sets (T2NSs) for Quality Evaluation of Project-Based Art and Design Courses: Evaluation Seven Projects……..27-40
Eman Hussein,Yousef Al-Qudah, HM Jaradat, Faisal Al-Sharqi, Saja Elnajar, Ali Jaradat, Similarity measure and sine exponential measure of possibility interval-valued neutrosophic hypersoft sets and their applications……....41-61
Xiao Ma, New Approach Single Valued Neutrosophic Sets for Teaching Quality Challenges Evaluation in College Public English Broad Impact……..62-78
Jimmy H. P., R. Vera-Ortega, A. M. Plaza and B. H. O. Correa, Global Gender Inequality in Industry: A Systematic Review with Neutrosophic Analysis……..79-88
Thanh- Ngan L., Quoc-Hieu P., Phi Hung N. and Lan-Anh Thi N., Rethinking Reverse Logistics: Neutrosophic Strategies for Warehouse Management Challenges…...89-131
Ying Du, Interval-Valued Neutrosophic Framework for Improving the Decision-Making on the Quality Evaluation of English Classics Translation…….132-144
Jun W., Xiong H., High-Quality Development Evaluation of Cultural and Tourism Integration Empowered by the Digital Economy under Double Valued Neutrosophic Sets……...……145-158
Benard O., O. Tunde Yakub, A. Abdulafeez O., O. Oluwapamilerin, Adenibuyan Micheal Tokunbo, Oluwatobi Balogun and Adekola Lanrewaju Olumide, Isotopic Properties of Neutrosophic Soft Quasigroup and Its Application in Decision-making……..159-185
Chen Yuan, Yongye Chen, Adaptability Evaluation of Vocational Education and Economic Development Under the Bipolar Neutrosophic Sets186-198
R. Priya, Nivetha Martin, Florentine Smarandache, Neutrosophic Linguistic Rhotrix in Managerial Decision Making……..199-217
Ying L.,Group Decision-Making Model Based on Triangular Neutrosophic Sets for Service Quality Evaluation in Tourism Mobile E-Commerce……218-229
Keyan Zhao, Bipolar Neutrosophic Driven Approach for Assessing the Teaching Quality of English Language and Literature Learning Outcomes ……230-245
Maikel Y. Leyva Vázquez, Florentin Smarandache, Andean Epistemology, Ch'ixi, and Neutrosophic Logic……....246-257
Murugan P., Nasreen k., Tonguc C., Inverse fractional function setting sine trigonometric neutrosophic set approach to interaction aggregating operators…258-272
Hussam Elbehiery, Hanaa fathi, Mohamed Eassa, Ahmed Abdelhafeez, Mohamed Refaat Abdellah, Hadeer Mahmoud, Advanced Machine Learning Approaches for Breast Cancer Detection with Neutrosophic Sets……..273-284
Raed Hatamleh, Ahmed Salem Heilat, M.Palanikumar and Abdallah Al-Husban, Characterization of interaction aggregating operators setting interval-valued Pythogorean neutrosophic set……..285-305
Phi-Hung Nguyen, Lan-Anh Thi Nguyen, The-Vu Pham, Hong-Quan Le, Thu-Hoai Thi Nguyen, Tra-Giang Vu, Huong-Giang Hoang Le, Optimizing Horizontal Collaboration in Logistics with Neutrosophic Z-Number Decision Models……..306-342
Xiuli L., Expanded Triangular Fuzzy Neutrosophic Numbers for Safety Risk Evaluation of Urban Rail Transit Engineering: An Innovative Method for Better Risk Assessment…343-360
Ketty Marilú Moscoso-Paucarchuco, Lucila Maribel Justiniano-Advincula, Monica Gisela Justiniano-Advincula, Manuel Michael Beraún-Espíritu, Inocente Feliciano Salazar-Rojas, Carlos Walter Mayta-Molina and Juan Dionicio Paz-Soldán Chávez, Classification of Organizational Behavior at the National Agrarian University of La Selva Based on Plithogenic Sets……..361-368
Xing She, Ying Liu, Qian Liu, Le Zhang, Quality Evaluation of Immersive Virtual Reality Interactive Art Design under Cultural Heritage Digitization under Interval Valued Neutrosophic Numbers……..369-381
María Gracia Paredes-Morales, Gino Joel Alberdy-Rodriguez, Susy Grey Morales-Cobos, Dessire Amandiz Castro-Valderrama, Gloria Angélica Valderrama-Barragán, and Edwin Favio Valderrama-Barragán, A Neutrosophic Model for Assessing the Impact of Artificial Intelligence on Civil Liability……..382-388
Hongxia Cui, Adaptive Triangular Linguistic Neutrosophic Cubic Fuzzy Sets for College English Blended Teaching Mode Evaluation Method ……389-406
Ahmed Salem Heilat, The Numerical Applications of (ABM) and (AMM) Numerical Methods on Some Neutrosophic and Dual Problems ……..407-421
Ye Liang, Landscape Design Quality Evaluation of Abandoned Coal Mine Sites Using Single-Valued Triangular Neutrosophic Numbers……..…422-437
Muhammad Saqlain, José M. Merigó, Poom Kumam, Neutrosophic Sets and Systems: A Decade of Scientific Contribution and Growth……..438-465
Fengfang Wu, Convolutional Interval-Valued Neutrosophic Network for Intelligent Evaluation of Smart Clothing Design Choices……466-478
D. A. A., A. A. Salama, Alaa A. A., H. K. M. M., Enhancing Missing Data Imputation for Migrants Data: A Neutrosophic Set-Based Machine Learning Approach…..479-502
Haiting Chen, Practical Overview of Triangular Bipolar Neutrosophic Numbers for Design Effect Evaluation of Ethnic Minority Clothing: Comprehensive Guide….503-517
Leonardo José España-Barraza, Efraín Javier De-la-Hoz-Granadillo, and Katia Ninozca Flores-Ledesma, Human Capital and Innovation in Developed and Developing Countries: A Systematic Literature Review Using Methodi Ordinatio and HyperSoft Sets……..……518-525
Qing Li, Wenya Chen, Clear Approach to Education Quality Evaluation of Multilingual Higher Education Internationalization under Probabilistic Simplified Neutrosophic Sets for Academic Improvement ……..526-539
Ahmed S. H., A Comparison between Euler's Method and 4-th Order Runge- Kutta Method for Numerical Solutions of Neutrosophic and Dual Differential Problems……540-559
Hanwei Qu, Caiyun Duan, Jiancheng Yu, Effective Approach to Shock Absorption Performance Evaluation of Mechanical Component Springs Using the Single Valued Trapezoidal Neutrosophic Numbers for Structural Stability……560-574
S.Ghosh, R. Bhardwaj, S. Narayan and S. Sonam, Fixed Point Theorems in Neutrosophic Fuzzy Metric Space and a Characterization to its Completeness…575-602
Qing Tan, Chengcheng Wu, Xuan Li, Yao Yao, Quality Assessment of Innovation and Entrepreneurship Talent Cultivation in Universities from the Perspective of Collaborative Education Under Triangular Neutrosophic Cubic Linguistic Hesitant Fuzzy Set……..603-621
K. Rajesh, S. Moorthy, A. Jeeva, D. Moganraj, Application of the Neutrosophic Fuzzy Sets in Profession Determination……622-637
Shengnan Zhang, Yuwei Wang, An Analysis of Bipolar Trapezoidal Neutrosophic Sets (BTNSs) for Examining the Income-Increasing and Poverty-Reduction Effects of Ecological Compensation in the Huai River Basin of Anhui Province: A Comprehensive Study……..638-654
J. Shirin Nejad, M. Saraj, Solving a Global-Mixed Integer Signomial Geometric Fractional Programming Problem……655-666
Edgardo Antonio Carmona-Arrieta, Katia Ninozca Flores-Ledesma, Álvaro Andrés Escobar-Espinoza, Abrahan Cesar Neri-Ayala, Pedro Barrientos-Gutiérrez, Pablo José Guerrero, The potential of alliances for regional productive development from the Tutorial Centers of the University of Cartagena measured with the support of Plithogenic SWOT Analysis……667-674
Walid Abdullah, Enhancing Medical X-Ray Image Classification with Neutrosophic Set Theory and Advanced Deep Learning Models……675-698
T. Fujita, F. Smarandache, and V. Christianto, A Concise Formalization of Hyperrealism-Antirealism in Physics Using Neutrosophic Logic……699-711
Huihui Gao, Kun Qian, Risk Assessment of Municipal Water Supply and Drainage Project Costs Using Neutrosophic Numbers……712-728
Aparna Tripathy, Amaresh Chandra Panada, Siva Prasad Behera, Bhabani S. Mohanty, Edge connectivity of a neutrosophic graph……729-740
M. Jdid, F. Smarandache, Takaaki Fujita, Neutrosophic augmented Lagrange multipliers method Nonlinear Programming Problems Constrained by Inequalities…….741-752
Z. Wen, J. Liu, Jianchang Huang, Hybrid SuperHyperSoft Model for Evaluation of Comprehensive Transportation Efficiency: Analysis and Results…..753-769
Lulu Shi, New Energy Vehicle Industry Chain Evaluation by Developing Decision-Making Methodology with HyperSoft Set and Neutrosophic Sets…..770-784
Ruijuan Zhang, Neutrosophic HyperSoft Set (NHSS) for Teaching Quality Assessment in University Interior Design Courses……785-798
Yaser Ahmad Alhasan, Asma Mohammed Alshallali and Qusay Alhassan, The double integrals of 2- refined neutrosophic functions and its applications…..799-811
Gao Huijuan, Fu Licheng, Quality Review of Computer Digital Media Art Design using Neutrosophic Logic: A Case Study……812-826
Vijaya V, Said Broumi and Manikandan H, Complex Fermatean Neutrosophic Sets and their Applications in Decision Making……827-839
Xinxiu Yang, Prioritization of Higher Education Management Strategies Based on the Neutrosophic Uncertainty Model for Sustainability……840-855

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From Di-alectics to N-alectics: Indigenous Cultures and Ancestral Philosophies in Latin America

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Abstract: This paper explores the n-alectic framework, which extends traditional di-alectic and tri-alectic models by integrating multiple subcomponents through refined neutrosophic logic. Drawing from pre-Columbian philosophies, particularly Mesoamerican, Andean, and Amazonian traditions, the study demonstrates how indigenous worldviews align with n-alectic structures. The figure of Quetzalcóatl/Kukulkán illustrates a triadic balance of sky, earth, and underworld, while Andean Yanantin and Pachakuti reflect complementary duality and transformation. Shuar cosmology further exemplifies a complex n-alectic system, where multiple spiritual and natural forces interact dynamically. The ethical decision-making model, applied to resource extraction and Pachamama principles, utilizes weighted metrics and neutrosophic distance calculations to evaluate sustainable choices. Results highlight n-alectic reasoning as a powerful tool for balancing economic, environmental, and social factors, promoting harmonious integration of opposing forces in decision-making.

Keywords:
n-alectic, neutrosophic logic, indigenous philosophy, ethical decision-making, Andean cosmology, Mesoamerican worldview

1. Introduction

While the **di-alectic** is the dynamic of opposites T and F (or Truth and Falsehood), the neutrosophy is a **tri-alectic**, the dynamic of opposites (T and F) and the neutrality/indeterminacy (I) between them [1].

This broader perspective aligns with the philosophical traditions of Latin America, where pre-Columbian thought systems embraced complex relationships between dualities and intermediate states rather than rigid dichotomies. In The Dawn of Everything, David Graeber and David Wengrow [2] explore how pre-Columbian societies have shaped Western philosophy, challenging Eurocentric narratives and offering alternative conceptual frameworks that enrich human thought. A striking example of tri-alectic reasoning is found in the Toltec and Aztec worldviews, particularly in the figure of Quetzalcóatl (or Kukulkán in Maya culture) [3]. The feathered serpent deity who embodies the dynamic interplay of opposites (T and F) and the mediating force of indeterminacy (I), an essential feature of trialectic thinking.

Figure 1. The effect of the descent of the serpent observed at the Temple of Kukulcán during the March 2009 equinox [4].

This scheme illustrates a temporal trialectic, where the balance between opposing forces (light and shadow) is not merely dualistic but includes an intermediate state. In the equinox phenomenon of Kukulkán, this state is the penumbra, where light and shadow coexist and transform each other.

The figure of Quetzalcóatl/Kukulkan represents a dynamic interaction between three forces:

$$(T = Sky, I = Earth, F = \text{Underworld})$$

- Sky (T - The quetzal, the divine, and pure wisdom).
- Earth (I - The serpent Coatl, transformation, and the balance between the material and the spiritual).
- Underworld (F - Death, change, and renewal through sacrifice).

Quetzalcóatl embodies the force that maintains the necessary balance to prevent the chaos of the underworld (e.g., Mictlán) from overtaking the human world. This model illustrates how trialectics were inherently embedded in the Mesoamerican worldview, where opposites were not mere contrasts but interdependent forces engaged in a continuous process of transformation.

Figure 1. Quetzalcóatl: The Tri-Alectic Harmony of Earth and Sky at Teotihuacan
Subtitle: Feathers of the heavens and serpent of the earth converge in indeterminacy, symbolizing the intermediary balance between opposing forces on the Pyramid of Quetzalcóatl at Teotihuacan [5].

In the Andean Dialectic [1, 2], the opposites Yanantin and Pachakuti, are not only contradictory but also complementary, as in our real life [6,7].

The woman is the complementary of the man, but sometimes contradictory as well (arguing and fighting).

We may consider, for example, the **Quadruple Neutrosophic Logic**,
where (T, F) is refined into (T, I_1, I_2, F), as follows:
T = man
F = woman
I_1 = a type of indeterminacy, called complementary relationship between man and woman (they help each other);
I_2 = another type of indeterminacy, called contradictory relationship between man and woman (they are fighting one with the other).
Therefore, herein one has a **quadr-alectic**, dynamic of four elements.

The most general dynamic is the **n-alectic**, which is the dynamic of n (sub)components of the
Refined Neutrosophic Logic [8]:

The Neutrosophic Components (T, I, F) were refined into n Neutrosophic SubComponents

($T_1, T_2, ...T_p; I_1, I_2, ..., I_r; F_1, F_2, ..., F_s$),
where p, r, s, n are positive integers and p + r + s = n.

In Shuar or Jíbaro, an amazonian culture from Ecuador and Peru (Figure 3), cosmology is based on three primary realms the Upper World (T - The Celestial and Spiritual Realm), the Middle World (I - The Human and Social Realm), and the Lower World (F - The Realm of Chaos, Transformation, and Darkness)—are closely associated with specific deities and spiritual forces that embody their respective characteristics [9]. These deities can be mapped onto the n-alectic structure as described in the refined neutrosophic logic framework. The Shuar worldview extends beyond a simple duality (e.g., good vs. evil) or even a basic triadic

structure (Upper, Middle, and Lower Worlds); instead, it represents a complex n-alectic system where multiple subcomponents interact dynamically.

For instance [10-12]:

- Tsunki (T_1 - The Water Spirit and Guardian of Healing) and Shakaim (F_1 - The Force of Strength and Chaos) are not simply opposites; they interact dynamically through the role of the shaman (I_3 - The Mediating Force Between Worlds), who seeks balance.
- Nunkui (T3 - The Spirit of Fertility and the Earth's Provider) ensures fertility, but Nekás (F_3 - The Shadow Spirit, the Uncontrollable Wild Forces) represents destruction—both are necessary for ecological balance.
- The physical warrior force (Iwia, I_1 - The Physical World and Warrior Spirit) struggles between the celestial wisdom of Etsa (T_2 - The Sun Spirit, Provider of Life and Strength) and the chaotic transformation of Kakarma (F_2 - The Force of Change and Rebirth).
- The social and cultural world (I_2 - The Collective Spirit and Tradition) serves as the repository of knowledge, customs, and community rituals, which regulate how humans interact with both the spiritual and natural worlds. Through cultural transmission, oral traditions, and collective ceremonies, I_2 stabilizes the balance between I_1 (the physical world), I_3 (spiritual mediation), and the broader cosmological forces (T and F).

This n-alectic framework closely aligns with refined neutrosophic logic, as each deity, force, or spirit does not exist in isolation or as part of a rigid binary system but rather within a relational network characterized by complementarity, contradiction, and transformation.

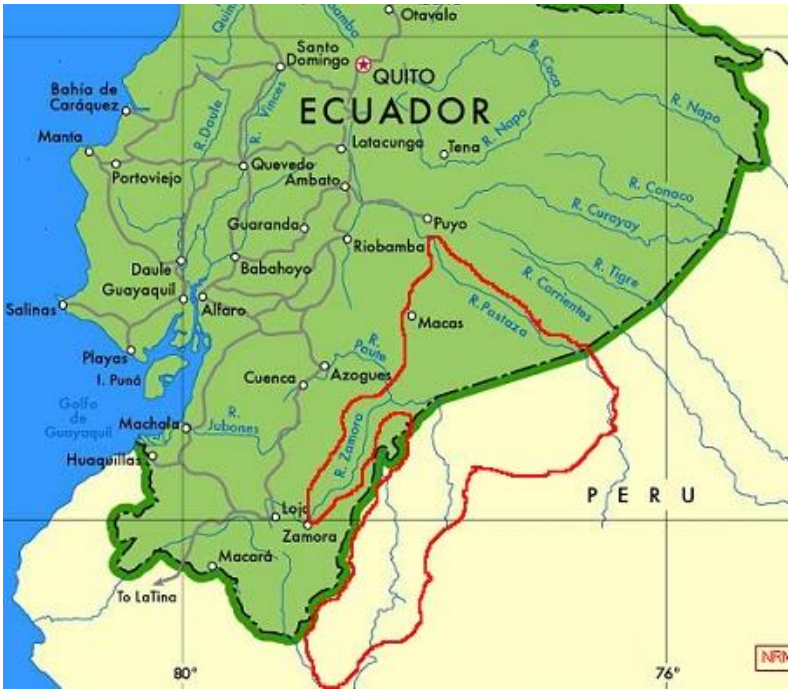

Figure 3. Jivaro Territory highlighted in red, between Ecuador and Peru [13]

The Neutrosophic Components (T, I, F) were refined into n Neutrosophic SubComponents [14]:

(T_1, T_2,T_3; I_1, I_2, I_3; F_1, F_2, F_3),

This n-alectic framework allows for a more dynamic and layered understanding of reality, where each subcomponent interacts with the others in complementary, contradictory, or indeterminate ways. In this way, the Shuar cosmology exemplifies an n-alectic structure, where the interplay of multiple subcomponents creates a rich and dynamic understanding of existence. This goes beyond simple dualities or triadic relationships, embracing a more complex and interconnected view of reality that aligns with the principles of refined neutrosophic logic.

2. Practical Application: Incorporation of Ethical Components of Pachamama in Decision-Making

Making an ethical decision within the framework of n-alectic and respect for Pachamama [15, 16] involves considering multiple interconnected dimensions that reflect both human and non-human aspects. The n-alectic, based on refined neutrosophic logic, allows for the integration of dynamic subcomponents (truth, indeterminacy, and falsity) to address complex problems where absolute truths are scarce, and perspectives are diverse. In this context, ethical decision-making must balance economic, social, environmental, and spiritual interests, especially in settings where indigenous worldviews recognize nature as a rights-bearing entity.

2.1. Define Neutrosophic Subcomponents

In the context of mining and respect for Pachamama, the neutrosophic subcomponents can be represented as follows:

- Truth (T): Economic benefits and social development derived from mining.

  Example: Job creation, income for local communities, investment in infrastructure.

- Indeterminacy (I): Ambiguous or uncertain aspects related to mining.
  I_T: Indeterminacy leaning toward truth (e.g., mining projects that promise benefits but have not yet demonstrated results).

  I: Pure indeterminacy (e.g., lack of comprehensive studies on environmental impact).

  I_F: Indeterminacy leaning toward falsity (e.g., potential risks to fragile ecosystems).

- Falseness (F): Negative impacts associated with mining.

  Example: Water pollution, deforestation, biodiversity loss, cultural and spiritual harm to indigenous communities.

These subcomponents can be further refined according to the specific needs of the case, following the n-alectic model**:**

$(\boldsymbol{T}, \boldsymbol{I_t}, \boldsymbol{I}, \boldsymbol{I_F}, \boldsymbol{F})$

These subcomponents can be further refined according to the specific needs of the case[17]

2.2. Assign Weights to Components

Weights are assigned to each subcomponent based on the desired emphasis. For example:

w_T = 0.33 (for Truth, T),

w_{IT} = 0.165 (for IT, indeterminacy leaning toward truth),

w_I = 0.165 (for I, pure indeterminacy),

w_{IF} = 0.165 (for IF, indeterminacy leaning toward falsity),

w_F = 0.175 (for Falseness, F).

The total sum of these weights is:

$$w_T + w_{IT} + w_I + w_{IF} + w_F = 0.33 + 0.165 + 0.165 + 0.165 + 0.175 = 1$$

This ensures that the weight distribution is balanced and that each subcomponent has the desired influence on the outcome.

The weights can vary depending on whether the goal is to minimize risks or maximize benefits.

The assignment of weights to components can be linked to the Andean principle of complementarity, which governs various aspects of life, including political, social, economic, and spiritual dimensions, by reflecting the coexistence and balance of opposing or different elements to achieve a greater purpose 1. For instance, "risks" and "benefits" can be viewed as complementary forces that need to be weighed according to context and priorities, mirroring the Andean approach of balancing opposing forces or interests. This concept aligns with the principle of relationality in Andean philosophy, which highlights the interconnectedness of all elements and the importance of considering how each factor, such as risk or benefit, impacts the entire system, ensuring decisions are made in harmony with Andean principles.

2.3. Determine the Ideal Option

Given a set of options, $X = \{x_1, x_2, \dots, x_n\}$, the ideal solution I is defined as [18]:

$$I = \left(max(T_x), max(IT_x), min(I_x), min(IF_x), min(F_x)\right) \quad (1)$$

Where:

- T_x: Truth associated with option x.
- IT_x: Indeterminacy leaning toward truth associated with option x.
- I_x: Pure Indeterminacy associated with option x.
- IF_x: Indeterminacy leaning toward falsity associated with option x.
- F_x: Minimum Falseness associated with option x.

This ideal serves as a reference point for evaluating alternatives.

The concept of the "ideal option," which involves maximizing truth (T), minimizing falseness (F), and managing indeterminacy (I_T, I, I_F), resonates with pre-Columbian philosophical principles, particularly those of the Shuar and other Andean and Amazonian cultures. These worldviews emphasize a dynamic balance between opposing yet complementary forces, such as the spiritual and earthly realms, reflecting an ongoing process of adjustment to achieve harmony. This principle of equilibrium aligns with the pursuit of

an ideal option that maximizes positive outcomes while minimizing negatives and addressing inherent uncertainties. The refined neutrosophic classification of indeterminacy into three types—I_T (leaning toward truth), I (pure indeterminacy), and I_F (leaning toward falsity)—parallels the Shuar cosmovision, where realities are not absolute or binary but complex and context-dependent. For instance, I_T may represent benevolent spiritual forces promoting well-being, I symbolize natural ambiguities like cycles of transformation, and I_F corresponds to chaotic or destructive forces that must be minimized.

This approach also aligns with the Andean principle of complementarity, where opposites coexist and integrate into a harmonious whole, as seen in the concept of Yanantin. Furthermore, the goal of minimizing falseness and maximizing truth in the neutrosophic framework reflects pre-Columbian ethical principles centered on respecting Pachamama (Mother Earth) and collective well-being. These cultures prioritized decisions that balanced human needs with ecological harmony, embodying a holistic ethic that values sustainability and respect for vital universal forces. Such an approach highlights the importance of integrating multiple dimensions to achieve optimal and balanced solutions 4

4. Calculate Neutrosophic Distance

The distance between two refined numbers can be calculated using a weighted metric, such as Euclidean or Hamming distance. The general formula for the weighted distance is [19,20]:

$$\boldsymbol{d_i^+ = \sum_{i=1}^{n}\left(w_T\left|T_{A(x_i)} - T_{B(x_i)}\right|^{\lambda} + w_{IT}\left|IT_{A(x_i)} - IT_{B(x_i)}\right|^{\lambda} + w_I\left|I_{A(x_i)} - I_{B(x_i)}\right|^{\lambda} + w_{IF}\left|IF_{A(x_i)} - IF_{B(x_i)}\right|^{\lambda} + w_F\left|F_{A(x_i)} - F_{B(x_i)}\right|^{\lambda}\right)} \quad \textbf{(2)}$$

Where:

λ>0 defines the type of distance (e.g., λ=1 for Hamming distance, λ=2 for Euclidean distance).

A smaller distance indicates greater similarity between the refined numbers. This approach is useful when identifying the option closest to an ideal or reference standard.

The chosen option will be the one closest to the ideal. The calculation of distance between refined numbers using weighted metrics, such as Euclidean or Hamming distance, can be associated with pre-Columbian philosophical principles, particularly the principle of balance and complementarity found in Andean and Amazonian worldviews. This principle emphasizes the dynamic interconnection between opposing or complementary forces to achieve harmony, which is reflected in the way neutrosophic subcomponents (truth, indeterminacy, and falsity) are weighted and compared to find an optimal equilibrium. In Andean philosophy, the principle of complementarity (Yanantin) posits that opposites coexist and balance each other to form a harmonious whole [21]. For instance, the subcomponents T (truth), IT (indeterminacy leaning toward truth), I (pure indeterminacy), IF (indeterminacy leaning toward falsity), and F (falsity) represent complementary forces. The assignment of weights mirrors how these forces are balanced according to context, akin to the adjustment of opposites in Andean thought to achieve a greater purpose.

This approach also aligns with the principle of relationality , central to indigenous worldviews, which underscores the interconnectedness of all elements and the need to consider relationships between components when making decisions 2. The distance calculation carefully evaluates how each subcomponent (T, IT, I, IF, F) impacts the entire system, ensuring coherence with a holistic framework. This reflects how pre-Columbian cultures prioritized decisions that integrated multiple dimensions—human, spiritual, and

environmental—to maintain ecological and social balance. Thus, the use of weighted metrics in neutrosophic logic resonates with traditional principles like complementarity and relationality, highlighting the importance of balancing opposing forces and considering systemic interconnections for sustainable and harmonious outcomes.

3 Illustrative Example: Ethical Decision-Making in Mining Using the N-Alectic Framework

A multinational mining company is proposing a large-scale mining project in a region inhabited by indigenous communities that adhere to Pachamama-based ethical principles. The decision-making process must integrate economic development, social impact, environmental sustainability, and cultural respect while acknowledging the uncertainties and contradictions inherent in mining activities.

Step 1: Defining Neutrosophic Subcomponents

Using n-alectic principles, we classify the potential impacts of mining as follows:

- Truth (T) – Positive Outcomes
- Indeterminacy (I) – Uncertainties and Ambiguities
    - IT (Indeterminacy leaning toward truth): The mining project promises economic benefits, but long-term success is uncertain.
    - I (Pure Indeterminacy): Lack of comprehensive environmental impact studies creates unknown risks.
    - IF (Indeterminacy leaning toward falsity): Unclear legal protections for indigenous land rights and potential policy changes.
- Falseness (F) – Negative Outcomes

Thus, the mining scenario can be structured as:

$$(T, IT, I, IF; F)$$

Step 2: Assigning Weights to Components

To ensure that ethical considerations align with Andean principles of complementarity and relationality, we assign the following weights:

- Economic and social benefits: w_T=0.30
- Indeterminate factors: w_{IT}=0.15 w_I==0.15 w_{IF}==0.15
- Environmental and cultural harms: w_F==0.25

$W = (0.30, 0.15, 0.15, 0.15, 0.25)$

These weights reflect an ethical balance where economic and social benefits (T) are important but do not outweigh cultural and environmental sustainability (F), consistent with Pachamama ethics.

Step 3: Identifying the Ideal Ethical Decision

The ideal decision should maximize economic and social truth (T), minimize falseness (F), and manage indeterminacies (IT, I, IF) responsibly.

Using the neutrosophic ideal solution formula:

we evaluate two decision alternatives:

Option A: Sustainable Mining with Indigenous Leadership

- **T:** High (T1=0.8)
- **IT:** Medium (IT=0.5)
- **I:** Low (I=0.3)
- **IF:** Low (IF=0.2)
- **F:** Very low (F=0.2)

$A = (0.8,0.50,0.3,0.2,0.2)$

Option B: Conventional Large-Scale Mining

- **T:** High (T=0.7)
- **IT:** High (IT=0.6)
- **I:** High (I=0.7)
- **IF:** High (IF=0.6)
- **F:** High (F1=0.7)

$B = (0.7, 0.6, 0.7, 0.6, 0.70)$

The Idel solution in calculated using Equation 1 (as follows:

$$I = \left(max(0.8, 0.7), max(0.5, 0.6), min(0.3, 0.7), min(0.2, 0.6), min(0.2, 0.7)\right)$$

$$I = (0.8, 0.6, 0.3, 0.2, 0.2)$$

Step 4: Calculating the Neutrosophic Distance

Using the weighted metric formula ($\boldsymbol{\lambda = 2}$), we compute the distance from the ideal ethical decision (I)

$d_A{}^+$= 0.0387

$d_B{}^+$= 0. 0.3369.

- Option A (Sustainable Mining with Indigenous Leadership) has a lower neutrosophic distance, making it the ethically preferable choice under Pachamama principles.

This result supports the idea that ethical decision-making should balance economic, environmental, and cultural sustainability, emphasizing harmony and complementarity over pure economic gains.

3. Conclusions

The n-alectic framework offers a sophisticated analytical tool that goes beyond binary opposition by incorporating multiple interdependent subcomponents. Whether in philosophical, cultural, or practical decision-making contexts, this model enhances our ability to navigate complexity and achieve harmonious integration of opposing forces—a principle deeply rooted in indigenous worldviews and refined neutrosophic logic.

Future research can further explore the practical applications of n-alectic models across various domains. In policymaking, the n-alectic framework can be applied to government and environmental policies, enhancing decision-making by integrating multiple dimensions of truth, indeterminacy, and falseness. In the realm of economic sustainability, these principles could contribute to the development of more holistic and adaptable sustainable development models. Additionally, in technology and AI ethics, refined neutrosophic logic may offer valuable insights for ethical decision-making in artificial intelligence, addressing the complexities of uncertainty and bias. From a cultural perspective, further research could examine how indigenous cosmologies align with n-alectic thought, providing deeper insights into their philosophical implications on a global scale. By continuing to refine and expand this approach, we can foster a more inclusive and comprehensive understanding of reality, bridging diverse worldviews and methodologies to promote a sustainable and ethical future.

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Neutrosophic normal vague set fitting to trigonometric concept via aggregation operators and its augmentation

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Abstract. Using a cotangent trigonometric neutrosophic normal vague set (CotNNVS), we begin this communication article with several novel techniques. Some trigonometric neutrosophic sets and imprecise neutrosophic sets are extended by the novel idea of CotNNVS. We will discuss the various aggregating processes that interpret Cotangent trigonometric neutrosophic normal vague weighted averaging (CotNNVWA), cotangent trigonometric neutrosophic normal vague weighted geometric (CotNNVWG), cotangent trigonometric generalized neutrosophic normal vague weighted averaging (CotGNNVWA), and cotangent trigonometric generalized neutrosophic normal vague weighted geometric (CotGNNVWG) are the new topics covered in this paper.

Keywords: CotNNVWA, CotNNVWG, CotGNNVWA, CotGNNVWG.

1. Introduction

In the majority of actual issues, uncertainty is present everywhere. Fuzzy set (FS) theory [1], intuitionistic fuzzy set (IFS) theory [2], Pythagorean fuzzy set (PFS) theory [3], and neutrosophic set (NSS) theory [4] are only a few of the uncertain theories that have been presented through to deal with the uncertainties. The membership value (MV) of an element in a FS is the degree to which each element of the universe belongs to the set, but with a grade or degree of belongingness that ranges from zero to one. Clustering algorithms are used in applications of FSs, such as fuzzy c-numbers and regression prediction for fuzzy time series [5] [6]. Afterwards, Atanassov introduced the idea of an IFS logic, which is categorized based on the

requirement that the total of its MV and non-membership value (NMV) be less than or equal to one [2]. When the total of the MV and NMVs exceeds one, we may occasionally run into difficulties making decisions. The novel idea of PFS logic, an extension of IFS, was proposed by Yager [3]. It is defined by the square sum of its MV and NMV, for which the value is less than or equal to one. [7–9] Akram et al. spoke about the several applications based on the PFS. We extend the logic for geometric aggregation operators under the concept of an interval-valued PFS to a group DM strategy, as explored by Rahman et al. [10]. An IVPFS with aggregation operator was examined by Peng et al. [11]. Some methods for MGCDM based on an induced IVPFS Einstein aggregation operator were suggested by Rahman et al. [12]. An application for PFS Einstein choquet integral operators was used by Khan [13].

A novel hypothesis called as NSS has just been proposed. The primary difference between FS and IFS are represented by the neutrality of mind, which is what the name "neutosophy" refers to. Florentin Smarandache introduced NSS [4]. Each claim is assessed to have a degree of truth, a degree of indeterminacy, and a degree of falsehood in this logic. NSS is a set that falls between $[0, 1]$ and contains degrees of truth, indeterminacy, and falsehood for every element in the universe. It has been demonstrated from a philosophical perspective that an NSS generalizes a classical set, an FS, an IVFS, etc. The Pythagorean neutrosophic IVS (PNSIVS) was first presented by Florentin Smarandache et al. [14]. The single-valued NSS is used based on context analysis [15] and medical diagnosis [16]. The distance measurements for IFSshamming, euclidean, normalized hamming, and normalized euclidean distancesas well as their similarities to PFSs were expanded by Ejegwa [17] and applied to both MCDM and MADM situations. Based on a review of the literature, we find that the majority of distance functions for PNSNIVSs are presented as PNSIVS generalizations. The concept of MADM for a novel class of NSS aggregation operators is discussed by Palanikumar et al. [18]. Applications of spherical ambiguous normal operators for farmer selection by Palanikumar et al. [19]. This study focuses on the novel idea of CotNNVS. The CotNNVS aggregation operators are where we begin. As stated, the paper is divided into five sections. 1 is referred to as an introduction. It is said that Section 2 refers to the fundamental ideas presented. In Section 3, fundamental algebraic operations are covered. In Section 4, the aggregation operations for CotNNVNs are discussed. The conclusion may be found in the last section 5.

2. **Basic concept**

In this section, we recall some of the basic definitions required for our additional studies.

Definition 2.1. [4] The NSS $\mathbb{k} = \left\{ u, \langle \aleph_{\mathbb{k}}^{\gtrdot}[\varkappa], \aleph_{\mathbb{k}}^{\beth}[\varkappa], \aleph_{\mathbb{k}}^{\daleth}[\varkappa] \rangle \middle| \varkappa \in \mathscr{U} \right\}$, where $\aleph_{\mathbb{k}}^{\gtrdot}, \aleph_{\mathbb{k}}^{\beth}, \aleph_{\mathbb{k}}^{\daleth} : \mathscr{U} \to [0, 1]$ is called the truth MV, indeterminacy MV and falsity MV of $\varkappa \in \mathscr{U}$ to $\mathbb{k}$, respectively and

$0 \preccurlyeq \aleph_{\Bbbk}^{\rangle}[\varkappa] + \aleph_{\Bbbk}^{\beth}[\varkappa] + \aleph_{\Bbbk}^{\lrcorner}[\varkappa] \preccurlyeq 3$. For comfortable, $\Bbbk = \langle \aleph_{\Bbbk}^{\rangle}, \aleph_{\Bbbk}^{\beth}, \aleph_{\Bbbk}^{\lrcorner} \rangle$ is said to be a neutrosophic number[NSN].

Definition 2.2. [11] Let $\Bbbk = \left\langle [\aleph^{\rangle -}, \aleph^{\rangle +}], [\aleph^{\lrcorner -}, \aleph^{\lrcorner +}] \right\rangle$, $\Bbbk_1 = \left\langle [\aleph_1^{\rangle -}, \aleph_1^{\rangle +}], [\aleph_1^{\lrcorner -}, \aleph_1^{\lrcorner +}] \right\rangle$ and $\Bbbk_2 = \left\langle [\aleph_2^{\rangle -}, \aleph_2^{\rangle +}], [\aleph_2^{\lrcorner -}, \aleph_2^{\lrcorner +}] \right\rangle$ be the PIVFNs, and $\varsigma > 0$. Then, the new basic operations are formed as below:

(1) $\Bbbk_1 \ddagger \Bbbk_2 = \left[\begin{matrix} \left[\sqrt{[\aleph_1^{\rangle -}]^2 + [\aleph_2^{\rangle -}]^2 - [\aleph_1^{\rangle -}]^2 \cdot [\aleph_2^{\rangle -}]^2}, \sqrt{[\aleph_1^{\rangle +}]^2 + [\aleph_2^{\rangle +}]^2 - [\aleph_1^{\rangle +}]^2 \cdot [\aleph_2^{\rangle +}]^2} \right], \\ \left[\aleph_1^{\lrcorner -} \cdot \aleph_2^{\lrcorner -}, \aleph_1^{\lrcorner +} \cdot \aleph_2^{\lrcorner +} \right] \end{matrix} \right]$,

(2) $\Bbbk_1 \amalg \Bbbk_2 = \left[\begin{matrix} \left[\aleph_1^{\rangle -} \cdot \aleph_2^{\rangle -}, \aleph_1^{\rangle +} \cdot \aleph_2^{\rangle +} \right], \\ \left[\sqrt{[\aleph_1^{\lrcorner -}]^2 + [\aleph_2^{\lrcorner -}]^2 - [\aleph_1^{\lrcorner -}]^2 \cdot [\aleph_2^{\lrcorner -}]^2}, \sqrt{[\aleph_1^{\lrcorner +}]^2 + [\aleph_2^{\lrcorner +}]^2 - [\aleph_1^{\lrcorner +}]^2 \cdot [\aleph_2^{\lrcorner +}]^2} \right] \end{matrix} \right]$,

(3) $\varsigma \cdot \Bbbk = \left[\left[\sqrt{1 - \left[1 - [\aleph^{\rangle -}]^2\right]^{\varsigma}}, \sqrt{1 - \left[1 - [\aleph^{\rangle +}]^2\right]^{\varsigma}} \right], \left[[\aleph^{\lrcorner -}]^{\varsigma}, [\aleph^{\lrcorner +}]^{\varsigma} \right] \right]$,

(4) $\Bbbk^{\varsigma} = \left[\left[[\aleph^{\rangle -}]^{\varsigma}, [\aleph^{\rangle +}]^{\varsigma} \right], \left[\sqrt{1 - \left[1 - [\aleph^{\lrcorner -}]^2\right]^{\varsigma}}, \sqrt{1 - \left[1 - [\aleph^{\lrcorner +}]^2\right]^{\varsigma}} \right] \right]$.

3. Basic Operation for CotNNVN

The novel concept of CotNNVN and its operations, where $\cot \pi/4 = \mho$, were constructed.

Definition 3.1. Let $[\wp, \ell] \in N$, $\Bbbk = \left\langle [\wp, \ell]; [\aleph^{\rangle -}, \aleph^{[1-\beth]+}], [\aleph^{\lrcorner -}, \aleph^{\lrcorner +}], [\aleph^{\lrcorner -}, \aleph^{[1-\rangle]+}] \right\rangle$ be the NSNVN. Then $\cot \Bbbk = \left\{ \left[\left[\mho \cdot [\aleph_{\Bbbk}^{\rangle -}[\varkappa]] \right], \left[\mho \cdot [\aleph_{\Bbbk}^{[1-\beth]+}[\varkappa]] \right] \right], \left[1 - \left[\mho \cdot 1 - [\aleph_{\Bbbk}^{\lrcorner -}[\varkappa]] \right], 1 - \left[\mho \cdot 1 - [\aleph_{\Bbbk}^{\lrcorner +}[\varkappa]] \right] \right], \left[1 - \left[\mho \cdot [1 - \aleph_{\Bbbk}^{\lrcorner -}[\varkappa]] \right], 1 - \left[\mho \cdot [1 - \aleph_{\Bbbk}^{[1-\rangle]+}[\varkappa]] \right] \right] \right\}$.
Thus, $\cot \Bbbk$ is a CotNNVN and
$\left[\mho \cdot \aleph_{\Bbbk}^{[1-\beth]+}[\varkappa] \right] \in [0,1]$, $\left[\mho \cdot \aleph_{\Bbbk}^{\lrcorner +}[\varkappa] \right] \in [0,1]$ and $1 - \left[\mho \cdot \left[1 - \aleph_{\Bbbk}^{[1-\rangle]+}[\varkappa] \right] \right] \in [0,1]$. Hence, $\cot \Bbbk = \left\{ \left[\left[\mho \cdot [\aleph_{\Bbbk}^{\rangle -}[\varkappa]] \right], \left[\mho \cdot [\aleph_{\Bbbk}^{[1-\beth]+}[\varkappa]] \right] \right], 1 - \left[\left[\mho \cdot [1 - [\aleph_{\Bbbk}^{\lrcorner -}[\varkappa]]] \right], 1 - \left[\mho \cdot [1 - [\aleph_{\Bbbk}^{\lrcorner +}[\varkappa]]] \right] \right], \left[1 - \left[\mho \cdot [[1 - \aleph_{\Bbbk}^{\lrcorner -}[\varkappa]]] \right], 1 - \left[\mho \cdot [[1 - \aleph_{\Bbbk}^{[1-\rangle]+}[\varkappa]]] \right] \right] \right\}$
is a CotNNVN, put $\left[\aleph_{\Bbbk}^{\rangle -}, \aleph_{\Bbbk}^{[1-\beth]+} \right] = \left[\aleph_{\Bbbk}^{\rangle -} e^{-\left[\frac{y-\wp}{\ell}\right]^2}, \aleph_{\Bbbk}^{[1-\beth]+} e^{-\left[\frac{y-\wp}{\ell}\right]^2} \right]$ and $\left[\aleph_{\Bbbk}^{\lrcorner -}, \aleph_{\Bbbk}^{\lrcorner +} \right] = \left[\aleph_{\Bbbk}^{\lrcorner -} e^{-\left[\frac{y-\wp}{\ell}\right]^2}, \aleph_{\Bbbk}^{\lrcorner +} e^{-\left[\frac{y-\wp}{\ell}\right]^2} \right]$ and $\left[\aleph_{\Bbbk}^{\lrcorner -}, \aleph_{\Bbbk}^{[1-\rangle]+} \right] = \left[\aleph_{\Bbbk}^{\lrcorner -} e^{-\left[\frac{y-\wp}{\ell}\right]^2}, \aleph_{\Bbbk}^{[1-\rangle]+} e^{-\left[\frac{y-\wp}{\ell}\right]^2} \right]$, $y \in Y$, Here Y is a non-empty set.

Definition 3.2. Let $\Bbbk = \left\langle [\wp, \ell]; [\aleph^{\rangle -}, \aleph^{[1-\beth]+}], [\aleph^{\lrcorner -}, \aleph^{\lrcorner +}], [\aleph^{\lrcorner -}, \aleph^{[1-\rangle]+}] \right\rangle$ be the CotNNVN. Then the score function of $\Bbbk$ is defined by $S[\Bbbk] = \frac{\wp}{2} \left[\frac{2 + [[\mho^2 \cdot \aleph^{\rangle -}]] + [[\mho^2 \cdot \aleph^{[1-\beth]+}]] - [[\mho^2 \cdot \aleph^{\lrcorner -}]] - [[\mho^2 \cdot \aleph^{\lrcorner +}]] - [[\mho^2 \cdot \aleph^{\lrcorner -}]] - [[\mho^2 \cdot \aleph^{[1-\rangle]+}]]}{2} \right]$, where $S[\Bbbk]$ lies between -1 and 1.

Definition

3.3. Let $\Bbbk = \left\langle [\wp, \ell]; [\aleph^{\rangle -}, \aleph^{[1-\beth]+}], [\aleph^{\lrcorner -}, \aleph^{\lrcorner +}], [\aleph^{\lrcorner -}, \aleph^{[1-\rangle]+}] \right\rangle$, $\Bbbk_1 = \left\langle [\wp_1, \ell_1]; [\aleph_1^{\rangle -}, \aleph_1^{[1-\beth]+}], [\aleph_1^{\lrcorner -}, \aleph_1^{\lrcorner +}], [\aleph_1^{\lrcorner -}, \aleph_1^{[1-\rangle]+}] \right\rangle$ and $\Bbbk_2 = \left\langle [\wp_2, \ell_2]; [\aleph_2^{\rangle -}, \aleph_2^{[1-\beth]+}], [\aleph_2^{\lrcorner -}, \aleph_2^{\lrcorner +}], [\aleph_2^{\lrcorner -}, \aleph_2^{[1-\rangle]+}] \right\rangle$ be the CotNNVNs and $\varsigma > 0$.

(1) $\cot \Bbbk_1 \ddagger \cot \Bbbk_2 =$

$$\left[\begin{array}{c} [\wp_1 \ddagger \wp_2, \ell_1 \ddagger \ell_2]; \\ \left[\begin{array}{c} [[\mho^2 \cdot \aleph_1^{\gimel-}]]^\varsigma + [[\mho^2 \cdot \aleph_2^{\gimel-}]]^\varsigma - [[\mho^2 \cdot \aleph_1^{\gimel-}]]^\varsigma \cdot [[\mho^2 \cdot \aleph_2^{\gimel-}]]^\varsigma, \\ [[\mho^2 \cdot \aleph_1^{[1-\beth]+}]]^\varsigma + [[\mho^2 \cdot \aleph_2^{[1-\beth]+}]]^\varsigma - [[\mho^2 \cdot \aleph_1^{[1-\beth]+}]]^\varsigma \cdot [[\mho^2 \cdot \aleph_2^{[1-\beth]+}]]^\varsigma \end{array}\right], \\ \left[[\mho^2 \cdot \aleph_1^{\daleth-}] \cdot [\mho^2 \cdot \aleph_2^{\daleth-}], [\mho^2 \cdot \aleph_1^{\daleth+}] \cdot [\mho^2 \cdot \aleph_2^{\daleth+}]\right], \\ \left[[\mho^2 \cdot \aleph_1^{\beth-}] \cdot [\mho^2 \cdot \aleph_2^{\beth-}], [\mho^2 \cdot \aleph_1^{[1-\gimel]+}] \cdot [\mho^2 \cdot \aleph_2^{[1-\gimel]+}]\right] \end{array}\right].$$

(2) $\cot \Bbbk_1 \amalg \cot \Bbbk_2 =$

$$\left[\begin{array}{c} [\wp_1 \amalg \wp_2, \ell_1 \amalg \ell_2]; \\ \left[[\mho^2 \cdot \aleph_1^{\gimel-}] \cdot [\mho^2 \cdot \aleph_2^{\gimel-}], [\mho^2 \cdot \aleph_1^{[1-\beth]+}] \cdot [\mho^2 \cdot \aleph_2^{[1-\beth]+}]\right], \\ \left[\begin{array}{c} [[\mho^2 \cdot \aleph_1^{\daleth-}]]^\varsigma + [[\mho^2 \cdot \aleph_2^{\daleth-}]]^\varsigma - [[\mho^2 \cdot \aleph_1^{\daleth-}]]^\varsigma \cdot [[\mho^2 \cdot \aleph_2^{\daleth-}]]^\varsigma, \\ [[\mho^2 \cdot \aleph_1^{\daleth+}]]^\varsigma + [[\mho^2 \cdot \aleph_2^{\daleth+}]]^\varsigma - [[\mho^2 \cdot \aleph_1^{\daleth+}]]^\varsigma \cdot [[\mho^2 \cdot \aleph_2^{\daleth+}]]^\varsigma \end{array}\right], \\ \left[\begin{array}{c} [[\mho^2 \cdot \aleph_1^{\beth-}]]^\varsigma + [[\mho^2 \cdot \aleph_2^{\beth-}]]^\varsigma - [[\mho^2 \cdot \aleph_1^{\beth-}]]^\varsigma \cdot [[\mho^2 \cdot \aleph_2^{\beth-}]]^\varsigma, \\ [[\mho^2 \cdot \aleph_1^{[1-\gimel]+}]]^\varsigma + [[\mho^2 \cdot \aleph_2^{[1-\gimel]+}]]^\varsigma - [[\mho^2 \cdot \aleph_1^{[1-\gimel]+}]]^\varsigma \cdot [[\mho^2 \cdot \aleph_2^{[1-\gimel]+}]]^\varsigma \end{array}\right] \end{array}\right].$$

(3) $\varsigma \cdot \cot \Bbbk = \left[\begin{array}{c} [\varsigma \cdot \wp, \varsigma \cdot \ell]; \\ \left[1 - \left[1 - [[\mho^2 \cdot \aleph^{\gimel-}]]^\varsigma\right]^\varsigma, 1 - \left[1 - [[\mho^2 \cdot \aleph^{[1-\beth]+}]]^\varsigma\right]^\varsigma\right], \\ \left[[[\mho^2 \cdot \aleph^{\daleth-}]]^\varsigma, [[\mho^2 \cdot \aleph^{\daleth+}]]^\varsigma\right], \\ \left[[[\mho^2 \cdot \aleph^{\beth-}]]^\varsigma, [[\mho^2 \cdot \aleph^{[1-\gimel]+}]]^\varsigma\right] \end{array}\right].$

(4) $[\cot \Bbbk]^\varsigma = \left[\begin{array}{c} [\wp^\varsigma, \ell^\varsigma]; \left[[[\mho^2 \cdot \aleph^{\gimel-}]]^\varsigma, [[\mho^2 \cdot \aleph^{[1-\beth]+}]]^\varsigma\right], \\ \left[1 - \left[1 - [[\mho^2 \cdot \aleph^{\daleth-}]]^\varsigma\right]^\varsigma, 1 - \left[1 - [[\mho^2 \cdot \aleph^{\daleth+}]]^\varsigma\right]^\varsigma\right], \\ \left[1 - \left[1 - [[\mho^2 \cdot \aleph^{\beth-}]]^\varsigma\right]^\varsigma, 1 - \left[1 - [[\mho^2 \cdot \aleph^{[1-\gimel]+}]]^\varsigma\right]^\varsigma\right] \end{array}\right].$

4. **Aggregation operators for CotNNVNs**

The novel operators based on CotNNVWA, CotNNVWG, CotGNNVWA and CotGNNVWG are introduced in this section.

4.1. *CotNNV weighted averaging[CotNNVWA]*

Definition 4.1. Let $\Bbbk_i = \left\langle [\wp_i, \ell_i]; [\aleph_i^{\gimel-}, \aleph_i^{[1-\beth]+}], [\aleph_i^{\daleth-}, \aleph_i^{\daleth+}], [\aleph_i^{\beth-}, \aleph_i^{[1-\gimel]+}] \right\rangle$ be a finite collection of CotNNVNs, $H = [\eth_1, \eth_2, ..., \eth_n]$ be the weight of $\Bbbk_i$, $\eth_i \succcurlyeq 0$ and $\ddagger_{i=1}^n \eth_i = 1$. Then CotNNVWA $[\Bbbk_1, \Bbbk_2, ..., \Bbbk_n] = \ddagger_{i=1}^n \eth_i \Bbbk_i$.

Theorem 4.2. *Let* $\Bbbk_i = \left\langle [\wp_i, \ell_i]; [\aleph_i^{\gimel-}, \aleph_i^{[1-\beth]+}], [\aleph_i^{\daleth-}, \aleph_i^{\daleth+}], [\aleph_i^{\beth-}, \aleph_i^{[1-\gimel]+}] \right\rangle$ *be a finite collection of CotNNVNs. Then prove that* $CotNNVWA[\Bbbk_1, \Bbbk_2, ..., \Bbbk_n] =$

$$\left[\begin{array}{c} \left[\ddagger_{i=1}^n \eth_i \wp_i, \ddagger_{i=1}^n \eth_i \ell_i\right]; \\ \left[1 - \amalg_{i=1}^n \left[1 - [[\mho^2 \cdot \aleph_i^{\gimel-}]]^\varsigma\right]^{\eth_i}, 1 - \amalg_{i=1}^n \left[1 - [[\mho^2 \cdot \aleph_i^{[1-\beth]+}]]^\varsigma\right]^{\eth_i}\right], \\ \left[\amalg_{i=1}^n [[\mho^2 \cdot \aleph_i^{\daleth-}]]^{\eth_i}, \amalg_{i=1}^n [[\mho^2 \cdot \aleph_i^{\daleth+}]]^{\eth_i}\right], \\ \left[\amalg_{i=1}^n [[\mho^2 \cdot \aleph_i^{\beth-}]]^{\eth_i}, \amalg_{i=1}^n [[\mho^2 \cdot \aleph_i^{[1-\gimel]+}]]^{\eth_i}\right] \end{array}\right].$$

Proof. If $n=2$, then $\mathrm{CotNNVWA}[\mathbb{k}_1,\mathbb{k}_2]=\eth_1\cot\mathbb{k}_1\ddagger\eth_2\cot\mathbb{k}_2$, put

$$\eth_1\cot\mathbb{k}_1=\begin{bmatrix}\Big[\eth_1\wp_1,\eth_1\ell_1\Big];\\ \Big[1-\Big[1-[[\mho^2\cdot\aleph_1^{\gtrdot-}]]^{\varsigma}\Big]^{\eth_1},1-\Big[1-[[\mho^2\cdot\aleph_1^{[1-\lrcorner]+}]]^{\varsigma}\Big]^{\eth_1}\Big],\\ \Big[[[\mho^2\cdot\aleph_1^{\lrcorner-}]]^{\eth_1},[[\mho^2\cdot\aleph_1^{\lrcorner+}]]^{\eth_1}\Big],\\ \Big[[[\mho^2\cdot\aleph_1^{\lrcorner-}]]^{\eth_1},[[\mho^2\cdot\aleph_1^{[1-\gtrdot]+}]]^{\eth_1}\Big]\end{bmatrix}$$

and

$$\eth_2\cot\mathbb{k}_2=\begin{bmatrix}\Big[\eth_2\wp_2,\eth_2\ell_2\Big];\\ \Big[1-\Big[1-[[\mho^2\cdot\aleph_2^{\gtrdot-}]]^{\varsigma}\Big]^{\eth_2},1-\Big[1-[[\mho^2\cdot\aleph_2^{[1-\lrcorner]+}]]^{\varsigma}\Big]^{\eth_2}\Big],\\ \Big[[[\mho^2\cdot\aleph_2^{\lrcorner-}]]^{\eth_2},[[\mho^2\cdot\aleph_2^{\lrcorner+}]]^{\eth_2}\Big],\\ \Big[[[\mho^2\cdot\aleph_2^{\lrcorner-}]]^{\eth_2},[[\mho^2\cdot\aleph_2^{[1-\gtrdot]+}]]^{\eth_2}\Big]\end{bmatrix}.$$

Now, $\eth_1\cot\mathbb{k}_1\ddagger\eth_2\cot\mathbb{k}_2$

$$=\begin{bmatrix}\Big[\eth_1\wp_1\ddagger\eth_2\wp_2,\eth_1\ell_1\ddagger\eth_2\ell_2\Big];\\ \begin{bmatrix}\Big[1-\Big[1-[[\mho^2\cdot\aleph_1^{\gtrdot-}]]^{\varsigma}\Big]^{\eth_1}\Big]+\Big[1-\Big[1-[[\mho^2\cdot\aleph_2^{\gtrdot-}]]^{\varsigma}\Big]^{\eth_2}\Big]\\ -\Big[1-\Big[1-[[\mho^2\cdot\aleph_1^{\gtrdot-}]]^{\varsigma}\Big]^{\eth_1}\Big]\cdot\Big[1-\Big[1-[[\mho^2\cdot\aleph_2^{\gtrdot-}]]^{\varsigma}\Big]^{\eth_2}\Big],\\ \Big[1-\Big[1-[[\mho^2\cdot\aleph_1^{[1-\lrcorner]+}]]^{\varsigma}\Big]^{\eth_1}\Big]+\Big[1-\Big[1-[[\mho^2\cdot\aleph_2^{[1-\lrcorner]+}]]^{\varsigma}\Big]^{\eth_2}\Big]\\ -\Big[1-\Big[1-[[\mho^2\cdot\aleph_1^{[1-\lrcorner]+}]]^{\varsigma}\Big]^{\eth_1}\Big]\cdot\Big[1-\Big[1-[[\mho^2\cdot\aleph_2^{[1-\lrcorner]+}]]^{\varsigma}\Big]^{\eth_2}\Big]\end{bmatrix},\\ \Big[[[\mho^2\cdot\aleph_1^{\lrcorner-}]]^{\eth_1}\cdot[[\mho^2\cdot\aleph_2^{\lrcorner-}]]^{\eth_2},[[\mho^2\cdot\aleph_1^{\lrcorner+}]]^{\eth_1}[[\mho^2\cdot\aleph_2^{\lrcorner+}]]^{\eth_2}\Big],\\ \Big[[[\mho^2\cdot\aleph_1^{\lrcorner-}]]^{\eth_1}\cdot[[\mho^2\cdot\aleph_2^{\lrcorner-}]]^{\eth_2},[[\mho^2\cdot\aleph_1^{[1-\gtrdot]+}]]^{\eth_1}[[\mho^2\cdot\aleph_2^{[1-\gtrdot]+}]]^{\eth_2}\Big]\end{bmatrix},$$

$$=\begin{bmatrix}\Big[\eth_1\wp_1\ddagger\eth_2\wp_2,\eth_1\ell_1\ddagger\eth_2\ell_2\Big];\\ \begin{bmatrix}1-\Big[1-[[\mho^2\cdot\aleph_1^{\gtrdot-}]]^{\varsigma}\Big]^{\eth_1}\cdot\Big[1-[[\mho^2\cdot\aleph_2^{\gtrdot-}]]^{\varsigma}\Big]^{\eth_2},\\ 1-\Big[1-[[\mho^2\cdot\aleph_1^{[1-\lrcorner]+}]]^{\varsigma}\Big]^{\eth_1}\cdot\Big[1-[[\mho^2\cdot\aleph_2^{[1-\lrcorner]+}]]^{\varsigma}\Big]^{\eth_2}\end{bmatrix},\\ \Big[[[\mho^2\cdot\aleph_1^{\lrcorner-}]]^{\eth_1}\cdot[[\mho^2\cdot\aleph_2^{\lrcorner-}]]^{\eth_2},[[\mho^2\cdot\aleph_1^{\lrcorner+}]]^{\eth_1}\cdot[[\mho^2\cdot\aleph_2^{\lrcorner+}]]^{\eth_2}\Big],\\ \Big[[[\mho^2\cdot\aleph_1^{\lrcorner-}]]^{\eth_1}\cdot[[\mho^2\cdot\aleph_2^{\lrcorner-}]]^{\eth_2},[[\mho^2\cdot\aleph_1^{[1-\gtrdot]+}]]^{\eth_1}\cdot[[\mho^2\cdot\aleph_2^{[1-\gtrdot]+}]]^{\eth_2}\Big]\end{bmatrix}.$$

Hence $CotNNVWA[\mathbb{k}_1,\mathbb{k}_2]$

$$=\begin{bmatrix}\Big[\ddagger_{i=1}^{2}\eth_i\wp_i,\ddagger_{i=1}^{2}\eth_i\ell_i\Big];\\ \Big[1-\amalg_{i=1}^{2}\Big[1-[[\mho^2\cdot\aleph_i^{\gtrdot-}]]^{\varsigma}\Big]^{\eth_i},1-\amalg_{i=1}^{2}\Big[1-[[\mho^2\cdot\aleph_i^{[1-\lrcorner]+}]]^{\varsigma}\Big]^{\eth_i}\Big],\\ \Big[\amalg_{i=1}^{2}[[\mho^2\cdot\aleph_i^{\lrcorner-}]]^{\eth_i},\amalg_{i=1}^{2}[[\mho^2\cdot\aleph_i^{\lrcorner+}]]^{\eth_i}\Big],\\ \Big[\amalg_{i=1}^{2}[[\mho^2\cdot\aleph_i^{\lrcorner-}]]^{\eth_i},\amalg_{i=1}^{2}[[\mho^2\cdot\aleph_i^{[1-\gtrdot]+}]]^{\eth_i}\Big]\end{bmatrix}.$$

Similarly, $CotNNVWA[\mathbb{k}_1, \mathbb{k}_2, ..., \mathbb{k}_n] =$

$$\begin{bmatrix} \left[\ddagger_{i=1}^{n} \eth_i \wp_i, \ddagger_{i=1}^{n} \eth_i \ell_i \right]; \\ \left[1 - \amalg_{i=1}^{n} \left[1 - [[\mho^2 \cdot \aleph_i^{\gtrdot -}]]^{\varsigma} \right]^{\eth_i}, 1 - \amalg_{i=1}^{n} \left[1 - [[\mho^2 \cdot \aleph_i^{[1-\lrcorner]+}]]^{\varsigma} \right]^{\eth_i} \right], \\ \left[\amalg_{i=1}^{n} [[\mho^2 \cdot \aleph_i^{\urcorner -}]]^{\eth_i}, \amalg_{i=1}^{n} [[\mho^2 \cdot \aleph_i^{\urcorner +}]]^{\eth_i} \right], \\ \left[\amalg_{i=1}^{n} [[\mho^2 \cdot \aleph_i^{\lrcorner -}]]^{\eth_i}, \amalg_{i=1}^{n} [[\mho^2 \cdot \aleph_i^{[1-\gtrdot]+}]]^{\eth_i} \right] \end{bmatrix}.$$

If $n = l + 1$, then CotNNVWA $[\mathbb{k}_1, \mathbb{k}_2, ..., \mathbb{k}_n, \mathbb{k}_{n+1}]$

$$= \begin{bmatrix} \left[\ddagger_{i=1}^{n} \eth_i \wp_i \ddagger \eth_{n+1} \wp_{n+1}, \ddagger_{i=1}^{n} \eth_i \ell_i \ddagger \eth_{n+1} \ell_{n+1} \right]; \\ \begin{bmatrix} \ddagger_{i=1}^{n} \left[1 - \left[1 - [[\mho^2 \cdot \aleph_i^{\gtrdot -}]]^{\varsigma} \right]^{\eth_i} \right] + \left[1 - \left[1 - [[\mho^2 \cdot \aleph_{n+1}^{\gtrdot -}]]^{\varsigma} \right]^{\eth_{n+1}} \right] \\ - \amalg_{i=1}^{n} \left[1 - \left[1 - [[\mho^2 \cdot \aleph_i^{\gtrdot -}]]^{\varsigma} \right]^{\eth_i} \right] \cdot \left[1 - \left[1 - [[\mho^2 \cdot \aleph_{n+1}^{\gtrdot -}]]^{\varsigma} \right]^{\eth_{n+1}} \right], \\ \ddagger_{i=1}^{n} \left[1 - \left[1 - [[\mho^2 \cdot \aleph_i^{[1-\lrcorner]+}]]^{\varsigma} \right]^{\eth_i} \right] + \left[1 - \left[1 - [[\mho^2 \cdot \aleph_{n+1}^{[1-\lrcorner]+}]]^{\varsigma} \right]^{\eth_{n+1}} \right] \\ - \amalg_{i=1}^{n} \left[1 - \left[1 - [[\mho^2 \cdot \aleph_i^{[1-\lrcorner]+}]]^{\varsigma} \right]^{\eth_i} \right] \cdot \left[1 - \left[1 - [[\mho^2 \cdot \aleph_{n+1}^{[1-\lrcorner]+}]]^{\varsigma} \right]^{\eth_{n+1}} \right] \end{bmatrix}, \\ \left[\amalg_{i=1}^{n} [[\mho^2 \cdot \aleph_i^{\urcorner -}]]^{\eth_i} \cdot [[\mho^2 \cdot \aleph_{n+1}^{\urcorner -}]]^{\eth_{n+1}}, \amalg_{i=1}^{n} [[\mho^2 \cdot \aleph_i^{\urcorner +}]]^{\eth_i} \cdot [[\mho^2 \cdot \aleph_{n+1}^{\urcorner +}]]^{\eth_{n+1}} \right], \\ \left[\amalg_{i=1}^{n} [[\mho^2 \cdot \aleph_i^{\lrcorner -}]]^{\eth_i} \cdot [[\mho^2 \cdot \aleph_{n+1}^{\lrcorner -}]]^{\eth_{n+1}}, \amalg_{i=1}^{n} [[\mho^2 \cdot \aleph_i^{[1-\gtrdot]+}]]^{\eth_i} \cdot [[\mho^2 \cdot \aleph_{n+1}^{[1-\gtrdot]+}]]^{\eth_{n+1}} \right] \end{bmatrix}$$

$$= \begin{bmatrix} \left[\ddagger_{i=1}^{n+1} \eth_i \wp_i, \ddagger_{i=1}^{n+1} \eth_i \ell_i \right]; \\ \left[1 - \amalg_{i=1}^{n+1} \left[1 - [[\mho^2 \cdot \aleph_i^{\gtrdot -}]]^{\varsigma} \right]^{\eth_i}, 1 - \amalg_{i=1}^{n+1} \left[1 - [[\mho^2 \cdot \aleph_i^{[1-\lrcorner]+}]]^{\varsigma} \right]^{\eth_i} \right], \\ \left[\amalg_{i=1}^{n+1} [[\mho^2 \cdot \aleph_i^{\urcorner -}]]^{\eth_i}, \amalg_{i=1}^{n+1} [[\mho^2 \cdot \aleph_i^{\urcorner +}]]^{\eth_i} \right], \\ \left[\amalg_{i=1}^{n+1} [[\mho^2 \cdot \aleph_i^{\lrcorner -}]]^{\eth_i}, \amalg_{i=1}^{n+1} [[\mho^2 \cdot \aleph_i^{[1-\gtrdot]+}]]^{\eth_i} \right] \end{bmatrix}.$$

Theorem 4.3. *If all* $\mathbb{k}_i = \left\langle [\wp_i, \ell_i]; \left[[[\mho \cdot \aleph_i^{\gtrdot -}]], [[\mho \cdot \aleph_i^{[1-\lrcorner]+}]] \right], \left[[[\mho \cdot \aleph_i^{\urcorner -}]], [[\mho \cdot \aleph_i^{\urcorner +}]] \right], \left[[[\mho \cdot \aleph_i^{\lrcorner -}]], [[\mho \cdot \aleph_i^{[1-\gtrdot]+}]] \right] \right\rangle$ *are equal and* $\mathbb{k}_i = \mathbb{k}$ *with* $\varsigma = 1$. *Prove that* $CotNNVWA[\mathbb{k}_1, \mathbb{k}_2, ..., \mathbb{k}_n] = \mathbb{k}$.

Proof. Since, $[\wp_i, \ell_i] = [\wp, \ell]$, $[[\mho \cdot \aleph_i^{\gtrdot -}], [\mho \cdot \aleph_i^{[1-\lrcorner]+}]] = [[\mho \cdot \aleph^{\gtrdot -}], [\mho \cdot \aleph^{[1-\lrcorner]+}]]$, $[[\mho \cdot \aleph_i^{\urcorner -}], [\mho \cdot \aleph_i^{\urcorner +}]] = [[\mho \cdot \aleph^{\urcorner -}], [\mho \cdot \aleph^{\urcorner +}]]$ and $[[\mho \cdot \aleph_i^{\lrcorner -}], [\mho \cdot \aleph_i^{[1-\gtrdot]+}]] = [[\mho \cdot \aleph^{\lrcorner -}], [\mho \cdot \aleph^{[1-\gtrdot]+}]]$, for $i = 1, 2, ..., n$ and $\ddagger_{i=1}^{n} \eth_i = 1$. Now, $CotNNVWA[\mathbb{k}_1, \mathbb{k}_2, ..., \mathbb{k}_n]$

$$= \begin{bmatrix} \left[\ddagger_{i=1}^{n} \eth_i \wp_i, \ddagger_{i=1}^{n} \eth_i \ell_i \right]; \\ \left[1 - \amalg_{i=1}^{n} \left[1 - [[\mho \cdot \aleph_i^{\gtrdot -}]]^{\varsigma} \right]^{\eth_i}, 1 - \amalg_{i=1}^{n} \left[1 - [[\mho \cdot \aleph_i^{[1-\lrcorner]+}]]^{\varsigma} \right]^{\eth_i} \right], \\ \left[\amalg_{i=1}^{n} [[\mho \cdot \aleph_i^{\urcorner -}]]^{\eth_i}, \amalg_{i=1}^{n} [[\mho \cdot \aleph_i^{\urcorner +}]]^{\eth_i} \right], \\ \left[\amalg_{i=1}^{n} [[\mho \cdot \aleph_i^{\lrcorner -}]]^{\eth_i}, \amalg_{i=1}^{n} [[\mho \cdot \aleph_i^{[1-\gtrdot]+}]]^{\eth_i} \right] \end{bmatrix}$$

$$
\begin{aligned}
&= \begin{bmatrix} \Big[\wp \ddagger_{i=1}^{n} \eth_i, \ell \ddagger_{i=1}^{n} \eth_i\Big]; \\ \Big[1-\Big[1-[[\mho\cdot\aleph^{\gtrdot-}]]^{\varsigma}\Big]^{\ddagger_{i=1}^{n}\eth_i}, 1\Big[1-[[\mho\cdot\aleph^{[1-\lrcorner]+}]]^{\varsigma}\Big]^{\ddagger_{i=1}^{n}\eth_i}\Big], \\ \Big[[[\mho\cdot\aleph^{\lrcorner-}]]^{\ddagger_{i=1}^{n}\eth_i}, [[\mho\cdot\aleph^{\lrcorner+}]]^{\ddagger_{i=1}^{n}\eth_i}\Big], \\ \Big[[[\mho\cdot\aleph^{\lrcorner-}]]^{\ddagger_{i=1}^{n}\eth_i}, [[\mho\cdot\aleph^{[1-\gtrdot]+}]]^{\ddagger_{i=1}^{n}\eth_i}\Big] \end{bmatrix} \\
&= \begin{bmatrix} [\wp,\ell]; \Big[1-\Big[1-[[\mho\cdot\aleph^{\gtrdot-}]]^{\varsigma}\Big], 1-\Big[1-[[\mho\cdot\aleph^{[1-\lrcorner]+}]]^{\varsigma}\Big]\Big], \\ \Big[[[\mho\cdot\aleph^{\lrcorner-}]], [[\mho\cdot\aleph^{\lrcorner+}]]\Big], \\ \Big[[[\mho\cdot\aleph^{\lrcorner-}]], [[\mho\cdot\aleph^{[1-\gtrdot]+}]]\Big] \end{bmatrix} \\
&= \Bbbk.
\end{aligned}
$$

Theorem 4.4. *Let* $\Bbbk_i = \Big\langle [\wp_{ij}, \ell_{ij}]; [\aleph_{ij}^{\gtrdot-}, \aleph_{ij}^{[1-\lrcorner]+}], [\aleph_{ij}^{\lrcorner-}, \aleph_{ij}^{\lrcorner+}][\aleph_{ij}^{\lrcorner-}, \aleph_{ij}^{[1-\gtrdot]+}]\Big\rangle$ *be a finite collection of CotNNVWA, where* $[i = 1,2,...,n]; [j = 1,2,...,i_j]$, $\underbrace{\wp} = \inf_{1\preccurlyeq i\preccurlyeq n,\ j=1,2,...,i_j} \wp_{ij}$, $\overbrace{\wp} = \sup_1 \wp_{ij}$, $\underbrace{\ell} = \sup_1 \ell_{ij}$, $\overbrace{\ell} = \inf_1 \ell_{ij}$, $\underbrace{\aleph^{\gtrdot-}} = \inf_1 \aleph_{ij}^{\gtrdot^l}$, $\overbrace{\aleph^{\gtrdot-}} = \sup_1 \aleph_{ij}^{\gtrdot^l}$, $\underbrace{\aleph^{[1-\lrcorner]+}} = \inf_1 \aleph_{ij}^{\gtrdot^u}$, $\overbrace{\aleph^{[1-\lrcorner]+}} = \sup_1 \aleph_{ij}^{\gtrdot^u}$, $\underbrace{\aleph^{\lrcorner-}} = \inf_1 \aleph_{ij}^{\lrcorner^l}$, $\overbrace{\aleph^{\lrcorner-}} = \sup_1 \aleph_{ij}^{\lrcorner^l}$, $\underbrace{\aleph^{\lrcorner+}} = \inf_1 \aleph_{ij}^{\lrcorner^u}$, $\overbrace{\aleph^{\lrcorner+}} = \sup_1 \aleph_{ij}^{\lrcorner^u}$, $\underbrace{\aleph^{\gtrdot-}} = \inf_1 \aleph_{ij}^{\gtrdot^l}$, $\overbrace{\aleph^{\gtrdot-}} = \sup_1 \aleph_{ij}^{\gtrdot^l}$, $\underbrace{\aleph^{[1-\gtrdot]+}} = \inf_1 \aleph_{ij}^{\gtrdot^u}$, $\overbrace{\aleph^{[1-\gtrdot]+}} = \sup_1 \aleph_{ij}^{\gtrdot^u}$.

Then, $\Big\langle [\underbrace{\wp}, \underbrace{\ell}]; [\underbrace{\aleph^{\gtrdot-}}, \underbrace{\aleph^{[1-\lrcorner]+}}], [\overbrace{\aleph^{\lrcorner-}}, \overbrace{\aleph^{\lrcorner+}}], [\overbrace{\aleph^{\gtrdot-}}, \overbrace{\aleph^{[1-\gtrdot]+}}]\Big\rangle$
$\preccurlyeq CotNNVWA[\Bbbk_1, \Bbbk_2, ..., \Bbbk_n]$
$\preccurlyeq \Big\langle [\overbrace{\wp}, \overbrace{\ell}]; [\overbrace{\aleph^{\gtrdot-}}, \overbrace{\aleph^{[1-\lrcorner]+}}], [\underbrace{\aleph^{\lrcorner-}}, \underbrace{\aleph^{\lrcorner+}}], [\underbrace{\aleph^{\gtrdot-}}, \underbrace{\aleph^{[1-\gtrdot]+}}]\Big\rangle$.

Proof. Since, $\underbrace{\aleph^{\gtrdot-}} = \inf_1 \aleph_{ij}^{\gtrdot^l}$, $\overbrace{\aleph^{\gtrdot-}} = \sup_1 \aleph_{ij}^{\gtrdot^l}$ $\underbrace{\aleph^{[1-\lrcorner]+}} = \inf_1 \aleph_{ij}^{\gtrdot^u}$, $\overbrace{\aleph^{[1-\lrcorner]+}} = \sup_1 \aleph_{ij}^{\gtrdot^u}$ and $\underbrace{\aleph^{\gtrdot-}} \preccurlyeq \aleph_{ij}^{\gtrdot^l} \preccurlyeq \overbrace{\aleph^{\gtrdot-}}$ and $\underbrace{\aleph^{[1-\lrcorner]+}} \preccurlyeq \aleph_{ij}^{\gtrdot^u} \preccurlyeq \overbrace{\aleph^{[1-\lrcorner]+}}$.
We have, $\underbrace{[\mho^2\cdot\aleph^{\gtrdot-}]} + \underbrace{[\mho^2\cdot\aleph^{[1-\lrcorner]+}]}$

$$
\begin{aligned}
&= 1-\amalg_{i=1}^{n}\Big[1-\underbrace{[[\mho^2\cdot\aleph^{\gtrdot-}]]^{\varsigma}}\Big]^{\eth_i} + 1-\amalg_{i=1}^{n}\Big[1-\underbrace{[[\mho^2\cdot\aleph^{[1-\lrcorner]+}]]^{\varsigma}}\Big]^{\eth_i} \\
&\preccurlyeq 1-\amalg_{i=1}^{n}\Big[1-[\mho^2\cdot\aleph_{ij}^{\gtrdot^l}]^{\varsigma}\Big]^{\eth_i} + 1-\amalg_{i=1}^{n}\Big[1-[\mho^2\cdot\aleph_{ij}^{\gtrdot^u}]^{\varsigma}\Big]^{\eth_i} \\
&\preccurlyeq 1-\amalg_{i=1}^{n}\Big[1-\overbrace{[[\mho^2\cdot\aleph^{\gtrdot-}]]^{\varsigma}}\Big]^{\eth_i} + 1-\amalg_{i=1}^{n}\Big[1-\overbrace{[[\mho^2\cdot\aleph^{[1-\lrcorner]+}]]^{\varsigma}}\Big]^{\eth_i} \\
&= \overbrace{[\mho^2\cdot\aleph^{\gtrdot-}]} + \overbrace{[\mho^2\cdot\aleph^{[1-\lrcorner]+}]}.
\end{aligned}
$$

Since, $\underbrace{\aleph^{\lrcorner-}} = \inf_1 \aleph_{ij}^{\lrcorner^l}$, $\overbrace{\aleph^{\lrcorner-}} = \sup_1 \aleph_{ij}^{\lrcorner^l}$ $\underbrace{\aleph^{\lrcorner+}} = \inf_1 \aleph_{ij}^{\lrcorner^u}$, $\overbrace{\aleph^{\lrcorner+}} = \sup_1 \aleph_{ij}^{\lrcorner^u}$ and $\underbrace{\aleph^{\lrcorner-}} \preccurlyeq \aleph_{ij}^{\lrcorner^l} \preccurlyeq \overbrace{\aleph^{\lrcorner-}}$ and $\underbrace{\aleph^{\lrcorner+}} \preccurlyeq \aleph_{ij}^{\lrcorner^u} \preccurlyeq \overbrace{\aleph^{\lrcorner+}}$. We have,

$$
\begin{aligned}
\underbrace{[\mho^2\cdot\aleph^{\lrcorner-}]} + \underbrace{[\mho^2\cdot\aleph^{\lrcorner+}]} &= \amalg_{i=1}^{n}\underbrace{[[\mho^2\cdot\aleph^{\lrcorner-}]]}^{\eth_i} + \amalg_{i=1}^{n}\underbrace{[[\mho^2\cdot\aleph^{\lrcorner+}]]}^{\eth_i} \\
&\preccurlyeq \amalg_{i=1}^{n}[\mho^2\cdot\aleph_{ij}^{\lrcorner^l}]^{\eth_i} + \amalg_{i=1}^{n}[\mho^2\cdot\aleph_{ij}^{\lrcorner^u}]^{\eth_i} \\
&\preccurlyeq \amalg_{i=1}^{n}\overbrace{[[\mho^2\cdot\aleph^{\lrcorner-}]]}^{\eth_i} + \amalg_{i=1}^{n}\overbrace{[[\mho^2\cdot\aleph^{\lrcorner+}]]}^{\eth_i} \\
&= \overbrace{[\mho^2\cdot\aleph^{\lrcorner-}]} + \overbrace{[\mho^2\cdot\aleph^{\lrcorner+}]}.
\end{aligned}
$$

Since, $\underbrace{\aleph^{\lrcorner-}} = \inf_1 \aleph_{ij}^{\lrcorner^l}$, $\overbrace{\aleph^{\lrcorner-}} = \sup_1 \aleph_{ij}^{\lrcorner^l}$ $\underbrace{\aleph^{[1-\curlywedge]+}} = \inf_1 \aleph_{ij}^{\lrcorner^u}$, $\overbrace{\aleph^{[1-\curlywedge]+}} = \sup_1 \aleph_{ij}^{\lrcorner^u}$ and $\underbrace{\aleph^{\lrcorner-}} \preccurlyeq \aleph_{ij}^{\lrcorner^l} \preccurlyeq \overbrace{\aleph^{\lrcorner-}}$ and $\underbrace{\aleph^{[1-\curlywedge]+}} \preccurlyeq \aleph_{ij}^{\lrcorner^u} \preccurlyeq \overbrace{\aleph^{[1-\curlywedge]+}}$. We have,

$$\begin{aligned}
\underbrace{[\mho^2 \cdot \aleph^{\lrcorner-}]} + \underbrace{[\mho^2 \cdot \aleph^{[1-\curlywedge]+}]} &= \amalg_{i=1}^n [\underbrace{[\mho^2 \cdot \aleph^{\lrcorner-}]}]^{\eth_i} + \amalg_{i=1}^n [\underbrace{[\mho^2 \cdot \aleph^{[1-\curlywedge]+}]}]^{\eth_i} \\
&\preccurlyeq \amalg_{i=1}^n [\mho^2 \cdot \aleph_{ij}^{\lrcorner^l}]^{\eth_i} + \amalg_{i=1}^n [\mho^2 \cdot \aleph_{ij}^{\lrcorner^u}]^{\eth_i} \\
&\preccurlyeq \amalg_{i=1}^n [\overbrace{[\mho^2 \cdot \aleph^{\lrcorner-}]}]^{\eth_i} + \amalg_{i=1}^n [\overbrace{[\mho^2 \cdot \aleph^{[1-\curlywedge]+}]}]^{\eth_i} \\
&= \overbrace{[\mho^2 \cdot \aleph^{\lrcorner-}]} + \overbrace{[\mho^2 \cdot \aleph^{[1-\curlywedge]+}]}.
\end{aligned}$$

Since, $\underbrace{\wp} = \inf_1 \wp_{ij}$, $\overbrace{\wp} = \sup_1 \wp_{ij}$, $\underbrace{\ell} = \sup_1 \ell_{ij}$, $\overbrace{\ell} = \inf_1 \ell_{ij}$ and $\underbrace{\wp} \preccurlyeq \wp_{ij} \preccurlyeq \overbrace{\wp}$ and $\overbrace{\ell} \preccurlyeq \ell_{ij} \preccurlyeq \underbrace{\ell}$.

Hence, $\ddagger_{i=1}^n \eth_i \underbrace{\wp} \preccurlyeq \ddagger_{i=1}^n \eth_i \wp_{ij} \preccurlyeq \ddagger_{i=1}^n \eth_i \overbrace{\wp}$ and $\ddagger_{i=1}^n \eth_i \overbrace{\ell} \preccurlyeq \ddagger_{i=1}^n \eth_i \ell_{ij} \preccurlyeq \ddagger_{i=1}^n \eth_i \underbrace{\ell}$.
Therefore,

$$\begin{aligned}
&\frac{\ddagger_{i=1}^n \eth_i \underbrace{\wp}}{2} \times \left[\begin{array}{c} \frac{2+\left[1-\amalg_{i=1}^n \left[1-[\underbrace{[\mho^2 \cdot \aleph^{\curlywedge-}]}]^{\varsigma}\right]^{\eth_i}\right]+\left[1-\amalg_{i=1}^n \left[1-[\underbrace{[\mho^2 \cdot \aleph^{[1-\lrcorner]+}]}]^{\varsigma}\right]^{\eth_i}\right]}{2} \\ -\frac{\left[\amalg_{i=1}^n [\overbrace{[\mho^2 \cdot \aleph^{\lrcorner-}]}]^{\eth_i}\right]+\left[\amalg_{i=1}^n [\overbrace{[\mho^2 \cdot \aleph^{\lrcorner+}]}]^{\eth_i}\right]}{2} \\ -\frac{\left[\amalg_{i=1}^n [\overbrace{[\mho^2 \cdot \aleph^{\lrcorner-}]}]^{\eth_i}\right]+\left[\amalg_{i=1}^n [\overbrace{[\mho^2 \cdot \aleph^{[1-\curlywedge]+}]}]^{\eth_i}\right]}{2} \end{array} \right] \\
\preccurlyeq\ &\frac{\ddagger_{i=1}^n \eth_i \wp_{ij}}{2} \times \left[\begin{array}{c} \frac{2+\left[1-\amalg_{i=1}^n \left[1-[\mho^2 \cdot \aleph_{ij}^{\curlywedge^l}]^{\varsigma}\right]^{\eth_i}\right]+\left[1-\amalg_{i=1}^n \left[1-[\mho^2 \cdot \aleph_{ij}^{\curlywedge^u}]^{\varsigma}\right]^{\eth_i}\right]}{2} \\ -\frac{\left[\amalg_{i=1}^n [\mho^2 \cdot \aleph_{ij}^{\lrcorner^l}]^{\eth_i}\right]+\left[\amalg_{i=1}^n [\mho^2 \cdot \aleph_{ij}^{\lrcorner^u}]^{\eth_i}\right]}{2} \\ -\frac{\left[\amalg_{i=1}^n [\mho^2 \cdot \aleph_{ij}^{\lrcorner^l}]^{\eth_i}\right]+\left[\amalg_{i=1}^n [\mho^2 \cdot \aleph_{ij}^{\lrcorner^u}]^{\eth_i}\right]}{2} \end{array} \right] \\
\preccurlyeq\ &\frac{\ddagger_{i=1}^n \eth_i \overbrace{\wp}}{2} \times \left[\begin{array}{c} \frac{2+\left[1-\amalg_{i=1}^n \left[1-[\overbrace{[\mho^2 \cdot \aleph^{\curlywedge-}]}]^{\varsigma}\right]^{\eth_i}\right]+\left[1-\amalg_{i=1}^n \left[1-[\overbrace{[\mho^2 \cdot \aleph^{[1-\lrcorner]+}]}]^{\varsigma}\right]^{\eth_i}\right]}{2} \\ -\frac{\left[\amalg_{i=1}^n [\underbrace{[\mho^2 \cdot \aleph^{\lrcorner-}]}]^{\eth_i}\right]+\left[\amalg_{i=1}^n [\underbrace{[\mho^2 \cdot \aleph^{\lrcorner+}]}]^{\eth_i}\right]}{2} \\ -\frac{\left[\amalg_{i=1}^n [\underbrace{[\mho^2 \cdot \aleph^{\lrcorner-}]}]^{\eth_i}\right]+\left[\amalg_{i=1}^n [\underbrace{[\mho^2 \cdot \aleph^{[1-\curlywedge]+}]}]^{\eth_i}\right]}{2} \end{array} \right].
\end{aligned}$$

Hence, $\Big\langle [\underbrace{\wp}, \underbrace{\ell}]; [\underbrace{[\mho^2 \cdot \aleph^{\curlywedge-}]}, \underbrace{[\mho^2 \cdot \aleph^{[1-\lrcorner]+}]}], [\overbrace{[\mho^2 \cdot \aleph^{\lrcorner-}]}, \overbrace{[\mho^2 \cdot \aleph^{\lrcorner+}]}],$
$[\overbrace{[\mho^2 \cdot \aleph^{\lrcorner-}]}, \overbrace{[\mho^2 \cdot \aleph^{[1-\curlywedge]+}]}]\Big\rangle \preccurlyeq CotNNVWA[\mathbb{k}_1, \mathbb{k}_2, ..., \mathbb{k}_n]$
$\preccurlyeq \Big\langle [\overbrace{\wp}, \overbrace{\ell}]; [\overbrace{[\mho^2 \cdot \aleph^{\curlywedge-}]}, \overbrace{[\mho^2 \cdot \aleph^{[1-\lrcorner]+}]}], [\underbrace{[\mho^2 \cdot \aleph^{\lrcorner-}]}, \underbrace{[\mho^2 \cdot \aleph^{\lrcorner+}]}],$
$[\underbrace{[\mho^2 \cdot \aleph^{\lrcorner-}]}, \underbrace{[\mho^2 \cdot \aleph^{[1-\curlywedge]+}]}]\Big\rangle.$

Theorem 4.5. *Let* $\Bbbk_i = \left\langle [\wp_{t_{ij}}, \ell_{t_{ij}}]; [\aleph_{t_{ij}}^{\gtrdot -}], \aleph_{t_{ij}}^{[1-\lrcorner]+}]], [\aleph_{t_{ij}}^{\beth -}], \aleph_{t_{ij}}^{\beth +}]], [\aleph_{t_{ij}}^{\lrcorner -}], \aleph_{t_{ij}}^{[1-\gtrdot]+}]] \right\rangle$ *and* $H_i = \left\langle [\wp_{h_{ij}}, \ell_{h_{ij}}]; [\aleph_{h_{ij}}^{\gtrdot -}], \aleph_{h_{ij}}^{[1-\lrcorner]+}]], [\aleph_{h_{ij}}^{\beth -}], \aleph_{h_{ij}}^{\beth +}]], [\aleph_{h_{ij}}^{\lrcorner -}], \aleph_{h_{ij}}^{[1-\gtrdot]+}]] \right\rangle$ *be the two families of CotNNVWAs. For any* i, *if there is* $\wp_{t_{ij}} \preccurlyeq \ell_{h_{ij}}$, $\left[[\mho^2 \cdot \aleph_{t_{ij}}^{\gtrdot -}]\right] + \left[[\mho^2 \cdot \aleph_{t_{ij}}^{[1-\lrcorner]+}]\right] \preccurlyeq \left[[\mho^2 \cdot \aleph_{h_{ij}}^{\gtrdot -}]\right] + \left[[\mho^2 \cdot \aleph_{h_{ij}}^{[1-\lrcorner]+}]\right]$ *and* $\left[[\mho^2 \cdot \aleph_{t_{ij}}^{\beth -}]\right] + \left[[\mho^2 \cdot \aleph_{t_{ij}}^{\beth +}]\right] \succcurlyeq \left[[\mho^2 \cdot \aleph_{h_{ij}}^{\beth -}]\right] + \left[[\mho^2 \cdot \aleph_{h_{ij}}^{\beth +}]\right]$ *and* $\left[[\mho^2 \cdot \aleph_{t_{ij}}^{\lrcorner -}]\right] + \left[[\mho^2 \cdot \aleph_{t_{ij}}^{[1-\gtrdot]+}]\right] \succcurlyeq \left[[\mho^2 \cdot \aleph_{h_{ij}}^{\lrcorner -}]\right] + \left[[\mho^2 \cdot \aleph_{h_{ij}}^{[1-\gtrdot]+}]\right]$ *or* $\Bbbk_i \preccurlyeq H_i$, *then* $CotNNVWA[\Bbbk_1, \Bbbk_2, ..., \Bbbk_n] \preccurlyeq CotNNVWA[H_1, H_2, ..., H_n]$.

Proof. For every i, $\wp_{t_{ij}} \preccurlyeq \ell_{h_{ij}}$. Thus, $\ddagger_{i=1}^n \wp_{t_{ij}} \preccurlyeq \ddagger_{i=1}^n \ell_{h_{ij}}$.

For any i, $\left[[\mho^2 \cdot \aleph_{t_{ij}}^{\gtrdot -}]\right] + \left[[\mho^2 \cdot \aleph_{t_{ij}}^{[1-\lrcorner]+}]\right] \preccurlyeq \left[[\mho^2 \cdot \aleph_{h_{ij}}^{\gtrdot -}]\right] + \left[[\mho^2 \cdot \aleph_{h_{ij}}^{[1-\lrcorner]+}]\right]$.

Therefore, $1 - \left[[\mho^2 \cdot \aleph_{t_i}^{\gtrdot -}]\right] + 1 - \left[[\mho^2 \cdot \aleph_{t_i}^{[1-\lrcorner]+}]\right] \succcurlyeq 1 - \left[[\mho^2 \cdot \aleph_{h_i}^{\gtrdot -}]\right] + 1 - \left[[\mho^2 \cdot \aleph_{h_i}^{[1-\lrcorner]+}]\right]$.

Hence, $\amalg_{i=1}^n \left[1 - \left[[\mho^2 \cdot \aleph_{t_i}^{\gtrdot -}]\right]\right]^{\eth_i} + \amalg_{i=1}^n \left[1 - \left[[\mho^2 \cdot \aleph_{t_i}^{[1-\lrcorner]+}]\right]\right]^{\eth_i} \succcurlyeq$

$\amalg_{i=1}^n \left[1 - \left[[\mho^2 \cdot \aleph_{h_i}^{\gtrdot -}]\right]\right]^{\eth_i} + \amalg_{i=1}^n \left[1 - \left[[\mho^2 \cdot \aleph_{h_i}^{[1-\lrcorner]+}]\right]\right]^{\eth_i}$

and $\left[1 - \amalg_{i=1}^n \left[1 - \left[[\mho^2 \cdot \aleph_{t_i}^{\gtrdot -}]\right]^{\varsigma}\right]^{\eth_i}\right] + \left[1 - \amalg_{i=1}^n \left[1 - \left[[\mho^2 \cdot \aleph_{t_i}^{[1-\lrcorner]+}]\right]^{\varsigma}\right]^{\eth_i}\right] \preccurlyeq$

$\left[1 - \amalg_{i=1}^n \left[1 - \left[[\mho^2 \cdot \aleph_{h_i}^{\gtrdot -}]\right]^{\varsigma}\right]^{\eth_i}\right] + \left[1 - \amalg_{i=1}^n \left[1 - \left[[\mho^2 \cdot \aleph_{h_i}^{[1-\lrcorner]+}]\right]^{\varsigma}\right]^{\eth_i}\right]$.

and $\dfrac{2 + \left[1 - \amalg_{i=1}^n \left[1 - \left[[\mho^2 \cdot \aleph_{t_i}^{\gtrdot -}]\right]^{\varsigma}\right]^{\eth_i}\right] + \left[1 - \amalg_{i=1}^n \left[1 - \left[[\mho^2 \cdot \aleph_{t_i}^{[1-\lrcorner]+}]\right]^{\varsigma}\right]^{\eth_i}\right]}{2} \preccurlyeq$

$\dfrac{2 + \left[1 - \amalg_{i=1}^n \left[1 - \left[[\mho^2 \cdot \aleph_{h_i}^{\gtrdot -}]\right]^{\varsigma}\right]^{\eth_i}\right] + \left[1 - \amalg_{i=1}^n \left[1 - \left[[\mho^2 \cdot \aleph_{h_i}^{[1-\lrcorner]+}]\right]^{\varsigma}\right]^{\eth_i}\right]}{2}$.

For every i,

$\left[[\mho^2 \cdot \aleph_{t_{ij}}^{\beth -}]\right]^{\varsigma} + \left[[\mho^2 \cdot \aleph_{t_{ij}}^{\beth +}]\right]^{\varsigma} \succcurlyeq \left[[\mho^2 \cdot \aleph_{h_{ij}}^{\beth -}]\right]^{\varsigma} + \left[[\mho^2 \cdot \aleph_{h_{ij}}^{\beth +}]\right]^{\varsigma}$.

Therefore,

$-\dfrac{\left[\amalg_{i=1}^n [\mho^2 \cdot \aleph_{t_{ij}}^{\beth -}]\right] + \left[\amalg_{i=1}^n [\mho^2 \cdot \aleph_{t_{ij}}^{\beth +}]\right]}{2} \preccurlyeq -\dfrac{\left[\amalg_{i=1}^n [\mho^2 \cdot \aleph_{h_{ij}}^{\beth -}]\right] + \left[\amalg_{i=1}^n [\mho^2 \cdot \aleph_{h_{ij}}^{\beth +}]\right]}{2}$.

For any i, $\left[[\mho^2 \cdot \aleph_{t_{ij}}^{\lrcorner -}]\right] + \left[[\mho^2 \cdot \aleph_{t_{ij}}^{[1-\gtrdot]+}]\right] \succcurlyeq \left[[\mho^2 \cdot \aleph_{h_{ij}}^{\lrcorner -}]\right] + \left[[\mho^2 \cdot \aleph_{h_{ij}}^{[1-\gtrdot]+}]\right]$.

Therefore,

$-\dfrac{\left[\amalg_{i=1}^n [\mho^2 \cdot \aleph_{t_{ij}}^{\lrcorner -}]\right] + \left[\amalg_{i=1}^n [\mho^2 \cdot \aleph_{t_{ij}}^{[1-\gtrdot]+}]\right]}{2} \preccurlyeq -\dfrac{\left[\amalg_{i=1}^n [\mho^2 \cdot \aleph_{h_{ij}}^{\lrcorner -}]\right] + \left[\amalg_{i=1}^n [\mho^2 \cdot \aleph_{h_{ij}}^{[1-\gtrdot]+}]\right]}{2}$.

$$\frac{\ddagger_{i=1}^n \wp_{tij}}{2} \times \begin{bmatrix} \dfrac{2 + \left[1 - \amalg_{i=1}^n \left[1 - [\mho^2 \cdot \aleph_{ti}^{\gtrdot\, l}]^{\varsigma}\right]^{\eth_i}\right] + \left[1 - \amalg_{i=1}^n \left[1 - [\mho^2 \cdot \aleph_{ti}^{\gtrdot\, u}]^{\varsigma}\right]^{\eth_i}\right]}{2} \\ - \dfrac{\left[\amalg_{i=1}^n [\mho^2 \cdot \aleph_{tij}^{\beth\, l}]\right] + \left[\amalg_{i=1}^n [\mho^2 \cdot \aleph_{tij}^{\beth\, u}]\right]}{2} \\ - \dfrac{\left[\amalg_{i=1}^n [\mho^2 \cdot \aleph_{tij}^{\lrcorner\, l}]\right] + \left[\amalg_{i=1}^n [\mho^2 \cdot \aleph_{tij}^{\lrcorner\, u}]\right]}{2} \end{bmatrix}$$

$$\preccurlyeq \frac{\ddagger_{i=1}^n \wp_{hij}}{2} \times \begin{bmatrix} \dfrac{2 + \left[1 - \amalg_{i=1}^n \left[1 - [\mho^2 \cdot \aleph_{hi}^{\gtrdot\, l}]^{\varsigma}\right]^{\eth_i}\right] + \left[1 - \amalg_{i=1}^n \left[1 - [\mho^2 \cdot \aleph_{hi}^{\gtrdot\, u}]^{\varsigma}\right]^{\eth_i}\right]}{2} \\ - \dfrac{\left[\amalg_{i=1}^n [\mho^2 \cdot \aleph_{hij}^{\beth\, l}]\right] + \left[\amalg_{i=1}^n [\mho^2 \cdot \aleph_{hij}^{\beth\, u}]\right]}{2} \\ - \dfrac{\left[\amalg_{i=1}^n [\mho^2 \cdot \aleph_{hij}^{\lrcorner\, l}]\right] + \left[\amalg_{i=1}^n [\mho^2 \cdot \aleph_{hij}^{\lrcorner\, u}]\right]}{2} \end{bmatrix}.$$

Hence, $CotNNVWA[\Bbbk_1, \Bbbk_2, ..., \Bbbk_n] \preccurlyeq CotNNVWA[H_1, H_2, ..., H_n]$.

4.2. *CotNNV weighted geometric[CotNNVWG]*

Definition 4.6. Let $\Bbbk_i = \left\langle [\wp_i, \ell_i]; [\aleph_i^{\gtrdot -}, \aleph_i^{[1-\lrcorner]+}], [\aleph_i^{\beth -}, \aleph_i^{\beth +}], [\aleph_i^{\lrcorner -}, \aleph_i^{[1-\gtrdot]+}] \right\rangle$ be the finite collection of CotNNVNs. The CotNNVWG operator is defined as CotNNVWG $[\Bbbk_1, \Bbbk_2, ..., \Bbbk_n] = \amalg_{i=1}^{n} [\cot \Bbbk_i]^{\eth_i}$ $[i = 1, 2, ..., n]$.

Corollary 4.7. *Let* $\Bbbk_i = \left\langle [\wp_i, \ell_i]; [\aleph_i^{\gtrdot -}, \aleph_i^{[1-\lrcorner]+}], [\aleph_i^{\beth -}, \aleph_i^{\beth +}], [\aleph_i^{\lrcorner -}, \aleph_i^{[1-\gtrdot]+}] \right\rangle$ *be a collection of CotNNVNs. Prove that* $CotNNVWG[\Bbbk_1, \Bbbk_2, ..., \Bbbk_n] =$

$$\begin{bmatrix} \left[\amalg_{i=1}^{n} \wp_i^{\eth_i}, \amalg_{i=1}^{n} \ell_i^{\eth_i} \right]; \left[\amalg_{i=1}^{n} [[\mho^2 \cdot \aleph_i^{\gtrdot -}]]^{\eth_i}, \amalg_{i=1}^{n} [[\mho^2 \cdot \aleph_i^{[1-\lrcorner]+}]]^{\eth_i} \right], \\ \left[1 - \amalg_{i=1}^{n} \left[1 - [[\mho^2 \cdot \aleph_i^{\beth -}]]^{\varsigma} \right]^{\eth_i}, 1 - \amalg_{i=1}^{n} \left[1 - [[\mho^2 \cdot \aleph_i^{\beth +}]]^{\varsigma} \right]^{\eth_i} \right], \\ \left[1 - \amalg_{i=1}^{n} \left[1 - [[\mho^2 \cdot \aleph_i^{\lrcorner -}]]^{\varsigma} \right]^{\eth_i}, 1 - \amalg_{i=1}^{n} \left[1 - [[\mho^2 \cdot \aleph_i^{[1-\gtrdot]+}]]^{\varsigma} \right]^{\eth_i} \right] \end{bmatrix}.$$

Corollary 4.8. *If all* $\Bbbk_i = \left\langle [\wp_i, \ell_i]; \left[[[\mho \cdot \aleph_i^{\gtrdot -}]], [[\mho \cdot \aleph_i^{[1-\lrcorner]+}]] \right], \left[[[\mho \cdot \aleph_i^{\beth -}]], [[\mho \cdot \aleph_i^{\beth +}]] \right], \left[[[\mho \cdot \aleph_i^{\lrcorner -}]], [[\mho \cdot \aleph_i^{[1-\gtrdot]+}]] \right] \right\rangle$ *are equal and* $\Bbbk_i = \Bbbk$ *with* $\varsigma = 1$. *Prove that* $CotNNVWG[\Bbbk_1, \Bbbk_2, ..., \Bbbk_n] = \Bbbk$.

4.3. *Generalized CotNNVWA [CotGNNVWA]*

Definition 4.9. Let $\Bbbk_i = \left\langle [\wp_i, \ell_i]; [\aleph_i^{\gtrdot -}, \aleph_i^{[1-\lrcorner]+}], [\aleph_i^{\beth -}, \aleph_i^{\beth +}], [\aleph_i^{\lrcorner -}, \aleph_i^{[1-\gtrdot]+}] \right\rangle$ be the finite collection of CotNNVN. Then CotGNNVWA $[\Bbbk_1, \Bbbk_2, ..., \Bbbk_n] = \left[\ddagger_{i=1}^{n} \eth_i [\cot \Bbbk_i]^{\varsigma} \right]^{1/\varsigma}$.

Theorem 4.10. *Let* $\Bbbk_i = \left\langle [\wp_i, \ell_i]; [\aleph_i^{\gtrdot -}, \aleph_i^{[1-\lrcorner]+}], [\aleph_i^{\beth -}, \aleph_i^{\beth +}], [\aleph_i^{\lrcorner -}, \aleph_i^{[1-\gtrdot]+}] \right\rangle$ *be the finite collection of CotNNVNs. Prove that CotGNNVWA* $[\Bbbk_1, \Bbbk_2, ..., \Bbbk_n] =$

$$\begin{bmatrix} \left[\left[\ddagger_{i=1}^{n} \eth_i \wp_i^{\varsigma} \right]^{1/\varsigma}, \left[\ddagger_{i=1}^{n} \eth_i \ell_i^{\varsigma} \right]^{1/\varsigma} \right]; \\ \left[\left[1 - \amalg_{i=1}^{n} \left[1 - \left[[[\mho^2 \cdot \aleph_i^{\gtrdot -}]]^{\varsigma} \right]^{\eth_i} \right]^{1/\varsigma}, \left[1 - \amalg_{i=1}^{n} \left[1 - \left[[[\mho^2 \cdot \aleph_i^{[1-\lrcorner]+}]]^{\varsigma} \right]^{\eth_i} \right]^{1/\varsigma} \right], \right. \\ \left[1 - \left[1 - \left[\amalg_{i=1}^{n} \left[1 - \left[1 - [[\mho^2 \cdot \aleph_i^{\beth -}]]^{\varsigma} \right]^{\eth_i} \right]^{\varsigma} \right]^{1/\varsigma}, \right. \right. \\ \left. \left. \left[1 - \left[1 - \left[\amalg_{i=1}^{n} \left[1 - \left[1 - [[\mho^2 \cdot \aleph_i^{\beth +}]]^{\varsigma} \right]^{\eth_i} \right]^{\varsigma} \right]^{1/\varsigma} \right] \right. \right. \\ \left. \left. \left[1 - \left[1 - \left[\amalg_{i=1}^{n} \left[1 - \left[1 - [[\mho^2 \cdot \aleph_i^{\lrcorner -}]]^{\varsigma} \right]^{\eth_i} \right]^{\varsigma} \right]^{1/\varsigma}, \right. \right. \right. \\ \left. \left. \left. \left. 1 - \left[1 - \left[\amalg_{i=1}^{n} \left[1 - \left[1 - [[\mho^2 \cdot \aleph_i^{[1-\gtrdot]+}]]^{\varsigma} \right]^{\eth_i} \right]^{\varsigma} \right]^{1/\varsigma} \right] \right] \right] \right] \end{bmatrix}.$$

Proof. To prove that, $\ddagger_{i=1}^{n}\eth_i[\cot \Bbbk_i]^{\varsigma} =$

$$\left[\begin{array}{c} \left[\left[\ddagger_{i=1}^{n}\ \eth_i \wp_i^{\varsigma}\right], \left[\ddagger_{i=1}^{n}\ \eth_i \ell_i^{\varsigma}\right]\right]; \\ \left[1-\amalg_{i=1}^{n}\left[1-\left[[[\mho^2\cdot \aleph_i^{\gtrdot -}]]^{\varsigma}\right]^{\eth_i}\right.\right., 1-\amalg_{i=1}^{n}\left[1-\left[[[\mho^2\cdot \aleph_i^{[1-\lrcorner]+}]]^{\varsigma}\right]^{\eth_i}\right.\right], \\ \left[\amalg_{i=1}^{n}\left[1-\left[1-[[\mho^2\cdot \aleph_i^{\lrcorner -}]]^{\varsigma}\right]^{\eth_i}\right.\right., \amalg_{i=1}^{n}\left[1-\left[1-[[\mho^2\cdot \aleph_i^{\lrcorner +}]]^{\varsigma}\right]^{\eth_i}\right.\right], \\ \left[\amalg_{i=1}^{n}\left[1-\left[1-[[\mho^2\cdot \aleph_i^{\lrcorner -}]]^{\varsigma}\right]^{\eth_i}\right.\right., \amalg_{i=1}^{n}\left[1-\left[1-[[\mho^2\cdot \aleph_i^{[1-\gtrdot]+}]]^{\varsigma}\right]^{\eth_i}\right.\right] \end{array}\right].$$

If $n=2$, then $\eth_1[\cot \Bbbk_1]^{\varsigma} \ddagger \eth_2[\cot \Bbbk_2]^{\varsigma}$

$$= \left[\begin{array}{c} \left[\eth_1\wp_1^{\varsigma} \ddagger \eth_2\wp_2^{\varsigma}, \eth_1\ell_1^{\varsigma} \ddagger \eth_2\ell_2^{\varsigma}\right]; \\ \left[\begin{array}{c} \left[1-\left[1-\left[[[\mho^2\cdot \aleph_1^{\gtrdot -}]]^{\varsigma}\right]^{\eth_1}\right.\right]^{\varsigma}+\left[1-\left[1-\left[[[\mho^2\cdot \aleph_2^{\gtrdot -}]]^{\varsigma}\right]^{\eth_1}\right.\right]^{\varsigma}, \\ -\left[1-\left[1-\left[[[\mho^2\cdot \aleph_1^{\gtrdot -}]]^{\varsigma}\right]^{\eth_1}\right.\right]^{\varsigma}\cdot\left[1-\left[1-\left[[[\mho^2\cdot \aleph_2^{\gtrdot -}]]^{\varsigma}\right]^{\eth_1}\right.\right]^{\varsigma} \end{array}\right. \\ \left[\begin{array}{c} \left[1-\left[1-\left[[[\mho^2\cdot \aleph_1^{[1-\lrcorner]+}]]^{\varsigma}\right]^{\eth_1}\right.\right]^{\varsigma}+\left[1-\left[1-\left[[[\mho^2\cdot \aleph_2^{[1-\lrcorner]+}]]^{\varsigma}\right]^{\eth_1}\right.\right]^{\varsigma}, \\ -\left[1-\left[1-\left[[[\mho^2\cdot \aleph_1^{[1-\lrcorner]+}]]^{\varsigma}\right]^{\eth_1}\right.\right]^{\varsigma}\cdot\left[1-\left[1-\left[[[\mho^2\cdot \aleph_2^{[1-\lrcorner]+}]]^{\varsigma}\right]^{\eth_1}\right.\right]^{\varsigma} \end{array}\right. \\ \left[\begin{array}{c} \left[\left[1-\left[1-[[\mho^2\cdot \aleph_1^{\lrcorner -}]]^{\varsigma}\right]^{\eth_1}\right.\right.\cdot\left[1-\left[1-[[\mho^2\cdot \aleph_2^{\lrcorner -}]]^{\varsigma}\right]^{\eth_1}\right., \\ \left[1-\left[1-[[\mho^2\cdot \aleph_1^{\lrcorner +}]]^{\varsigma}\right]^{\eth_1}\right.\cdot\left[1-\left[1-[[\mho^2\cdot \aleph_2^{\lrcorner +}]]^{\varsigma}\right]^{\eth_1}\right] \end{array}\right. \\ \left[\begin{array}{c} \left[1-\left[1-[[\mho^2\cdot \aleph_1^{\lrcorner -}]]^{\varsigma}\right]^{\eth_1}\right.\cdot\left[1-\left[1-[[\mho^2\cdot \aleph_2^{\lrcorner -}]]^{\varsigma}\right]^{\eth_1}\right., \\ \left[1-\left[1-[[\mho^2\cdot \aleph_1^{[1-\gtrdot]+}]]^{\varsigma}\right]^{\eth_1}\right.\cdot\left[1-\left[1-[[\mho^2\cdot \aleph_2^{[1-\gtrdot]+}]]^{\varsigma}\right]^{\eth_1}\right] \end{array}\right. \end{array}\right],$$

$$= \left[\begin{array}{c} \left[\ddagger_{i=1}^{2}\ \eth_i\wp_i^{\varsigma}, \ddagger_{i=1}^{2}\eth_i\ell_i^{\varsigma}\right]; \\ \left[1-\amalg_{i=1}^{2}\left[1-\left[[[\mho^2\cdot \aleph_1^{\gtrdot -}]]^{\varsigma}\right]^{\eth_i}\right.\right., 1-\amalg_{i=1}^{2}\left[1-\left[[[\mho^2\cdot \aleph_1^{[1-\lrcorner]+}]]^{\varsigma}\right]^{\eth_i}\right.\right], \\ \left[\amalg_{i=1}^{2}\left[1-\left[1-[[\mho^2\cdot \aleph_i^{\lrcorner -}]]^{\varsigma}\right]^{\eth_i}\right.\right., \amalg_{i=1}^{2}\left[1-\left[1-[[\mho^2\cdot \aleph_i^{\lrcorner +}]]^{\varsigma}\right]^{\eth_i}\right.\right], \\ \left[\amalg_{i=1}^{2}\left[1-\left[1-[[\mho^2\cdot \aleph_i^{\lrcorner -}]]^{\varsigma}\right]^{\eth_i}\right.\right., \amalg_{i=1}^{2}\left[1-\left[1-[[\mho^2\cdot \aleph_i^{[1-\gtrdot]+}]]^{\varsigma}\right]^{\eth_i}\right.\right] \end{array}\right].$$

In general, the form $\ddagger_{i=1}^{n}\eth_i[\cot \Bbbk_i]^{\varsigma} =$

$$\begin{bmatrix} \left[\ddagger_{i=1}^{n}\ \eth_i\wp_i^{\varsigma}, \ddagger_{i=1}^{n}\eth_i\ell_i^{\varsigma}\right]; \\ \left[1-\amalg_{i=1}^{n}\left[1-\left[[[\mho^2\cdot\aleph_1^{\gtrdot-}]]^{\varsigma}\right]^{\eth_i}, 1-\amalg_{i=1}^{n}\left[1-\left[[[\mho^2\cdot\aleph_1^{[1-\lrcorner]+}]]^{\varsigma}\right]^{\eth_i}\right], \right. \\ \left[\amalg_{i=1}^{n}\left[1-\left[1-[[\mho^2\cdot\aleph_i^{\lrcorner-}]]^{\varsigma}\right]^{\varsigma}\right]^{\eth_i}, \amalg_{i=1}^{n}\left[1-\left[1-[[\mho^2\cdot\aleph_i^{\lrcorner-}]]^{\varsigma}\right]^{\varsigma}\right]^{\eth_i}\right], \\ \left.\left[\amalg_{i=1}^{n}\left[1-\left[1-[[\mho^2\cdot\aleph_i^{\lrcorner-}]]^{\varsigma}\right]^{\varsigma}\right]^{\eth_i}, \amalg_{i=1}^{n}\left[1-\left[1-[[\mho^2\cdot\aleph_i^{\lrcorner-}]]^{\varsigma}\right]^{\varsigma}\right]^{\eth_i}\right]\right. \end{bmatrix}.$$

If $n=l+1$, then $\ddagger_{i=1}^{n}\eth_i[\cot \Bbbk_i]^{\varsigma} \ddagger \eth_{n+1}[\cot \Bbbk_{n+1}]^{\varsigma} = \ddagger_{i=1}^{n+1}\eth_i[\cot \Bbbk_i]^{\varsigma}$.

Now,$\ddagger_{i=1}^{n}\eth_i[\cot \Bbbk_i]^{\varsigma} \ddagger \eth_{n+1}[\cot \Bbbk_{n+1}]^{\varsigma} =$

$\eth_1[\cot \Bbbk_1]^{\varsigma} \ddagger \eth_2[\cot \Bbbk_2]^{\varsigma} \ddagger ... \ddagger \eth_l[\cot \Bbbk_n]^{\varsigma} \ddagger \eth_{n+1}[\cot \Bbbk_{n+1}]^{\varsigma}$

$$= \begin{bmatrix} \left[\ddagger_{i=1}^{n}\ \eth_i\wp_i^{\varsigma} \ddagger \eth_{n+1}\wp_{n+1}^{\varsigma}, \ddagger_{i=1}^{n}\eth_i\ell_i^{\varsigma} \ddagger \eth_{n+1}\ell_{n+1}^{\varsigma}\right]; \\ \left[\begin{array}{l} \left[1-\amalg_{i=1}^{n}\left[1-\left[[[\mho^2\cdot\aleph_i^{\gtrdot-}]]^{\varsigma}\right]^{\eth_i}\right]^{\varsigma}+\left[1-\left[1-\left[[[\mho^2\cdot\aleph_{n+1}^{\gtrdot-}]]^{\varsigma}\right]^{\eth_1}\right]^{\varsigma}, \right. \\ -\left[1-\amalg_{i=1}^{n}\left[1-\left[[[\mho^2\cdot\aleph_i^{\gtrdot-}]]^{\varsigma}\right]^{\eth_i}\right]^{\varsigma}\cdot\left[1-\left[1-\left[[[\mho^2\cdot\aleph_{n+1}^{\gtrdot-}]]^{\varsigma}\right]^{\eth_1}\right]^{\varsigma}, \\ \left[1-\amalg_{i=1}^{n}\left[1-\left[[[\mho^2\cdot\aleph_i^{[1-\lrcorner]+}]]^{\varsigma}\right]^{\eth_i}\right]^{\varsigma}+\left[1-\left[1-\left[[[\mho^2\cdot\aleph_{n+1}^{[1-\lrcorner]+}]]^{\varsigma}\right]^{\eth_1}\right]^{\varsigma}, \\ \left.-\left[1-\amalg_{i=1}^{n}\left[1-\left[[[\mho^2\cdot\aleph_i^{[1-\lrcorner]+}]]^{\varsigma}\right]^{\eth_i}\right]^{\varsigma}\cdot\left[1-\left[1-\left[[[\mho^2\cdot\aleph_{n+1}^{[1-\lrcorner]+}]]^{\varsigma}\right]^{\eth_1}\right]^{\varsigma}\right. \end{array}\right], \\ \left[\begin{array}{l} \left[\amalg_{i=1}^{n}\left[1-\left[1-[[\mho^2\cdot\aleph_i^{\lrcorner-}]]^{\varsigma}\right]^{\varsigma}\right]^{\eth_i}\cdot\left[1-\left[1-[[\mho^2\cdot\aleph_{n+1}^{\lrcorner-}]]^{\varsigma}\right]^{\varsigma}\right]^{\eth_1}, \right. \\ \left.\left[\amalg_{i=1}^{n}\left[1-\left[1-[[\mho^2\cdot\aleph_i^{\lrcorner+}]]^{\varsigma}\right]^{\varsigma}\right]^{\eth_i}\cdot\left[1-\left[1-[[\mho^2\cdot\aleph_{n+1}^{\lrcorner+}]]^{\varsigma}\right]^{\varsigma}\right]^{\eth_1}\right] \end{array}\right], \\ \left[\begin{array}{l} \left[\amalg_{i=1}^{n}\left[1-\left[1-[[\mho^2\cdot\aleph_i^{\lrcorner-}]]^{\varsigma}\right]^{\varsigma}\right]^{\eth_i}\cdot\left[1-\left[1-[[\mho^2\cdot\aleph_{n+1}^{\lrcorner-}]]^{\varsigma}\right]^{\varsigma}\right]^{\eth_1}, \right. \\ \left.\left[\amalg_{i=1}^{n}\left[1-\left[1-[[\mho^2\cdot\aleph_i^{[1-\gtrdot]+}]]^{\varsigma}\right]^{\varsigma}\right]^{\eth_i}\cdot\left[1-\left[1-[[\mho^2\cdot\aleph_{n+1}^{[1-\gtrdot]+}]]^{\varsigma}\right]^{\varsigma}\right]^{\eth_1}\right] \end{array}\right] \end{bmatrix}.$$

Thus, $\ddagger_{i=1}^{n+1}\eth_i[\cot \Bbbk_i]^{\varsigma} =$

$$\begin{bmatrix} \left[\ddagger_{i=1}^{n+1}\ \eth_i\wp_i^{\varsigma}, \ddagger_{i=1}^{n+1}\eth_i\ell_i^{\varsigma}\right]; \\ \left[1-\amalg_{i=1}^{n+1}\left[1-\left[[[\mho^2\cdot\aleph_1^{\gtrdot-}]]^{\varsigma}\right]^{\eth_i}, 1-\amalg_{i=1}^{n+1}\left[1-\left[[[\mho^2\cdot\aleph_1^{[1-\lrcorner]+}]]^{\varsigma}\right]^{\eth_i}\right], \right. \\ \left[\amalg_{i=1}^{n+1}\left[1-\left[1-[[\mho^2\cdot\aleph_i^{\lrcorner-}]]^{\varsigma}\right]^{\varsigma}\right]^{\eth_i}, \amalg_{i=1}^{n+1}\left[1-\left[1-[[\mho^2\cdot\aleph_i^{\lrcorner-}]]^{\varsigma}\right]^{\varsigma}\right]^{\eth_i}\right], \\ \left.\left[\amalg_{i=1}^{n+1}\left[1-\left[1-[[\mho^2\cdot\aleph_i^{\lrcorner-}]]^{\varsigma}\right]^{\varsigma}\right]^{\eth_i}, \amalg_{i=1}^{n+1}\left[1-\left[1-[[\mho^2\cdot\aleph_i^{\lrcorner-}]]^{\varsigma}\right]^{\varsigma}\right]^{\eth_i}\right]\right. \end{bmatrix}.$$

Hence, $\left[\ddagger_{i=1}^{n+1}\eth_i[\cot \Bbbk_i]^{\varsigma}\right]^{1/\varsigma}$

$$\left[\begin{array}{c}\left[\left[\ddagger_{i=1}^{n+1}\ \eth_i\wp_i^{\varsigma}\right]^{1/\varsigma},\left[\ddagger_{i=1}^{n+1}\ \eth_i\ell_i^{\varsigma}\right]^{1/\varsigma}\right];\\ \left[\left[1-\amalg_{i=1}^{n+1}\left[1-\left[[[\mho^2\cdot\aleph_i^{\gtrdot-}]]^{\varsigma}\right]^{\eth_i}\right]^{1/\varsigma},\left[1-\amalg_{i=1}^{n+1}\left[1-\left[[[\mho^2\cdot\aleph_i^{[1-\lrcorner]+}]]^{\varsigma}\right]^{\eth_i}\right]^{1/\varsigma}\right],\right.\right.\\ \left[\begin{array}{c}1-\left[1-\left[\amalg_{i=1}^{n+1}\left[1-\left[1-[[\mho^2\cdot\aleph_i^{\beth-}]]^{\varsigma}\right]^{\eth_i}\right]\right]\right]^{1/\varsigma},\\ 1-\left[1-\left[\amalg_{i=1}^{n+1}\left[1-\left[1-[[\mho^2\cdot\aleph_i^{\beth+}]]^{\varsigma}\right]^{\eth_i}\right]\right]\right]^{1/\varsigma}\end{array}\right],\\ \left[\begin{array}{c}1-\left[1-\left[\amalg_{i=1}^{n+1}\left[1-\left[1-[[\mho^2\cdot\aleph_i^{\lrcorner-}]]^{\varsigma}\right]^{\eth_i}\right]\right]\right]^{1/\varsigma},\\ 1-\left[1-\left[\amalg_{i=1}^{n+1}\left[1-\left[1-[[\mho^2\cdot\aleph_i^{[1-\gtrdot]+}]]^{\varsigma}\right]^{\eth_i}\right]\right]\right]^{1/\varsigma}\end{array}\right]\end{array}\right].$$

It is true for any n.

In the situation of $\varsigma=1$, the CotGNNVWA operator is compressed to the CotNNVWA operator.

Theorem 4.11. *If all* $\Bbbk_i=\left\langle[\wp_i,\ell_i];\left[[[\mho\cdot\aleph_i^{\gtrdot-}]],[[\mho\cdot\aleph_i^{[1-\lrcorner]+}]]\right],\left[[[\mho\cdot\aleph_i^{\beth-}]],[[\mho\cdot\aleph_i^{\beth+}]]\right],\left[[[\mho\cdot\aleph_i^{\lrcorner-}]],[[\mho\cdot\aleph_i^{[1-\gtrdot]+}]]\right]\right\rangle$ *are equal and* $\Bbbk_i=\Bbbk$ *with* $\varsigma=1$. *Prove that CotGNNVWA*$[\Bbbk_1,\Bbbk_2,...,\Bbbk_n]=\Bbbk$.

For Theorem 4.4 and Theorem 4.5, the CotGNNVWA operator is used to satisfy the boundedness and monotonicity properties.

4.4. *Generalized CotNNVWG [CotGNNVWG]*

Definition 4.12. Let $\Bbbk_i=\left\langle[\wp_i,\ell_i];[\aleph_i^{\gtrdot-},\aleph_i^{[1-\lrcorner]+}],[\aleph_i^{\beth-},\aleph_i^{\beth+}],[\aleph_i^{\lrcorner-},\aleph_i^{[1-\gtrdot]+}]\right\rangle$ be the finite collection of CotNNVNs. Then CotGNNVWG $[\Bbbk_1,\Bbbk_2,...,\Bbbk_n]=\frac{1}{\varsigma}\left[\amalg_{i=1}^{n}\left[\varsigma\cot\Bbbk_i\right]^{\eth_i}\right]$, $[i=1,2,...,n]$.

Corollary 4.13. *Let* $\Bbbk_i=\left\langle[\wp_i,\ell_i];[\aleph_i^{\gtrdot-},\aleph_i^{[1-\lrcorner]+}],[\aleph_i^{\beth-},\aleph_i^{\beth+}],[\aleph_i^{\lrcorner-},\aleph_i^{[1-\gtrdot]+}]\right\rangle$ *be the finite collection of CotNNVNs. Prove that CotGNNVWG*$[\Bbbk_1,\Bbbk_2,...,\Bbbk_n]=$

$$\left[\begin{array}{c}\left[\frac{1}{\varsigma}\amalg_{i=1}^{n}\left[\varsigma\wp_i\right]^{\eth_i},\frac{1}{\varsigma}\amalg_{i=1}^{n}\left[\varsigma\ell_i\right]^{\eth_i}\right];\\ \left[\begin{array}{c}1-\left[1-\left[\amalg_{i=1}^{n}\left[1-\left[1-[[\mho^2\cdot\aleph_i^{\gtrdot-}]]^{\varsigma}\right]^{\eth_i}\right]^{\varsigma}\right]^{1/\varsigma},\right.\\ 1-\left[1-\left[\amalg_{i=1}^{n}\left[1-\left[1-[[\mho^2\cdot\aleph_i^{[1-\lrcorner]+}]]^{\varsigma}\right]^{\eth_i}\right]^{\varsigma}\right]^{1/\varsigma}\right]\end{array}\right.,\\ \left[\left[1-\amalg_{i=1}^{n}\left[1-\left[[[\mho^2\cdot\aleph_i^{\beth-}]]^{\varsigma}\right]^{\eth_i}\right]^{1/\varsigma},\left[1-\amalg_{i=1}^{n}\left[1-\left[[[\mho^2\cdot\aleph_i^{\beth+}]]^{\varsigma}\right]^{\eth_i}\right]^{1/\varsigma}\right],\right.\right.\\ \left.\left.\left[1-\amalg_{i=1}^{n}\left[1-\left[[[\mho^2\cdot\aleph_i^{\lrcorner-}]]^{\varsigma}\right]^{\eth_i}\right]^{1/\varsigma},\left[1-\amalg_{i=1}^{n}\left[1-\left[[[\mho^2\cdot\aleph_i^{[1-\gtrdot]+}]]^{\varsigma}\right]^{\eth_i}\right]^{1/\varsigma}\right]\right]\end{array}\right].$$

The CotGNNVWG operator is changed to the CotNNVWG operator based on $\varsigma=1$.

The CotGNNVWG operator is used to satisfy the boundedness and monotonicity properties, which are supported by Theorem 4.4 and Theorem 4.5.

Corollary 4.14. *If all* $\mathbb{k}_i = \left\langle [\wp_i, \ell_i]; \left[[\mho \cdot \aleph_i^{\gimel -}]], [[\mho \cdot \aleph_i^{[1-\gimel]+}]]\right], \left[[[\mho \cdot \aleph_i^{\beth -}]], [[\mho \cdot \aleph_i^{\beth +}]]\right], \left[[[\mho \cdot \aleph_i^{\daleth -}]], [[\mho \cdot \aleph_i^{[1-\daleth]+}]]\right]\right\rangle$ *are equal and* $\mathbb{k}_i = \mathbb{k}$ *with* $\varsigma = 1$. *Prove that* $CotGNNVWG[\mathbb{k}_1, \mathbb{k}_2, ..., \mathbb{k}_n] = \mathbb{k}$.

5. Conclusion:

The euclidean and hamming distance measurements for CotNNVSs that we have presented in this research are advantageous due to their mathematical simplicity. Appropriate numerical examples demonstrate the superiority of the euclidean and hamming distance metrics. Examples from everyday life demonstrate the usefulness of the euclidean and hamming distance metrics. Better aggregation operation rules for CotNNVWA, CotNNVWG, CotGNNVWA, and CotGNNVWG have been suggested by us. Additionally, we have spoken about a few attributes and provided some instances of how we create these operators. Since this is a new field of study, the author is sure that future academics who are interested in this topic will find the talks in this article useful.

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Type-2 Neutrosophic Sets (T2NSs) for Quality Evaluation of Project-Based Art and Design Courses: Evaluation Seven Projects

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Abstract: Conflicting objectives for alternatives must be valued in a decision-making process, and the optimal option must be chosen based on the decision-makers requirements. For this selection, multi-objective optimization techniques might offer a solution. The purpose of this study is to introduce two relatively new multi-objective optimization techniques: multi-objective optimization based on simple ratio analysis (MOOSRA) and Entropy methods. This study uses the multi-criteria decision-making method (MCDM) to evaluate the project-based art and design courses. The entropy method is used to compute the criteria weights and the MOOSRA method is used to rank the alternatives. The MCDM method is used under the type-2 neutrosophic sets (T2NSs) to deal with vague and uncertain information. A case study with 14 criteria and seven projects are used in this study. The results show that Graphic Design Projects have the highest rank and the Product & Industrial Design has the lowest rank. Sensitivity analysis was conducted to show the different ranks of alternatives.

Keywords: Type-2 Neutrosophic Sets; Projects Arts and Design Courses; Uncertainty; Entropy; MOOSRA method.

1. Introduction

Zadeh suggested fuzzy set theory as an extension of classical sets. Following Zadeh's work, scholars became interested in fuzzy set theory, and the number of academic papers on the topic has grown significantly across a range of fields, including the social sciences, engineering, and economics[1], [2]. A membership function is used to describe a fuzzy set. The idea of linguistic truth and fuzzy sets, whose membership levels are described using language terms like low, medium, high, very low, not low, not high, etc., are closely related. Conversely, one could think of a fuzzy set A as a mapping from discourse universe U to subsets of the interval [0, 1] that is defined by a membership function containing linguistic factors.

As a generalization of fuzzy sets, Atanassov presented the idea of intuitionistic fuzzy sets. Two functions known as membership and non-membership functions define an intuitionistic fuzzy

set. An essential tool for modeling situations involving reluctance is an intuitionistic fuzzy set[3], [4]. Studies on the theoretical elements of intuitionistic fuzzy sets and their applications in decision-making problems have been growing quickly since Atanassov established the idea.

Important tools for modeling uncertainty-related problems include the fuzzy set, intuitionistic fuzzy set, type-2 fuzzy set, and type-2 intuitionistic fuzzy set. These ideas help model uncertainty, but they might not be enough to explain some problems with contradictory data. The primary cause of this deficiency is the distinction between the ideas of uncertainty and inconsistency. The idea of inconsistency conveys meaninglessness among outcomes produced under the same conditions, but the idea of uncertainty conveys deficiencies or fuzziness of results[5], [6]. For instance, a decision-maker may respond to the same question in a "positive" or "negative" manner at various times.

This demonstrates how inconsistent the decision-maker is. The statement "it is positive between 0.6 and 0.8" or "0.7 positive" indicates that the person making the decision is unsure of the answer to the question, indicating that there is some hesitation. Therefore, Smarandache created the neutrosophic set theory to solve inconsistency-related difficulties. An extension of the intuitionistic, classical, fuzzy, paraconsistent, dialetheist, paradoxist, and tautological sets based on neutrosophy is known as a neutrosophic set[7], [8]. The field of philosophy known as "neutrosophy" examines the genesis, characteristics, and application of neutralities.

According to neutrosophy, a statement, theory, event, concept, or entity "A" is compared to its opposite "Anti-A," something that is not A, "Non-A," and anything that is neither "A" nor "Anti-A," represented by "Neut-A." From a universal set to a real or non-real standard subset, a neutrosophic set is defined by three functions[9], [10]. They are known as the truth-membership function, falsity-membership function, and indeterminacy membership function, and they are independent of one another. While the neutrosophic set theory is useful for modeling some problems, it is not always a good fit for modeling engineering challenges[11], [12].

Abdel Basset et al. [13] recently defined the concept of type-2 neutrosophic numbers (T2NNs) by defining arithmetical and set-theoretical operations between T2NNs and assigning truth-membership, indeterminacy membership, and falsity-membership values to components of a single-valued neutrosophic number[14], [15]. They also introduced several ideas, including T2NN aggregating operators and score and accuracy functions.

Among the multi-objective optimization techniques is the MOOSRA approach. When the MOOSRA and MOORA methods are compared, the MOOSRA technique is less sensitive to significant fluctuations in the criterion values and does not exhibit the negative performance scores of the MOORA approach. To rank the options, it was utilized to create the MCDM framework[16], [17].

The MOORA approach and the MOOSRA method have comparable application steps. Specifically, the problem's decision matrix is constructed in the first stage, and the decision matrix

is then normalized in the second. The MOOSRA technique uses a straightforward ratio of the total of the normalized performance values for advantageous criteria to the total of the normalized performance values for non-beneficial criteria to determine the overall performance score of each alternative[18], [19].

The main contributions of this study are organized as follows:

— This study uses two MCDM approaches to select the best project. The entropy method is used to compute the criteria weights and the MOOSRA method is used to rank the alternatives.
— This study evaluated the criteria and alternatives of projects based on art design. We use 14 criteria and seven alternatives in this study.
— The T2NSs are used to deal with uncertainty and vague information in the evaluation process between the criteria and alternatives.

This is how the remainder of the paper is structured: First, the T2NSs and their operations are conducted. Secondly, the Entropy and MOOSRA approaches' respective methodological underpinnings are presented. Then, using seven projects to be evaluated under different criteria. Then we show the sensitivity analysis. Finally, the applications' outcomes are shown, and suggestions for further research are talked about.

2. Type-2 Neutrosophic Sets (T2NSs)

Let two type-2 neutrosophic numbers (T2NNs) such as[13]:

$$A_1 = \left(\left(T_{T_{A_1}}(x), T_{I_{A_1}}(x), T_{F_{A_1}}(x)\right), \left(I_{T_{A_1}}(x), I_{I_{A_1}}(x), I_{F_{A_1}}(x)\right), \left(F_{T_{A_1}}(x), F_{I_{A_1}}(x), F_{F_{A_1}}(x)\right)\right) \text{ and}$$

$$A_2 = \left(\left(T_{T_{A_2}}(x), T_{I_{A_2}}(x), T_{F_{A_2}}(x)\right), \left(I_{T_{A_2}}(x), I_{I_{A_2}}(x), I_{F_{A_2}}(x)\right), \left(F_{T_{A_2}}(x), F_{I_{A_2}}(x), F_{F_{A_2}}(x)\right)\right)$$

$$A_1 \oplus A_2 = \left(\begin{array}{c} \begin{pmatrix} T_{T_{A_1}}(x) + T_{T_{A_2}}(x) - T_{T_{A_1}}(x)T_{T_{A_2}}(x), \\ T_{I_{A_1}}(x) + T_{I_{A_2}}(x) - T_{I_{A_1}}(x)T_{I_{A_2}}(x), \\ T_{F_{A_1}}(x) + T_{F_{A_2}}(x) - T_{F_{A_1}}(x)T_{F_{A_2}}(x) \end{pmatrix}, \\ \left(I_{T_{A_1}}(x)I_{T_{A_2}}(x), I_{I_{A_1}}(x)I_{I_{A_2}}(x), I_{F_{A_1}}(x)I_{F_{A_2}}(x)\right), \\ \left(F_{T_{A_1}}(x)F_{T_{A_2}}(x), F_{I_{A_1}}(x)F_{I_{A_2}}(x), F_{F_{A_1}}(x)F_{F_{A_2}}(x)\right) \end{array}\right) \quad (1)$$

$$A_1 \otimes A_2 = \begin{pmatrix} \left(T_{T_{A_1}}(x)T_{T_{A_2}}(x), T_{I_{A_1}}(x)T_{I_{A_2}}(x), T_{F_{A_1}}(x)T_{F_{A_2}}(x)\right), \\ \begin{pmatrix} I_{T_{A_1}}(x) + I_{T_{A_2}}(x) - I_{T_{A_1}}(x)I_{T_{A_2}}(x), \\ I_{I_{A_1}}(x) + I_{I_{A_2}}(x) - I_{I_{A_1}}(x)I_{I_{A_2}}(x), \\ I_{F_{A_1}}(x) + I_{F_{A_2}}(x) - I_{F_{A_1}}(x)I_{F_{A_2}}(x) \end{pmatrix}, \\ \begin{pmatrix} F_{T_{A_1}}(x) + F_{T_{A_2}}(x) - F_{T_{A_1}}(x)F_{T_{A_2}}(x), \\ F_{I_{A_1}}(x) + F_{I_{A_2}}(x) - F_{I_{A_1}}(x)F_{I_{A_2}}(x), \\ F_{F_{A_1}}(x) + F_{F_{A_2}}(x) - F_{F_{A_1}}(x)F_{F_{A_2}}(x) \end{pmatrix} \end{pmatrix} \quad (2)$$

$$\curlywedge A_1 = \begin{pmatrix} \begin{pmatrix} \left(1 - \left(1 - T_{T_{A_1}}(x)\right)^{\curlywedge}\right), \\ \left(1 - \left(1 - T_{T_{A_1}}(x)\right)^{\curlywedge}\right), \\ \left(1 - \left(1 - T_{T_{A_1}}(x)\right)^{\curlywedge}\right) \end{pmatrix}, \\ \left(\left(I_{T_{A_1}}(x)\right)^{\curlywedge}, \left(I_{T_{A_1}}(x)\right)^{\curlywedge}, \left(I_{T_{A_1}}(x)\right)^{\curlywedge}\right), \\ \left(\left(F_{T_{A_1}}(x)\right)^{\curlywedge}, \left(F_{T_{A_1}}(x)\right)^{\curlywedge}, \left(F_{T_{A_1}}(x)\right)^{\curlywedge}\right) \end{pmatrix} \quad (3)$$

$$A_1^{\curlywedge} = \begin{pmatrix} \left(\left(T_{T_{A_1}}(x)\right)^{\curlywedge}, \left(T_{T_{A_1}}(x)\right)^{\curlywedge}, \left(T_{T_{A_1}}(x)\right)^{\curlywedge}\right), \\ \begin{pmatrix} \left(1 - \left(1 - I_{T_{A_1}}(x)\right)^{\curlywedge}\right), \\ \left(1 - \left(1 - I_{T_{A_1}}(x)\right)^{\curlywedge}\right), \\ \left(1 - \left(1 - I_{T_{A_1}}(x)\right)^{\curlywedge}\right) \end{pmatrix}, \\ \begin{pmatrix} \left(1 - \left(1 - F_{T_{A_1}}(x)\right)^{\curlywedge}\right), \\ \left(1 - \left(1 - F_{T_{A_1}}(x)\right)^{\curlywedge}\right), \\ \left(1 - \left(1 - F_{T_{A_1}}(x)\right)^{\curlywedge}\right) \end{pmatrix} \end{pmatrix} \quad (4)$$

3. Materials and Methods

This section shows the proposed approach under the T2NSs to compute the criteria weights and rank the alternatives. This study uses two MCDM methods such as the Entropy method to compute the criteria weights and the MOOSRA method to rank the alternatives. We used the T2NSs to deal with vague and uncertain data. Figure 1 shows the proposed approach to computing the criteria weights and ranking the alternatives.

Table 1. The T2NNs.

Terms of T2NSs	T2NNs
Very Low	((0.20,0.20,0.10), (0.65,0.80,0.85), (0.45,0.80,0.70))
Low	((0.35,0.35,0.10), (0.50,0.75,0.80), (0.50,0.75,0.65))
Medium Low	((0.50,0.30,0.50), (0.50,0.35,0.45), (0.45,0.30,0.60))
Medium	((0.40,0.45,0.50), (0.40,0.45,0.50), (0.35,0.40,0.45))
Medum High	((0.60,0.45,0.50), (0.20,0.15,0.25), (0.10,0.25,0.15))
High	((0.70,0.75,0.80), (0.15,0.20,0.25), (0.10,0.15,0.20))
Very High	((0.95,0.90,0.95), (0.10,0.10,0.05), (0.05,0.05,0.05))

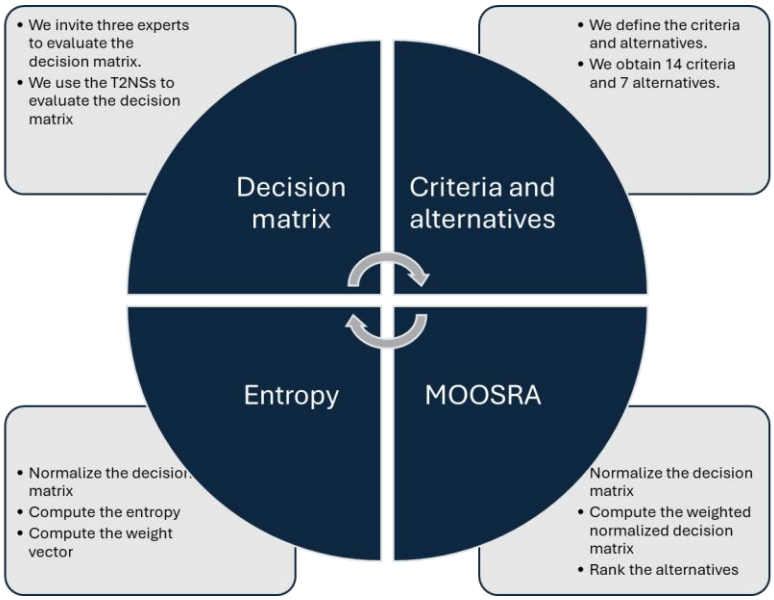
• We invite three experts to evaluate the decision matrix.
• We use the T2NSs to evaluate the decision matrix
Decision matrix
Criteria and alternatives
• We define the criteria and alternatives.
• We obtain 14 criteria and 7 alternatives.
Entropy
MOOSRA
• Normalize the decision matrix
• Compute the entropy
• Compute the weight vector
Normalize the decision matrix
• Compute the weighted normalized decision matrix
• Rank the alternatives

Figure 1. The process of MCDM methods.

3.1 T2NSs-Entropy Method

This part uses the Entropy method under the T2NSs to compute the criteria weights. The steps of the proposed approach are organized as follows:

Step 1. Build the decision matrix.
We used the linguistic terms of the T2NSs to build the decision matrix by the experts. Then we replace these terms by using the T2NNs as shown in Table 1. Then we apply the score function to obtain the one value. Then we aggrege the decision matrix between the criteria and alternatives.

$$X = \begin{bmatrix} x_{11} & \cdots & x_{1n} \\ \vdots & \ddots & \vdots \\ x_{m1} & \cdots & x_{mn} \end{bmatrix} \tag{5}$$

Step 2. Normalize the decision matrix.
The decision matrix is normalized between the criteria and alternatives such as:

$$P_{ij} = \frac{x_{ij}}{\sum_{i=1}^{m} x_{ij}}; i = 1, \dots, m; j = 1, \dots, n \quad (6)$$

Step 3. Compute the entropy
The entropy number is computed such as:

$$n_j = -h \sum_{i=1}^{m} p_{ij} \ln p_{ij} \quad (7)$$

$$h = \frac{1}{\ln(m)} \quad (8)$$

Step 4. Compute the weight vector.
The weight vector between the criteria is computed such as:

$$w_j = \frac{1-n_j}{\sum_{j=1}^{n}(1-n_j)} \quad (9)$$

3.2 T2NSs-MOOSRA Method

The MOOSRA method is the MCDM method used to rank the alternatives. It is a multi-objective optimization method. The steps of the MOOSRA method are organized as:

Step 5. Normalize the decision matrix.
We start with the combined decision matrix. Then we normalize the decision matrix between the criteria and the alternatives such as:

$$q_{ij} = \frac{x_{ij}}{\sqrt{\sum_{i=1}^{m} x_{ij}^2}} \quad (10)$$

Step 6. Compute the weighted normalized decision matrix.

The weighted normalized decision matrix between the criteria and the alternatives is computed as:

$$d_{ij} = w_j q_{ij} \quad (11)$$

Step 7. Rank the alternatives.
The alternatives are ranked based on the highest score in each row in the weighted normalized decision matrix.

$$f_i = \frac{\sum benefit\ value}{\sum non\ benefit\ value} \quad (12)$$

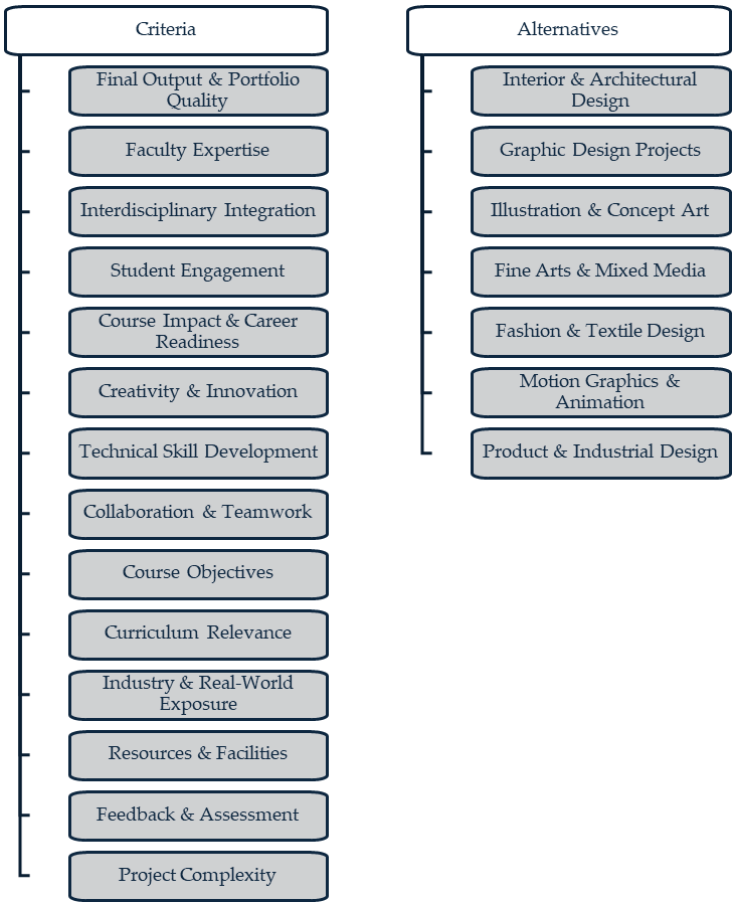

Figure 2. The criteria and types of projects.

4. Results and Analysis

This section shows the results of the evaluation of art and design project courses. This study collects 14 criteria and seven projects to be evaluated. Three experts have evaluated the criteria and alternatives. Figure 2 shows the criteria and alternatives.

4.1 Results T2NSs-Entropy Method

Three experts initially evaluated the criteria and alternatives using the terms listed in Table 1. These qualitative terms were then converted into their corresponding T2NN values, as detailed in Table 2, and subsequently combined into a single comprehensive matrix. In Step 2, Equation (6) is applied to normalize this decision matrix, with the normalized values presented in Table 3. Following this, Step 3 involves using Equation (7) to compute the entropy, which helps in understanding the degree of variation in the data. Finally, in Step 4, Equation (9) is used to calculate the weight vector, as illustrated in Figure 3.

Table 2. The T2NNs between criteria and alternatives.

	A_1	A_2	A_3	A_4	A_5	A_6	A_7
C_1	((0.20,0.20,0.10), (0.65,0.80,0.85), (0.45,0.80,0.70))	((0.20,0.20,0.10), (0.65,0.80,0.85), (0.45,0.80,0.70))	((0.95,0.90,0.95), (0.10,0.10,0.05), (0.05,0.05,0.05))	((0.95,0.90,0.95), (0.10,0.10,0.05), (0.05,0.05,0.05))	((0.70,0.75,0.80), (0.15,0.20,0.25), (0.10,0.15,0.20))	((0.60,0.45,0.50), (0.20,0.15,0.25), (0.10,0.25,0.15))	((0.40,0.45,0.50), (0.40,0.45,0.50), (0.35,0.40,0.45))

C_2	((0.35,0.35,0.10), (0.50,0.75,0.80), (0.50,0.75,0.65))	((0.70,0.75,0.80), (0.15,0.20,0.25), (0.10,0.15,0.20))	((0.95,0.90,0.95), (0.10,0.10,0.05), (0.05,0.05,0.05))	((0.95,0.90,0.95), (0.10,0.10,0.05), (0.05,0.05,0.05))	((0.20,0.20,0.10), (0.65,0.80,0.85), (0.45,0.80,0.70))	((0.35,0.35,0.10), (0.50,0.75,0.80), (0.50,0.75,0.65))	((0.50,0.30,0.50), (0.50,0.35,0.45), (0.45,0.30,0.60))
C_3	((0.50,0.30,0.50), (0.50,0.35,0.45), (0.45,0.30,0.60))	((0.70,0.75,0.80), (0.15,0.20,0.25), (0.10,0.15,0.20))	((0.20,0.20,0.10), (0.65,0.80,0.85), (0.45,0.80,0.70))	((0.40,0.45,0.50), (0.40,0.45,0.50), (0.35,0.40,0.45))	((0.60,0.45,0.50), (0.20,0.15,0.25), (0.10,0.25,0.15))	((0.50,0.30,0.50), (0.50,0.35,0.45), (0.45,0.30,0.60))	((0.35,0.35,0.10), (0.50,0.75,0.80), (0.50,0.75,0.65))
C_4	((0.40,0.45,0.50), (0.40,0.45,0.50), (0.35,0.40,0.45))	((0.60,0.45,0.50), (0.20,0.15,0.25), (0.10,0.25,0.15))	((0.35,0.35,0.10), (0.50,0.75,0.80), (0.50,0.75,0.65))	((0.50,0.30,0.50), (0.50,0.35,0.45), (0.45,0.30,0.60))	((0.70,0.75,0.80), (0.15,0.20,0.25), (0.10,0.15,0.20))	((0.40,0.45,0.50), (0.40,0.45,0.50), (0.35,0.40,0.45))	((0.20,0.20,0.10), (0.65,0.80,0.85), (0.45,0.80,0.70))
C_5	((0.60,0.45,0.50), (0.20,0.15,0.25), (0.10,0.25,0.15))	((0.40,0.45,0.50), (0.40,0.45,0.50), (0.35,0.40,0.45))	((0.50,0.30,0.50), (0.50,0.35,0.45), (0.45,0.30,0.60))	((0.35,0.35,0.10), (0.50,0.75,0.80), (0.50,0.75,0.65))	((0.95,0.90,0.95), (0.10,0.10,0.05), (0.05,0.05,0.05))	((0.60,0.45,0.50), (0.20,0.15,0.25), (0.10,0.25,0.15))	((0.70,0.75,0.80), (0.15,0.20,0.25), (0.10,0.15,0.20))
C_6	((0.70,0.75,0.80), (0.15,0.20,0.25), (0.10,0.15,0.20))	((0.50,0.30,0.50), (0.50,0.35,0.45), (0.45,0.30,0.60))	((0.40,0.45,0.50), (0.40,0.45,0.50), (0.35,0.40,0.45))	((0.20,0.20,0.10), (0.65,0.80,0.85), (0.45,0.80,0.70))	((0.95,0.90,0.95), (0.10,0.10,0.05), (0.05,0.05,0.05))	((0.70,0.75,0.80), (0.15,0.20,0.25), (0.10,0.15,0.20))	((0.60,0.45,0.50), (0.20,0.15,0.25), (0.10,0.25,0.15))
C_7	((0.95,0.90,0.95), (0.10,0.10,0.05), (0.05,0.05,0.05))	((0.35,0.35,0.10), (0.50,0.75,0.80), (0.50,0.75,0.65))	((0.60,0.45,0.50), (0.20,0.15,0.25), (0.10,0.25,0.15))	((0.95,0.90,0.95), (0.10,0.10,0.05), (0.05,0.05,0.05))	((0.20,0.20,0.10), (0.65,0.80,0.85), (0.45,0.80,0.70))	((0.95,0.90,0.95), (0.10,0.10,0.05), (0.05,0.05,0.05))	((0.40,0.45,0.50), (0.40,0.45,0.50), (0.35,0.40,0.45))
C_8	((0.95,0.90,0.95), (0.10,0.10,0.05), (0.05,0.05,0.05))	((0.20,0.20,0.10), (0.65,0.80,0.85), (0.45,0.80,0.70))	((0.70,0.75,0.80), (0.15,0.20,0.25), (0.10,0.15,0.20))	((0.95,0.90,0.95), (0.10,0.10,0.05), (0.05,0.05,0.05))	((0.35,0.35,0.10), (0.50,0.75,0.80), (0.50,0.75,0.65))	((0.95,0.90,0.95), (0.10,0.10,0.05), (0.05,0.05,0.05))	((0.50,0.30,0.50), (0.50,0.35,0.45), (0.45,0.30,0.60))
C_9	((0.20,0.20,0.10), (0.65,0.80,0.85), (0.45,0.80,0.70))	((0.95,0.90,0.95), (0.10,0.10,0.05), (0.05,0.05,0.05))	((0.95,0.90,0.95), (0.10,0.10,0.05), (0.05,0.05,0.05))	((0.70,0.75,0.80), (0.15,0.20,0.25), (0.10,0.15,0.20))	((0.50,0.30,0.50), (0.50,0.35,0.45), (0.45,0.30,0.60))	((0.20,0.20,0.10), (0.65,0.80,0.85), (0.45,0.80,0.70))	((0.35,0.35,0.10), (0.50,0.75,0.80), (0.50,0.75,0.65))
C_{10}	((0.35,0.35,0.10), (0.50,0.75,0.80), (0.50,0.75,0.65))	((0.95,0.90,0.95), (0.10,0.10,0.05), (0.05,0.05,0.05))	((0.95,0.90,0.95), (0.10,0.10,0.05), (0.05,0.05,0.05))	((0.60,0.45,0.50), (0.20,0.15,0.25), (0.10,0.25,0.15))	((0.40,0.45,0.50), (0.40,0.45,0.50), (0.35,0.40,0.45))	((0.35,0.35,0.10), (0.50,0.75,0.80), (0.50,0.75,0.65))	((0.20,0.20,0.10), (0.65,0.80,0.85), (0.45,0.80,0.70))
C_{11}	((0.50,0.30,0.50), (0.50,0.35,0.45), (0.45,0.30,0.60))	((0.70,0.75,0.80), (0.15,0.20,0.25), (0.10,0.15,0.20))	((0.20,0.20,0.10), (0.65,0.80,0.85), (0.45,0.80,0.70))	((0.40,0.45,0.50), (0.40,0.45,0.50), (0.35,0.40,0.45))	((0.95,0.90,0.95), (0.10,0.10,0.05), (0.05,0.05,0.05))	((0.50,0.30,0.50), (0.50,0.35,0.45), (0.45,0.30,0.60))	((0.95,0.90,0.95), (0.10,0.10,0.05), (0.05,0.05,0.05))
C_{12}	((0.40,0.45,0.50), (0.40,0.45,0.50), (0.35,0.40,0.45))	((0.60,0.45,0.50), (0.20,0.15,0.25), (0.10,0.25,0.15))	((0.35,0.35,0.10), (0.50,0.75,0.80), (0.50,0.75,0.65))	((0.50,0.30,0.50), (0.50,0.35,0.45), (0.45,0.30,0.60))	((0.95,0.90,0.95), (0.10,0.10,0.05), (0.05,0.05,0.05))	((0.40,0.45,0.50), (0.40,0.45,0.50), (0.35,0.40,0.45))	((0.95,0.90,0.95), (0.10,0.10,0.05), (0.05,0.05,0.05))
C_{13}	((0.60,0.45,0.50), (0.20,0.15,0.25), (0.10,0.25,0.15))	((0.40,0.45,0.50), (0.40,0.45,0.50), (0.35,0.40,0.45))	((0.50,0.30,0.50), (0.50,0.35,0.45), (0.45,0.30,0.60))	((0.35,0.35,0.10), (0.50,0.75,0.80), (0.50,0.75,0.65))	((0.20,0.20,0.10), (0.65,0.80,0.85), (0.45,0.80,0.70))	((0.60,0.45,0.50), (0.20,0.15,0.25), (0.10,0.25,0.15))	((0.70,0.75,0.80), (0.15,0.20,0.25), (0.10,0.15,0.20))
C_{14}	((0.70,0.75,0.80), (0.15,0.20,0.25), (0.10,0.15,0.20))	((0.95,0.90,0.95), (0.10,0.10,0.05), (0.05,0.05,0.05))	((0.95,0.90,0.95), (0.10,0.10,0.05), (0.05,0.05,0.05))	((0.20,0.20,0.10), (0.65,0.80,0.85), (0.45,0.80,0.70))	((0.35,0.35,0.10), (0.50,0.75,0.80), (0.50,0.75,0.65))	((0.70,0.75,0.80), (0.15,0.20,0.25), (0.10,0.15,0.20))	((0.40,0.45,0.50), (0.40,0.45,0.50), (0.35,0.40,0.45))
	A_1	A_2	A_3	A_4	A_5	A_6	A_7
C_1	((0.20,0.20,0.10), (0.65,0.80,0.85), (0.45,0.80,0.70))	((0.70,0.75,0.80), (0.15,0.20,0.25), (0.10,0.15,0.20))	((0.95,0.90,0.95), (0.10,0.10,0.05), (0.05,0.05,0.05))	((0.95,0.90,0.95), (0.10,0.10,0.05), (0.05,0.05,0.05))	((0.20,0.20,0.10), (0.65,0.80,0.85), (0.45,0.80,0.70))	((0.60,0.45,0.50), (0.20,0.15,0.25), (0.10,0.25,0.15))	((0.40,0.45,0.50), (0.40,0.45,0.50), (0.35,0.40,0.45))
C_2	((0.35,0.35,0.10), (0.50,0.75,0.80), (0.50,0.75,0.65))	((0.70,0.75,0.80), (0.15,0.20,0.25), (0.10,0.15,0.20))	((0.95,0.90,0.95), (0.10,0.10,0.05), (0.05,0.05,0.05))	((0.70,0.75,0.80), (0.15,0.20,0.25), (0.10,0.15,0.20))	((0.95,0.90,0.95), (0.10,0.10,0.05), (0.05,0.05,0.05))	((0.95,0.90,0.95), (0.10,0.10,0.05), (0.05,0.05,0.05))	((0.20,0.20,0.10), (0.65,0.80,0.85), (0.45,0.80,0.70))
C_3	((0.50,0.30,0.50), (0.50,0.35,0.45), (0.45,0.30,0.60))	((0.70,0.75,0.80), (0.15,0.20,0.25), (0.10,0.15,0.20))	((0.70,0.75,0.80), (0.15,0.20,0.25), (0.10,0.15,0.20))	((0.95,0.90,0.95), (0.10,0.10,0.05), (0.05,0.05,0.05))	((0.95,0.90,0.95), (0.10,0.10,0.05), (0.05,0.05,0.05))	((0.20,0.20,0.10), (0.65,0.80,0.85), (0.45,0.80,0.70))	((0.35,0.35,0.10), (0.50,0.75,0.80), (0.50,0.75,0.65))
C_4	((0.40,0.45,0.50), (0.40,0.45,0.50), (0.35,0.40,0.45))	((0.70,0.75,0.80), (0.15,0.20,0.25), (0.10,0.15,0.20))	((0.95,0.90,0.95), (0.10,0.10,0.05), (0.05,0.05,0.05))	((0.95,0.90,0.95), (0.10,0.10,0.05), (0.05,0.05,0.05))	((0.20,0.20,0.10), (0.65,0.80,0.85), (0.45,0.80,0.70))	((0.40,0.45,0.50), (0.40,0.45,0.50), (0.35,0.40,0.45))	((0.20,0.20,0.10), (0.65,0.80,0.85), (0.45,0.80,0.70))
C_5	((0.60,0.45,0.50), (0.20,0.15,0.25), (0.10,0.25,0.15))	((0.70,0.75,0.80), (0.15,0.20,0.25), (0.10,0.15,0.20))	((0.95,0.90,0.95), (0.10,0.10,0.05), (0.05,0.05,0.05))	((0.95,0.90,0.95), (0.10,0.10,0.05), (0.05,0.05,0.05))	((0.20,0.20,0.10), (0.65,0.80,0.85), (0.45,0.80,0.70))	((0.60,0.45,0.50), (0.20,0.15,0.25), (0.10,0.25,0.15))	((0.70,0.75,0.80), (0.15,0.20,0.25), (0.10,0.15,0.20))
C_6	((0.70,0.75,0.80), (0.15,0.20,0.25), (0.10,0.15,0.20))	((0.95,0.90,0.95), (0.10,0.10,0.05), (0.05,0.05,0.05))	((0.95,0.90,0.95), (0.10,0.10,0.05), (0.05,0.05,0.05))	((0.20,0.20,0.10), (0.65,0.80,0.85), (0.45,0.80,0.70))	((0.95,0.90,0.95), (0.10,0.10,0.05), (0.05,0.05,0.05))	((0.70,0.75,0.80), (0.15,0.20,0.25), (0.10,0.15,0.20))	((0.60,0.45,0.50), (0.20,0.15,0.25), (0.10,0.25,0.15))
C_7	((0.95,0.90,0.95), (0.10,0.10,0.05), (0.05,0.05,0.05))	((0.70,0.75,0.80), (0.15,0.20,0.25), (0.10,0.15,0.20))	((0.95,0.90,0.95), (0.10,0.10,0.05), (0.05,0.05,0.05))	((0.95,0.90,0.95), (0.10,0.10,0.05), (0.05,0.05,0.05))	((0.20,0.20,0.10), (0.65,0.80,0.85), (0.45,0.80,0.70))	((0.95,0.90,0.95), (0.10,0.10,0.05), (0.05,0.05,0.05))	((0.40,0.45,0.50), (0.40,0.45,0.50), (0.35,0.40,0.45))
C_8	((0.95,0.90,0.95), (0.10,0.10,0.05), (0.05,0.05,0.05))	((0.20,0.20,0.10), (0.65,0.80,0.85), (0.45,0.80,0.70))	((0.70,0.75,0.80), (0.15,0.20,0.25), (0.10,0.15,0.20))	((0.95,0.90,0.95), (0.10,0.10,0.05), (0.05,0.05,0.05))	((0.95,0.90,0.95), (0.10,0.10,0.05), (0.05,0.05,0.05))	((0.20,0.20,0.10), (0.65,0.80,0.85), (0.45,0.80,0.70))	((0.50,0.30,0.50), (0.50,0.35,0.45), (0.45,0.30,0.60))
C_9	((0.20,0.20,0.10), (0.65,0.80,0.85), (0.45,0.80,0.70))	((0.95,0.90,0.95), (0.10,0.10,0.05), (0.05,0.05,0.05))	((0.95,0.90,0.95), (0.10,0.10,0.05), (0.05,0.05,0.05))	((0.70,0.75,0.80), (0.15,0.20,0.25), (0.10,0.15,0.20))	((0.95,0.90,0.95), (0.10,0.10,0.05), (0.05,0.05,0.05))	((0.95,0.90,0.95), (0.10,0.10,0.05), (0.05,0.05,0.05))	((0.20,0.20,0.10), (0.65,0.80,0.85), (0.45,0.80,0.70))

C_{10}	((0.35,0.35,0.10), (0.50,0.75,0.80), (0.50,0.75,0.65))	((0.95,0.90,0.95), (0.10,0.10,0.05), (0.05,0.05,0.05))	((0.70,0.75,0.80), (0.15,0.20,0.25), (0.10,0.15,0.20))	((0.95,0.90,0.95), (0.10,0.10,0.05), (0.05,0.05,0.05))	((0.95,0.90,0.95), (0.10,0.10,0.05), (0.05,0.05,0.05))	((0.20,0.20,0.10), (0.65,0.80,0.85), (0.45,0.80,0.70))	((0.20,0.20,0.10), (0.65,0.80,0.85), (0.45,0.80,0.70))
C_{11}	((0.50,0.30,0.50), (0.50,0.35,0.45), (0.45,0.30,0.60))	((0.70,0.75,0.80), (0.15,0.20,0.25), (0.10,0.15,0.20))	((0.20,0.20,0.10), (0.65,0.80,0.85), (0.45,0.80,0.70))	((0.70,0.75,0.80), (0.15,0.20,0.25), (0.10,0.15,0.20))	((0.95,0.90,0.95), (0.10,0.10,0.05), (0.05,0.05,0.05))	((0.95,0.90,0.95), (0.10,0.10,0.05), (0.05,0.05,0.05))	((0.20,0.20,0.10), (0.65,0.80,0.85), (0.45,0.80,0.70))
C_{12}	((0.40,0.45,0.50), (0.40,0.45,0.50), (0.35,0.40,0.45))	((0.60,0.45,0.50), (0.20,0.15,0.25), (0.10,0.25,0.15))	((0.70,0.75,0.80), (0.15,0.20,0.25), (0.10,0.15,0.20))	((0.95,0.90,0.95), (0.10,0.10,0.05), (0.05,0.05,0.05))	((0.95,0.90,0.95), (0.10,0.10,0.05), (0.05,0.05,0.05))	((0.20,0.20,0.10), (0.65,0.80,0.85), (0.45,0.80,0.70))	((0.95,0.90,0.95), (0.10,0.10,0.05), (0.05,0.05,0.05))
C_{13}	((0.60,0.45,0.50), (0.20,0.15,0.25), (0.10,0.25,0.15))	((0.70,0.75,0.80), (0.15,0.20,0.25), (0.10,0.15,0.20))	((0.95,0.90,0.95), (0.10,0.10,0.05), (0.05,0.05,0.05))	((0.95,0.90,0.95), (0.10,0.10,0.05), (0.05,0.05,0.05))	((0.20,0.20,0.10), (0.65,0.80,0.85), (0.45,0.80,0.70))	((0.60,0.45,0.50), (0.20,0.15,0.25), (0.10,0.25,0.15))	((0.70,0.75,0.80), (0.15,0.20,0.25), (0.10,0.15,0.20))
C_{14}	((0.70,0.75,0.80), (0.15,0.20,0.25), (0.10,0.15,0.20))	((0.95,0.90,0.95), (0.10,0.10,0.05), (0.05,0.05,0.05))	((0.95,0.90,0.95), (0.10,0.10,0.05), (0.05,0.05,0.05))	((0.20,0.20,0.10), (0.65,0.80,0.85), (0.45,0.80,0.70))	((0.35,0.35,0.10), (0.50,0.75,0.80), (0.50,0.75,0.65))	((0.70,0.75,0.80), (0.15,0.20,0.25), (0.10,0.15,0.20))	((0.40,0.45,0.50), (0.40,0.45,0.50), (0.35,0.40,0.45))
	A_1	A_2	A_3	A_4	A_5	A_6	A_7
C_1	((0.70,0.75,0.80), (0.15,0.20,0.25), (0.10,0.15,0.20))	((0.50,0.30,0.50), (0.50,0.35,0.45), (0.45,0.30,0.60))	((0.40,0.45,0.50), (0.40,0.45,0.50), (0.35,0.40,0.45))	((0.20,0.20,0.10), (0.65,0.80,0.85), (0.45,0.80,0.70))	((0.70,0.75,0.80), (0.15,0.20,0.25), (0.10,0.15,0.20))	((0.60,0.45,0.50), (0.20,0.15,0.25), (0.10,0.25,0.15))	((0.40,0.45,0.50), (0.40,0.45,0.50), (0.35,0.40,0.45))
C_2	((0.35,0.35,0.10), (0.50,0.75,0.80), (0.50,0.75,0.65))	((0.70,0.75,0.80), (0.15,0.20,0.25), (0.10,0.15,0.20))	((0.50,0.30,0.50), (0.50,0.35,0.45), (0.45,0.30,0.60))	((0.40,0.45,0.50), (0.40,0.45,0.50), (0.35,0.40,0.45))	((0.20,0.20,0.10), (0.65,0.80,0.85), (0.45,0.80,0.70))	((0.35,0.35,0.10), (0.50,0.75,0.80), (0.50,0.75,0.65))	((0.50,0.30,0.50), (0.50,0.35,0.45), (0.45,0.30,0.60))
C_3	((0.50,0.30,0.50), (0.50,0.35,0.45), (0.45,0.30,0.60))	((0.70,0.75,0.80), (0.15,0.20,0.25), (0.10,0.15,0.20))	((0.70,0.75,0.80), (0.15,0.20,0.25), (0.10,0.15,0.20))	((0.50,0.30,0.50), (0.50,0.35,0.45), (0.45,0.30,0.60))	((0.40,0.45,0.50), (0.40,0.45,0.50), (0.35,0.40,0.45))	((0.20,0.20,0.10), (0.65,0.80,0.85), (0.45,0.80,0.70))	((0.35,0.35,0.10), (0.50,0.75,0.80), (0.50,0.75,0.65))
C_4	((0.40,0.45,0.50), (0.40,0.45,0.50), (0.35,0.40,0.45))	((0.60,0.45,0.50), (0.20,0.15,0.25), (0.10,0.25,0.15))	((0.70,0.75,0.80), (0.15,0.20,0.25), (0.10,0.15,0.20))	((0.50,0.30,0.50), (0.50,0.35,0.45), (0.45,0.30,0.60))	((0.40,0.45,0.50), (0.40,0.45,0.50), (0.35,0.40,0.45))	((0.20,0.20,0.10), (0.65,0.80,0.85), (0.45,0.80,0.70))	((0.20,0.20,0.10), (0.65,0.80,0.85), (0.45,0.80,0.70))
C_5	((0.60,0.45,0.50), (0.20,0.15,0.25), (0.10,0.25,0.15))	((0.70,0.75,0.80), (0.15,0.20,0.25), (0.10,0.15,0.20))	((0.50,0.30,0.50), (0.50,0.35,0.45), (0.45,0.30,0.60))	((0.40,0.45,0.50), (0.40,0.45,0.50), (0.35,0.40,0.45))	((0.20,0.20,0.10), (0.65,0.80,0.85), (0.45,0.80,0.70))	((0.60,0.45,0.50), (0.20,0.15,0.25), (0.10,0.25,0.15))	((0.70,0.75,0.80), (0.15,0.20,0.25), (0.10,0.15,0.20))
C_6	((0.70,0.75,0.80), (0.15,0.20,0.25), (0.10,0.15,0.20))	((0.50,0.30,0.50), (0.50,0.35,0.45), (0.45,0.30,0.60))	((0.40,0.45,0.50), (0.40,0.45,0.50), (0.35,0.40,0.45))	((0.20,0.20,0.10), (0.65,0.80,0.85), (0.45,0.80,0.70))	((0.95,0.90,0.95), (0.10,0.10,0.05), (0.05,0.05,0.05))	((0.70,0.75,0.80), (0.15,0.20,0.25), (0.10,0.15,0.20))	((0.60,0.45,0.50), (0.20,0.15,0.25), (0.10,0.25,0.15))
C_7	((0.95,0.90,0.95), (0.10,0.10,0.05), (0.05,0.05,0.05))	((0.35,0.35,0.10), (0.50,0.75,0.80), (0.50,0.75,0.65))	((0.70,0.75,0.80), (0.15,0.20,0.25), (0.10,0.15,0.20))	((0.50,0.30,0.50), (0.50,0.35,0.45), (0.45,0.30,0.60))	((0.40,0.45,0.50), (0.40,0.45,0.50), (0.35,0.40,0.45))	((0.20,0.20,0.10), (0.65,0.80,0.85), (0.45,0.80,0.70))	((0.40,0.45,0.50), (0.40,0.45,0.50), (0.35,0.40,0.45))
C_8	((0.70,0.75,0.80), (0.15,0.20,0.25), (0.10,0.15,0.20))	((0.50,0.30,0.50), (0.50,0.35,0.45), (0.45,0.30,0.60))	((0.40,0.45,0.50), (0.40,0.45,0.50), (0.35,0.40,0.45))	((0.20,0.20,0.10), (0.65,0.80,0.85), (0.45,0.80,0.70))	((0.35,0.35,0.10), (0.50,0.75,0.80), (0.50,0.75,0.65))	((0.95,0.90,0.95), (0.10,0.10,0.05), (0.05,0.05,0.05))	((0.50,0.30,0.50), (0.50,0.35,0.45), (0.45,0.30,0.60))
C_9	((0.20,0.20,0.10), (0.65,0.80,0.85), (0.45,0.80,0.70))	((0.70,0.75,0.80), (0.15,0.20,0.25), (0.10,0.15,0.20))	((0.50,0.30,0.50), (0.50,0.35,0.45), (0.45,0.30,0.60))	((0.40,0.45,0.50), (0.40,0.45,0.50), (0.35,0.40,0.45))	((0.20,0.20,0.10), (0.65,0.80,0.85), (0.45,0.80,0.70))	((0.20,0.20,0.10), (0.65,0.80,0.85), (0.45,0.80,0.70))	((0.35,0.35,0.10), (0.50,0.75,0.80), (0.50,0.75,0.65))
C_{10}	((0.35,0.35,0.10), (0.50,0.75,0.80), (0.50,0.75,0.65))	((0.95,0.90,0.95), (0.10,0.10,0.05), (0.05,0.05,0.05))	((0.70,0.75,0.80), (0.15,0.20,0.25), (0.10,0.15,0.20))	((0.50,0.30,0.50), (0.50,0.35,0.45), (0.45,0.30,0.60))	((0.40,0.45,0.50), (0.40,0.45,0.50), (0.35,0.40,0.45))	((0.20,0.20,0.10), (0.65,0.80,0.85), (0.45,0.80,0.70))	((0.20,0.20,0.10), (0.65,0.80,0.85), (0.45,0.80,0.70))
C_{11}	((0.50,0.30,0.50), (0.50,0.35,0.45), (0.45,0.30,0.60))	((0.70,0.75,0.80), (0.15,0.20,0.25), (0.10,0.15,0.20))	((0.50,0.30,0.50), (0.50,0.35,0.45), (0.45,0.30,0.60))	((0.40,0.45,0.50), (0.40,0.45,0.50), (0.35,0.40,0.45))	((0.20,0.20,0.10), (0.65,0.80,0.85), (0.45,0.80,0.70))	((0.50,0.30,0.50), (0.50,0.35,0.45), (0.45,0.30,0.60))	((0.95,0.90,0.95), (0.10,0.10,0.05), (0.05,0.05,0.05))
C_{12}	((0.70,0.75,0.80), (0.15,0.20,0.25), (0.10,0.15,0.20))	((0.50,0.30,0.50), (0.50,0.35,0.45), (0.45,0.30,0.60))	((0.40,0.45,0.50), (0.40,0.45,0.50), (0.35,0.40,0.45))	((0.20,0.20,0.10), (0.65,0.80,0.85), (0.45,0.80,0.70))	((0.95,0.90,0.95), (0.10,0.10,0.05), (0.05,0.05,0.05))	((0.40,0.45,0.50), (0.40,0.45,0.50), (0.35,0.40,0.45))	((0.95,0.90,0.95), (0.10,0.10,0.05), (0.05,0.05,0.05))
C_{13}	((0.60,0.45,0.50), (0.20,0.15,0.25), (0.10,0.25,0.15))	((0.70,0.75,0.80), (0.15,0.20,0.25), (0.10,0.15,0.20))	((0.50,0.30,0.50), (0.50,0.35,0.45), (0.45,0.30,0.60))	((0.40,0.45,0.50), (0.40,0.45,0.50), (0.35,0.40,0.45))	((0.20,0.20,0.10), (0.65,0.80,0.85), (0.45,0.80,0.70))	((0.60,0.45,0.50), (0.20,0.15,0.25), (0.10,0.25,0.15))	((0.70,0.75,0.80), (0.15,0.20,0.25), (0.10,0.15,0.20))
C_{14}	((0.70,0.75,0.80), (0.15,0.20,0.25), (0.10,0.15,0.20))	((0.50,0.30,0.50), (0.50,0.35,0.45), (0.45,0.30,0.60))	((0.40,0.45,0.50), (0.40,0.45,0.50), (0.35,0.40,0.45))	((0.20,0.20,0.10), (0.65,0.80,0.85), (0.45,0.80,0.70))	((0.35,0.35,0.10), (0.50,0.75,0.80), (0.50,0.75,0.65))	((0.70,0.75,0.80), (0.15,0.20,0.25), (0.10,0.15,0.20))	((0.40,0.45,0.50), (0.40,0.45,0.50), (0.35,0.40,0.45))

Table 3. The normalized decision matrix by the Entropy method.

	A_1	A_2	A_3	A_4	A_5	A_6	A_7
C_1	0.098933	0.121242	0.18558	0.162625	0.14258	0.164888	0.124151
C_2	0.075768	0.196587	0.195222	0.185324	0.115017	0.126621	0.105461

C_3	0.132399	0.201751	0.154466	0.167075	0.182487	0.084063	0.077758
C_4	0.140095	0.19409	0.178402	0.173294	0.137541	0.114192	0.062386
C_5	0.152466	0.153064	0.142003	0.127055	0.100747	0.152466	0.172197
C_6	0.166667	0.137442	0.1386	0.049479	0.193576	0.166667	0.147569
C_7	0.203035	0.103187	0.177542	0.173596	0.073445	0.152656	0.11654
C_8	0.202862	0.076312	0.162798	0.159936	0.117965	0.159936	0.120191
C_9	0.06019	0.224569	0.201338	0.180218	0.142907	0.11862	0.072158
C_{10}	0.077758	0.234326	0.212609	0.181786	0.167776	0.065849	0.059895
C_{11}	0.121037	0.184438	0.076849	0.143452	0.161063	0.152097	0.161063
C_{12}	0.133135	0.138484	0.117088	0.120654	0.198811	0.093016	0.198811
C_{13}	0.160428	0.161057	0.149418	0.13369	0.053791	0.160428	0.181189
C_{14}	0.187317	0.186016	0.186667	0.05561	0.072195	0.187317	0.124878

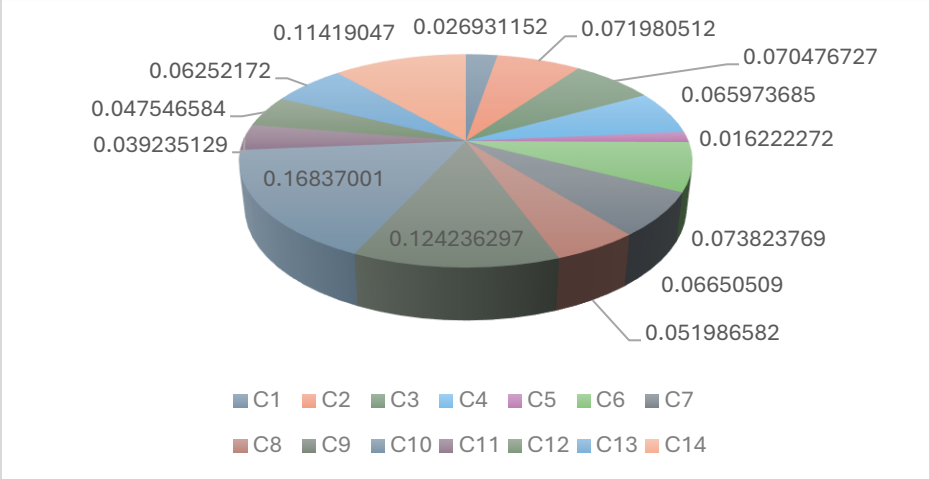

Figure 3. The criteria weights.

4.2 Results T2NSs-MOOSRA Method

In step 5, Equation (10) is used to normalize the decision matrix, ensuring that all criteria values are standardized for effective comparison, as illustrated in Table 4. Next, in step 6, Equation (11) computes the weighted normalized decision matrix by incorporating the relative importance of each criterion, with the resulting values presented in Table 5. Finally, in step 7, Equation (12) is applied to rank the alternatives based on these weighted values, and the final rankings are displayed in Table 6.

Table 4. The normalized decision matrix by the MOOSRA method.

	A_1	A_2	A_3	A_4	A_5	A_6	A_7
C_1	0.155288	0.190304	0.291292	0.255261	0.223797	0.258813	0.194871
C_2	0.118419	0.307249	0.305115	0.289646	0.179762	0.197898	0.164826
C_3	0.21346	0.325272	0.249036	0.269366	0.294213	0.13553	0.125365
C_4	0.237411	0.328913	0.302327	0.293672	0.233083	0.193514	0.105722
C_5	0.224733	0.225614	0.20931	0.187277	0.1485	0.224733	0.253816
C_6	0.223506	0.184315	0.185867	0.066353	0.259593	0.223506	0.197896
C_7	0.284748	0.144715	0.248995	0.243462	0.103003	0.214093	0.163443
C_8	0.303542	0.114185	0.243595	0.239313	0.176511	0.239313	0.179842

C_9	0.091468	0.341267	0.305963	0.273869	0.21717	0.180262	0.109655
C_{10}	0.111879	0.33715	0.305905	0.261556	0.241397	0.094745	0.086177
C_{11}	0.185825	0.283161	0.117984	0.220237	0.247275	0.23351	0.247275
C_{12}	0.186341	0.193828	0.16388	0.168871	0.278263	0.130189	0.278263
C_{13}	0.236963	0.237892	0.220701	0.197469	0.079452	0.236963	0.267628
C_{14}	0.26822	0.266357	0.267289	0.079628	0.103376	0.26822	0.178813

Table 5. The weighted normalized decision matrix by the MOOSRA method.

	A_1	A_2	A_3	A_4	A_5	A_6	A_7
C_1	0.004182	0.005125	0.007845	0.006874	0.006027	0.00697	0.005248
C_2	0.008524	0.022116	0.021962	0.020849	0.012939	0.014245	0.011864
C_3	0.015044	0.022924	0.017551	0.018984	0.020735	0.009552	0.008835
C_4	0.015663	0.0217	0.019946	0.019375	0.015377	0.012767	0.006975
C_5	0.003646	0.00366	0.003395	0.003038	0.002409	0.003646	0.004117
C_6	0.0165	0.013607	0.013721	0.004898	0.019164	0.0165	0.014609
C_7	0.018937	0.009624	0.016559	0.016191	0.00685	0.014238	0.01087
C_8	0.01578	0.005936	0.012664	0.012441	0.009176	0.012441	0.009349
C_9	0.011364	0.042398	0.038012	0.034025	0.02698	0.022395	0.013623
C_{10}	0.018837	0.056766	0.051505	0.044038	0.040644	0.015952	0.01451
C_{11}	0.007291	0.01111	0.004629	0.008641	0.009702	0.009162	0.009702
C_{12}	0.00886	0.009216	0.007792	0.008029	0.01323	0.00619	0.01323
C_{13}	0.014815	0.014873	0.013799	0.012346	0.004967	0.014815	0.016733
C_{14}	0.030628	0.030415	0.030522	0.009093	0.011805	0.030628	0.020419

Table 6. The rank of alternatives.

	Values	Rank
A_1	0.190071	3
A_2	0.26947	6
A_3	0.259903	5
A_4	0.218823	4
A_5	0.200007	3
A_6	0.189501	2
A_7	0.160085	1

5. Sensitivity analysis

A sensitivity analysis was conducted to assess how changes in criteria weights affect the ranking of alternatives. The study examined fifteen different weighting scenarios, as illustrated in Figure 4. In the first scenario, equal weights were assigned to all criteria. In the second scenario, the weight of the first criterion was increased by 10%, while the weights of the other criteria remained unchanged. Similarly, in the third scenario, a 10% increase was applied to the weight of the second criterion, with the other criteria maintaining their original values. The MOOSRA method was then applied to each of the fifteen cases to determine the rankings of the alternatives. For every case, both the normalized decision matrix and the weighted normalized decision matrix were

computed before ranking the alternatives. Figure 5 displays the rankings across these cases, consistently showing that alternative 2 ranks as the best option while alternative 7 ranks as the worst. Overall, the results indicate that the ranking of alternatives remains stable despite the variations in criteria weights.

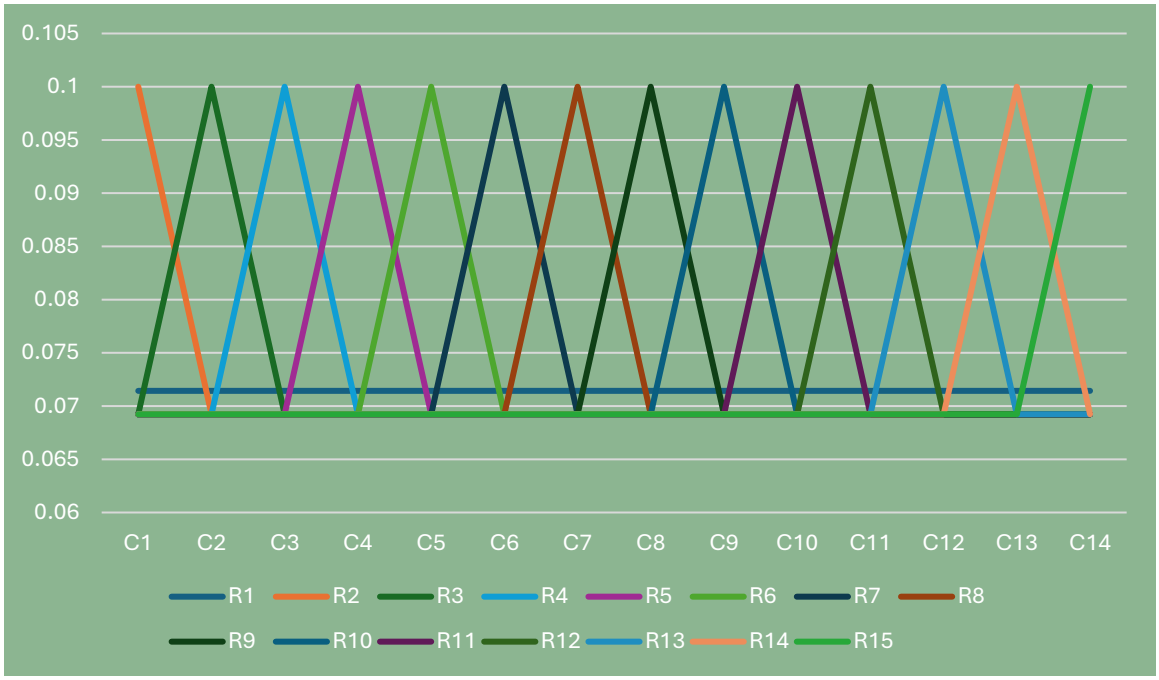

Figure 4. The criteria weights with different cases.

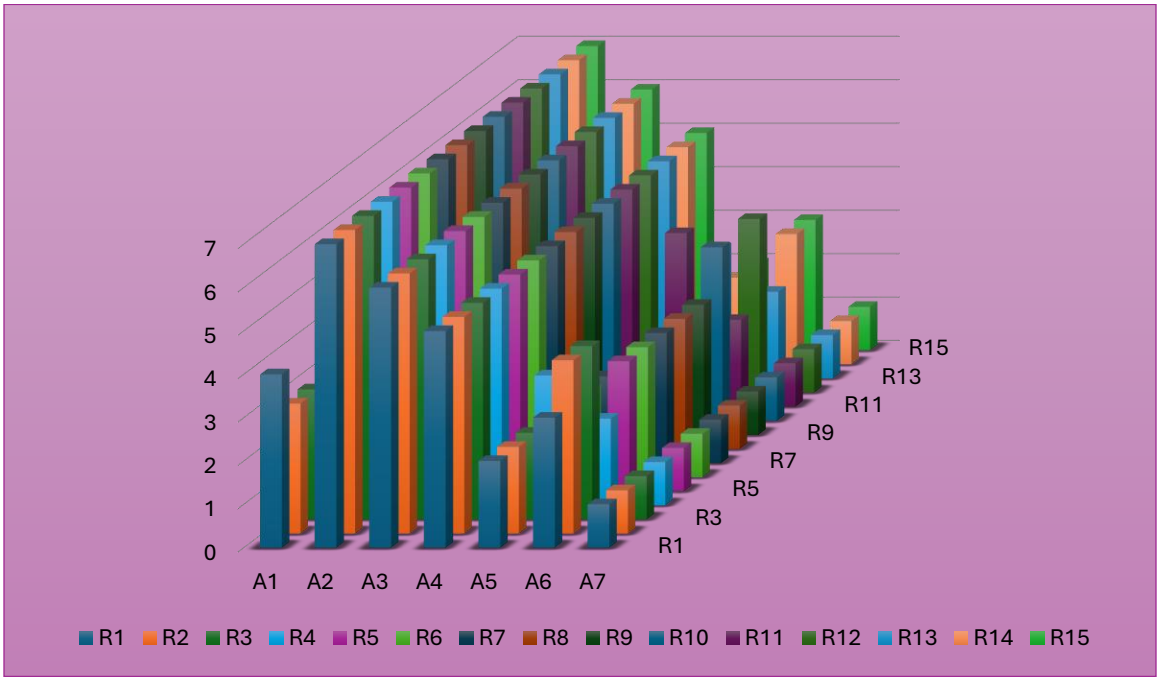

Figure 5. The rank of alternatives under different cases.

6. Conclusions

This study has made several significant contributions. First, it introduced distinct definitions and operations for type-2 neutrosophic sets to effectively handle uncertainty and vague data. Second, the research applied various Multi-Criteria Decision-Making (MCDM) methods, such as the Entropy and MOOSRA methods, to calculate criteria weights and rank the alternatives. Third, a detailed case study was conducted to identify the best project based on art design and different courses. Fourth, a sensitivity analysis was performed to examine the ranking of alternatives under various criteria weight scenarios, demonstrating that the rankings remain stable regardless of the changes. Looking ahead, future research will investigate additional techniques for type-2 neutrosophic sets, including the CRITIC method, correlation coefficient, and similarity metrics. This study is the first of its kind in the field to rank art design projects using these innovative approaches.

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Similarity measure and sine exponential measure of possibility interval-valued neutrosophic hypersoft sets and their applications

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Abstract. The Hypersoft set (HSS) has been created as an extension of a soft set (SS) in order to address limitations for the thought of disjoint attribute-valued sets corresponding to distinct attributes. The HSSs melted with several fuzziness structures present in literature to handle real-world scenarios with complexity and uncertainty. One such extension is the interval-valued neutrosophic hypersoft set (iv-NHSS), which focuses on the partitioning of each attribute into its attribute-valued set with these attribute-valued sets, three iv-NHS-memberships degree line closed interval [0,1]. However, the iv-NHSS model is short of assessing the vague nature of parameters and sub-parameters that cause some doubt in DM opinions. This work focuses primarily on show the impact the degree of fuzzy possibility on efficient work of iv-NHSS when we introduce novel concepts of possibility interval valued neutrosophic hyper soft set (in short piv-NHSS) by giving all iv-NHSs-values, the fuzzy possibility degree increases the efficiency for the multi-argument approximately. Therefore, the axioms of set theory such as piv-NHS-subset, piv-NHS-null set, piv-NHS-absolute set and piv-NHSS-complement defined on this concept, as well as we investigated the union, intersection, AND, OR of two piv-NHSSs, and relevant laws, with the help of several numerical examples and piv-NHS-matrix representations. Moreover, we discovered both similarities measure and sine exponential measure between two piv-NHSSs and these measures are successfully applied in decision-making to judge choose the best tourist place. In the end, a conclusion of this work is presented with some suggestions for future studies of this work.

Keywords: Interval-valued neutrosophic set; soft set; hypersoft set; interval-valued neutrosophic hypersoft set; possibility interval-valued neutrosophic hypersoft set; similarity measures; decision-making.

1. **Introduction**

Neutrosophic set (NS) [1] is conceptualized by Smarandache as a new mathematical generalization of fuzzy sets (FSs) [2] and intuitionistic fuzzy sets (IFSs) [3] to tackle many intricate problems involving numerous inconsistent, uncertain, indeterminate, and incomplete information. The NS structures are characterized by three membership functions, i.e. true membership (T-membership), indeterminacy-membership (I-membership), and non-membership (F-membership) respectively. The mechanism of action of these structures (T-membership, I-membership, and F-membership) is characterized by their ability to represent the components of the universal set to a degree ranging between 0 and 1. In life applications, which the universal set can represent, it is necessary to define the three functions of NS; therefore Wang et al. [4] proposed the idea of single value-NS (sv-NS) and show its operations and some algebraic properties. Then, in order to take advantage of the flexibility provided by the interval structure, Wang et al. [5] develop sv-NS to interval-valued -NS (iv-NS) where every NS-structure expresses as an interval value. The research works on sv-NS and iv-NS and their related algebraic structures have been employed in a lot of different real-life fields. For example ahin and Kk [6] introduced a neutrosophic subset hood measure for sv-NS to deal with neutrosophic information. Huang [7] defined new distance measures of sv-NS and checked its applications. Luo et al. [8] explored the VIKOR design-making method on sv-NSs. Hashim et al. [9] proposed two kinds of entropy measures on iv-NSs and showed their application. Al-sharqi et al [10]- [15] presented many works on both sv-NS and iv-NSs and explained the mechanism for using these concepts in solving some life problems related to the mechanism for choosing the best option. Dealing with these tools has shown a barrier in dealing with uncertain data and unclear objects that character parameterization tools to handle this issue, Molodtsov [16] offered a new parameterization tool named soft set (SS). From a mathematical perspective, the SS provides a clear description of the components of the universal set, for example, if we suggest that the universal set X contains three houses x_1, x_2, x_3 then SS provide a description of the characteristics that distinguish these three houses. The SSs and iv-NSs inspired many mathematicians to present many research works. Deli [17] smelted the iv-NS in the SS to form a iv-N-soft set (iv-NSS) and gossiping about its important applications. Ihsan et al. [18] depicted iv-NSS as a hybrid model under an expert system and utilized this model in instruction supply chain management to select a suitable supplier for a construction project.

Al-Sharqi et al. [19]- [20] proposed several ideas on iv-NSS under complex environments and showed how to use these ideas in real-life situations. And many research works of high scientific value that dealt with these concepts with high quality as shown [21]– [25] .

On the other hand, in 2018, Smarandache explained that each attribute has sub-attributes, for example, the attribute of size, which is divided into large, medium, or small. This means that the concept of the soft set is no longer sufficient to cover the attributes of the elements of the universal set. Accordingly, in order to cover this gap, an expanded set of the SS was launched, called the hypersoft set (HSS) [26], which is characterized by containing the sub-elements of the SS. As mathematical works for this HSS: The essential operation of HSSs is defined by Saeed et al. [27]. As a developed of the Musa and Asaad [28] idea Al-Quran et al. [29] introduced the idea of BFHSSs. The distance measures employed between two IVFHSSs [30] and tests in the application of suitable medication. The concept of neutrosophic HSSs proposed by Zulqarnain et al. [31]. In addition, there are a lot successful studies to combine the HSS with NS-structures see [32]- [37] . In addition to all of the above, the preference level for the parameter value is 0, i.e., because there is no function that cares about this procedure. Accordingly, the probability degree, whose value ranges between 0 and 1, is considered a suitable tool to clarify the degree of interest of the user. Accordingly, this idea has received wide attention from researchers, who have provided many fuzzy tools, including: Possibility of polar fuzzy soft sets (PPFSSs) characterized by Khalil et al. [38] and the possibility of IVFS-set (PIVFS-set) investigated by Fu et al. [39]. Similarity measures between two possibility neutrosophic soft expert sets (PNSESs) and some elementary axioms and algebraic operations. Possibility degree of single-valued neutrosophic hypersoft set (psv-NHSS) integrated by Rahman et al. [40] and used resolve the solid waste site selection problem. Recently, Romdhini et al. [41] discussed three decision-making algorithms based on possibility-iv-FSSs (piv-FSSs). Based on what was mentioned above and what was mentioned in these previous studies, the main contributions that were given to this work as following:

1.Construct piv-NHSS as integrating both iv-NSs and HSS with the probability degree of all sub-parameters of iv-NHSS.

2.Define all the basic set operations on piv-NHSS and explain them with numerical examples.

3.Introduce some mathematical properties of all set operations and work on proving them.

4.Define similarity measures between two piv-NHSSs and build a new multi step algorithm based on these measures to handle one real-life application.

5. Define a sine exponential measure between two piv-NHSSs and build a new multi step algorithm based on these measures to handle one real-life application.

All the rest of this article's results are systematized in the following Figure 1:

Section 2
• We revisit some critical terminologies and properties regarding our idea.
Section 3
• We presented the main definition of pin-NHSS with properties.
Section 4
• The set-theoretic operations of piv-NHSSs are characterized in this section.
Section 5
• In this section we proposes a similarity between two piv-NHSS with application.
Section 6
• In this section we proposes a sine exponential measure between two piv-NHSS with application.
Section 7
• The conclusion was discussed in Section 7

Figure 1: Represents of paper organization.

2. **Preliminaries**

In this part, we revisit some critical definitions and properties for our idea

Definition 2.1. [5] An iv-NS $\widehat{\Gamma}$ on a fixed set $\ddot{\mathcal{Z}} = \left\{\ddot{z}^1, \ddot{z}^2, \ddot{z}^3, ..., \ddot{z}^n\right\}$ is stated as $\widehat{\Gamma}(\ddot{z}^n) = \langle \dagger_{\mathbb{T}}(\ddot{z}^n), \dagger_{\mathbb{I}}(\ddot{z}^n), \dagger_{\mathbb{F}}(\ddot{z}^n)\rangle$ where $\dagger_{\mathbb{T}}(\ddot{z}^n) = \left\langle\left[\dagger^l_{\mathbb{T}}(\ddot{z}^n), \dagger^u_{\mathbb{T}}(\ddot{z}^n)\right]\right\rangle$,$\dagger_{\mathbb{I}}(\ddot{z}^n) = \left\langle\left[\dagger^l_{\mathbb{I}}(\ddot{z}^n), \dagger^u_{\mathbb{I}}(\ddot{z}^n)\right]\right\rangle$ and $\dagger_{\mathbb{F}}(\ddot{z}^n) = \left\langle\left[\dagger^l_{\mathbb{F}}(\ddot{z}^n), \dagger^u_{\mathbb{F}}(\ddot{z}^n)\right]\right\rangle$ respectively, with the condition $0 \leq \dagger^l_{\mathbb{T}}(\ddot{z}^n) + \dagger^l_{\mathbb{I}}(\ddot{z}^n) + \dagger^l_{\mathbb{F}}(\ddot{z}^n) \leq 3$ and $0 \leq \dagger^u_{\mathbb{T}}(\ddot{z}^n) + \dagger^u_{\mathbb{I}}(\ddot{z}^n) + \dagger^u_{\mathbb{F}}(\ddot{z}^n) \leq 3$. Here the terms $\dagger^l_{\mathbb{T}}(\ddot{z}^n), \dagger^l_{\mathbb{I}}(\ddot{z}^n), \dagger^l_{\mathbb{F}}(\ddot{z}^n), \dagger^u_{\mathbb{T}}(\ddot{z}^n), \dagger^u_{\mathbb{I}}(\ddot{z}^n), \dagger^u_{\mathbb{F}}(\ddot{z}^n)$ are declared as lower and upper three NS-memberships.

Definition 2.2. [5] **(Properties of iv-NSs)**

Let $\widehat{\Gamma}^1 = \left\{\left\langle \dagger^1_{\mathbb{T}}(\ddot{z}^n), \dagger^1_{\mathbb{I}}(\ddot{z}^n), \dagger^1_{\mathbb{F}}(\ddot{z}^n)\right\rangle\right\}$ and $\widehat{\Gamma}^2 = \left\{\left\langle \dagger^2_{\mathbb{T}}(\ddot{z}^n), \dagger^2_{\mathbb{I}}(\ddot{z}^n), \dagger^2_{\mathbb{F}}(\ddot{z}^n)\right\rangle\right\}$ be two iv-NSs on on a fixed set $\ddot{\mathcal{Z}} = \left\{\ddot{z}^1, \ddot{z}^2, \ddot{z}^3, ..., \ddot{z}^n\right\}$ as stated in the Definition 1. Then the following basic operations on iv-NSs are given as following:

(i) $\widehat{\Gamma}^1 \subseteq \widehat{\Gamma}^2$ iff $\dagger^1_{\mathbb{T}}(\ddot{z}^n) \leq \dagger^2_{\mathbb{T}}(\ddot{z}^n)$, $\dagger^1_{\mathbb{I}}(\ddot{z}^n) \geq \dagger^2_{\mathbb{I}}(\ddot{z}^n)$ and $\dagger^1_{\mathbb{F}}(\ddot{z}^n) \geq \dagger^2_{\mathbb{F}}(\ddot{z}^n)$.

(ii) $\widehat{\Gamma}^1 = \widehat{\Gamma}^2$ iff $\dagger^1_{\mathbb{T}}(\ddot{z}^n) = \dagger^2_{\mathbb{T}}(\ddot{z}^n)$, $\dagger^1_{\mathbb{I}}(\ddot{z}^n) = \dagger^2_{\mathbb{I}}(\ddot{z}^n)$ and $\dagger^1_{\mathbb{F}}(\ddot{z}^n) = \dagger^2_{\mathbb{F}}(\ddot{z}^n)$.

(iii) The complement of $\widehat{\Gamma}^1$ denotes $c\left(\widehat{\Gamma}^1\right)=\left\{\left\langle c\,\dagger_{\mathbb{T}}^1(\ddot{z}^n),c\,\dagger_{\mathbb{I}}^1(\ddot{z}^n),c\,\dagger_{\mathbb{F}}^1(\ddot{z}^n)\right\rangle\right\}$, such that $c\,\dagger_{\mathbb{T}}^1(\ddot{z}^n)=\left[\dagger_{\mathbb{F}}^{1,l}(\ddot{z}^n),\dagger_{\mathbb{F}}^{1,u}(\ddot{z}^n)\right]$,$c\,\dagger_{\mathbb{I}}^1(\ddot{z}^n)=\left[1-\dagger_{\mathbb{F}}^{1,u}(\ddot{z}^n),1-\dagger_{\mathbb{F}}^{1,i}(\ddot{z}^n)\right]$ and $c\,\dagger_{\mathbb{F}}^1(\ddot{z}^n)=\left[\dagger_{\mathbb{T}}^{1,l}(\ddot{z}^n),\dagger_{\mathbb{T}}^{1,u}(\ddot{z}^n)\right]$.

(iv) The union between two iv-NSs is also iv-NS and defined as the following

$\widehat{\Gamma}^3=\widehat{\Gamma}^1\cup\widehat{\Gamma}^2$ where $\widehat{\Gamma}^3=\left\{\left\langle\dagger_{\mathbb{T}}^3(\ddot{z}^n),\dagger_{\mathbb{I}}^3(\ddot{z}^n),\dagger_{\mathbb{F}}^3(\ddot{z}^n)\right\rangle\right\}$ such that

$\dagger_{\mathbb{T}}^{3,l}(\ddot{z}^n)=\left(\max\left[\dagger_{\mathbb{T}}^{1,l}(\ddot{z}^n),\dagger_{\mathbb{T}}^{2,l}(\ddot{z}^n)\right]\right)$,$\dagger_{\mathbb{T}}^{3,u}(\ddot{z}^n)=\left(\max\left[\dagger_{\mathbb{T}}^{1,u}(\ddot{z}^n),\dagger_{\mathbb{T}}^{2,u}(\ddot{z}^n)\right]\right)$

$\dagger_{\mathbb{I}}^{3,l}(\ddot{z}^n)=\left(\min\left[\dagger_{\mathbb{I}}^{1,l}(\ddot{z}^n),\dagger_{\mathbb{I}}^{2,l}(\ddot{z}^n)\right]\right)$,$\dagger_{\mathbb{I}}^{3,u}(\ddot{z}^n)=\left(\min\left[\dagger_{\mathbb{I}}^{1,u}(\ddot{z}^n),\dagger_{\mathbb{I}}^{2,u}(\ddot{z}^n)\right]\right)$

$\dagger_{\mathbb{F}}^{3,l}(\ddot{z}^n)=\left(\min\left[\dagger_{\mathbb{F}}^{1,l}(\ddot{z}^n),\dagger_{\mathbb{F}}^{2,l}(\ddot{z}^n)\right]\right)$,$\dagger_{\mathbb{I}}^{3,u}(\ddot{z}^n)=\left(\min\left[\dagger_{\mathbb{I}}^{1,u}(\ddot{z}^n),\dagger_{\mathbb{F}}^{2,u}(\ddot{z}^n)\right]\right)$

(v) The intersection between two iv-NSs is also iv-NS and defined as the following

$\widehat{\Gamma}^4=\widehat{\Gamma}^1\cap\widehat{\Gamma}^2$ where $\widehat{\Gamma}^4=\left\{\left\langle\dagger_{\mathbb{T}}^4(\ddot{z}^n),\dagger_{\mathbb{I}}^4(\ddot{z}^n),\dagger_{\mathbb{F}}^4(\ddot{z}^n)\right\rangle\right\}$ such that

$\dagger_{\mathbb{T}}^{3,l}(\ddot{z}^n)=\left(\min\left[\dagger_{\mathbb{T}}^{1,l}(\ddot{z}^n),\dagger_{\mathbb{T}}^{2,l}(\ddot{z}^n)\right]\right)$,$\dagger_{\mathbb{T}}^{4,u}(\ddot{z}^n)=\left(\min\left[\dagger_{\mathbb{T}}^{1,u}(\ddot{z}^n),\dagger_{\mathbb{T}}^{2,u}(\ddot{z}^n)\right]\right)$

$\dagger_{\mathbb{I}}^{4,l}(\ddot{z}^n)=\left(\max\left[\dagger_{\mathbb{I}}^{1,l}(\ddot{z}^n),\dagger_{\mathbb{I}}^{2,l}(\ddot{z}^n)\right]\right)$,$\dagger_{\mathbb{I}}^{4,u}(\ddot{z}^n)=\left(\max\left[\dagger_{\mathbb{I}}^{1,u}(\ddot{z}^n),\dagger_{\mathbb{I}}^{2,u}(\ddot{z}^n)\right]\right)$

$\dagger_{\mathbb{F}}^{4,l}(\ddot{z}^n)=\left(\max\left[\dagger_{\mathbb{F}}^{1,l}(\ddot{z}^n),\dagger_{\mathbb{F}}^{2,l}(\ddot{z}^n)\right]\right)$,$\dagger_{\mathbb{I}}^{4,u}(\ddot{z}^n)=\left(\max\left[\dagger_{\mathbb{I}}^{1,u}(\ddot{z}^n),\dagger_{\mathbb{F}}^{2,u}(\ddot{z}^n)\right]\right)$.

Definition 2.3. [16]Assume that $\Lambda=\{\eta_1,\eta_2,\eta_3,...,\eta_r\}$ for $i=1,2,3,...,r$ define as a set of attributes (features) and $\ddot{\mathcal{Z}}=\left\{\ddot{z}^1,\ddot{z}^2,\ddot{z}^3,...,\ddot{z}^n\right\}$ as a fixed set then a SS $\ddot{\Theta}$ on $\ddot{\mathcal{Z}}$ is declared as $\ddot{\Theta}=\left\{\left(\mathbf{S}_{\ddot{\Theta}}(\eta_r),\Lambda\right),\eta_r\in\Lambda\right\}$ where $\mathbf{S}_{\ddot{\Theta}}:\Lambda\to P^{(\ddot{Z})}$ and $S_{\ddot{\Theta}}(\eta_r)\subseteq P^{(\ddot{Z})}$ is denoted as η_r-approximate element of $\ddot{\Theta}$.

Definition 2.4. [26] Assume that $\Lambda=\{\eta_1,\eta_2,\eta_3,...,\eta_r\}$ for $i=1,2,3,...,r$ be a set of parameters (attributes (features)) that have relevant sub-parametric values given as Λ_1,Λ_2,Λ_3...,Λ_m then the HSS is declared as $\ddot{\ddot{\Theta}}=\{(\mathbf{S}_{\ddot{\ddot{\Theta}}}(\eta_r),\Lambda),\eta_r\in\Lambda\subseteq\Omega\}$ where $\mathbf{S}_{\ddot{\ddot{\Theta}}}:\Lambda\to P^{(\ddot{Z})}$ and $S_{\ddot{\ddot{\Theta}}}(\eta_r)\subseteq P^{(\ddot{Z})}$ is denoted as η_r-approximate element of $\ddot{\ddot{\Theta}}$ and $\Omega=\prod_{i=1}^{n}\Lambda_i$.

3. Possibility interval-valued neutrosophic hypersoft set (piv-NHSS)

This part explores and discusses the elementary characterization of a possibility interval-valued neutrosophic hypersoft set (piv-NHSS) along with the basic set operations of piv-NHSS and some appropriate examples for all these operations.

Definition 3.1. Let $\ddot{\mathcal{Z}}=\left\{\ddot{z}^1,\ddot{z}^2,\ddot{z}^3,...,\ddot{z}^n\right\}$be a non-empty softhyper fixed set and $\Lambda_1,\Lambda_2,\Lambda_3,...,\Lambda_n$ be sets of defferent traits that have $n\geq 1$ sub-value and $\Omega=\prod_{i=1}^{n}\Lambda_i$ such that every $\Lambda_i=\{\eta_1,\eta_2,\eta_3,...,\eta_r\}$ for $i=1,2,3,...,r$ Then a piv-NHSS $\widehat{\Upsilon}_{\hat{\mu}}$ define over a non-empty hypersoft fixed set $\left(\ddot{\mathcal{Z}},\Omega\right)$ as the following form $(\Omega,\dagger_\Omega)$ where

(i) $\dagger_\Omega:\Omega\to iv-NS\times\mathcal{I}_{\hat{Z}}$ such that it define as

$\dagger_\Omega(\eta)=\left\{\left(\frac{\ddot{z}^n}{\widehat{\Gamma}(\eta)(\ddot{z}^n)},\hat{\mu}(\eta)(\ddot{z}^n)\right):\ddot{z}^n\in\ddot{Z}\ \&\ \eta\in\Omega\right\}$ and

$\widehat{\Gamma}(\eta)(\ddot{z}^n)=\langle\dagger_{\mathbb{T}}(\eta)(\ddot{z}^n),\dagger_{\mathbb{I}}(\eta)(\ddot{z}^n),\dagger_{\mathbb{F}}(\eta)(\ddot{z}^n)\rangle$

where $\dagger_{\mathbb{T}}(\eta)(\ddot{z}^n) = \left\langle \left[\dagger^l_{\mathbb{T}}(\eta)(\ddot{z}^n), \dagger^u_{\mathbb{T}}(\eta)(\ddot{z}^n)\right]\right\rangle$,$\dagger_{\mathbb{I}}(\eta)(\ddot{z}^n) = \left\langle \left[\dagger^l_{\mathbb{I}}(\eta)(\ddot{z}^n), \dagger^u_{\mathbb{I}}(\eta)(\ddot{z}^n)\right]\right\rangle$ and $\dagger_{\mathbb{F}}(\eta)(\ddot{z}^n) = \left\langle \left[\dagger^l_{\mathbb{F}}(\eta)(\ddot{z}^n), \dagger^u_{\mathbb{F}}(\eta)(\ddot{z}^n)\right]\right\rangle$ respectively, with the condition $0 \leq \dagger^l_{\mathbb{T}}(\eta)(\ddot{z}^n) + \dagger^l_{\mathbb{I}}(\eta)(\ddot{z}^n) + \dagger^l_{\mathbb{F}}(\eta)(\ddot{z}^n) \leq 3$ and $0 \leq \dagger^u_{\mathbb{T}}(\eta)(\ddot{z}^n) + \dagger^u_{\mathbb{I}}(\eta)(\ddot{z}^n) + \dagger^u_{\mathbb{F}}(\eta)(\ddot{z}^n) \leq 3$.
(ii)$I_{\hat{Z}} : \hat{Z} \to [0,1]$ and $\hat{\mu}(\eta)(\ddot{z}^n) \in I_{\hat{Z}}$ denotes to the degree of possibility of $\ddot{z}^n \in \hat{Z}$ in $\widehat{\Upsilon}(\ddot{z}^n)$. Based on the above a piv-NHSS $\widehat{\Upsilon}_{\hat{\mu}}$ define over a non-empty hypersoft fixed set $\left(\ddot{\mathcal{Z}}, \Omega\right)$ as the following structure:

$$\widehat{\Upsilon}_{\hat{\mu}} = \left\{\left(\eta, \left\{\left(\frac{\ddot{z}^n}{\Gamma(\eta)(\ddot{z}^n)}, \hat{\mu}(\eta)(\ddot{z}^n)\right) : \ddot{z}^n \in \ddot{Z}\right\}\right) \eta \in \Omega\right\}$$

Example 3.2. The Human Resources (HR) department announces available jobs for applicants to apply for and fill. In this context, a lot number of Contains(He/She) apply for these jobs, and accordingly, a committee of experts is formed to analyze each applicant's skills and qualifications to choose the most suitable one for these positions. Therefore to analyze the information of all candidates using our proposed model, piv-NHSS, we assume that the fixed set contains four candidates, i.e $\ddot{\mathcal{Z}} = \left\{\ddot{z}^1, \ddot{z}^2, \ddot{z}^3, \ddot{z}^4\right\}$ with members of the attributes (Criteria) i.e η_1 = Academic Certificate,η_2 = English language level and η_3 = Computer skills, these attributes (Criteria) having their sub-attribute values given as following: $\Lambda_1 = \{\eta_{1,1} = Diploma, \eta_{1,2} = BA\}$, $\Lambda_2 = \{\eta_{2,1} = intermediate, \eta_{2,2} = High\}$, $\Lambda_2 = \{\eta_{3,1} = High\}$ then $\Omega = \Lambda_1 \times \Lambda_2 \times \Lambda_3 = \{\eta_1, \eta_2, \eta_3, \eta_4\}$ where every component η_i has a 3-tuple entity. Then the analyze based on piv-NHSS $\widehat{\Upsilon}_{\hat{\mu}}$ are given as following:
$\widehat{\Upsilon}_{\hat{\mu}}(\eta_1) = \left\{\left(\frac{\ddot{z}^1}{\langle[0.2,0.5],[0.1,0.3],[0.4,0.4]\rangle}, 0.4\right), \left(\frac{\ddot{z}^2}{\langle[0.5,0.6],[0.3,0.5],[0.3,0.8]\rangle}, 0.7\right),\right.$
$\left.\left(\frac{\ddot{z}^3}{\langle[0.2,0.4],[0.1,0.3],[0.5,0.6]\rangle}, 0.6\right), \left(\frac{\ddot{z}^4}{\langle[0.3,0.6],[0.4,0.5],[0.2,0.5]\rangle}, 0.8\right)\right\}$
$\widehat{\Upsilon}_{\hat{\mu}}(\eta_2) = \left\{\left(\frac{\ddot{z}^1}{\langle[0.3,0.6],[0.2,0.5],[0.2,0.3]\rangle}, 0.2\right), \left(\frac{\ddot{z}^2}{\langle[0.4,0.7],[0.4,0.8],[0.1,0.5]\rangle}, 0.5\right),\right.$
$\left.\left(\frac{\ddot{z}^3}{\langle[0.6,0.6],[0.3,0.7],[0.5,0.6]\rangle}, 0.3\right), \left(\frac{\ddot{z}^4}{\langle[0.5,0.7],[0.8,0.6],[0.7,0.7]\rangle}, 0.6\right)\right\}$
$\widehat{\Upsilon}_{\hat{\mu}}(\eta_3) = \left\{\left(\frac{\ddot{z}^1}{\langle[0.1,0.8],[0.7,0.9],[0.6,0.8]\rangle}, 0.3\right), \left(\frac{\ddot{z}^2}{\langle[0.2,0.4],[0.6,0.9],[0.5,0.7]\rangle}, 0.5\right),\right.$
$\left.\left(\frac{\ddot{z}^3}{\langle[0.4,0.7],[0.8,0.8],[0.6,0.6]\rangle}, 0.9\right), \left(\frac{\ddot{z}^4}{\langle[0.5,0.5],[0.7,0.7],[0.3,0.3]\rangle}, 0.5\right)\right\}$
$\widehat{\Upsilon}_{\hat{\mu}}(\eta_4) = \left\{\left(\frac{\ddot{z}^1}{\langle[0.1,0.8],[0.5,0.7],[0.2,0.2]\rangle}, 0.4\right), \left(\frac{\ddot{z}^2}{\langle[0.1,0.1],[0.6,0.6],[0.7,0.8]\rangle}, 0.1\right),\right.$
$\left.\left(\frac{\ddot{z}^3}{\langle[0.6,0.6],[0.7,0.7],[0.9,0.9]\rangle}, 0.5\right), \left(\frac{\ddot{z}^4}{\langle[0.2,0.3],[0.4,0.8],[0.4,0.7]\rangle}, 0.6\right)\right\}$

The piv-NHSS is formed as

$\widehat{\Upsilon}_{\hat{\mu}} = \Bigg\{\widehat{\Upsilon}_{\hat{\mu}}(\eta_1) = \left\{\left(\frac{\ddot{z}^1}{\langle[0.2,0.5],[0.1,0.3],[0.4,0.4]\rangle}, 0.4\right), \left(\frac{\ddot{z}^2}{\langle[0.5,0.6],[0.3,0.5],[0.3,0.8]\rangle}, 0.7\right),\right.$
$\left.\left(\frac{\ddot{z}^3}{\langle[0.2,0.4],[0.1,0.3],[0.5,0.6]\rangle}, 0.6\right), \left(\frac{\ddot{z}^4}{\langle[0.3,0.6],[0.4,0.5],[0.2,0.5]\rangle}, 0.8\right)\right\}$
$\widehat{\Upsilon}_{\hat{\mu}}(\eta_2) = \left\{\left(\frac{\ddot{z}^1}{\langle[0.3,0.6],[0.2,0.5],[0.2,0.3]\rangle}, 0.2\right), \left(\frac{\ddot{z}^2}{\langle[0.4,0.7],[0.4,0.8],[0.1,0.5]\rangle}, 0.5\right),\right.$
$\left.\left(\frac{\ddot{z}^3}{\langle[0.6,0.6],[0.3,0.7],[0.5,0.6]\rangle}, 0.3\right), \left(\frac{\ddot{z}^4}{\langle[0.5,0.7],[0.8,0.6],[0.7,0.7]\rangle}, 0.6\right)\right\}$

$$\widehat{\Upsilon}_{\hat{\mu}}(\eta_3) = \left\{ \left(\frac{\ddot{z}^1}{\langle [0.1,0.8],[0.7,0.9],[0.6,0.8] \rangle}, 0.3 \right), \left(\frac{\ddot{z}^2}{\langle [0.2,0.4],[0.6,0.9],[0.5,0.7] \rangle}, 0.5 \right), \left(\frac{\ddot{z}^3}{\langle [0.4,0.7],[0.8,0.8],[0.6,0.6] \rangle}, 0.9 \right), \left(\frac{\ddot{z}^4}{\langle [0.5,0.5],[0.7,0.7],[0.3,0.3] \rangle}, 0.5 \right) \right\}$$
$$\widehat{\Upsilon}_{\hat{\mu}}(\eta_4) = \left\{ \left(\frac{\ddot{z}^1}{\langle [0.1,0.8],[0.5,0.7],[0.2,0.2] \rangle}, 0.4 \right), \left(\frac{\ddot{z}^2}{\langle [0.1,0.1],[0.6,0.6],[0.7,0.8] \rangle}, 0.1 \right), \left(\frac{\ddot{z}^3}{\langle [0.6,0.6],[0.7,0.7],[0.9,0.9] \rangle}, 0.5 \right), \left(\frac{\ddot{z}^4}{\langle [0.2,0.3],[0.4,0.8],[0.4,0.7] \rangle}, 0.6 \right) \right\} \Bigg\}$$

Here, each part $\widehat{\Upsilon}_{\hat{\mu}}(\eta_{i=1,2,3,4})$ indicates the degree of satisfaction, dissatisfaction, and neutrality for each candidate, in addition to the degree of acceptance of this evaluation, represented by the degree of probability.

Furthermore, the values mentioned above can be displayed as a matrix as follows:

$$\widehat{\Upsilon}_{\hat{\mu}} = \begin{pmatrix} \Omega/\ddot{Z} & \ddot{z}^1 & \ddot{z}^2 & \ddot{z}^3 & \ddot{z}^4 \\ \eta_1 & (\langle [.2,.5],[.1,.3],[.4,.4] \rangle, .4) & (\langle [.5,.6],[.3,.5],[.3,.8] \rangle, .7) & (\langle [.2,.4],[.1,.3],[.5,.6] \rangle, .6) & (\langle [.3,.6],[.4,.5],[.2,.5] \rangle, .8) \\ \eta_2 & (\langle [.3,.6],[.2,.5],[.2,.3] \rangle, .2) & (\langle [.4,.7],[.4,.8],[.1,.5] \rangle, .5) & (\langle [.6,.6],[.3,.7],[.5,.6] \rangle, .3) & (\langle [.5,.7],[.6,.8],[.7,.7] \rangle, .6) \\ \eta_3 & (\langle [.1,.8],[.7,.9],[.6,.8] \rangle, .3) & (\langle [.2,.4],[.6,.9],[.5,.7] \rangle, .5) & (\langle [.4,.7],[.8,.8],[.6,.6] \rangle, .9) & (\langle [.5,.5],[.7,.7],[.3,.3] \rangle, .5) \\ \eta_4 & (\langle [.1,.8],[.5,.7],[.2,.2] \rangle, .4) & (\langle [.1,.1],[.6,.6],[.7,.8] \rangle, .1) & (\langle [.6,.6],[.7,.7],[.9,.9] \rangle, .5) & (\langle [.2,.3],[.4,.8],[.4,.7] \rangle, .6) \end{pmatrix}_{4\times 4}$$

Definition 3.3. (**piv-NHS-nullset**) A piv-NHSS $\widehat{\Upsilon}^1_{\hat{\mu}^1}$ on $\left(\ddot{\mathcal{Z}}, \Omega\right)$ is said to be piv-NHS-nullset and denotes $\widehat{\Phi}^0_{\hat{\mu}^0}$ if
$\widehat{\Upsilon}^1_{\hat{\mu}}(\eta)(\ddot{z}^n) = ([0,0],[0,0],[0,0])$ and for possibility membership part $\hat{\mu}^1(\eta)(\ddot{z}^n) = 0$.

Example 3.4. If $\widehat{\Phi}^0_{\hat{\mu}^0}$ given as following matrix:

$$\widehat{\Phi}^0_{\mu^0} = \begin{pmatrix} (\langle [0.0],[0.0],[0.0] \rangle, 0) & (\langle [0.0],[0.0],[0.0] \rangle, 0) \\ (\langle [0.0],[0.0],[0.0] \rangle, 0) & (\langle [0.0],[0.0],[0.0] \rangle, 0) \end{pmatrix}$$

Then here $\widehat{\Phi}^0_{\hat{\mu}^0}$ known piv-NHS-nullset.

Definition 3.5. (**piv-NHS-absolute set**) A piv-NHSS $\widehat{\Upsilon}^1_{\hat{\mu}^1}$ on $\left(\ddot{\mathcal{Z}}, \Omega\right)$ is said to be piv-NHS-nullset and denotes $\widehat{\Phi}^1_{\hat{\mu}^1}$ if
$\widehat{\Upsilon}^1_{\hat{\mu}}(\eta)(\ddot{z}^n) = ([1,1],[1,1],[1,1])$ and for possibility membership part $\hat{\mu}^1(\eta)(\ddot{z}^n) = 1$.

Example 3.6. If $\widehat{\Phi}^1_{\hat{\mu}^1}$ given as following matrix:

$$\widehat{\Phi}^1_{\mu^1} = \begin{pmatrix} (\langle [1.1],[1.1],[1.1] \rangle, 1) & (\langle [1.1],[1.1],[1.1] \rangle, 1) \\ (\langle [1.1],[1.1],[1.1] \rangle, 1) & (\langle [1.1],[1.1],[1.1] \rangle, 1) \end{pmatrix}$$

Then here $\widehat{\Phi}^1_{\hat{\mu}^1}$ known piv-NHS-absolute set.

Definition 3.7. (**piv-NHS-subset**) A piv-NHSS $\widehat{\Upsilon}^1_{\hat{\mu}^1} \sqsubseteq \widehat{\Upsilon}^2_{\hat{\mu}^2}$ on $\left(\ddot{\mathcal{Z}}, \Omega\right)$ if
$\widehat{\Upsilon}^1_{\hat{\mu}}(\eta)(\ddot{z}^n) \leq \widehat{\Upsilon}^2_{\hat{\mu}}(\eta)(\ddot{z}^n)$ such that $\widehat{\Upsilon}^{1,l}_{\mathbb{T}}(\eta)(\ddot{z}^n) \leq \widehat{\Upsilon}^{2,l}_{\mathbb{T}}(\eta)(\ddot{z}^n)$ and $\widehat{\Upsilon}^{1,u}(\eta)(\ddot{z}^n) \leq \widehat{\Upsilon}^{2,u}(\eta)(\ddot{z}^n), \widehat{\Upsilon}^{1,l}_{\mathbb{I}}(\eta)(\ddot{z}^n) \geq \widehat{\Upsilon}^{2,l}_{\mathbb{I}}(\eta)(\ddot{z}^n), \widehat{\Upsilon}^{1,u}_{\mathbb{I}}(\eta)(\ddot{z}^n) \geq \widehat{\Upsilon}^{2,u}_{\mathbb{I}}(\eta)(\ddot{z}^n), \widehat{\Upsilon}^{1,l}_{\mathbb{F}}(\eta)(\ddot{z}^n) \geq$

$\widehat{\Upsilon}^{2,l}_{\mathbb{F}}(\eta)(\ddot{z}^n), \widehat{\Upsilon}^{1,u}_{\mathbb{F}}(\eta)(\ddot{z}^n) \geq \widehat{\Upsilon}^{2,u}_{\mathbb{F}}(\eta)(\ddot{z}^n)$ and for possibility membership part $\hat{\mu}^1(\eta)(\ddot{z}^n) \leq \hat{\mu}^2(\eta)(\ddot{z}^n)$.

Example 3.8. Consider the two piv-NHSs $\widehat{\Upsilon}^1_{\hat{\mu}^1}$ $\widehat{\Upsilon}^2_{\hat{\mu}^2}$ given as following:

$$\widehat{\Upsilon}^1_{\hat{\mu}^1} = \left\{ \left(\frac{\ddot{z}^1}{\langle [0.2,0.5],[0.1,0.3],[0.4,0.4] \rangle}, 0.4 \right), \left(\frac{\ddot{z}^2}{\langle [0.3,0.6],[0.4,0.5],[0.2,0.5] \rangle}, 0.7 \right) \right\}$$

and

$$\widehat{\Upsilon}^2_{\hat{\mu}^2} = \left\{ \left(\frac{\ddot{z}^1}{\langle [0.4,0.8],[0.1,0.1],[0.2,0.3] \rangle}, 0.7 \right), \left(\frac{\ddot{z}^2}{\langle [0.7,0.7],[0.2,0.2],[0.1,0.3] \rangle}, 0.9 \right) \right\}$$

Then its clear here $\widehat{\Upsilon}^1_{\hat{\mu}^1} \sqsubseteq \widehat{\Upsilon}^2_{\hat{\mu}^2}$.

Definition 3.9. (**Equality in piv-NHSSs**) A piv-NHSS $\widehat{\Upsilon}^1_{\hat{\mu}^1} = \widehat{\Upsilon}^2_{\hat{\mu}^2}$ on $\left(\ddot{\mathcal{Z}}, \Omega\right)$ if $\widehat{\Upsilon}^1_{\hat{\mu}}(\eta)(\ddot{z}^n) = \widehat{\Upsilon}^2_{\hat{\mu}}(\eta)(\ddot{z}^n)$ such that $\widehat{\Upsilon}^1_{\mathbb{T}}(\eta)(\ddot{z}^n) = \widehat{\Upsilon}^2_{\mathbb{T}}(\eta)(\ddot{z}^n)$ and $\widehat{\Upsilon}^{1,u}(\eta)(\ddot{z}^n) = \widehat{\Upsilon}^{2,u}(\eta)(\ddot{z}^n), \widehat{\Upsilon}^I_{\mathbb{R}}(\eta)(\ddot{z}^n) = \widehat{\Upsilon}^2_{\mathbb{I}}(\eta)(\ddot{z}^n), \widehat{\Upsilon}^F_{\mathbb{R}}(\eta)(\ddot{z}^n) = \widehat{\Upsilon}^2_{\mathbb{F}}(\eta)(\ddot{z}^n)$ and for possibility membership part $\hat{\mu}^1(\eta)(\ddot{z}^n) = \hat{\mu}^2(\eta)(\ddot{z}^n)$.

Example 3.10. Consider the two piv-NHSs $\widehat{\Upsilon}^1_{\hat{\mu}^1}$ $\widehat{\Upsilon}^2_{\hat{\mu}^2}$ given as following:

$$\widehat{\Upsilon}^1_{\hat{\mu}^1} = \left\{ \left(\frac{\ddot{z}^1}{\langle [0.2,0.5],[0.1,0.3],[0.4,0.4] \rangle}, 0.4 \right), \left(\frac{\ddot{z}^2}{\langle [0.3,0.6],[0.4,0.5],[0.2,0.5] \rangle}, 0.7 \right) \right\}$$

and

$$\widehat{\Upsilon}^2_{\hat{\mu}^2} = \left\{ \left(\frac{\ddot{z}^1}{\langle [0.2,0.5],[0.1,0.3],[0.4,0.4] \rangle}, 0.4 \right), \left(\frac{\ddot{z}^2}{\langle [0.3,0.6],[0.4,0.5],[0.2,0.5] \rangle}, 0.7 \right) \right\}$$

Then its clear here $\widehat{\Upsilon}^1_{\hat{\mu}^1} = \widehat{\Upsilon}^2_{\hat{\mu}^2}$.

4. The set-theoretic operations of piv-NHSSs

In this section, we will present the set-theoretic operations of piv-NHSSs as well as some Illustrative examples and some properties.

Definition 4.1. (**Complement in piv-NHSSs**) A piv-NHSS $\widehat{\Upsilon}_{\hat{\mu}}$ on $\left(\ddot{\mathcal{Z}}, \Omega\right)$ have a complement in the following form $\widehat{\Upsilon}^c_{\hat{\mu}^c}$and define as

$$\widehat{\Upsilon}^c_{\hat{\mu}^c} = \left\{ \left(\eta, \left\{ \left(\frac{\ddot{z}^n}{\dagger^c(\eta)(\ddot{z}^n)}, \hat{\mu^c}(\eta)(\ddot{z}^n) \right) : \ddot{z}^n \in \ddot{Z} \right\} \right) \eta \in \Omega \right\}$$

where

$$\dagger^c_{\mathbb{T}}(\eta)(\ddot{z}^n) = \left\langle \left[\dagger^l_{\mathbb{F}}(\eta)(\ddot{z}^n), \dagger^u_{\mathbb{F}}(\eta)(\ddot{z}^n) \right] \right\rangle, \dagger^c_{\mathbb{I}}(\eta)(\ddot{z}^n) = \left\langle \left[1 - \dagger^u_{\mathbb{I}}(\eta)(\ddot{z}^n), 1 - \dagger^l_{\mathbb{I}}(\eta)(\ddot{z}^n) \right] \right\rangle \text{ and}$$

$$\dagger^c_{\mathbb{F}}(\eta)(\ddot{z}^n) = \left\langle \left[\dagger^l_{\mathbb{T}}(\eta)(\ddot{z}^n), \dagger^u_{\mathbb{T}}(\eta)(\ddot{z}^n) \right] \right\rangle$$

and for possibility fuzzy membership part $\hat{\mu}^c(\eta)(\ddot{z}^n) = 1 - \hat{\mu}(\eta)(\ddot{z}^n)$.

Example 4.2. Consider the two piv-NHSs $\left(\widehat{\Upsilon}^1_{\hat{\mu}^1}\right)^c \widehat{\Upsilon}^1_{\hat{\mu}^1}$ given as following:

$$\widehat{\Upsilon}^1_{\hat{\mu}^1} = \left\{ \left(\frac{\ddot{z}^1}{\langle [0.2, 0.5], [0.1, 0.3], [0.4, 0.4] \rangle}, 0.4 \right), \left(\frac{\ddot{z}^2}{\langle [0.3, 0.6], [0.4, 0.5], [0.2, 0.5] \rangle}, 0.7 \right) \right\}$$

then the $\left(\widehat{\Upsilon}^1_{\hat{\mu}^1}\right)^c$ is given based on above definition as following:

$$\left(\widehat{\Upsilon}^1_{\hat{\mu}^1}\right)^c = \left\{ \left(\frac{\ddot{z}^1}{\langle [0.4, 0.4], [0.7, 0.9], [0.2, 0.5] \rangle}, 0.6 \right), \left(\frac{\ddot{z}^2}{\langle [0.2, 0.5], [0.5, 0.6], [0.3, 0.6] \rangle}, 0.3 \right) \right\}$$

Definition 4.3. (**Union in piv-NHSSs**) Let $\widehat{\Upsilon}^1_{\hat{\mu}^1}$ and $\widehat{\Upsilon}^2_{\hat{\mu}^2}$ be two piv-NHSSs. Then the their union denotes $\widehat{\Upsilon}^1_{\hat{\mu}^1} \ddot{\sqcup} \widehat{\Upsilon}^2_{\hat{\mu}^2}$ is also piv-HSS $\widehat{\Upsilon}^3_{\hat{\mu}^3}$ and given as following:

$$\widehat{\Upsilon}^3_{\mathbb{T}}(\eta)(\ddot{z}^n) = \max\left\{ \widehat{\Upsilon}^1_{\mathbb{T}}(\eta)(\ddot{z}^n), \widehat{\Upsilon}^2_{\mathbb{T}}(\eta)(\ddot{z}^n) \right\}$$
$$= \left\{ \max\left[\widehat{\Upsilon}^{1,l}_{\mathbb{T}}(\eta)(\ddot{z}^n), \widehat{\Upsilon}^{2,l}_{\mathbb{T}}(\eta)(\ddot{z}^n) \right], \max\left[\widehat{\Upsilon}^{1,u}_{\mathbb{T}}(\eta)(\ddot{z}^n), \widehat{\Upsilon}^{2,u}_{\mathbb{T}}(\eta)(\ddot{z}^n) \right] \right\},$$
$$\widehat{\Upsilon}^3_{\mathbb{I}}(\eta)(\ddot{z}^n) = \min\left\{ \widehat{\Upsilon}^1_{\mathbb{I}}(\eta)(\ddot{z}^n), \widehat{\Upsilon}^2_{\mathbb{I}}(\eta)(\ddot{z}^n) \right\}$$
$$= \left\{ \min\left[\widehat{\Upsilon}^{1,l}_{\mathbb{I}}(\eta)(\ddot{z}^n), \widehat{\Upsilon}^{2,l}_{\mathbb{T}}(\eta)(\ddot{z}^n) \right], \min\left[\widehat{\Upsilon}^{1,u}_{\mathbb{I}}(\eta)(\ddot{z}^n), \widehat{\Upsilon}^{2,u}_{\mathbb{I}}(\eta)(\ddot{z}^n) \right] \right\},$$
$$\widehat{\Upsilon}^3_{\mathbb{F}}(\eta)(\ddot{z}^n) = \min\left\{ \widehat{\Upsilon}^1_{\mathbb{F}}(\eta)(\ddot{z}^n), \widehat{\Upsilon}^2_{\mathbb{F}}(\eta)(\ddot{z}^n) \right\}$$
$$= \left\{ \min\left[\widehat{\Upsilon}^{1,l}_{\mathbb{F}}(\eta)(\ddot{z}^n), \widehat{\Upsilon}^{2,l}_{\mathbb{F}}(\eta)(\ddot{z}^n) \right], \min\left[\widehat{\Upsilon}^{1,u}_{\mathbb{F}}(\eta)(\ddot{z}^n), \widehat{\Upsilon}^{2,u}_{\mathbb{F}}(\eta)(\ddot{z}^n) \right] \right\},$$
and for possibility membership part
$$\hat{\mu}^3(\eta)(\ddot{z}^n) = \max\left\{ \hat{\mu}^1(\eta)(\ddot{z}^n), \hat{\mu}^2(\eta)(\ddot{z}^n) \right\}$$

Example 4.4. Consider the two piv-NHSs $\widehat{\Upsilon}^1_{\hat{\mu}^1}$ $\widehat{\Upsilon}^2_{\hat{\mu}^2}$ given as following:

$$\widehat{\Upsilon}^1_{\hat{\mu}^1} = \left\{ \left(\frac{\ddot{z}^1}{\langle [0.2, 0.5], [0.1, 0.3], [0.4, 0.4] \rangle}, 0.4 \right), \left(\frac{\ddot{z}^2}{\langle [0.3, 0.6], [0.4, 0.5], [0.2, 0.5] \rangle}, 0.7 \right) \right\}$$

and

$$\widehat{\Upsilon}^2_{\hat{\mu}^2} = \left\{ \left(\frac{\ddot{z}^1}{\langle [0.4, 0.8], [0.1, 0.1], [0.2, 0.3] \rangle}, 0.7 \right), \left(\frac{\ddot{z}^2}{\langle [0.7, 0.7], [0.2, 0.2], [0.1, 0.3] \rangle}, 0.9 \right) \right\}$$

Then the $\widehat{\Upsilon}^1_{\hat{\mu}^1} \ddot{\sqcup} \widehat{\Upsilon}^2_{\hat{\mu}^2}$.

$$= \left\{ \left(\frac{\ddot{z}^1}{\langle [0.4, 0.8], [0.1, 0.1], [0.2, 0.3] \rangle}, 0.7 \right), \left(\frac{\ddot{z}^2}{\langle [0.7, 0.7], [0.2, 0.2], [0.1, 0.3] \rangle}, 0.9 \right) \right\}$$

Definition 4.5. (**Intersection in piv-NHSSs**) Let $\widehat{\Upsilon}^1_{\hat{\mu}^1}$ and $\widehat{\Upsilon}^2_{\hat{\mu}^2}$ be two piv-NHSSs. Then the their intersection denotes $\widehat{\Upsilon}^1_{\hat{\mu}^1} \ddot{\sqcap} \widehat{\Upsilon}^2_{\hat{\mu}^2}$ is also piv-HSS $\widehat{\Upsilon}^4_{\hat{\mu}^4}$ and given as following:

$$\widetilde{\Upsilon}^{4}_{\mathbb{T}}(\eta)(\ddot{z}^n) = \min\left\{\widetilde{\Upsilon}^{1}_{\mathbb{T}}(\eta)(\ddot{z}^n), \widetilde{\Upsilon}^{2}_{\mathbb{T}}(\eta)(\ddot{z}^n)\right\}$$
$$= \left\{\min\left[\widetilde{\Upsilon}^{1,l}_{\mathbb{T}}(\eta)(\ddot{z}^n), \widetilde{\Upsilon}^{2,l}_{\mathbb{T}}(\eta)(\ddot{z}^n)\right], \min\left[\widetilde{\Upsilon}^{1,u}_{\mathbb{T}}(\eta)(\ddot{z}^n), \widetilde{\Upsilon}^{2,u}_{\mathbb{T}}(\eta)(\ddot{z}^n)\right]\right\},$$
$$\widetilde{\Upsilon}^{4}_{\mathbb{I}}(\eta)(\ddot{z}^n) = \max\left\{\widetilde{\Upsilon}^{1}_{\mathbb{I}}(\eta)(\ddot{z}^n), \widetilde{\Upsilon}^{2}_{\mathbb{I}}(\eta)(\ddot{z}^n)\right\}$$
$$= \left\{\max\left[\widetilde{\Upsilon}^{1,l}_{\mathbb{I}}(\eta)(\ddot{z}^n), \widetilde{\Upsilon}^{2,l}_{\mathbb{I}}(\eta)(\ddot{z}^n)\right], \max\left[\widetilde{\Upsilon}^{1,u}_{\mathbb{I}}(\eta)(\ddot{z}^n), \widetilde{\Upsilon}^{2,u}_{\mathbb{I}}(\eta)(\ddot{z}^n)\right]\right\},$$
$$\widetilde{\Upsilon}^{4}_{\mathbb{F}}(\eta)(\ddot{z}^n) = \max\left\{\widetilde{\Upsilon}^{1}_{\mathbb{F}}(\eta)(\ddot{z}^n), \widetilde{\Upsilon}^{2}_{\mathbb{F}}(\eta)(\ddot{z}^n)\right\}$$
$$= \left\{\max\left[\widetilde{\Upsilon}^{1,l}_{\mathbb{F}}(\eta)(\ddot{z}^n), \widetilde{\Upsilon}^{2,l}_{\mathbb{F}}(\eta)(\ddot{z}^n)\right], \max\left[\widetilde{\Upsilon}^{1,u}_{\mathbb{F}}(\eta)(\ddot{z}^n), \widetilde{\Upsilon}^{2,u}_{\mathbb{F}}(\eta)(\ddot{z}^n)\right]\right\},$$
and for possibility membership part is given as
$$\hat{\mu}^{4}(\eta)(\ddot{z}^n) = \min\left\{\hat{\mu}^{1}(\eta)(\ddot{z}^n), \hat{\mu}^{2}(\eta)(\ddot{z}^n)\right\}$$

Example 4.6. Consider the two piv-NHSs $\widetilde{\Upsilon}^{1}_{\hat{\mu}^1}$ $\widetilde{\Upsilon}^{2}_{\hat{\mu}^2}$ given as following:

$$\widetilde{\Upsilon}^{1}_{\hat{\mu}^1} = \left\{\left(\frac{\ddot{z}^1}{\langle[0.2,0.5],[0.1,0.3],[0.4,0.4]\rangle}, 0.4\right), \left(\frac{\ddot{z}^2}{\langle[0.3,0.6],[0.4,0.5],[0.2,0.5]\rangle}, 0.7\right)\right\}$$

and

$$\widetilde{\Upsilon}^{2}_{\hat{\mu}^2} = \left\{\left(\frac{\ddot{z}^1}{\langle[0.4,0.8],[0.1,0.1],[0.2,0.3]\rangle}, 0.7\right), \left(\frac{\ddot{z}^2}{\langle[0.7,0.7],[0.2,0.2],[0.1,0.3]\rangle}, 0.9\right)\right\}$$

Then the $\widetilde{\Upsilon}^{1}_{\hat{\mu}^1} \ddot{\sqcap} \widetilde{\Upsilon}^{2}_{\hat{\mu}^2} =$.

$$\left\{\left(\frac{\ddot{z}^1}{\langle[0.2,0.5],[0.1,0.3],[0.4,0.4]\rangle}, 0.4\right), \left(\frac{\ddot{z}^2}{\langle[0.3,0.6],[0.4,0.5],[0.2,0.5]\rangle}, 0.7\right)\right\}$$

Proposition 4.7. *Assume that $\widetilde{\Upsilon}^{1}_{\hat{\mu}^1}$ be a piv-NHSSs on hypersoft fixed set $\left(\ddot{\mathcal{Z}}, \Omega\right)$. Then the next properties are satisfied:*
(i). $\widetilde{\Upsilon}^{1}_{\hat{\mu}^1} \ddot{\sqcup} \widetilde{\Upsilon}^{1}_{\hat{\mu}^1} = \widetilde{\Upsilon}^{1}_{\hat{\mu}^1}$.
(ii). $\widetilde{\Upsilon}^{1}_{\hat{\mu}^1} \ddot{\sqcap} \widetilde{\Upsilon}^{1}_{\hat{\mu}^1} = \widetilde{\Upsilon}^{1}_{\hat{\mu}^1}$.
(iii). $\widetilde{\Upsilon}^{1}_{\hat{\mu}^1} \ddot{\sqcup} \widetilde{\Phi}^{0}_{\hat{\mu}^0} = \widetilde{\Upsilon}^{1}_{\hat{\mu}^1}$.
(iv). $\widetilde{\Upsilon}^{1}_{\hat{\mu}^1} \ddot{\sqcap} \widetilde{\Phi}^{0}_{\hat{\mu}^0} = \widetilde{\Phi}^{0}_{\hat{\mu}^0}$.
(v). $\widetilde{\Upsilon}^{1}_{\hat{\mu}^1} \ddot{\sqcup} \widetilde{\Phi}^{1}_{\hat{\mu}^1} = \widetilde{\Phi}^{1}_{\hat{\mu}^1}$.
(vi). $\widetilde{\Upsilon}^{1}_{\hat{\mu}^1} \ddot{\sqcap} \widetilde{\Phi}^{1}_{\hat{\mu}^1} = \widetilde{\Upsilon}^{1}_{\hat{\mu}^1}$.

Proposition 4.8. *Assume that $\widetilde{\Upsilon}^{1}_{\hat{\mu}^1}$, $\widetilde{\Upsilon}^{2}_{\hat{\mu}^2}$ and $\widetilde{\Upsilon}^{3}_{\hat{\mu}^3}$ be three piv-NHSSs on hypersoft fixed set $\left(\ddot{\mathcal{Z}}, \Omega\right)$. Then the next properties are satisfied:*
(i). $\widetilde{\Upsilon}^{1}_{\hat{\mu}^1} \ddot{\sqcup} \widetilde{\Upsilon}^{2}_{\hat{\mu}^2} = \widetilde{\Upsilon}^{2}_{\hat{\mu}^2} \ddot{\sqcup} \widetilde{\Upsilon}^{1}_{\hat{\mu}^1}$.
(ii). $\widetilde{\Upsilon}^{1}_{\hat{\mu}^1} \ddot{\sqcap} \widetilde{\Upsilon}^{2}_{\hat{\mu}^2} = \widetilde{\Upsilon}^{2}_{\hat{\mu}^2} \ddot{\sqcap} \widetilde{\Upsilon}^{1}_{\hat{\mu}^1}$.
(iii). $\widetilde{\Upsilon}^{1}_{\hat{\mu}^1} \ddot{\sqcup} \left(\widetilde{\Upsilon}^{2}_{\hat{\mu}^2} \ddot{\sqcup} \widetilde{\Upsilon}^{3}_{\hat{\mu}^3}\right) = \left(\widetilde{\Upsilon}^{1}_{\hat{\mu}^1} \ddot{\sqcup} \widetilde{\Upsilon}^{2}_{\hat{\mu}^2}\right) \ddot{\sqcup} \widetilde{\Upsilon}^{3}_{\hat{\mu}^3}$.

(iv). $\widehat{\Upsilon}^1_{\hat{\mu}^1}\ddot{\sqcap}\left(\widehat{\Upsilon}^2_{\hat{\mu}^2}\ddot{\sqcap}\widehat{\Upsilon}^3_{\hat{\mu}^3}\right)=\left(\widehat{\Upsilon}^1_{\hat{\mu}^1}\ddot{\sqcap}\widehat{\Upsilon}^2_{\hat{\mu}^2}\right)\ddot{\sqcap}\widehat{\Upsilon}^3_{\hat{\mu}^3}$.

Proposition 4.9. *Assume that* $\widehat{\Upsilon}^1_{\hat{\mu}^1}$, $\widehat{\Upsilon}^2_{\hat{\mu}^2}$ *and* $\widehat{\Upsilon}^3_{\hat{\mu}^3}$ *be three piv-NHSSs on hypersoft fixed set* $\left(\ddot{\mathcal{Z}},\Omega\right)$*.Then the next properties are satisfied:*

(i). $\widehat{\Upsilon}^1_{\hat{\mu}^1}\ddot{\sqcup}\left(\widehat{\Upsilon}^2_{\hat{\mu}^2}\ddot{\sqcap}\widehat{\Upsilon}^3_{\hat{\mu}^3}\right)=\left(\widehat{\Upsilon}^1_{\hat{\mu}^1}\ddot{\sqcup}\widehat{\Upsilon}^2_{\hat{\mu}^2}\right)\ddot{\sqcap}\left(\widehat{\Upsilon}^1_{\hat{\mu}^1}\ddot{\sqcup}\widehat{\Upsilon}^3_{\hat{\mu}^3}\right)$.

(ii). $\widehat{\Upsilon}^1_{\hat{\mu}^1}\ddot{\sqcap}\left(\widehat{\Upsilon}^2_{\hat{\mu}^2}\ddot{\sqcup}\widehat{\Upsilon}^3_{\hat{\mu}^3}\right)=\left(\widehat{\Upsilon}^1_{\hat{\mu}^1}\ddot{\sqcap}\widehat{\Upsilon}^2_{\hat{\mu}^2}\right)\ddot{\sqcup}\left(\widehat{\Upsilon}^1_{\hat{\mu}^1}\ddot{\sqcap}\widehat{\Upsilon}^3_{\hat{\mu}^3}\right)$.

Proof.

(i). For all $\eta,\ddot{z}^n\in\Omega$ *and* $\ddot{\mathcal{Z}}$ respectively

$\chi_{\widehat{\Upsilon}^1(\eta)(\ddot{z}^n)\ddot{\sqcup}\left(\widehat{\Upsilon}^2(\eta)(\ddot{z}^n)\ddot{\sqcap}\widehat{\Upsilon}^3(\eta)(\ddot{z}^n)\right)}=\ddot{\sqcup}\left\{\chi_{\widehat{\Upsilon}^1(\eta)(\ddot{z}^n)},\chi_{\left(\widehat{\Upsilon}^2(\eta)(\ddot{z}^n)\ddot{\sqcap}\widehat{\Upsilon}^3(\eta)(\ddot{z}^n)\right)}\right\}$

$=\ddot{\sqcup}\left\{\chi_{\widehat{\Upsilon}^1(\eta)(\ddot{z}^n)},\ddot{\sqcap}\left(\chi_{\widehat{\Upsilon}^2(\eta)(\ddot{z}^n)},\chi_{\widehat{\Upsilon}^3(\eta)(\ddot{z}^n)}\right)\right\}$

$=\{\langle\eta,\ \max\left(\widehat{\Upsilon}^1_{\mathbb{T}}(\ddot{z}^n),\min\left(\widehat{\Upsilon}^2_{\mathbb{T}}(\ddot{z}^n),\widehat{\Upsilon}^3_{\mathbb{T}}(\ddot{z}^n)\right)\right),$

$\min\left(\widehat{\Upsilon}^1_{\mathbb{I}}(\ddot{z}^n),\max\left(\widehat{\Upsilon}^2_{\mathbb{I}}(\ddot{z}^n),\widehat{\Upsilon}^3_{\mathbb{I}}(\ddot{z}^n)\right)\right),$

$\min\left(\widehat{\Upsilon}^1_{\mathbb{F}}(\ddot{z}^n),\max\left(\widehat{\Upsilon}^2_{\mathbb{F}}(\ddot{z}^n),\widehat{\Upsilon}^3_{\mathbb{F}}(\ddot{z}^n)\right)\right)\Big\}$

$\Big\{\Big\langle\eta,\min\left(\max\left(\widehat{\Upsilon}^1_{\mathbb{T}}(\ddot{z}^n),\widehat{\Upsilon}^2_{\mathbb{T}}(\ddot{z}^n)\right)\ \max\left(\widehat{\Upsilon}^1_{\mathbb{T}}(\ddot{z}^n),\widehat{\Upsilon}^3_{\mathbb{T}}(\ddot{z}^n)\right)\right),$

$\max\left(\min\left(\widehat{\Upsilon}^1_{\mathbb{I}}(\ddot{z}^n),\widehat{\Upsilon}^2_{\mathbb{I}}(\ddot{z}^n)\right)\ \min\left(\widehat{\Upsilon}^1_{\mathbb{I}}(\ddot{z}^n),\widehat{\Upsilon}^3_{\mathbb{I}}(\ddot{z}^n)\right)\right)$

$\max\left(\min\left(\widehat{\Upsilon}^1_{\mathbb{F}}(\ddot{z}^n),\widehat{\Upsilon}^2_{\mathbb{F}}(\ddot{z}^n)\right)\ \min\left(\widehat{\Upsilon}^1_{\mathbb{F}}(\ddot{z}^n),\widehat{\Upsilon}^3_{\mathbb{F}}(\ddot{z}^n)\right)\right)\Big\rangle\Big\}$

$=\ddot{\sqcap}\left(\ddot{\sqcup}\left(\chi_{\widehat{\Upsilon}^1(\eta)(\ddot{z}^n)\ddot{\sqcup}\widehat{\Upsilon}^2(\eta)(\ddot{z}^n)},\chi_{\widehat{\Upsilon}^1(\eta)(\ddot{z}^n)\cup\widehat{\Upsilon}^3(\eta)(\ddot{z}^n)}\right)\right)$

$=\chi_{\left(\widehat{\Upsilon}^1(\eta)(\ddot{z}^n)\ddot{\sqcup}\widehat{\Upsilon}^2(\eta)(\ddot{z}^n)\right)\ddot{\sqcap}\widehat{\Upsilon}^1(\eta)(\ddot{z}^n)\ddot{\sqcup}\widehat{\Upsilon}^3(\eta)(\ddot{z}^n)}$.

and for possibility membership par

$\chi_{\hat{\mu}^1(\eta)(\ddot{z}^n)\cup(\hat{\mu}^2(\eta)(\ddot{z}^n)\cap\hat{\mu}^3(\eta)(\ddot{z}^n))}=\max\left\{\chi_{\hat{\mu}^1(\eta)(\ddot{z}^n)},\chi_{(\hat{\mu}^2(\eta)(\ddot{z}^n)\cap\hat{\mu}^3(\eta)(\ddot{z}^n))}\right\}$

$=\max\left\{\chi_{\hat{\mu}^1(\eta)(\ddot{z}^n)},\min\left(\chi_{\hat{\mu}^2(\eta)(\ddot{z}^n)},\chi_{\hat{\mu}^3(\eta)(\ddot{z}^n)}\right)\right\}$

$=\min\left\{\max\left(\chi_{\hat{\mu}^1(\eta)(\ddot{z}^n)},\chi_{\hat{\mu}^2(\eta)(\ddot{z}^n)}\right),\max\left(\chi_{\hat{\mu}^1(\eta)(\ddot{z}^n)},\chi_{\hat{\mu}^3(\eta)(\ddot{z}^n)}\right)\right\}$

$=\min\left\{\chi_{\hat{\mu}^1(\eta)(\ddot{z}^n)\cup\hat{\mu}^2(\eta)(\ddot{z}^n)},\chi_{\hat{\mu}^1(\eta)(\ddot{z}^n)\cup\hat{\mu}^3(\eta)(\ddot{z}^n)}\right\}$

$=\chi_{(\hat{\mu}^1(\eta)(\ddot{z}^n)\ddot{\sqcup}\hat{\mu}^2(\eta)(\ddot{z}^n))\ddot{\sqcap}(\hat{\mu}^1(\eta)(\ddot{z}^n)\ddot{\sqcup}\hat{\mu}^3(\eta)(\ddot{z}^n))}$ □

(ii) can be proved similar to (i).

5. **Similarity Measure on piv-NHSs**

Similarity techniques play a major role in linking two or more mathematical structures into a mathematical system whose outputs are a fixed degree that can be relied upon to resolve

the controversy over many issues of uncertainty. Accordingly, researchers have been interested in linking mathematical concepts through similarity tools. In this section, we will develop some of the previous techniques [42]– [44] to find a mathematical formula that aims to find the similarity ratio between two piv-NHSs.

Definition 5.1. Assume that $\widehat{\Upsilon}^1_{\hat{\mu}^1}$ and $\widehat{\Upsilon}^2_{\hat{\mu}^2}$ be two piv-NHSSs.Then the Similarity measure between $\widehat{\Upsilon}^1_{\hat{\mu}^1}$ and $\widehat{\Upsilon}^2_{\hat{\mu}^2}$ set by $\hat{\hat{\mathbb{S}}}\left(\widehat{\Upsilon}^1_{\hat{\mu}^1}, \widehat{\Upsilon}^2_{\hat{\mu}^2}\right)$ is defined as follows:

$$\hat{\hat{\mathbb{S}}}\left(\widehat{\Upsilon}^1_{\hat{\mu}^1}, \widehat{\Upsilon}^2_{\hat{\mu}^2}\right) = \dddot{\mathbb{M}}\left(\widehat{\Upsilon}^1, \widehat{\Upsilon}^2\right) \times \dddot{\mathbb{M}}\left(\hat{\mu}^1, \hat{\mu}^2\right),$$

such that

$$\ddot{M}\left(\widehat{\Upsilon}^1(\eta)(\ddot{z}^n), \widehat{\Upsilon}^2(\eta)(\ddot{z}^n)\right) = Max\left(\dddot{M}_i\left(\widehat{\Upsilon}^1(\eta)(\ddot{z}^n), \widehat{\Upsilon}^2(\eta)(\ddot{z}^n)\right)\right)$$

$$\dddot{M}\left(\hat{\mu}^1(\eta)(\ddot{z}^n), \hat{\mu}^2(\eta)(\ddot{z}^n)\right) = Max\left(\dddot{M}_i\left(\hat{\mu}^1(\eta)(\ddot{z}^n), \hat{\mu}^2(\eta)(\ddot{z}^n)\right)\right)$$

where

$$\ddot{M}\left(\widehat{\Upsilon}^1(\eta)(\ddot{z}^n), \widehat{\Upsilon}^2(\eta)(\ddot{z}^n)\right) = 1 - \frac{1}{\sqrt{n}} \times \sqrt{\sum_{i=1}^{n}\left(\phi'_{\widehat{\Upsilon}^1(\eta)(\ddot{z}^n)} - \phi'_{\widehat{\Upsilon}^2(\eta)(\ddot{z}^n)}\right)^2}$$

such that,

$$\phi'_{\widehat{\Upsilon}^1(\eta)(\ddot{z}^n)} = \frac{\dagger^{1,l}_{\mathbb{T}}(\eta)(\ddot{z}^n) + \dagger^{1,u}_{\mathbb{T}}(\eta)(\ddot{z}^n) + \dagger^{1,l}_{\mathbb{I}}(\eta)(\ddot{z}^n) + \dagger^{1,u}_{\mathbb{I}}(\eta)(\ddot{z}^n) + \dagger^{1,l}_{\mathbb{F}}(\eta)(\ddot{z}^n) + \dagger^{1,u}_{\mathbb{F}}(\eta)(\ddot{z}^n)}{6}$$

$$\phi'_{\widehat{\Upsilon}^2(\eta)(\ddot{z}^n)} = \frac{\dagger^{2,l}_{\mathbb{T}}(\eta)(\ddot{z}^n) + \dagger^{2,u}_{\mathbb{T}}(\eta)(\ddot{z}^n) + \dagger^{2,l}_{\mathbb{I}}(\eta)(\ddot{z}^n) + \dagger^{2,u}_{\mathbb{I}}(\eta)(\ddot{z}^n) + \dagger^{2,l}_{\mathbb{F}}(\eta)(\ddot{z}^n) + \dagger^{2,u}_{\mathbb{F}}(\eta)(\ddot{z}^n)}{6}$$

and for fuzzy possibility degree

$$\dddot{M}_i\left(\hat{\mu}^1(\eta)(\ddot{z}^n), \hat{\mu}^2(\eta)(\ddot{z}^n)\right) = 1 - \frac{\sum_{i=1}^{m}\left|\hat{\mu}^1_i(\eta)(\ddot{z}^n) - \hat{\mu}^2_i(\eta)(\ddot{z}^n)\right|}{\sum_{i=1}^{m}\left|\hat{\mu}^1_i(\eta)(\ddot{z}^n) + \hat{\mu}^2_i(\eta)(\ddot{z}^n)\right|}$$

Definition 5.2. Assume that $\widehat{\Upsilon}^1_{\hat{\mu}^1}$ and $\widehat{\Upsilon}^2_{\hat{\mu}^2}$ be two piv-NHSSs.We say that $\widehat{\Upsilon}^1_{\hat{\mu}^1}$ and $\widehat{\Upsilon}^2_{\hat{\mu}^2}$ are significantly similar if $\hat{\hat{\mathbb{S}}}\left(\widehat{\Upsilon}^1_{\hat{\mu}^1}, \widehat{\Upsilon}^2_{\hat{\mu}^2}\right) \geq \frac{1}{2}$.

Algorithm 1. Engage similarity measures of piv-NHSs to selection the optimal tourist location

Step 1. Build piv-NHSS $\widehat{\Upsilon}^{1}_{\hat{\mu}^1}$ based on expert opinion.

Step 2. Build piv-NHSS $\widehat{\Upsilon}^{2}_{\hat{\mu}^2}$ based on external expert opinion.

Step 3. Determine $\hat{\hat{\mathbb{S}}}\left(\widehat{\Upsilon}^{1}_{\hat{\mu}^1}, \widehat{\Upsilon}^{2}_{\hat{\mu}^2}\right)$ starting Definition 5.1.

Step 4. Check similarity score and select the maxim value by $\tilde{N} = Max\left\{\hat{\hat{\mathbb{S}}}_1, \hat{\hat{\mathbb{S}}}_2, ..., \hat{\hat{\mathbb{S}}}_m\right\}$ who consider the optimal tourist location.

Step 5. End Algorithm 1.

Algorithm 1 is shown in Figure 2.

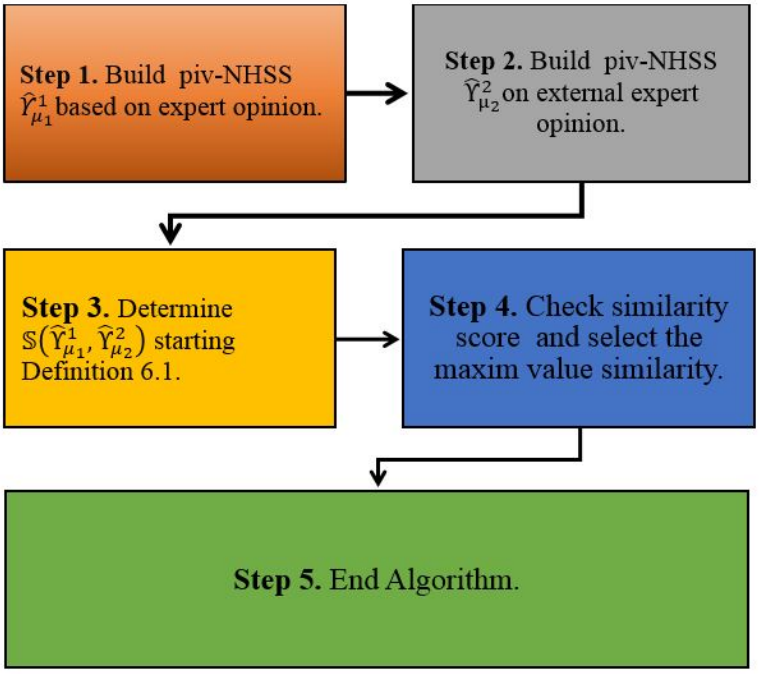
Step 1. Build piv-NHSS $\widehat{\Upsilon}^{1}_{\mu_1}$ based on expert opinion.
Step 2. Build piv-NHSS $\widehat{\Upsilon}^{2}_{\mu_2}$ on external expert opinion.
Step 3. Determine $\mathbb{S}(\widehat{\Upsilon}^{1}_{\mu_1}, \widehat{\Upsilon}^{2}_{\mu_2})$ starting Definition 6.1.
Step 4. Check similarity score and select the maxim value similarity.
Step 5. End Algorithm.

Figure 2: Algorithm 1

5.1. *Application of piv-NHSS in the most qualified employee selection*

In this section, we deal with one of the daily life DM scenarios, specifically the process of selecting a competent employee to fill a position, by presenting a multi-step algorithm, which is an extension or expansion of the algorithm presented in [52].

5.2. *Cuse studty*

The Human Resources (HR) department announces available jobs for applicants to apply for and fill. In this context, a lot number of Contains(He/She) apply for these jobs, and accordingly, a committee of experts is formed to analyze each applicant's skills and

qualifications to choose the most suitable one for these positions. Therefore to analyze the information of all candidates using our proposed model, piv-NHSS, we assume that the fixed set contains four candidates, i.e $\ddot{\mathcal{Z}} = \left\{\ddot{z}^1, \ddot{z}^2, \ddot{z}^3, \ddot{z}^4\right\}$ with members of the attributes (Criteria) i.e η_1 = Academic Certificate,η_2 = English language level and η_3 = Computer skills, these attributes (Criteria) having their sub-attribute values given as following: $\Lambda_1 = \{\eta_{1,1} = Diploma, \eta_{1,2} = BA\}$, $\Lambda_2 = \{\eta_{2,1} = intermediate, \eta_{2,2} = High\}$, $\Lambda_2 = \{\eta_{3,1} = High\}$ then $\Omega = \Lambda_1 \times \Lambda_2 \times \Lambda_3 = \{\eta_1, \eta_2, \eta_3, \eta_4\}$ where every component η_i has a 3-tuple entity. Then accordingly, a committee was formed to interview each candidate individually, and the committee adopted the following steps to accomplish its work:

Step 1. Analyze the information of each candidate based on piv-NHSS $\widehat{\Upsilon}^1_{\hat{\mu}^1}$ are given as following:

$$\widehat{\Upsilon}^1_{\hat{\mu}^1} = \left\{ \widehat{\Upsilon}^1_{\hat{\mu}}(\eta_1) = \left\{ \left(\frac{\ddot{z}^1}{\langle [0.2,0.5],[0.1,0.3],[0.4,0.4] \rangle}, 0.4 \right), \left(\frac{\ddot{z}^2}{\langle [0.5,0.6],[0.3,0.5],[0.3,0.8] \rangle}, 0.7 \right), \left(\frac{\ddot{z}^3}{\langle [0.2,0.4],[0.1,0.3],[0.5,0.6] \rangle}, 0.6 \right), \left(\frac{\ddot{z}^4}{\langle [0.3,0.6],[0.4,0.5],[0.2,0.5] \rangle}, 0.8 \right) \right\} \right.$$

$$\widehat{\Upsilon}^1_{\hat{\mu}}(\eta_2) = \left\{ \left(\frac{\ddot{z}^1}{\langle [0.3,0.6],[0.2,0.5],[0.2,0.3] \rangle}, 0.2 \right), \left(\frac{\ddot{z}^2}{\langle [0.4,0.7],[0.4,0.8],[0.1,0.5] \rangle}, 0.5 \right), \left(\frac{\ddot{z}^3}{\langle [0.6,0.6],[0.3,0.7],[0.5,0.6] \rangle}, 0.3 \right), \left(\frac{\ddot{z}^4}{\langle [0.5,0.7],[0.8,0.6],[0.7,0.7] \rangle}, 0.6 \right) \right\}$$

$$\widehat{\Upsilon}^1_{\hat{\mu}}(\eta_3) = \left\{ \left(\frac{\ddot{z}^1}{\langle [0.1,0.8],[0.7,0.9],[0.6,0.8] \rangle}, 0.3 \right), \left(\frac{\ddot{z}^2}{\langle [0.2,0.4],[0.6,0.9],[0.5,0.7] \rangle}, 0.5 \right), \left(\frac{\ddot{z}^3}{\langle [0.4,0.7],[0.8,0.8],[0.6,0.6] \rangle}, 0.9 \right), \left(\frac{\ddot{z}^4}{\langle [0.5,0.5],[0.7,0.7],[0.3,0.3] \rangle}, 0.5 \right) \right\}$$

$$\left. \widehat{\Upsilon}^1_{\hat{\mu}}(\eta_4) = \left\{ \left(\frac{\ddot{z}^1}{\langle [0.1,0.8],[0.5,0.7],[0.2,0.2] \rangle}, 0.4 \right), \left(\frac{\ddot{z}^2}{\langle [0.1,0.1],[0.6,0.6],[0.7,0.8] \rangle}, 0.1 \right), \left(\frac{\ddot{z}^3}{\langle [0.6,0.6],[0.7,0.7],[0.9,0.9] \rangle}, 0.5 \right), \left(\frac{\ddot{z}^4}{\langle [0.2,0.3],[0.4,0.8],[0.4,0.7] \rangle}, 0.6 \right) \right\} \right\}$$

Step 2. Build piv-NHSS $\widehat{\Upsilon}^2_{\hat{\mu}^2}$ based on external expert opinion.

$$\widehat{\Upsilon}^2_{\hat{\mu}^2} = \left\{ \widehat{\Upsilon}^2_{\hat{\mu}}(\eta_1) = \left\{ \left(\frac{\ddot{z}^1}{\langle [0.4,0.5],[0.7,0.7],[0.3,0.8] \rangle}, 0.8 \right), \left(\frac{\ddot{z}^2}{\langle [0.3,0.5],[0.6,0.7],[0.8,0.9] \rangle}, 0.4 \right), \left(\frac{\ddot{z}^3}{\langle [0.7,0.9],[0.1,0.1],[0.5,0.5] \rangle}, 0.8 \right), \left(\frac{\ddot{z}^4}{\langle [0.4,0.4],[0.4,0.6],[0.1,0.3] \rangle}, 0.7 \right) \right\} \right.$$

$$\widehat{\Upsilon}^2_{\hat{\mu}}(\eta_2) = \left\{ \left(\frac{\ddot{z}^1}{\langle [0.4,0.6],[0.2,0.4],[0.3,0.3] \rangle}, 0.1 \right), \left(\frac{\ddot{z}^2}{\langle [0.3,0.4],[0.5,0.6],[0.3,0.5] \rangle}, 0.8 \right), \left(\frac{\ddot{z}^3}{\langle [0.2,0.5],[0.3,0.7],[0.5,0.7] \rangle}, 0.9 \right), \left(\frac{\ddot{z}^4}{\langle [0.6,0.6],[0.4,0.5],[0.6,0.8] \rangle}, 0.9 \right) \right\}$$

$$\widehat{\Upsilon}^2_{\hat{\mu}}(\eta_3) = \left\{ \left(\frac{\ddot{z}^1}{\langle [0.1,0.8],[0.7,0.9],[0.6,0.8] \rangle}, 0.8 \right), \left(\frac{\ddot{z}^2}{\langle [0.2,0.4],[0.6,0.9],[0.5,0.7] \rangle}, 0.7 \right), \left(\frac{\ddot{z}^3}{\langle [0.4,0.7],[0.8,0.8],[0.6,0.6] \rangle}, 0.5 \right), \left(\frac{\ddot{z}^4}{\langle [0.5,0.5],[0.7,0.7],[0.3,0.3] \rangle}, 0.8 \right) \right\}$$

$$\left. \widehat{\Upsilon}^2_{\hat{\mu}}(\eta_4) = \left\{ \left(\frac{\ddot{z}^1}{\langle [0.1,0.8],[0.5,0.7],[0.2,0.2] \rangle}, 0.4 \right), \left(\frac{\ddot{z}^2}{\langle [0.1,0.1],[0.6,0.6],[0.3,0.5] \rangle}, 0.7 \right), \left(\frac{\ddot{z}^3}{\langle [0.6,0.6],[0.7,0.7],[0.9,0.9] \rangle}, 0.8 \right), \left(\frac{\ddot{z}^4}{\langle [0.2,0.3],[0.8,0.9],[0.4,0.9] \rangle}, 0.5 \right) \right\} \right\}$$

Determine $\hat{\ddot{\mathbb{S}}}\left(\widehat{\Upsilon}^1_{\hat{\mu}^1}, \widehat{\Upsilon}^2_{\hat{\mu}^2}\right)$ starting Definition 5.1 as following:

$$\hat{\ddot{\mathbb{S}}}\left(\widehat{\Upsilon}^1_{\hat{\mu}}\left(\ddot{z}^1\right), \widehat{\Upsilon}^2_{\hat{\mu}}\left(\ddot{z}^1\right)\right) = 0.547$$
$$\hat{\ddot{\mathbb{S}}}\left(\widehat{\Upsilon}^1_{\hat{\mu}}\left(\ddot{z}^2\right), \widehat{\Upsilon}^2_{\hat{\mu}}\left(\ddot{z}^2\right)\right) = 0.703$$
$$\hat{\ddot{\mathbb{S}}}\left(\widehat{\Upsilon}^1_{\hat{\mu}}\left(\ddot{z}^3\right), \widehat{\Upsilon}^2_{\hat{\mu}}\left(\ddot{z}^3\right)\right) = 0.549$$
$$\hat{\ddot{\mathbb{S}}}\left(\widehat{\Upsilon}^1_{\hat{\mu}}\left(\ddot{z}^4\right), \widehat{\Upsilon}^2_{\hat{\mu}}\left(\ddot{z}^4\right)\right) = 0.761$$

Step 4. Check similarity score and select the maxim value by $\tilde{N} = Max\{0.547, 0.703, 0.549, 0.761\}$ who consider the optimal tourist location who $\ddot{z}^4$.

6. Sine Exponential Measure of piv-NHSs

In this section, we reveal sine exponential measure of of piv-NHSSs and employ this type measure in medical diagnosis.

Definition 6.1. Assume that $\widehat{\Upsilon}^1_{\hat{\mu}^1}$ and $\widehat{\Upsilon}^2_{\hat{\mu}^2}$ be two piv-NHSSs. Then the sine exponential measure is given as follows:

$$\ddot{\mathbb{S}}\left(\widehat{\Upsilon}^1_{\hat{\mu}^1}, \widehat{\Upsilon}^2_{\hat{\mu}^2}\right) = \frac{1}{n}\left[\sum_{i=1}^{n} \sin e^{-\left(\left[\left|\dagger^1_{\mathbb{T}}(\eta)(\ddot{z}^n)-\dagger^2_{\mathbb{T}}(\eta)(\ddot{z}^n)\right|+\left|\dagger^1_{\mathbb{I}}(\eta)(\ddot{z}^n)-\dagger^2_{\mathbb{I}}(\eta)(\ddot{z}^n)\right|+\left|\dagger^1_{\mathbb{F}}(\eta)(\ddot{z}^n)-\dagger^2_{\mathbb{F}}(\eta)(\ddot{z}^n)\right|\right]\times\left(\hat{\mu}^1(\eta)(\ddot{z}^n)\times\hat{\mu}^2(\eta)(\ddot{z}^n)\right)\right)}\right]$$

$$= \frac{1}{n}\left[\sum_{i=1}^{n} \sin e^{-\left(\begin{bmatrix} \left|\left(\dagger^{1,l}_{\mathbb{R}}(\eta)(\ddot{z}^n)\times\dagger^{2,l}_{\mathbb{R}}(\eta)(\ddot{z}^n)\right)-\left(\dagger^{1,u}_{\mathbb{R}}(\eta)(\ddot{z}^n)\times\dagger^{2,u}_{\mathbb{R}}(\eta)(\ddot{z}^n)\right)\right|+ \\ \left|\left(\dagger^{1,l}_{\mathbb{R}}(\eta)(\ddot{z}^n)\times\dagger^{2,l}_{\mathbb{R}}(\eta)(\ddot{z}^n)\right)-\left(\dagger^{1,u}_{\mathbb{R}}(\eta)(\ddot{z}^n)\times\dagger^{2,u}_{\mathbb{R}}(\eta)(\ddot{z}^n)\right)\right|+ \\ \left|\left|\left(\dagger^{1,l}_{\mathbb{R}}(\eta)(\ddot{z}^n)\times\dagger^{2,l}_{\mathbb{R}}(\eta)(\ddot{z}^n)\right)-\left(\dagger^{1,u}_{\mathbb{R}}(\eta)(\ddot{z}^n)\times\dagger^{2,u}_{\mathbb{R}}(\eta)(\ddot{z}^n)\right)\right|\right| \end{bmatrix}\times\left(\hat{\mu}^1(\eta)(\ddot{z}^n)\times\hat{\mu}^2(\eta)(\ddot{z}^n)\right)\right)}\right]$$

Here, based on this formula, it can be guaranteed that the value of these measure is greater than 1.

Example 6.2. The new couples planned to spend their New Year's holiday in one of the following tourist destinations: ancient Chinese ruins , the highlands of Penang Island, or Langkawi Island $\ddot{\mathcal{Z}} = \left\{\ddot{z}^1, \ddot{z}^2, \ddot{z}^3, \ddot{z}^4\right\}$ where $\ddot{z}^1 = ancient\ Chinese\ ruins$, $\ddot{z}^2 = highlands\ of\ Penang\ Island$ and $\ddot{z}^3 = Langkawi\ Island$. The couple asked a friend who works for a travel agency to consult them. He asked them to specify the criteria that would determine the selection process. They answered that the criteria were distance and the means of transportation available for travel that can be expressed by i.e η_1 = distance,η_2 = transportation these attributes (Criteria) having their sub-attribute values given as following: $\Lambda_1 = \{\eta_{1,1} = 1000km, \eta_{1,2} = 2000km\}$, $\Lambda_2 = \{\eta_{2,1} = Airplane, \eta_{2,2} = Car\}$, then

$\Omega = \Lambda_1 \times \Lambda_2 = \{\eta_1, \eta_2, \eta_3, \eta_4\}$ where every component η_i has a 2-tuple entity.

In order to solve this problem and help couples choose the best tourist place to spend their vacation, we suggest the following algorithm:

Step 1. Build piv-NHSS $\widehat{\Upsilon}^1_{\hat{\mu}^1}$ and $\widehat{\Upsilon}^2_{\hat{\mu}^2}$ based on experts opinion.

Step 2. Determine $\dddot{\mathbb{S}}\left(\widehat{\Upsilon}^1_{\hat{\mu}^1}, \widehat{\Upsilon}^2_{\hat{\mu}^2}\right)$ starting Definition 6.1.

Step 4. Check Sine Exponential Measure score and select the maxim value by $\tilde{N} = Max\left\{\dddot{\mathbb{S}}_1, \hat{\dddot{\mathbb{S}}}_2, ..., \hat{\dddot{\mathbb{S}}}_m\right\}$ who consider the optimal tourist location.

Step 5. End Algorithm 2.

Algorithm 2 is shown in Figure 3.

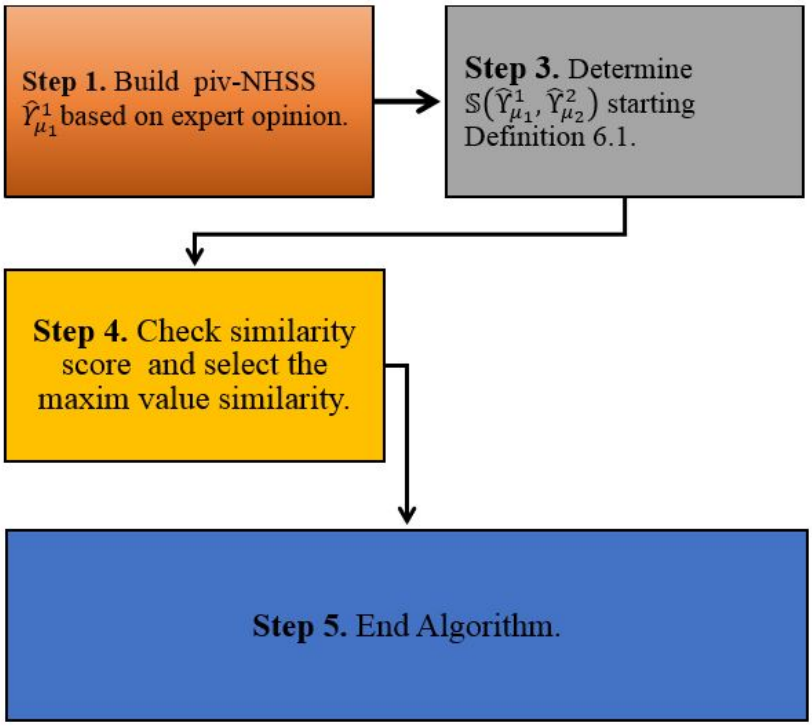
Step 1. Build piv-NHSS $\widehat{\Upsilon}^1_{\mu_1}$ based on expert opinion.
Step 3. Determine $\mathbb{S}\left(\widehat{\Upsilon}^1_{\mu_1}, \widehat{\Upsilon}^2_{\mu_2}\right)$ starting Definition 6.1.
Step 4. Check similarity score and select the maxim value similarity.
Step 5. End Algorithm.

Figure 3: Algorithm 2

Now to implement this algorithm, we start by applying the first step as follows:

$$\widehat{\Upsilon}^1_{\hat{\mu_1}} = \left\{ \widehat{\Upsilon}^1{}_{\hat{\mu_1}}(\eta_1) = \left\{ \left(\frac{\ddot{z}^1}{\langle[0.2,0.5],[0.1,0.3],[0.4,0.4]\rangle}, 0.4 \right), \left(\frac{\ddot{z}^2}{\langle[0.5,0.6],[0.3,0.5],[0.3,0.8]\rangle}, 0.7 \right), \left(\frac{\ddot{z}^3}{\langle[0.2,0.4],[0.1,0.3],[0.5,0.6]\rangle}, 0.6 \right) \right\} \right.$$

$$\widehat{\Upsilon}^1{}_{\hat{\mu_1}}(\eta_2) = \left\{ \left(\frac{\ddot{z}^1}{\langle[0.3,0.6],[0.2,0.5],[0.2,0.3]\rangle}, 0.2 \right), \left(\frac{\ddot{z}^2}{\langle[0.4,0.7],[0.4,0.8],[0.1,0.5]\rangle}, 0.5 \right), \left(\frac{\ddot{z}^3}{\langle[0.6,0.6],[0.3,0.7],[0.5,0.6]\rangle}, 0.3 \right) \right\}$$

$$\widehat{\Upsilon}^1{}_{\hat{\mu_1}}(\eta_3) = \left\{ \left(\frac{\ddot{z}^1}{\langle[0.1,0.8],[0.7,0.9],[0.6,0.8]\rangle}, 0.3 \right), \left(\frac{\ddot{z}^2}{\langle[0.2,0.4],[0.6,0.9],[0.5,0.7]\rangle}, 0.5 \right), \right.$$

$\left(\frac{\ddot{z}^3}{\langle[0.4,0.7],[0.8,0.8],[0.6,0.6]\rangle}, 0.9\right)\}$

$\widehat{\Upsilon}^1_{\hat{\mu_1}}(\eta_4) = \left\{\left(\frac{\ddot{z}^1}{\langle[0.1,0.8],[0.5,0.7],[0.2,0.2]\rangle}, 0.4\right), \left(\frac{\ddot{z}^2}{\langle[0.1,0.1],[0.6,0.6],[0.7,0.8]\rangle}, 0.1\right),\right.$

$\left.\left(\frac{\ddot{z}^3}{\langle[0.5,0.8],[0.5,0.9],[0.9,0.9]\rangle}, 0.7\right)\right\}\Bigg\}$

$\widehat{\Upsilon}^2_{\hat{\mu_2}} = \Bigg\{\widehat{\Upsilon}^2_{\hat{\mu_1}}(\eta_1) = \left\{\left(\frac{\ddot{z}^1}{\langle[0.1,0.2],[0.4,0.4],[0.2,0.6]\rangle}, 0.5\right), \left(\frac{\ddot{z}^2}{\langle[0.3,0.5],[0.6,0.6],[0.3,0.8]\rangle}, 0.4\right),\right.$

$\left.\left(\frac{\ddot{z}^3}{\langle[0.3,0.6],[0.5,0.5],[0.7,0.8]\rangle}, 0.9\right)\right\}$

$\widehat{\Upsilon}^2_{\hat{\mu_2}}(\eta_2) = \left\{\left(\frac{\ddot{z}^1}{\langle[0.5,0.7],[0.8,0.8],[0.7,0.9]\rangle}, 0.6\right), \left(\frac{\ddot{z}^2}{\langle[0.4,0.7],[0.4,0.8],[0.1,0.5]\rangle}, 0.7\right),\right.$

$\left.\left(\frac{\ddot{z}^3}{\langle[0.1,0.4],[0.5,0.7],[0.5,0.6]\rangle}, 0.8\right)\right\}$

$\widehat{\Upsilon}^2_{\hat{\mu_2}}(\eta_3) = \left\{\left(\frac{\ddot{z}^1}{\langle[0.3,0.5],[0.6,0.6],[0.9,0.9]\rangle}, 0.9\right), \left(\frac{\ddot{z}^2}{\langle[0.2,0.4],[0.6,0.9],[0.5,0.7]\rangle}, 0.5\right),\right.$

$\left.\left(\frac{\ddot{z}^3}{\langle[0.5,0.5],[0.6,0.7],[0.6,0.9]\rangle}, 0.1\right)\right\}$

$\widehat{\Upsilon}^2_{\hat{\mu_2}}(\eta_4) = \left\{\left(\frac{\ddot{z}^1}{\langle[0.5,0.7],[0.4,0.4],[0.1,0.6]\rangle}, 0.5\right), \left(\frac{\ddot{z}^2}{\langle[0.5,0.7],[0.3,0.5],[0.7,0.9]\rangle}, 0.5\right),\right.$

$\left.\left(\frac{\ddot{z}^3}{\langle[0.6,0.8],[0.6,0.9],[0.4,0.6]\rangle}, 0.8\right)\right\}\Bigg\}$

Then the one exponential measures shown in table 1 as following

TABLE 1. Exponential measures values of piv-NHSSs $\widehat{\Upsilon}^1_{\hat{\mu}^1}, \widehat{\Upsilon}^2_{\hat{\mu}^2}$

$\eta_j/\ddot{Z}^i$	z^1	z^2	z^3
η_1	0.0186	0.0241	0.0092
η_2	0.0143	0.0314	0.0157
η_3	0.0261	0.0283	0.01947
η_4	0.0181	0.0381	0.1025
Sum($\ddot{Z}^i$)	0.0771	0.1219	0.1468

From Table 3 we get the best choice which is $\ddot{z}^3$ who is Langkawi Island.

7. **Conclusions**

In this work, we offer an innovative piv-NHSS by combining both iv-NS and HSS under possibility possible environment.This novel notion is characterized by the ability to deal with complexity and uncertainty issues aptly. The elementary properties and set-theoretical like null piv-NHSS, absolute piv-NHSS, complement of piv-NHSS, union piv-NHSSs, intersection piv-NHSSs, AND-piv-NHSSs, and OR-piv-NHSS was successfully conceptualized as well as support them with examples. Moreover, we discovered both similarities measure and sine

exponential measure between two piv-NHSSs and these measures are successfully applied in decision-making to judge choose the best tourist place. As for future studies of this work, the authors recommend developing this concept by integrating it with a number of modern parameters tools [45]– [48] as well as integrating these results with some algebraic [49]– [54] and topological structures [55]– [58].

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New Approach Single Valued Neutrosophic Sets for Teaching Quality Challenges Evaluation in College Public English Broad Impacts

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Abstract
Previous research on teaching quality challenges in college public English has relied on methods that accept vague or fuzzy inputs. Single-valued neutrosophic Sets (SVNSs) are used to handle uncertain or unclear expert assessments by allowing both precise and imprecise data to be analyzed together. This approach ensures that valuable information is not lost during the evaluation process. To improve decision-making in multi-criteria decision-making (MCDM) problems, this study introduces a new integrated framework that combines the EDAS method with the MEREC method for criteria weight computation under the SVNS environment. The MEREC method is applied to determine the importance of each criterion, while the EDAS method ranks the alternatives. A case study on evaluating teaching quality challenges in college public English is conducted to demonstrate the effectiveness of the proposed framework. In this study, three experts evaluated 11 criteria and 7 alternatives. Sensitivity analysis was also performed to test the stability of the rankings under different criteria weight variations. The results show that the ranking of alternatives remains stable, confirming the reliability of the proposed approach.

Keywords: Single Valued Neutrosophic Sets; MEREC; EDAS Approach; Teaching Quality challenges in College Public English; Uncertainty.

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1. Introduction

The two most important elements in researchers' typical procedural steps for effective and efficient ranking alternatives are screening and ranking. Multi-criteria decision-making (MCDM) techniques are frequently used to systematically assess complex and interdependent situations with numerous actors, criteria, and conflicting objectives and are used to screen alternatives with superior qualities[1], [2]. In the framework of ranking the alternatives incorporating quantitative and qualitative criteria employing clear, basic inputs, previous researchers have thoroughly examined the importance and application of the MCDM approaches[3], [4].

Experts encounter issues when decision-makers judgments must be relied upon when inputs are not clearly defined. Depending on the decision-makers' perspectives, the complexity of the situation, and their differing levels of knowledge, the decisions may produce inaccurate or ambiguous information[5], [6]. In these situations, theories that handle uncertainties during the decision-making process include fuzzy sets (FSs), intuitionistic fuzzy sets (IFSs), and neutrosophic sets (NSs). Single-valued neutrosophic sets (SVNSs), a subset of NS created by Wang et al. and based on the basic idea of combining several inputs from several decision-makers into a single value, are used in this work[7], [7].

Since SVNSs also address ambiguities or inconsistencies, they have gained a great deal of relevance for solving problems in real-time. Real-time problem-solving with SVNSs has produced practical and helpful solutions for a variety of applications[8], [9].

Objective, subjective, or a combination of both methods are used to evaluate the relative importance of criteria for a particular application. The objective weighting techniques most frequently employed in real-time applications are entropy and CRITIC[10], [11]. A criterion is given more weight when its removal has a substantial impact on the overall performance, according to MEREC, a novel objective weighing method proposed by Keshavarz-Ghorabaee et al. that evaluates the criteria weighting from the exclusion perspective. Therefore, to derive criteria weights, MEREC is integrated into an MCDM issue for the first time under the SVNS environment[12], [13].

This study provides a novel integrated framework that combines a method based on the removal impacts of criteria (MEREC), measurement alternatives, and EDAS in the context of SVNSs. This builds on previous work.

The study's main aims and objectives are as follows:

[1] To offer a novel framework that examines both accurate and imprecise or undecided input parameters at the same time, offering a more practical method of addressing a multi-criteria problem.
[2] To use a unique integrated SVN-MEREC-EDAS approach compute the criteria weights and rank the alternatives.
[3] To employ MEREC, a novel criterion-weighing technique in SVNSs that evaluates the impact of removing criteria on alternative performances.
[4] The sensitivity analysis is conducted to show the stability of the rank of alternatives under different criteria weights.

The following describes the scope of this work. The initial definitions and mathematical procedures for solving SVNSs are given in Section 2, and Section 3 then presents the suggested methodology. A sensitivity study of the proposed method's application to compute the criteria weights and rank the alternatives is provided in Section 4. Lastly, Section 5 concludes the research project.

2. Preliminaries

This section shows the operations of the single valued neutrosophic sets (SVNSs) such as[14], [15], [16], [17]:

Definition 1.

Neutrosophic sets can be defined as:

$$N = \{(T_N(q_i), I_N(q_i), F_N(q_i))|q_i \in Q\} \quad (1)$$

Where Q refers to the universal set with q_i elements and three functions of truth, indeterminacy, and falsity of $T_N(q_i), I_N(q_i), and F_N(q_i)$

$$-0 \leq T_N(q_i) + I_N(q_i) + F_N(q_i) \leq 3 + \quad (2)$$

Definition 2.

The sum of the membership functions of SVNS between 0 and 3 such as:

$$0 \leq t_N(q_i) + i_N(q_i) + f_N(q_i) \leq 3 \quad (3)$$

Definition 3.

Let two single valued neutrosophic numbers (SVNNs) such as:

$a_1 = t_{a_1}(q), i_{a_1}(q), f_{a_1}(q)$ and $a_2 = t_{a_2}(q), i_{a_3}(q), f_{a_4}(q)$

$$a_1^c = \left(f_{a_1}(q), 1 - i_{a_1}(q), t_{a_1}(q)\right)$$

$$a_1 \cup a_2 = \begin{pmatrix} \max\{t_{a_1}(q), t_{a_2}(q)\}, \\ \min\{i_{a_1}(q), i_{a_2}(q)\}, \\ \min\{f_{a_1}(q), f_{a_2}(q)\} \end{pmatrix} \quad (4)$$

$$a_1 \cap a_2 = \begin{pmatrix} \min\{t_{a_1}(q), t_{a_2}(q)\}, \\ \max\{i_{a_1}(q), i_{a_2}(q)\}, \\ \max\{f_{a_1}(q), f_{a_2}(q)\} \end{pmatrix} \quad (5)$$

$$a_1 + a_2 = \begin{pmatrix} t_{a_1}(q) + t_{a_2}(q) - t_{a_1}(q)t_{a_2}(q), \\ i_{a_1}(q)i_{a_2}(q), \\ f_{a_1}(q)f_{a_2}(q) \end{pmatrix} \quad (6)$$

$$a_1 a_2 = \begin{pmatrix} t_{a_1}(q)t_{a_2}(q), \\ i_{a_1}(q) + i_{a_2}(q) - i_{a_1}(q)i_{a_2}(q), \\ f_{a_1}(q) + f_{a_2}(q) - f_{a_1}(q)f_{a_2}(q) \end{pmatrix} \quad (7)$$

$$ha_1 = \begin{pmatrix} 1-\left(1-t_{a_1}(q)\right)^h, \\ \left(i_{a_1}(q)\right)^h, \\ \left(f_{a_1}(q)\right)^h \end{pmatrix} \tag{8}$$

$$a_1^h = \begin{pmatrix} \left(t_{a_1}(q)\right)^h, \\ 1-\left(1-i_{a_1}(q)\right)^h, \\ 1-\left(1-f_{a_1}(q)\right)^h \end{pmatrix} \tag{9}$$

Definition 4.

The score function of the SVNSs is obtained by:

$$s(a_1) = \frac{2+t_{a_1}(q)-i_{a_1}(q)-f_{a_1}(q)}{3} \tag{10}$$

2.1 Research Gap

Despite extensive research on evaluating teaching quality in College Public English, many existing methods rely on traditional decision-making approaches that struggle with handling uncertainty and ambiguity in expert assessments. Most previous studies have used fuzzy logic, intuitionistic fuzzy sets, or conventional multi-criteria decision-making (MCDM) techniques, which, while effective in structured decision environments, often fail to capture the complexity of human judgment in uncertain conditions. These methods either oversimplify expert opinions or lack the flexibility to integrate both qualitative and quantitative data seamlessly.

This research fills this gap by introducing a novel framework that integrates Single Valued Neutrosophic Sets (SVNSs) with the MEREC and EDAS methods. The strength of SVNSs lies in their ability to preserve uncertainty without discarding valuable information, allowing for a more precise and adaptable evaluation process. The combination of MEREC for criteria weighing and EDAS for ranking alternatives provides a structured yet flexible solution that ensures reliable decision-making. This study offers a new perspective on handling imprecise and uncertain educational evaluations while enhancing the accuracy of ranking alternatives in multi-criteria decision-making problems.

3. Proposed method

A novel and comprehensive framework is introduced that integrates the MEREC and EDAS methods within the context of SVNSs to systematically evaluate the challenges affecting teaching quality in College Public English. In this framework, the MEREC method is meticulously employed to compute the weights of the evaluation criteria, ensuring that each factor is objectively assessed for its relative importance. Simultaneously, the EDAS method is utilized to rank the alternatives, thereby establishing a clear hierarchy of potential solutions based on their

overall performance. The integration of these two methods under the SVNSs paradigm effectively addresses the inherent uncertainties and ambiguities of the evaluation process, resulting in a robust and reliable analytical approach. The following sections detail the organized steps of the proposed methodology:

Step 1. – Construct the combined Crisp-Decision Matrix (C-DM) and SVN-Decision Matrix (SVN-DM)

The decision matrix is built based on a set of criteria such as $C_1, \dots, C_n$ and alternatives $A_1, \dots, A_m$. Then we invited a set of experts and decision makers to evaluate the criteria and alternatives. Then we used the SVNNs to evaluate the criteria and alternatives. Then we applied the score function to obtain crisp values. Then we combine the decision matrices into a single matrix.

Step 2. – Compute the criteria weights by the MEREC method

We start with the decision matrix. Then we normalize the decision matrix such as:

$$\left(N_{ij}\right)_{m\times n} = \begin{cases} \dfrac{\min x_{ij}}{x_{ij}} \; for\ positive\ criteria \\ \dfrac{x_{ij}}{\max x_{ij}} \; for\ negative\ criteria \end{cases} \tag{11}$$

Compute the overall performance

The MEREC method uses the non-linear function to compute the overall performance values of each alternative.

$$S_i = \ln\left(1 + \left(\frac{1}{m}\sum_j \left|\ln\left(N_{ij}\right)\right|\right)\right) \tag{12}$$

Assess the performance based on removing criteria

$$S_{ij}^* = \ln\left(1 + \left(\frac{1}{m}\sum_{k,k\neq j} \left|\ln(N_{ik})\right|\right)\right) \tag{13}$$

Compute the absolute deviations for criteria.

$$Q_j = \sum_i \left|S_{ij}^* - S_i\right| \tag{14}$$

Compute the final weights.

$$w_j = \frac{Q_j}{\sum_j Q_j} \tag{15}$$

Step 3. – Rank the alternatives using the EDAS method[18], [19]

This step ranks the alternatives using the EDAS method under the SVNSs.

Compute the average solution

$$V_j = \frac{\sum_{i=1}^{m} x_{ij}}{m} \tag{16}$$

Compute the positive and negative distances from average solution

$$Y_{ij} = \frac{\max\left(0, \left(x_{ij} - V_j\right)\right)}{V_j} \tag{17}$$

$$Z_{ij} = \frac{\max\left(0, \left(V_j - x_{ij}\right)\right)}{V_j} \tag{18}$$

For negative criteria

$$Y_{ij} = \frac{\max\left(0, \left(V_j - x_{ij}\right)\right)}{V_j} \tag{19}$$

$$Z_{ij} = \frac{\max\left(0, \left(x_{ij} - V_j\right)\right)}{V_j} \tag{20}$$

Compute the weighted Y_{ij} and Z_{ij}

$$SY_i = \sum_{j=1}^{n} Y_{ij} w_j \tag{21}$$

$$SZ_i = \sum_{j=1}^{n} Z_{ij} w_j \tag{22}$$

Compute the weighted normalized Y_{ij} and Z_{ij}

$$NSY_i = \frac{SY_i}{\max(SY_i)} \tag{23}$$

$$NSZ_i = \frac{SZ_i}{\max(SZ_i)} \tag{24}$$

Compute the appraisal value

$$AS_i = 0.5 * (NSY_i + NSZ_i) \tag{25}$$

Rank the alternatives.

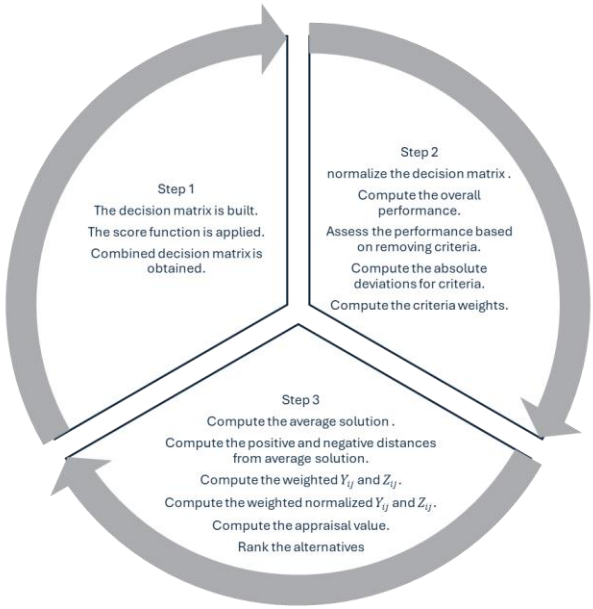
Step 1
The decision matrix is built.
The score function is applied.
Combined decision matrix is obtained.
Step 2
normalize the decision matrix .
Compute the overall performance.
Assess the performance based on removing criteria.
Compute the absolute deviations for criteria.
Compute the criteria weights.
Step 3
Compute the average solution .
Compute the positive and negative distances from average solution.
Compute the weighted Y_{ij} and Z_{ij}.
Compute the weighted normalized Y_{ij} and Z_{ij}.
Compute the appraisal value.
Rank the alternatives

Fig 1. A Comprehensive Illustration of the Research Framework and Its Underlying Components

4. Case study: Challenge Areas in College Public English Teaching Quality Evaluation

This section presents the results of our proposed approach. Figure 1 shows the steps of the SVN-MEREC-EDAS method, which we use to calculate the criteria and rank the alternatives. We began by collecting the opinions of experts and decision-makers, which helped us gather the list of criteria and alternatives (see Figure 2).

Step 1, To start, we invited three experts who each have over 20 years of experience in this field to evaluate the criteria and alternatives. We used Single-Valued Neutrosophic Numbers (SVNNs) to assess them, as detailed in Table 1. Next, we applied a score function to reduce the three values from the SVNNs into a single value for each evaluation. Finally, we averaged these values to determine the criteria weights and to rank the alternatives. This step-by-step process allowed us to identify the importance of each criterion and determine the best alternative simply and reliably.

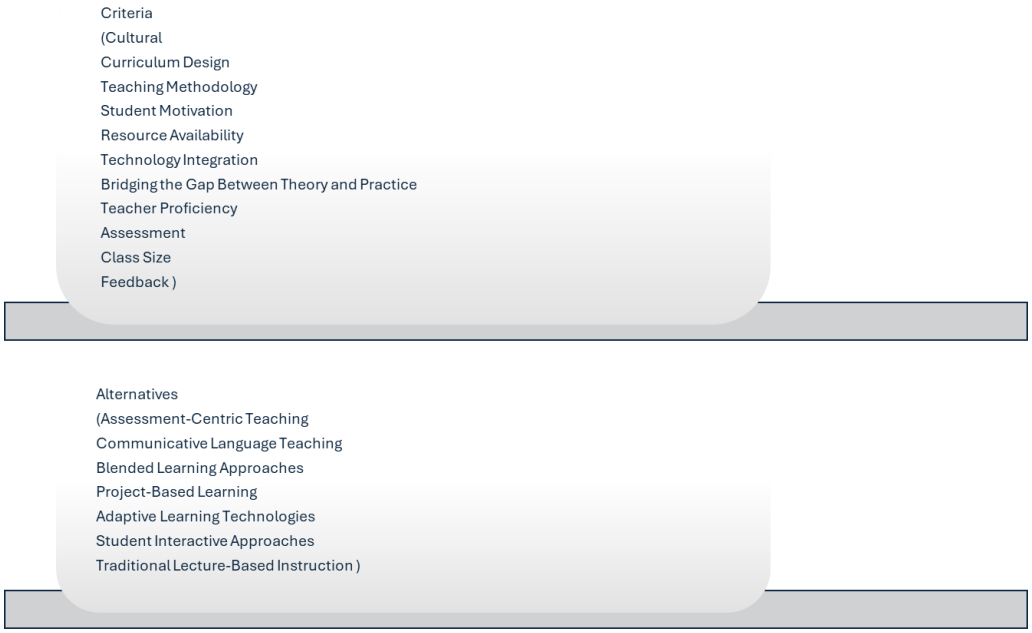

Fig. 2. A Comprehensive Illustration of the Complete Set of Criteria and the Spectrum of Alternative Options.

Table 1. A Detailed Overview of the SVNNs Matrix and Its Constituent Elements.

	$\mathbf{A_1}$	$\mathbf{A_2}$	$\mathbf{A_3}$	$\mathbf{A_4}$	$\mathbf{A_5}$	$\mathbf{A_6}$	$\mathbf{A_7}$
$\mathbf{C_1}$	(0.2,0.7,0.8)	(0.1,0.8,0.9)	(0.2,0.7,0.8)	(0.3,0.6,0.7)	(0.4,0.5,0.6)	(0.5,0.5,0.5)	(0.6,0.4,0.5)
$\mathbf{C_2}$	(0.8,0.2,0.3)	(0.2,0.7,0.8)	(0.1,0.8,0.9)	(0.2,0.7,0.8)	(0.9,0.1,0.2)	(0.8,0.2,0.3)	(0.7,0.3,0.4)
$\mathbf{C_3}$	(0.7,0.3,0.4)	(0.4,0.5,0.6)	(0.9,0.1,0.2)	(0.2,0.7,0.8)	(0.1,0.8,0.9)	(0.7,0.3,0.4)	(0.8,0.2,0.3)
$\mathbf{C_4}$	(0.6,0.4,0.5)	(0.5,0.5,0.5)	(0.8,0.2,0.3)	(0.2,0.7,0.8)	(0.1,0.8,0.9)	(0.6,0.4,0.5)	(0.9,0.1,0.2)
$\mathbf{C_5}$	(0.5,0.5,0.5)	(0.6,0.4,0.5)	(0.2,0.7,0.8)	(0.1,0.8,0.9)	(0.3,0.6,0.7)	(0.5,0.5,0.5)	(0.4,0.5,0.6)
$\mathbf{C_6}$	(0.4,0.5,0.6)	(0.7,0.3,0.4)	(0.2,0.7,0.8)	(0.1,0.8,0.9)	(0.2,0.7,0.8)	(0.4,0.5,0.6)	(0.5,0.5,0.5)
$\mathbf{C_7}$	(0.3,0.6,0.7)	(0.8,0.2,0.3)	(0.5,0.5,0.5)	(0.2,0.7,0.8)	(0.2,0.7,0.8)	(0.1,0.8,0.9)	(0.6,0.4,0.5)
$\mathbf{C_8}$	(0.2,0.7,0.8)	(0.9,0.1,0.2)	(0.4,0.5,0.6)	(0.2,0.7,0.8)	(0.1,0.8,0.9)	(0.2,0.7,0.8)	(0.7,0.3,0.4)
$\mathbf{C_9}$	(0.2,0.7,0.8)	(0.1,0.8,0.9)	(0.3,0.6,0.7)	(0.4,0.5,0.6)	(0.7,0.3,0.4)	(0.9,0.1,0.2)	(0.8,0.2,0.3)
$\mathbf{C_{10}}$	(0.8,0.2,0.3)	(0.3,0.6,0.7)	(0.2,0.7,0.8)	(0.1,0.8,0.9)	(0.6,0.4,0.5)	(0.8,0.2,0.3)	(0.9,0.1,0.2)
$\mathbf{C_{11}}$	(0.7,0.3,0.4)	(0.4,0.5,0.6)	(0.9,0.1,0.2)	(0.2,0.7,0.8)	(0.1,0.8,0.9)	(0.7,0.3,0.4)	(0.2,0.7,0.8)
	A_1	A_2	A_3	A_4	A_5	A_6	A_7
$\mathbf{C_1}$	(0.2,0.7,0.8)	(0.1,0.8,0.9)	(0.3,0.6,0.7)	(0.2,0.7,0.8)	(0.9,0.1,0.2)	(0.5,0.5,0.5)	(0.6,0.4,0.5)
$\mathbf{C_2}$	(0.8,0.2,0.3)	(0.2,0.7,0.8)	(0.1,0.8,0.9)	(0.4,0.5,0.6)	(0.3,0.6,0.7)	(0.2,0.7,0.8)	(0.9,0.1,0.2)
$\mathbf{C_3}$	(0.7,0.3,0.4)	(0.4,0.5,0.6)	(0.2,0.7,0.8)	(0.1,0.8,0.9)	(0.2,0.7,0.8)	(0.9,0.1,0.2)	(0.8,0.2,0.3)
$\mathbf{C_4}$	(0.6,0.4,0.5)	(0.4,0.5,0.6)	(0.3,0.6,0.7)	(0.2,0.7,0.8)	(0.1,0.8,0.9)	(0.6,0.4,0.5)	(0.9,0.1,0.2)
$\mathbf{C_5}$	(0.5,0.5,0.5)	(0.4,0.5,0.6)	(0.3,0.6,0.7)	(0.2,0.7,0.8)	(0.9,0.1,0.2)	(0.5,0.5,0.5)	(0.4,0.5,0.6)
$\mathbf{C_6}$	(0.4,0.5,0.6)	(0.3,0.6,0.7)	(0.2,0.7,0.8)	(0.1,0.8,0.9)	(0.2,0.7,0.8)	(0.4,0.5,0.6)	(0.5,0.5,0.5)
$\mathbf{C_7}$	(0.3,0.6,0.7)	(0.4,0.5,0.6)	(0.3,0.6,0.7)	(0.2,0.7,0.8)	(0.9,0.1,0.2)	(0.3,0.6,0.7)	(0.6,0.4,0.5)
$\mathbf{C_8}$	(0.2,0.7,0.8)	(0.9,0.1,0.2)	(0.4,0.5,0.6)	(0.3,0.6,0.7)	(0.2,0.7,0.8)	(0.9,0.1,0.2)	(0.7,0.3,0.4)
$\mathbf{C_9}$	(0.9,0.1,0.2)	(0.2,0.7,0.8)	(0.2,0.7,0.8)	(0.1,0.8,0.9)	(0.3,0.6,0.7)	(0.2,0.7,0.8)	(0.9,0.1,0.2)
$\mathbf{C_{10}}$	(0.8,0.2,0.3)	(0.2,0.7,0.8)	(0.1,0.8,0.9)	(0.3,0.6,0.7)	(0.2,0.7,0.8)	(0.9,0.1,0.2)	(0.9,0.1,0.2)

C11	(0.7,0.3,0.4)	(0.4,0.5,0.6)	(0.2,0.7,0.8)	(0.1,0.8,0.9)	(0.3,0.6,0.7)	(0.2,0.7,0.8)	(0.9,0.1,0.2)
	A1	A2	A3	A4	A5	A6	A7
C1	(0.4,0.5,0.6)	(0.7,0.3,0.4)	(0.2,0.7,0.8)	(0.1,0.8,0.9)	(0.4,0.5,0.6)	(0.5,0.5,0.5)	(0.6,0.4,0.5)
C2	(0.8,0.2,0.3)	(0.2,0.7,0.8)	(0.1,0.8,0.9)	(0.6,0.4,0.5)	(0.9,0.1,0.2)	(0.8,0.2,0.3)	(0.7,0.3,0.4)
C3	(0.7,0.3,0.4)	(0.4,0.5,0.6)	(0.2,0.7,0.8)	(0.1,0.8,0.9)	(0.6,0.4,0.5)	(0.9,0.1,0.2)	(0.8,0.2,0.3)
C4	(0.6,0.4,0.5)	(0.2,0.7,0.8)	(0.1,0.8,0.9)	(0.7,0.3,0.4)	(0.6,0.4,0.5)	(0.9,0.1,0.2)	(0.9,0.1,0.2)
C5	(0.2,0.7,0.8)	(0.1,0.8,0.9)	(0.7,0.3,0.4)	(0.6,0.4,0.5)	(0.9,0.1,0.2)	(0.5,0.5,0.5)	(0.4,0.5,0.6)
C6	(0.4,0.5,0.6)	(0.7,0.3,0.4)	(0.6,0.4,0.5)	(0.9,0.1,0.2)	(0.2,0.7,0.8)	(0.4,0.5,0.6)	(0.5,0.5,0.5)
C7	(0.3,0.6,0.7)	(0.2,0.7,0.8)	(0.1,0.8,0.9)	(0.7,0.3,0.4)	(0.6,0.4,0.5)	(0.9,0.1,0.2)	(0.6,0.4,0.5)
C8	(0.2,0.7,0.8)	(0.1,0.8,0.9)	(0.6,0.4,0.5)	(0.9,0.1,0.2)	(0.8,0.2,0.3)	(0.2,0.7,0.8)	(0.7,0.3,0.4)
C9	(0.9,0.1,0.2)	(0.2,0.7,0.8)	(0.1,0.8,0.9)	(0.6,0.4,0.5)	(0.9,0.1,0.2)	(0.9,0.1,0.2)	(0.8,0.2,0.3)
C10	(0.8,0.2,0.3)	(0.3,0.6,0.7)	(0.4,0.5,0.6)	(0.7,0.3,0.4)	(0.6,0.4,0.5)	(0.9,0.1,0.2)	(0.9,0.1,0.2)
C11	(0.7,0.3,0.4)	(0.2,0.7,0.8)	(0.1,0.8,0.9)	(0.6,0.4,0.5)	(0.9,0.1,0.2)	(0.7,0.3,0.4)	(0.2,0.7,0.8)

Step 2, In this step, we use the MEREC method to calculate the criteria weights for our study. First, we normalize the decision matrix using Eq. (11), as shown in Table 2.

Next, we apply Eq. (12) to determine the overall performance. After that, we evaluate the performance when each criterion is removed using Eq. (13) (refer to Table 3).

Then, we compute the absolute deviations for each criterion with Eq. (14).

Finally, we derive the criteria weights, which are illustrated in Fig. 3.

Table 2. Normalized Decision Matrix.

	A1	**A2**	**A3**	**A4**	**A5**	**A6**	**A7**
C1	0.777778	0.75	0.875	1	0.403846	0.466667	0.411765
C2	0.173913	0.571429	1	0.324324	0.193548	0.226415	0.181818
C3	0.25	0.384615	0.375	1	0.535714	0.208333	0.217391
C4	0.490196	0.714286	0.675676	0.735294	1	0.416667	0.320513
C5	0.756757	0.823529	0.756757	1	0.451613	0.622222	0.717949
C6	0.538462	0.42	0.677419	0.617647	1	0.538462	0.466667
C7	0.966667	0.674419	1	0.852941	0.58	0.725	0.568627
C8	1	0.375	0.488372	0.488372	0.617647	0.525	0.35
C9	0.305085	1	0.857143	0.529412	0.321429	0.305085	0.25
C10	0.347826	0.888889	1	0.705882	0.585366	0.32	0.307692
C11	0.466667	0.848485	0.756757	1	0.7	0.595745	0.7

Table 3. Performance Post Criteria Removal.

	A1	**A2**	**A3**	**A4**	**A5**	**A6**	**A7**
C1	0.535188	0.347456	0.254684	0.295066	0.442063	0.589475	0.646537
C2	0.452093	0.329836	0.26405	0.215797	0.398138	0.552327	0.60683
C3	0.472868	0.303617	0.193119	0.295066	0.458438	0.547962	0.615645
C4	0.510307	0.344317	0.236299	0.274036	0.493685	0.583744	0.634534
C5	0.533728	0.353444	0.2444	0.295066	0.448573	0.603875	0.672669

C_6	0.515419	0.309505	0.236484	0.261912	0.493685	0.596664	0.65248
C_7	0.546695	0.34061	0.26405	0.284243	0.462993	0.611444	0.661792
C_8	0.548477	0.301916	0.212722	0.245346	0.466585	0.595396	0.638767
C_9	0.484086	0.365764	0.25323	0.251069	0.428636	0.567812	0.622487
C_{10}	0.491405	0.358308	0.26405	0.271211	0.46352	0.570268	0.632565
C_{11}	0.507619	0.355348	0.2444	0.295066	0.473695	0.601712	0.671493

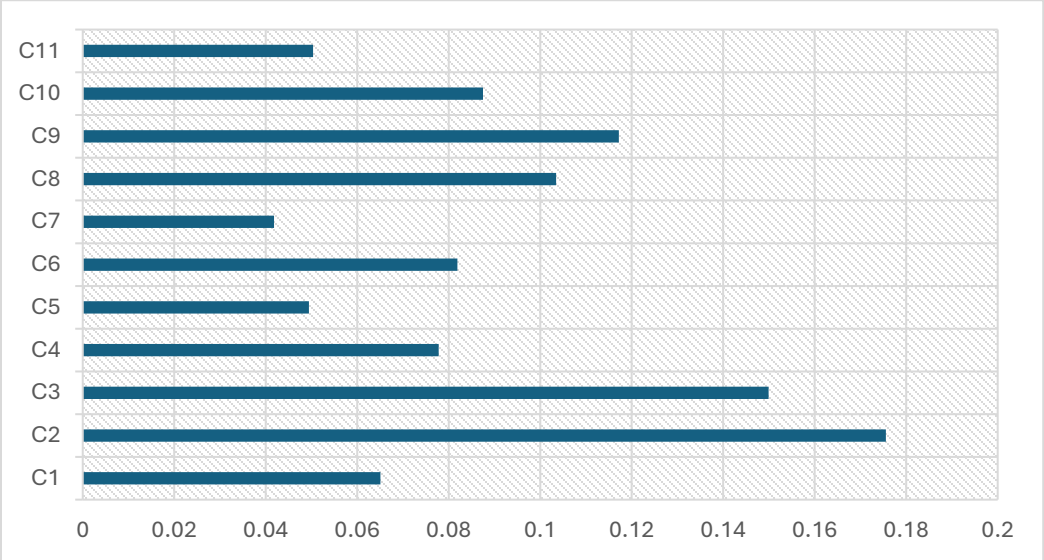

Fig.3. Criteria weight.

Step 3, In this step, we rank the alternatives using the EDAS method. First, we calculate the positive and negative distances of each alternative from the average solution using Equations (17) and (18), as shown in Tables 4 and 5. Since all criteria are positive, this calculation is straightforward. Next, we use Equations (21) and (22) to compute the weighted values. Y_{ij} and Z_{ij} , with the results displayed in Tables 6 and 7. After that, we obtain the weighted normalized values (see Table 8) and then calculate an overall appraisal value for each alternative. Finally, based on these appraisal values, we rank the alternatives, as illustrated in Figure 4.

Table 4. Positive Distances for Alternatives.

	A_1	A_2	A_3	A_4	A_5	A_6	A_7
C_1	0	0	0	0	0.467742	0.270161	0.439516
C_2	0.509375	0	0	0	0.35625	0.159375	0.44375
C_3	0.30031	0	0	0	0	0.560372	0.495356
C_4	0.115625	0	0	0	0	0.3125	0.70625
C_5	0	0	0	0	0.539007	0.117021	0
C_6	0.054054	0.351351	0	0	0	0.054054	0.216216
C_7	0	0.086643	0	0	0.263538	0.01083	0.288809
C_8	0	0.319865	0.013468	0.013468	0	0	0.414141
C_9	0.294671	0	0	0	0.22884	0.294671	0.579937
C_{10}	0.387931	0	0	0	0	0.508621	0.568966

C_{11}	0.473684	0	0	0	0	0.154386	0

Table 5. Negative Distances for Alternatives.

	A_1	A_2	A_3	A_4	A_5	A_6	A_7
C_1	0.237903	0.209677	0.322581	0.407258	0	0	0
C_2	0	0.540625	0.7375	0.190625	0	0	0
C_3	0	0.154799	0.133127	0.674923	0.393189	0	0
C_4	0	0.234375	0.190625	0.25625	0.453125	0	0
C_5	0.08156	0.156028	0.08156	0.304965	0	0	0.031915
C_6	0	0	0.162162	0.081081	0.432432	0	0
C_7	0.241877	0	0.267148	0.140794	0	0	0
C_8	0.505051	0	0	0	0.198653	0.057239	0
C_9	0	0.605016	0.539185	0.253918	0	0	0
C_{10}	0	0.456897	0.517241	0.316092	0.175287	0	0
C_{11}	0	0.189474	0.091228	0.312281	0.017544	0	0.017544

Table 6. The positive weighted matrix.

	A_1	A_2	A_3	A_4	A_5	A_6	A_7
C_1	0	0	0	0	0.030443	0.017584	0.028606
C_2	0.089424	0	0	0	0.062542	0.027979	0.077903
C_3	0.04502	0	0	0	0	0.084006	0.07426
C_4	0.008996	0	0	0	0	0.024313	0.054948
C_5	0	0	0	0	0.026645	0.005785	0
C_6	0.004426	0.028771	0	0	0	0.004426	0.017705
C_7	0	0.003624	0	0	0.011022	0.000453	0.012079
C_8	0	0.033098	0.001394	0.001394	0	0	0.042853
C_9	0.034526	0	0	0	0.026813	0.034526	0.067951
C_{10}	0.033946	0	0	0	0	0.044508	0.049788
C_{11}	0.02385	0	0	0	0	0.007773	0

Table 7. The negative weighted matrix.

	A_1	A_2	A_3	A_4	A_5	A_6	A_7
C_1	0.015484	0.013647	0.020995	0.026507	0	0	0
C_2	0	0.09491	0.129473	0.033465	0	0	0
C_3	0	0.023206	0.019957	0.101179	0.058944	0	0
C_4	0	0.018235	0.014831	0.019937	0.035254	0	0
C_5	0.004032	0.007713	0.004032	0.015075	0	0	0.001578
C_6	0	0	0.013279	0.006639	0.035411	0	0
C_7	0.010116	0	0.011173	0.005889	0	0	0
C_8	0.052259	0	0	0	0.020555	0.005923	0
C_9	0	0.070889	0.063176	0.029751	0	0	0
C_{10}	0	0.039981	0.045262	0.02766	0.015339	0	0
C_{11}	0	0.00954	0.004593	0.015724	0.000883	0	0.000883

Table 8. The EDAS values.

	SY_i	SZ_i	NSY_i	NSZ_i	AS_i
A_1	0.240189	0.081891	0.563702	0.250607	0.407155
A_2	0.065493	0.278122	0.153705	0.85112	0.502412
A_3	0.001394	0.326772	0.003271	1	0.501635
A_4	0.001394	0.281826	0.003271	0.862456	0.432863
A_5	0.157465	0.166386	0.369555	0.509182	0.439368
A_6	0.251354	0.005923	0.589904	0.018125	0.304014
A_7	0.426093	0.002461	1	0.007531	0.503766

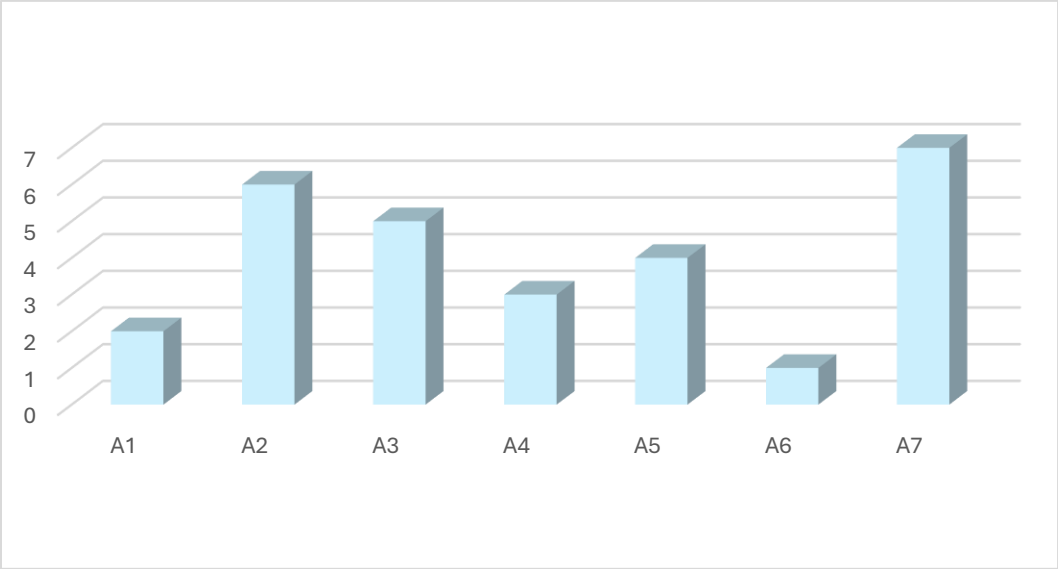

Fig. 4. The rank of alternatives.

We performed a sensitivity analysis to see how changes in criteria weights affect the ranking of the alternatives. In this study, we considered 12 different cases for assigning criteria weights (see Fig. 5). In the first case, all criteria were given equal weights. In the second case, we increased the weight of the first criterion by 12% while keeping the other criteria unchanged. In the third case, we increased the weight of the second criterion by 12%, and we continued this pattern for the remaining cases.

After adjusting the weights, we applied the EDAS method to calculate the appraisal values and rank the alternatives. In cases 2, 3, 4, 8, and 11, the analysis showed that alternative 7 was the best option and alternative 6 was the worst. In cases 1, 5, 6, 7, 9, 10, and 12, alternative 2 came out on top, with alternative 6 still being the worst option. This approach helps us understand how different weighting schemes can change the ranking, providing a clearer representation of the decision-making process.

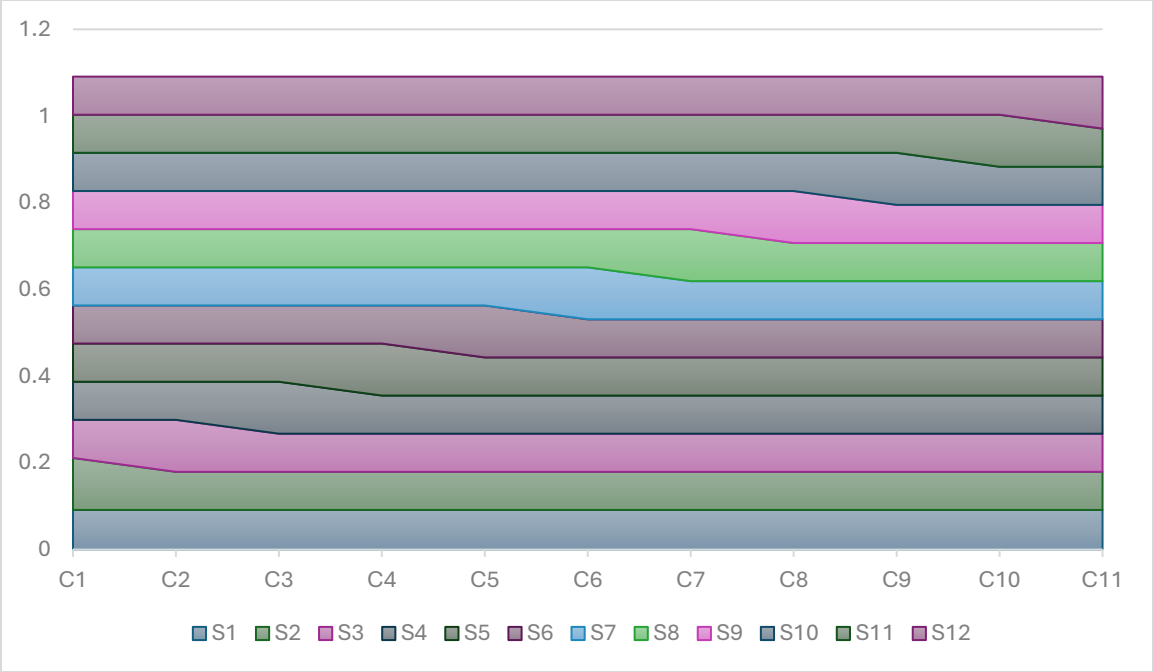

Fig 5. The different criteria weights.

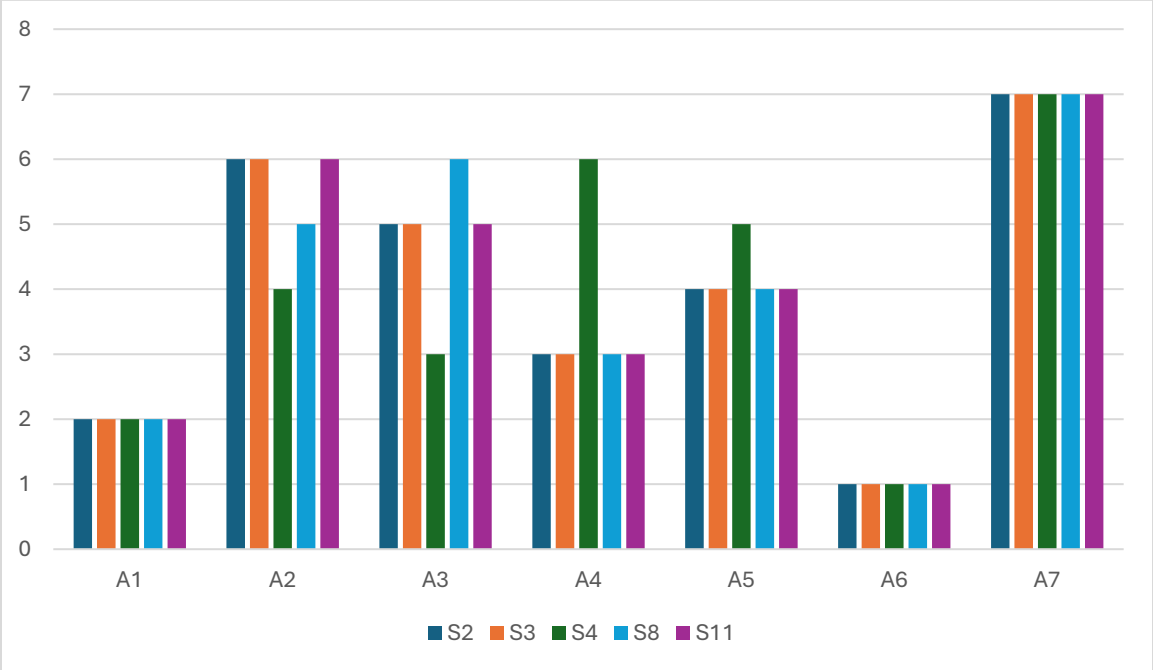

Fig. 6. Ranking of Cases 2, 3, 4, 8, and 11.

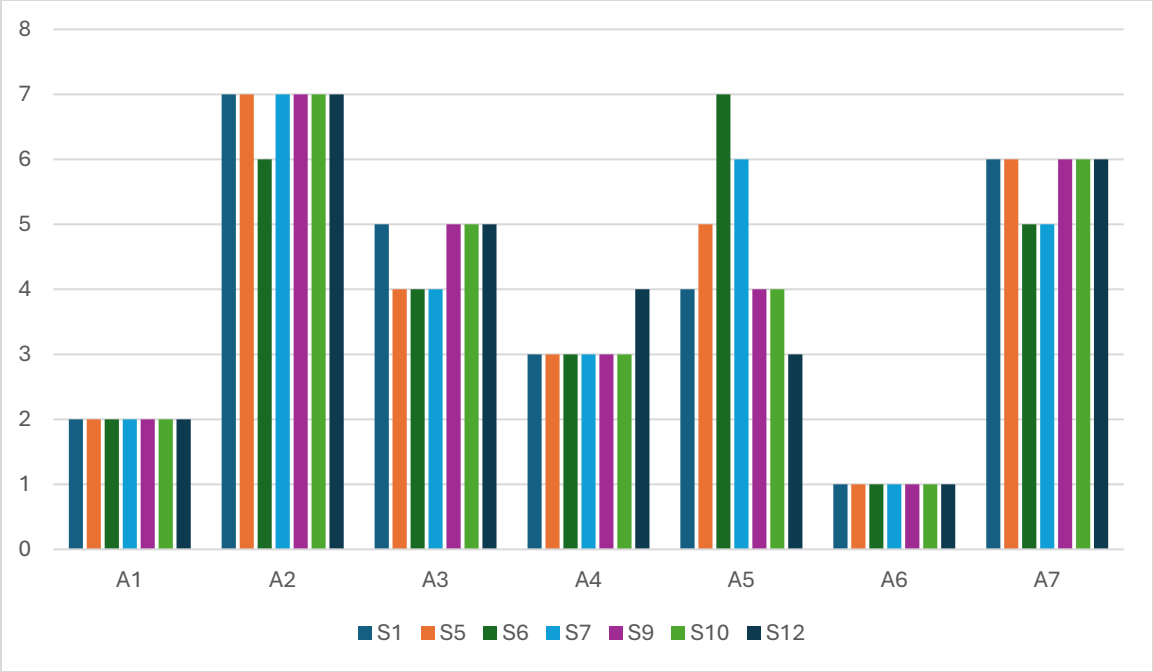

Fig. 7. The rank of cases 1,5,6,7,9,10 and 12.

4.1 Applications of the Proposed Framework

The proposed SVN-MEREC-EDAS framework is not limited to evaluating teaching quality; it has the potential to be applied in various fields that require decision-making under uncertainty.

[1] Beyond teaching quality evaluation, the framework can be used to assess faculty performance, curriculum effectiveness, and institutional ranking based on multiple uncertain and subjective criteria.

[2] Hospitals and medical institutions can use this approach to rank treatment plans, evaluate patient care quality, and optimize resource allocation based on expert assessments that involve uncertainty.

[3] Companies can apply this framework to select suppliers, manage risk in procurement decisions, and optimize logistics operations where uncertainty is a key factor.

[4] The method can assist in evaluating green energy solutions, sustainability initiatives, and environmental impact assessments, where conflicting criteria and uncertain data are common.

[5] Governments and policymakers can use this framework to rank urban development projects, allocate resources to public services, and assess policy impacts with uncertain expert input.

5. Conclusions

The purpose of this research study is to use a new framework to solve the issues surrounding the teaching of structures in college public English. After determining the objective criterion weights using MEREC in an SVN environment, the alternatives are ranked using EDAS. To create the

decision matrix, SVNSs are used to combine the views of three different experts. These views are then de-neutrosophied to produce clear values. By applying the concept to a case study on college public English teaching, its applicability is illustrated. The purpose of the sensitivity analysis is to show how much better the framework is than other MCDM techniques. Additionally, MEREC's advantages are demonstrated by contrasting its outcomes with those of other objective weighting techniques. This study also aims to assist decision-makers in making appropriate and accurate choices. Furthermore, this approach addresses the uncertainties and indeterminacies in the data acquired while enabling group aggregation to integrate the viewpoints of several decision-makers to get the final findings.

5.1 Future Work

While this research establishes a strong methodological foundation for evaluating teaching quality using SVNSs, there are several opportunities for future development and enhancement. One promising direction is the integration of machine learning techniques to refine criteria weighing and ranking processes. By analyzing expert historical decisions and identifying patterns, machine learning can improve the accuracy and efficiency of the decision-making framework. Another area for improvement is the incorporation of real-time data implementation. Currently, the framework relies on expert assessments, but integrating real-time student feedback, automated teaching performance analytics, and live tracking of educational outcomes could make the evaluation process more dynamic and responsive to changing conditions.

To validate the framework's effectiveness, future studies should also focus on broader case studies, applying the method across different educational institutions, academic subjects, and even beyond the education sector. This would help assess its adaptability and refine the model for more general applications. Additionally, integrating hybrid decision-making models by combining SVN-MEREC-EDAS with other established techniques, such as TOPSIS, VIKOR, or AHP, could provide a comparative analysis, offering more robust validation and strengthening the reliability of ranking results. Another key area for future work is dynamic criteria weighting, where the importance of each criterion could adjust over time in response to external influences, such as changes in educational policies, institutional priorities, or evolving teaching methodologies. By developing an adaptive model, the framework would remain relevant and applicable in long-term educational decision-making.

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Global Gender Inequality in Industry: A Systematic Review with Neutrosophic Analysis

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Abstract: This study examines gender inequalities in the worldwide sector via a systematic review employing a neutrosophic approach, focusing on income discrepancies, workplace harassment, preventative strategies, professional hurdles, and leadership representation. The neutrosophic methodology categorizes findings into components of truth, falsehood, and indeterminacy, assessing evidence via four metrics: neutrosophic score, reliability index, certainty level, and information quality index. In accordance with PRISMA 2020 standards, 282 studies published from 2020 to 2024 were examined. The findings indicate that workplace harassment exhibits the most substantial evidence (reliability=0.828, IQI=0.496), succeeded by wage disparities (reliability=0.813, IQI=0.400), whereas prevention policies reveal the greatest uncertainty (reliability=0.696). Robust negative correlations between Truth-Indeterminacy (-0.861) and Truth-Falsity (-0.858) substantiate the analytical approach. The analysis identifies substantial research deficiencies in the efficacy of prevention policies and the representation of leadership, especially within North American contexts. These findings establish a thorough framework for comprehending the elements of gender inequality in the workplace, presenting explicit guidance for research objectives and policy measures to mitigate gender disparities in industry.

Keywords: neutrosophic analysis, single-valued neutrosophic sets, neutrosophic correlation, gender inequality, workplace harassment

1. Introduction

Gender parity has emerged as a significant concern for the productive, sustainable, and ethical advancement of modern society. In the industrial sector, gender equality is both an ethical imperative and a strategic approach to enhance innovation, economic performance, and social cohesion [1,2]. Nonetheless, despite advancements in recent decades, gender inequities endure as a complicated and diverse issue. These inequities are evident in multiple domains, including income discrepancies, obstacles to female leadership, workplace harassment, and challenges in achieving work-life balance.From a neutrosophic perspective, these inequalities cannot be addressed solely from traditional approaches. Neutrosophic logic, developed by Smarandache in 1999, allows the analysis of phenomena characterized by the coexistence of the true, the false, and the indeterminate, offering a powerful tool to study complex issues that include multiple perspectives and levels of uncertainty [3]. In this sense, gender inequality in industry is an ideal topic to apply this approach, as existing

studies often present contradictory results or are limited by specific cultural, economic and sectoral contexts.

To systematically capture this complexity, we employ mathematical foundations that enable a more nuanced representation of uncertainty. Next, we present basic definitions and concepts concerning neutrosophic sets and single-valued neutrosophic sets.

Definition 1 ([4]). Let U be a discourse universe. $N = \{(x, T(x), I(x), F(x)) : x \in U\}$ is a neutrosophic set, denoted by a truth-membership function, $TN : U \rightarrow]0-, 1+[$; an indeterminacy-membership function, $IN : U \rightarrow]0-, 1+[$; and a falsity-membership function, $FN : U \rightarrow]0-, 1+[$.

The discourse universe U can be represented and examined using single-valued neutrosophic sets.

Definition 2 ([5]). Let U be a discourse universe. A single-valued neutrosophic set is defined as $N = \{(x, T(x), I(x), F(x)) : x \in U\}$, which is identified by a truth-membership function, $TN : U \rightarrow [0, 1]$; indeterminacy-membership function, $IN : U \rightarrow [0, 1]$; and falsity-membership function, $FN : U \rightarrow [0, 1]$, with $0 \leq TN(x) + IN(x) + FN(x) \leq 3$.

A organized mathematical framework for managing imprecision, contradictory information, and uncertainty in complex systems is offered by single-valued neutrosophic sets. They are useful in domains like decision-making and the social sciences where conventional statistical models are ineffective because of their capacity to depict truth, indeterminacy, and falsity. This method improves the study of complex issues such as wage discrepancies, where a number of interrelated elements, such as regional laws and cultural norms, lead to a range of inconsistent results.

The existing literature demonstrates the persistence of wage disparities. Hoff and Lee found that, even in highly qualified sectors such as the medical sector, women receive significantly lower incomes than men with similar profiles [6]. In a complementary manner, Gramiscelli et al. used statistical tools to show a 36% wage gap over a decade [7]. However, these gaps are not uniform and, in some contexts, the results are more complex to interpret due to factors such as local policies, cultural stereotypes, and sectoral differences.

Likewise, barriers to female leadership, known as the "glass ceiling", remain a significant obstacle to women's professional advancement. According to Lnal-Cavian and Sahil, this phenomenon, supported by gender stereotypes and prejudices, creates invisible barriers that limit women's participation in managerial roles [8]. On the other hand, Noronha and Aithal point out that the difficulty in balancing professional and personal life exacerbates these limitations, underlining the importance of more inclusive organizational policies [9].

The neutrosophic approach not only facilitates the analysis of these barriers but also allows for the identification of areas where evidence is contradictory or insufficient. For example, the impact of workplace harassment on women's productivity and well-being has been widely documented. Studies such as those by Yu Jung Jo, highlight that verbal harassment and other forms of violence significantly affect the performance of female workers [10]. However, organizational responses to this problem vary considerably, generating areas of indeterminacy that the neutrosophic approach can effectively address.

In this context, this research aims to carry out a systematic review with a neutrosophic approach. Based on this review, gender inequalities in the industry worldwide will be analyzed, focusing on identifying gaps, contradictions and areas of uncertainty in key issues such as:

- Wage disparities between men and women in different sectors.
- Impact of workplace harassment on productivity and well-being.
- Effectiveness of policies to address gender-based violence in the workplace.
- Barriers to women's career advancement.
- Gender representation in leadership roles and hierarchical levels.

To guide this research, five main areas of analysis have been identified, each with its specific subfields, as detailed in Table 1:

Table 1.Main research areas and their subfields

1. Gender Wage Disparities	- Wage disparities in different sectors (e.g. services, tourism, medicine). - Factors contributing to the wage gap (education, region, occupation, age, gender stereotypes).
2. Impact of Sexual Harassment in the Workplace	- Effects of sexual harassment on women's work performance. - Most common types of sexual harassment in the workplace (verbal, physical, psychological).
3. Policies and Measures against Gender Violence	- Effectiveness of policies and measures to prevent gender-based violence at work. - Analysis of specific policies implemented by companies and institutions.
4. Barriers to Women's Professional Advancement	- The "glass ceiling" phenomenon and its causes (gender stereotypes, prejudices, discriminatory practices). - Work-life balance and its impact on women's careers.
5.Gender Representation in Leadership Positions	- Studies of gender representation in leadership positions in various industries and sectors. - Factors that influence the low representation of women in managerial and executive roles.

2. Materials and Methods

This study follows the PRISMA 2020 guidelines to ensure transparency, completeness, and reproducibility[11-13]. The search strategy was carefully designed using specific terms related to gender inequalities in the workplace, such as “wage gaps”, “sexual harassment”, “gender policies”, “gender barriers”, “gender representation” and “industry”. Boolean operators (AND, OR and AND NOT) were used to optimize the results. The selected databases included Scopus, Science Direct, and Emerald Insight, which are recognized for their wide coverage of academic publications in social and business areas [14-16]. To ensure that the selected studies met the research objectives, inclusion and exclusion criteria were defined (see Table 2). These criteria consider factors such as language, thematic relevance, methodological quality, and timeliness of the data, prioritizing research published between 2020 and 2024.

Table 2. *Inclusion and exclusion criteria for the analysis*

Inclusion criteria	Exclusion criteria
• Studies published in English or Spanish.	• Research that is not available in English or Spanish.
• Studies that address specific areas of interest are defined in the introduction, such as wage disparities, sexual harassment, gender policies, professional barriers and gender representation in the business world.	• Studies that are not directly related to the areas of interest of the research.
• Research using systematic review approaches, policy analysis or related empirical studies.	• Documents that do not specifically address gender equity in the industry.

• Studies that present relevant and updated data, 2020-2024 post-COVID era, on gender inequalities in the workplace.	• Non-academic or questionable quality publications.
• Documents available in academic databases and specialized repositories.	• Obsolete or outdated studies that do not contribute significantly to the analysis of gender inequalities in the business world.
• Open access	
• Published mainly in business or management magazines.	

The methodology incorporates a neutrosophic approach to address the complexity and indeterminacy inherent to gender inequalities, following these steps:.

1-Study selection

During the study selection process, 10,567 records were initially identified, which were subjected to a thorough screening process. In the first stage, automated tools, such as Rayyan AI software, were used to eliminate duplicate records, resulting in the exclusion of 46 documents. Subsequently, a second filtering was carried out by reviewing titles, abstracts and keywords, discarding those studies that did not meet the previously established inclusion criteria, which reduced the set to 528 studies. Finally, a thorough evaluation of the full texts was carried out, obtaining a final sample of 282 studies that met the parameters defined for the analysis [17,18].

2. Neutrosophic Value Determination

For each dimension analyzed (Table 1), the neutrosophic components (T, I, F) based valiables on Table were determined by [19, 20]:

- Truth (T): Proportion of studies that present consistent and well-supported evidence.
- Indeterminacy (I): Proportion of studies with inconclusive, ambiguous, or insufficient evidence.
- Falsity (F): Proportion of studies that contradict or refute.

2. Correlation Analysis

To understand relationships between neutrosophic components, correlation coefficients were calculated using:

$$r(T,I,F) = \frac{\sum(Ti-\bar{T})(Ii-\bar{I})(Fi-\bar{F})}{\sqrt{\sum(Ti-\bar{T})^2\sum(Ii-\bar{I})^2\sum(Fi-\bar{F})^2}} \quad (1)$$

Where:

T_i, I_i, F_i are the values of the components of neutrosophic numbers.

$\bar{T}, \bar{I}, \bar{F}$ are the averages of the corresponding values.

This analysis helps identify patterns and dependencies between truth, indeterminacy, and falsity across dimensions.

3. Neutrosophic Metrics Calculation

To evaluate the quality and reliability of evidence across dimensions, four complementary metrics were computed:

a) Neutrosophic Score (S) [21, 22]:

$$S = T - F - I, \quad (2)$$

where S ∈ [-1, 1]

This metric evaluates the net balance between truth, falsity, and indeterminacy, with higher scores indicating a stronger evidence base. The score provides an overall assessment of evidence strength for each dimension.

b) Reliability Index (R):

$$R = T/(T + F) \quad (3)$$

where R ∈ [0, 1]

This index assesses the consistency of findings by measuring the proportion of true evidence relative to the sum of true and false evidence, excluding indeterminate cases. Higher values indicate more reliable findings.

c) Certainty Level (C)[23]:

$$C = 1 - I, \quad (4)$$

where C ∈ [0, 1]

This measure evaluates how well-defined the findings are by quantifying the degree of determinacy in the evidence. Higher values indicate less uncertainty and more clearly defined results.

d) Information Quality Index (IQI):

$$IQI = (T - F)(1 - I), \quad (5)$$

where IQI ∈ [-1, 1]

This comprehensive metric combines the strength of evidence with its level of certainty, providing an integrated measure of evidence quality. Higher values indicate both strong and certain evidence.

Risk of bias was evaluated considering:

- Selection bias in criteria application
- Data quality and reliability
- Potential bias in evidence synthesis

This comprehensive methodology provides a structured framework for analyzing gender inequalities while accounting for uncertainty and contradiction in the evidence base, enabling the identification of both well-established findings and areas requiring further investigation.

3. Results

Using the neutrosophic approach, the results were classified into three dimensions: true, false, and indeterminate. Wage disparities (65%) and the impact of workplace harassment (72%) stand out as the areas with the greatest consistency in the findings, reflecting solid evidence supported by multiple studies. In contrast, prevention policies show less robustness, with 55% of evidence classified as true, 25% considered false, and 20% indeterminate. Similarly, barriers to female leadership and representation in leadership roles show considerable levels of uncertainty, with 22% of evidence indeterminate in both categories. The results reflect significant gaps in research, especially in aspects related to the effectiveness of organizational policies and women's equal access to managerial positions. (See Table 3).

Table 3. *A Neutrosophic Analysis of Gender Inequality Dimensions*

Gender Dimensions	(T, I, F)	Score	Reliability T/(T+F)	Certainty (1-I)	IQI (T-F)(1-I)

Wage Disparities	(0.65, 0.20, 0.15)	0.30	0.813	0.80	0.400
Impact of Workplace Harassment	(0.72, 0.13, 0.15)	0.44	0.828	0.87	0.496
Prevention Policies	(0.55, 0.21, 0.24)	0.10	0.696	0.79	0.245
Barriers to Female Leadership	(0.61, 0.22, 0.17)	0.20	0.782	0.78	0.344
Representation in Female Leadership	(0.59, 0.22, 0.19)	18	0.756	0.80	0.312

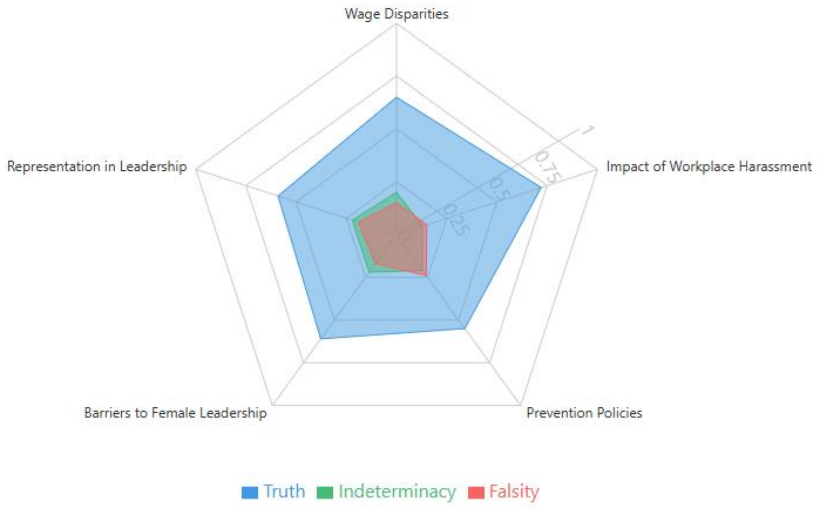

Figure 1. Neutrosophic Distribution of Gender Dimensions

Although the analysis included a limited number of dimensions (n=5), the robustness of the correlations is substantiated by the comprehensive foundation of 282 studies underlying each neutrosophic value, the consistency with theoretical expectations, the low standard deviations in component values (SDtruth=0.058, SDindeterminacy=0.034, SDfalsity=0.033), and the coherence across multiple neutrosophic metrics, collectively validating the identified relationships as genuine patterns rather than random associations.. The correlation analysis of the neutrosophic components (T, I, and F) reveals significant trends in the evaluation of gender inequality dimensions (Figure 2). The negative correlations between Truth and Falsity (-0.858) and between Truth and Indeterminacy (-0.861) showed the highest associations, suggesting that dimensions with higher truth values consistently exhibit lower degrees of falsity and indeterminacy. Although they tend to rise together, the somewhat positive correlation (0.477) between Indeterminacy and Falsity indicates that both components have some degree of independence. These conclusions are supported by the

descriptive statistics, which reveal that Truth has the greatest mean (0.624) and the least variance (SD=0.058, range: 0.55-0.72), whereas Falsity and Indeterminacy have lower and more consistent values (mean=0.180, SD=0.033) and mean=0.196, SD=0.034, respectively.

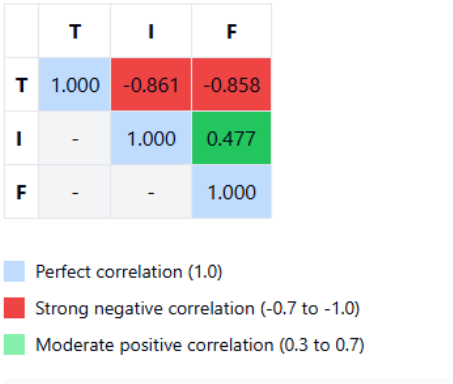

Figure 2. Neutrosophic correlation matrix

The neutrosophic approach is validated by the substantial negative T-I connection, and I-F's relative independence supports treating them as distinct components. Additionally, while the moderate I-F correlation indicates that indeterminacy and contradictions may require different approaches, the strong correlations point to a coherent evaluation framework that offers insightful information for research prioritization and policy development. This suggests that reducing indeterminacy could be a useful strategy for strengthening the quality of the evidence.

The neutrosophic analysis of gender inequality dimensions reveals distinct patterns in the intensity and reliability of evidence. The impact of workplace harassment demonstrates the strongest evidence with the highest neutrosophic score (0.44), reliability (0.828), certainty level (0.87), and IQI (0.496), followed by wage disparities with moderate to high metrics (score=0.30, reliability=0.813, certainty=0.80, IQI=0.400). In contrast, prevention policies show the weakest evidence (score=0.10, reliability=0.696, IQI=0.245), while leadership dimensions maintain intermediate values across all metrics.

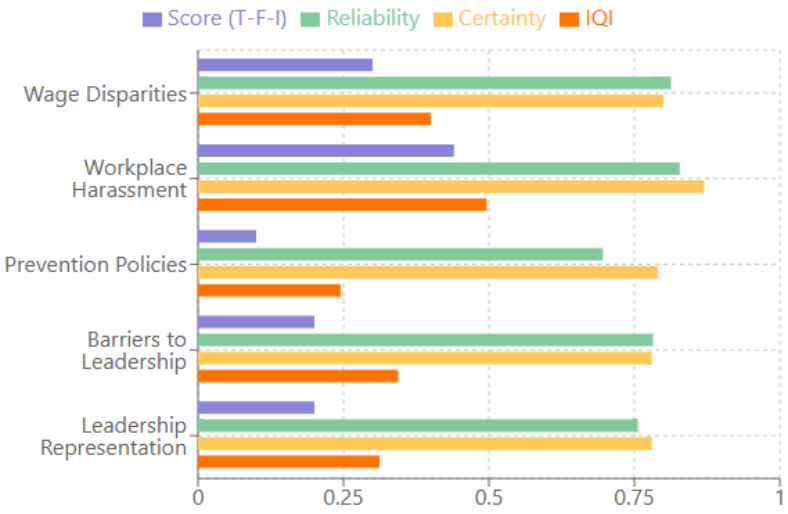

Figure 3. Neutrosphics MetricsComparison

The comprehensive neutrosophic analysis of gender inequality dimensions provides significant insights for both research and policy development. Workplace harassment emerges as the dimension with the most robust evidence across all metrics (reliability=0.828, IQI=0.496), suggesting immediate actionable opportunities, while prevention policies demonstrate the highest uncertainty and lowest reliability (0.696), indicating a critical need for improved evaluation methods and policy revision. Leadership dimensions, while showing moderate but consistent patterns across metrics, highlight the necessity for an integrated approach to address both barriers and representation issues. These findings establish clear research priorities: primary attention should focus on strengthening prevention policy frameworks, followed by developing more comprehensive approaches to leadership challenges, while maintaining robust monitoring systems for workplace harassment and wage disparities. The methodological approach demonstrates high consistency between different neutrosophic measures, effectively differentiating between well-documented and uncertain areas, thus providing a reliable framework for prioritizing both research initiatives and policy interventions in addressing workplace gender inequalities.

The findings from our neutrosophic analysis align with several international studies on gender inequality dimensions. The high reliability found in workplace harassment evidence (R=0.828) corresponds with findings from Jung Jo [10] and Russen et al. [25], who documented consistent patterns of workplace harassment affecting 45-48% of female workers. Similarly, the wage disparity reliability index (R=0.813) aligns with Gramiscelli et al.'s [7] decade-long analysis showing persistent gender pay gaps. The lower reliability in prevention policies (R=0.696) mirrors concerns raised by Kang et al. [26] and Bouwmeester et al. [27] regarding the inconsistent implementation of gender equality measures. Our findings on leadership barriers (R=0.782) and representation (R=0.756) corroborate studies by Baghdadi et al. [28]], who identified similar patterns in the persistence of the "glass ceiling" phenomenon. The correlation patterns between truth and indeterminacy components (-0.861) provide new methodological insights that complement traditional approaches to gender inequality research, as suggested by Kalpazidou Schmidt and Ovseiko [24].

5. Conclusions

This study demonstrates the effectiveness of the neutrosophic approach in analyzing gender inequalities in industry, providing a robust framework for evaluating evidence quality and reliability across different dimensions. The analysis reveals clear patterns in the strength and consistency of evidence, with workplace harassment showing the most robust metrics (reliability=0.828, IQI=0.496), followed by wage disparities (reliability=0.813, IQI=0.400), while prevention policies demonstrate the highest uncertainty and lowest reliability (0.696). The strong negative correlations between Truth-Indeterminacy (-0.861) and Truth-Falsity (-0.858) validate the methodological approach, while the moderate positive correlation between Indeterminacy-Falsity (0.477) suggests the independence of these components, providing a comprehensive framework for understanding gender inequality dimensions in the workplace.

Future research should focus on addressing the identified gaps and areas of high indeterminacy. Priority should be given to strengthening the evidence base for prevention policies, where current findings show the highest uncertainty and lowest reliability. Additionally, more comprehensive studies are needed to understand the complex interactions between leadership barriers and representation, particularly in different cultural and economic contexts. We recommend developing standardized evaluation methods for prevention policies, conducting longitudinal studies to track the effectiveness of gender equality initiatives. Furthermore, future studies should employ the neutrosophic approach to examine the intersectionality of gender inequalities with other forms of discrimination, providing a more nuanced understanding of workplace disparities..

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Rethinking Reverse Logistics: Neutrosophic Strategies for Warehouse Management Challenges

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Abstract: Implementing warehouse management platforms (WMPs) in Vietnam's smart reverse logistics (SRL) sector faces numerous barriers, hindering operational efficiency and competitiveness. This study develops a robust multi-stage decision-making framework to identify, analyze, and address these barriers, filling a critical research gap in SRL practices. The proposed model integrates Neutrosophic Sets (NS) for managing uncertainties in expert evaluations with the Decision-Making Trial and Evaluation Laboratory (DEMATEL) method to examine cause-and-effect relationships among barriers. Additionally, the Neutrosophic Combined Compromise Solution (NS COCOSO) method ranks solutions for overcoming these challenges. In the first stage, key barriers—such as insufficient strategic planning, lack of expertise in reverse logistics, inadequate performance metrics, and a shortage of skilled personnel—are identified through an extensive literature review and expert validation using the NS Delphi technique. The second stage applies NS DEMATEL to analyze the interdependencies and significance of these barriers. The final stage prioritizes solutions to mitigate these challenges using NS COCOSO. This integrated approach provides three key contributions: (i) a systematic exploration of inter-barrier relationships, (ii) an advanced mechanism for addressing uncertainties in expert judgments, and (iii) a comprehensive ranking of actionable strategies for stakeholders. The findings underscore the need for long-term strategic planning, regulatory alignment, and the adoption of cutting-edge technologies to enhance WMP implementation in SRL. Sensitivity analyses validate the model's robustness, and comparative assessments demonstrate its superiority over traditional methods. This study offers a novel and practical decision-support framework for logistics firms, policymakers, and stakeholders to drive sustainable advancements in reverse logistics.

Keywords: Warehouse management platforms, Smart reverse logistics, MCDM, Neutrosophic sets, DEMATEL, Delphi, COCOSO.

1. Introduction

1.1 Problem Statements

Reverse logistics (RL) involves transporting items from their final destination to the producer or another point in the supply chain for returns, recycling, remanufacturing, or disposal [1]. With the advent of the Fourth Industrial Revolution, the concept of Logistics 4.0 has emerged, leveraging state-of-the-art technologies such as big data analytics, artificial intelligence (AI), and the Internet of Things (IoT) to create cyber-physical systems (CPS). These systems integrate computational intelligence with intelligent physical assets to enhance resource allocation, communication responsiveness, real-time monitoring, and decision-making [2], thereby

enabling more efficient material flows [3]. Within this context, RL aims to maximize residual value recovery from end-of-life (EOL) products by improving internet-based connectivity, smartness, intelligence, and autonomous operations in manufacturing and logistics systems. Through proper design, operation, control, and maintenance, RL promotes cost-effective and efficient flows that begin with customers and end with manufacturers and suppliers. Integrating innovative technologies also fosters sustainability in RL systems, encompassing economic, ecological, and social dimensions [4].

As companies increasingly prioritize customer satisfaction, cost savings, and sustainability, RL has become a critical component of modern supply chain management. Efficient RL enables organizations to recover value from returned items, minimize waste, and comply with environmental regulations. Additionally, it enhances a company's reputation by showcasing its commitment to environmental responsibility. Adopting innovative practices in RL provides significant advantages, including cost savings, improved operational efficiency, and increased customer loyalty [5]. While RL has traditionally been viewed as a cost center focused on minimizing financial losses from returns and reducing environmental liabilities, recent studies demonstrate its potential for achieving substantial cost reductions and operational improvements. For instance, Letunovska [6] revealed that incorporating sustainable practices into RL can result in a 30% reduction in costs, enhanced operational performance, and strengthened customer loyalty.

In Vietnam, the rapid growth of e-commerce and manufacturing sectors underscores the importance of smart reverse logistics (SRL) for improving efficiency, reducing costs, and promoting sustainability. SRL facilitates the management of product returns, waste reduction, and compliance with environmental regulations. As Vietnam modernizes its logistics infrastructure, adopting SRL is vital to fostering a circular economy and enhancing its competitiveness in global supply chains. As integral components of supply chains, warehouses provide secure, temporary storage for materials while maintaining their quality. Key warehouse operations—such as receiving, stocking, inventory counting, order processing, and delivery—are central to effective inventory management, enabling accurate demand forecasting and efficient stock control [7]. Warehouse management systems (WMSs), or warehouse management platforms (WMPs), support these operations by improving energy efficiency, enabling real-time tracking, optimizing inventory management, and facilitating seamless data integration [7]. Despite significant growth in Vietnam’s logistics industry—expanding at an annual rate of 14-16% between 2017 and 2022 and contributing 4-5% to GDP—WMP adoption remains limited [8]. According to the Vietnam Logistics Business Association, only 30% of companies have implemented WMPs, with most small and medium-sized enterprises (SMEs) relying on manual processes [9].

Given the rapid growth of Vietnam’s logistics sector, particularly in warehousing, there is an urgent need for a systematic and holistic framework to address the barriers hindering WMP adoption. Existing research highlights significant gaps in identifying and analyzing these barriers, especially in fast-evolving economies like Vietnam [10]. Current models often fail to capture the intricate interdependencies among barriers, leaving critical relationships unexplored [11]. This necessitates the development of an integrated framework capable of assessing these challenges comprehensively while considering Vietnam’s unique infrastructure and economic context.

1.2 Research Motivation

The motivation for this study stems from the need to address critical barriers to adoption in Vietnam's RL sector. Vietnam's logistics industry has grown rapidly, driven by advancements in e-commerce, manufacturing, and supply chain modernization. However, despite this growth, limited adoption of advanced technologies such as WMPs hinders efficiency, cost-effectiveness, and sustainability [9]. Existing frameworks often fail to capture the intricate relationships among these barriers or account for the uncertainty and indeterminacy in expert judgments. This study aims to bridge this gap by leveraging the power of NS to create a robust and adaptive decision-making framework for evaluating RL barriers and prioritizing strategies.

Neutrosophic Sets (NS), an advanced extension of fuzzy set theory, represent a groundbreaking approach to managing uncertain, incomplete, imprecise, and indeterminate information in complex real-world scenarios [12]. Introduced by Smarandache [13], NS goes beyond the limitations of traditional fuzzy sets by incorporating three independent membership functions: truth, indeterminacy, and falsity [14]. This triadic structure provides a more nuanced and flexible representation of uncertainty, making it particularly effective in handling the vagueness and indeterminacy inherent in expert evaluations and decision-making processes. By enabling a comprehensive quantification of uncertainties, NS enhances the reliability and validity of research findings, ensuring that the calculated outcomes are closer to real-world dynamics [15].

Integrating NS into multicriteria decision-making (MCDM) methods offers significant advantages for addressing the challenges of RL in Vietnam. NS allows for more precise analysis of expert responses by transforming linguistic terms into neutrosophic values, enabling computational analysis that captures the data's inherent uncertainty, imprecision, and variability[12]. This capability is critical for complex decision-making scenarios in logistics, where data reliability and accuracy often vary due to the industry's dynamic nature[16].

Traditional fuzzy set theory and its numerous extensions, including Ordinary Fuzzy Sets (FS) [15], Intuitionistic Fuzzy Sets (IFS) [16], Pythagorean Fuzzy Sets (PFS) [17], and Picture Fuzzy Sets [18] and Spherical Fuzzy Sets [19], have been widely employed for modeling uncertainty in complex decision-making scenarios. However, NS significantly evolved in handling uncertainty, imprecision, and indeterminacy by incorporating three independent membership functions: truth, indeterminacy, and falsity.

For instance, the FS model uncertainty uses a single membership degree to represent the degree of truth for an element in a set. Despite their simplicity and ease of application, FS faces limitations when modeling scenarios involving hesitation or conflicting information [15]. This limitation arises because FS cannot explicitly represent non-membership or indeterminacy, which is common in real-world problems [12].

The IFS, introduced by Atanassov [16], extends FS by incorporating non-membership and membership degrees. However, IFS fails to account for indeterminacy explicitly, which is critical in scenarios involving incomplete or contradictory information. NS addresses this limitation by introducing an independent indeterminacy component, allowing for a more nuanced representation of uncertainty [20].

PFS enhances flexibility by permitting the squared sum of membership and non-membership degrees to remain within the interval [0, 1]. While PFS improves on the limitations of IFS, they impose mathematical constraints that restrict real-world applicability in situations with high degrees of uncertainty [17]. In contrast, NS provides a more flexible framework by allowing the truth, indeterminacy, and falsity components to vary independently between 0 and 1, with their sum ranging from 0 to 3 [21].

Picture Fuzzy Sets and Spherical Fuzzy Sets introduce hesitation degrees, representing the extent of uncertainty between membership and non-membership. However, their components are constrained to summation or squared conditions, limiting their flexibility in modeling diverse scenarios [20]. In contrast, NS overcomes these limitations by introducing three independent membership functions: truth, indeterminacy, and falsity. This triadic structure allows NS to handle incomplete, imprecise, and contradictory information more effectively. Unlike the fixed relationships between components in other fuzzy extensions, NS's independent variation of truth, indeterminacy, and falsity provides unparalleled flexibility and a more comprehensive representation of uncertainty. This adaptability makes NS suitable for addressing complex, real-world problems characterized by high ambiguity and conflicting information [20].

Overall, the unique capabilities of NS make them a superior choice for addressing complex and uncertain scenarios compared to traditional fuzzy set extensions. Their application spans various domains, including supply chain management, logistics, engineering, and sustainability. NS significantly enhances decision-making processes by providing a more flexible and comprehensive framework for modeling uncertainty, particularly in environments characterized by indeterminacy and conflicting information. As a result, NS serves as a powerful tool for advancing research and practice in uncertain and dynamic systems. This study introduces a multi-level MCDM framework tailored to the Vietnamese logistics context, combining the Neutrosophic Delphi (NS Delphi), Neutrosophic DEMATEL (NS DEMATEL), and Neutrosophic Combined Compromise Solution (NS COCOSO) methods. The framework systematically addresses the following:

(i) The Delphi method, integrated with NS, is employed to validate the importance and relevance of barriers. Expert responses are analyzed in a neutrosophic context to ensure consistency and reliability.

(ii) The NS DEMATEL method analyzes the interdependencies and cause-and-effect relationships among identified barriers. This approach highlights the key barriers that act as root causes, enabling targeted interventions.

(iii) The NS COCOSO method evaluates and ranks potential strategies to overcome these barriers. This ensures that the recommended solutions are actionable, practical, and aligned with the specific challenges faced by stakeholders in Vietnam’s RL sector.

1.3 Research Questions

To address the challenges of SRL influenced by WMPs in Vietnam, this study is guided by the following research questions:

(i) What is the current situation of SRL affected by WMPs in Vietnam?

(ii) What WMP barriers affect the SRL chain in Vietnam?

(iii) What is the cause-and-effect relationship between those WMP barriers, and where is the root cause factor?

(iv) What is the impact of the proposed strategies on those WMP barriers, and what is the priority of those strategies?

1.4 Research Significance

This research is significant because it provides a targeted evaluation of WMP barriers within the context of Vietnam's SRL sector, filling a critical gap in existing literature. While previous studies have focused on general logistics and supply chain challenges, this study takes a more specialized approach by addressing the nuanced and sector-specific issues related to WMP adoption in SRL. The insights gained from this research are crucial for stakeholders seeking to modernize logistics operations, improve efficiency, and enhance Vietnam's economic competitiveness in global supply chains.

This study makes several key contributions that are both practical and theoretical. First, it provides tailored recommendations for stakeholders, offering actionable insights and strategies to address the barriers to WMP adoption. These recommendations support informed decision-making and contribute to developing a modern SRL system in Vietnam. Second, the study introduces an advanced and innovative MCDM framework that integrates NS to handle uncertainties and ambiguities inherent in expert evaluations effectively. This framework not only enhances the precision and reliability of decision-making processes but also demonstrates the practical applicability of NS-based MCDM methods to real-world logistics challenges. Third, the research provides sector-specific insights by conducting a detailed analysis of the unique challenges and opportunities within Vietnam's SRL sector, highlighting the critical role of WMP adoption. It underscores the importance of addressing root causes and prioritizing strategies to improve efficiency, reduce costs, and enhance sustainability. By addressing these contributions, the study advances theoretical knowledge in logistics and decision-making and delivers practical tools and insights to drive significant improvements in Vietnam's logistics industry.

The remainder of this paper is organized as follows: Section 2 reviews relevant literature, Section 3 details the methodology, Sections 4 and 5 discuss empirical findings and insights, and Section 6 concludes with implications, limitations, and directions for future research.

2. Literature Review

Advancements in digitalization and Industry 4.0 technologies have significantly transformed WMPs by integrating automation, real-time tracking, and data-driven decision-making. These advancements offer enhanced operational efficiency, cost reduction, and environmental sustainability in logistics operations. However, the implementation of WMPs within SRL remains underexplored, particularly in developing economies such as Vietnam. Addressing this gap is crucial as SRL is vital in supporting circular economy practices, fostering sustainability, and improving competitiveness in global supply chains.

Advanced decision-making methodologies, including NS MCDM techniques, have emerged as practical tools for managing the uncertainties and complexities inherent in logistics and supply chain management. As introduced by Smarandache [13], NS extends fuzzy set theory by incorporating truth, indeterminacy, and falsity functions, allowing for a more nuanced handling of uncertainty and incomplete information. These methodologies have been widely applied in various logistics domains, such as supplier selection, sustainability assessment, and outsourcing decisions. However, their application to addressing WMP barriers within SRL contexts remains limited.

Table 1: Related Work

No.	Ref	Context	NS
1	Yang et al.[22]	"Strategic outsourcing in reverse logistics"	ν
2	Simic et al.[23]	"Smart and sustainable warehouse management systems"	ν
3	Kara et al. [7]	"Warehouse management software in sustainable logistics systems."	ν
4	Mishra et al.[24]	"Sustainable third-party reverse logistic provider."	ν
6	Görçün et al.[25]	"Sustainable supplier selection"	ν

7	Yazdani et al.[26]	"Sustainable management of end-of-life tires"	v
8	Simic et al.[27]	"Green-resilient model for smartphone closed-loop supply chain network design"	v
9	Shams et al.[28]	"Cosine Similarity and Neutrosophic Distance"	v
10	Ji et al. [29]	"selection of third-party logistics "	v
11	Salama et al.[30]	"Digital transformation using the Neutrosophic balanced` scorecard."	v
12	Mershia et al.[31]	"Neutrosophic Soft Open Sets in Decision-Making"	v
13	Mohamed et al.[32]	"Motivation and Job Satisfaction in the Logistics Sector"	v
14	Lu & Luo[33]	"Emergency Transportation and Logistics"	v
15	Mondal et al. [34]	"Inventory Policies for Seasonal Items"	v
16	Zhang et al.[35]	"Service Quality Evaluation in Cross-Border Logistics"	v

A review of existing literature, summarized in **Table 1**, underscores the application of advanced decision-making methods in logistics and supply chain management while highlighting gaps in addressing WMP barriers within SRL. Mishra and Rani [24] utilized a Single-Valued Neutrosophic Set (SVNS) framework with CRITIC and COCOSO approaches to prioritize sustainable third-party reverse logistics providers, effectively addressing multiple conflicting criteria and integrating sustainability into decision-making. Similarly, Ji et al. [29] applied neutrosophic Bonferroni operators to enhance decision reliability for third-party logistics provider (TPL) selection under uncertainty. These studies highlighted the adaptability of neutrosophic methods in optimizing logistics decisions but did not extend their focus to systemic challenges in SRL.

Building on sustainability-driven logistics, Görçün et al.[25] applied Neutrosophic methodologies to evaluate green supply chain practices and fresh food suppliers, demonstrating their utility in addressing uncertainty in sustainability-focused contexts. However, their work concentrated on supplier-specific issues and overlooked the broader operational barriers in SRL.

Lu and Luo [33] employed SVNS for emergency logistics, showcasing their application in real-time decision-making during disaster responses. Despite their contribution to critical logistics operations, the study did not explore systemic WMS challenges in SRL. Meanwhile, Simic et al.[27] explored Industry 4.0 technologies for WMS, advancing automation and digitalization in developed economies but neglecting the unique infrastructure and resource constraints of emerging markets like Vietnam. Similarly, Das et al. [36] addressed location-allocation problems in green logistics using a type-2 neutrosophic multi-objective model but failed to account for operational barriers specific to SRL environments.

The contextual gaps in SRL are further emphasized by Yang et al. [22] and Simic et al. [23], who highlighted that WMS technologies and Industry 4.0 applications are often developed for mature markets, overlooking the challenges prevalent in developing economies. Kara et al. [7] and Yazdani et al. [26] focused on advanced decision-making techniques like CRITIC and COCOSO for logistics provider selection but did not address the systemic and interdependent barriers critical to WMS adoption in SRL.

Theoretical advancements by Mershia et al.[31] introduced neutrosophic soft γ-open sets for addressing uncertainty and imprecision in decision-making. While foundational, their research lacked practical application in logistics, leaving opportunities to explore their potential in SRL. Similarly, Mohamed et al.[32] used SWARA with SVNS to analyze job satisfaction in logistics, addressing uncertainties in motivational factors but not operational challenges in SRL. Zhang et al.[35] utilized Interval Neutrosophic Sets (INS) with the INN-LogTODIM-GRA framework to assess service quality in cross-border logistics, focusing on uncertainty management but overlooking WMS and SRL-specific barriers.

Recent contributions by Salama et al. [30] and Shams et al. [28] expanded the application of Neutrosophic methodologies to strategic performance evaluation and computational biology, respectively, demonstrating their robustness in managing uncertainty. However, these studies remain theoretical or limited to niche applications, failing to address practical challenges in SRL and WMS adoption.

This review underscores the urgent need for a comprehensive framework integrating Neutrosophic methodologies—such as Delphi, DEMATEL, and COCOSO—to address the systemic and interdependent barriers in WMS adoption for SRL. Developing economies like Vietnam face distinct logistical constraints, including limited infrastructure and resource scarcity, requiring tailored solutions that effectively manage uncertainties and interdependencies. Addressing these gaps through an integrated framework can provide

actionable strategies and robust decision-making tools, advancing SRL operations and promoting sustainable logistics practices, particularly in volatile, uncertain, complex, and ambiguous (VUCA) environments.

3. Proposed Methods

3.1 Preliminaries

The NS theory extends the intuitionistic fuzzy set (IFS) theory [21]. In this theory, incomplete information is represented through three degrees: truth (T), falsity (F), and indeterminacy (I). Unlike the IFS theory, the sum of the membership degrees in NS does not necessarily have to equal 1. In NS, indeterminacy is quantified, and the truth, indeterminacy, and falsity membership degrees are assigned independently, with their total sum reaching a maximum of 3. The mathematical definitions of NS are as follows:

Definition 1. Let X be a space of points (objects), where each point is denoted as $x \in X$. An NS A in X is defined by three membership functions:

Truth-membership function $T_A(x)$

Indeterminacy-membership function $I_A(x)$

Falsity-membership function $F_A(x)$

These functions map elements of X to values within the extended range]0–, 1+[, meaning they can represent values slightly below 0 or above 1.

Specifically:

$T_A(x)$: $X \rightarrow]0^-, 1^+[$

$I_A(x)$: $X \rightarrow]0^-, 1^+[$

$F_A(x)$: $X \rightarrow]0^-, 1^+[$

There is no strict condition on the sum of these values, but the total of the highest values for $T_A(x)$, $I_A(x)$ and $F_A(x)$ must fall between 0 and 3

$0^- \leq supT_A(x) + supI_A(x) + supF_A(x) \leq 3^+$.

This framework allows flexibility in handling uncertainty, truth, and falsity within a given system without being constrained by fixed limitations.

Definition 2. [37] Let X represent a collection of objects, where each object is denoted as x. A Single-Valued.

SVNS denoted as $\tilde{A}$, can be expressed as follows:

$$\tilde{A} = \{ (x, T_{\tilde{A}}(x), I_{\tilde{A}}(x), F_{\tilde{A}}(x)) : x \in X \} \quad (1)$$

In this definition:

$T_{\tilde{A}}(x)$ indicates the truth membership function, reflecting the extent to which the object x belongs to the set.

$I_{\tilde{A}}(x)$ represents the indeterminacy membership function, capturing the uncertainty regarding x's membership in the set.

$F_{\tilde{A}}(x)$ denotes the falsity membership function, measuring how much x does not belong to the set.

Each of these functions produces values in the range $[0,1]$. The sum of these three membership values for any object x follows the inequality:

$0 \leq T_{\tilde{A}}(x), I_{\tilde{A}}(x), F_{\tilde{A}}(x) \leq 3$

When we refer to an object x within the SVNS $\tilde{A}$, we can call it a Single-Valued Neutrosophic Number (SVNN). For convenience, we can write this as:

$x = (T_{\tilde{A}}(x), I_{\tilde{A}}(x), F_{\tilde{A}}(x))$

Definition 3. Suppose we have two SVNNs represented as $a = (T_a, I_a, F_a)$ and $b = (T_b, I_b, F_b)$, where $k > 0$ is a positive constant. The following operations can be performed on these numbers:

$$a \supseteq b \Leftrightarrow T_a \geq T_b, I_a \leq I_b, F_a \leq F_b \quad (2)$$

$$a = b \Leftrightarrow a \supseteq b \; and \; b \supseteq a \quad (3)$$

$$a \cup b = \langle T_a \vee T_a, I_a \wedge I_b, F_a \wedge F_b \rangle \quad (4)$$

$$a \cap b = \langle T_a \wedge T_a, I_a \vee I_b, F_a \vee F_b \rangle \quad (5)$$

$$a^c = \langle F_a, 1 - I_a, T_a \rangle \text{ (Complement of } a) \quad (6)$$

Addition of two SVNNs:

$$a \oplus b = (T_a + T_b - T_a T_b, I_a I_b, F_a F_b) \tag{7}$$

This combines the truth, indeterminacy, and falsity values of both a and b.

Multiplication of two SVNNs:

$$a \otimes b = (T_a T_b, I_a + I_b - I_a I_b, F_a + F_b - F_a F_b) \tag{8}$$

This operation multiplies the truth values and adjusts the indeterminacy and falsity values accordingly.

Scaling an SVNN by a positive constant k:

$$ka = (1 - (1 - T_a)^k, I_a^k, F_a^k) \tag{9}$$

This scales the truth, indeterminacy, and falsity values of a using the constant k.

Raising an SVNN to the power of k:

$$a^k = (T_a^k, 1 - (1 - I_a)^k, 1 - (1 - F_a)^k) \tag{10}$$

Here, each element of a is raised to the power k.

Definition 4. This explains how to aggregate multiple SVNNs using a weighted approach. Suppose we have a collection of SVNNs, denoted as $\tilde{A}_j = \left(T_{\tilde{A}_j}, I_{\tilde{A}_j}, F_{\tilde{A}_j}\right)$, where $j = 1,2, \ldots . n$. Each SVNN has three components: truth, indeterminacy, and falsity membership functions.

The Single-Valued Neutrosophic Weighted Aggregation Arithmetic (SVNWAA) operator for these SVNNs is calculated as follows:

$$\boldsymbol{SVNWAA}\left(\tilde{A}_1, \tilde{A}_2, \ldots, \tilde{A}_n\right) = \sum_{j=1}^{n} w_j \tilde{A}_j \tag{11}$$
$$= [\, 1 - \prod_{j=1}^{n} \left(1 - T_{\tilde{A}_j}\right)^{w_j}, \prod_{j=1}^{n} \left(I_{\tilde{A}_j}\right)^{w_j}, \prod_{j=1}^{n} \left(F_{\tilde{A}_j}\right)^{w_j}]$$

The Single-Valued Neutrosophic Weighted Aggregation Geometric (SVNWAG) operator for these SVNNs is calculated as follows:

$$\boldsymbol{SVNWAG}\left(\tilde{A}_1, \tilde{A}_2, \ldots, \tilde{A}_n\right) = \prod_{j=1}^{n} \left(\tilde{A}_j\right)^{w_j} \tag{12}$$
$$= [\prod_{j=1}^{n} \left(T_{\tilde{A}_j}\right)^{w_j}, 1 - \prod_{j=1}^{n} \left(1 - I_{\tilde{A}_j}\right)^{w_j}, 1 - \prod_{j=1}^{n} \left(1 - F_{\tilde{A}_j}\right)^{w_j}]$$

In this formula:

w_j represents the weight for each SVNN $\tilde{A}_j$, and the weights satisfy $w_j > 0$ and $\sum_{j=1}^{n} w_j = 1$.

The truth component $T_{\tilde{A}_j}$ is aggregated using a complement product-based formula.

The indeterminacy $I_{\tilde{A}_j}$ and falsity $F_{\tilde{A}_j}$ components are combined through weighted geometric means.

Definition 5. This introduces the concept of deneutrosophication, simplifying an SVNN by converting it into a real number. Let us break it down:

Given an SVNN, $\tilde{A} = \{ (x, T_{\tilde{A}}(x), I_{\tilde{A}}(x), F_{\tilde{A}}(x)) : x \in X \}$

The goal is to simplify this set into a single real number using the following formula:

$$E(\tilde{A}) = \frac{3 + T_{\tilde{A}} - 2I_{\tilde{A}} - F_{\tilde{A}}}{4} \tag{13}$$

Illustrative example 1: Let us work with two SVNNs: $a = (0.8, 0.15, 0.2)$ and $b = (0.6, 0.35, 0.4)$, $k = 0.7$, $w_a = 0.6$ and $w_b = 0.4$ an example of Eqs. (2)–(7) are shown below:

$a \oplus b = (0.8, 0.15, 0.2) \oplus (0.6, 0.35, 0.4)$ = (0.92, 0.0525, 0.08)

$a \otimes b = (0.8, 0.15, 0.2) \otimes (0.6, 0.35, 0.4) = (\, 0.48, 0.4475, 0.52)$

$ka = 0.7 \cdot (0.8, 0.15, 0.2) = (0.6759, 0.2650, 0.3241)$

$a^k = (0.8554, 0.1075, 0.1446)$

$SVNWAA(a, b) = (0.7361, 0.2105, 0.2639)$

$SVNWAG = (0.7130, 0.2365, 0.2870)$

$$E(a,b) = \frac{3 + 0.7361 - 2 \cdot 0.2105 - 0.2639}{4} = 0.7628$$

3.2 Research Flowchart

The proposed model consists of three sequential phases: NS Delphi, NS DEMATEL, and NS COCOSO, as illustrated in **Figure 1**. Each phase addresses specific aspects of evaluating and mitigating barriers to WMPs in SRL.

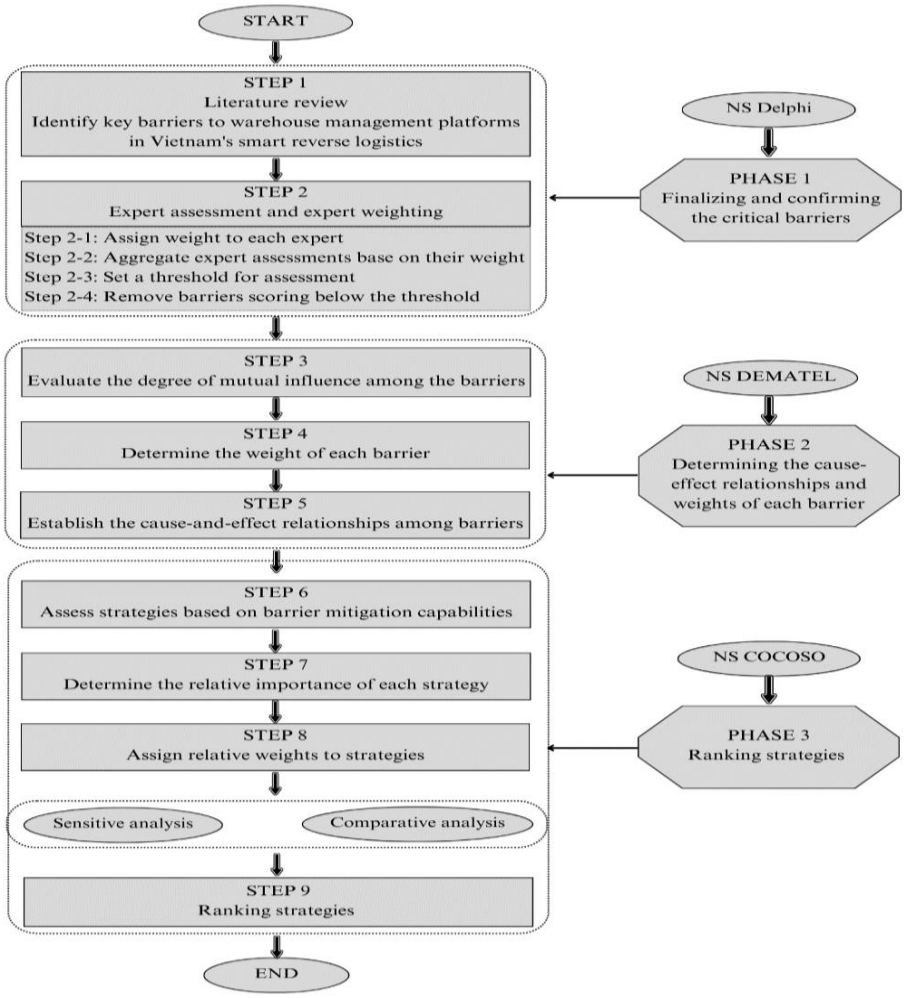
START
STEP 1
Literature review
Identify key barriers to warehouse management platforms in Vietnam's smart reverse logistics
STEP 2
Expert assessment and expert weighting
Step 2-1: Assign weight to each expert
Step 2-2: Aggregate expert assessments base on their weight
Step 2-3: Set a threshold for assessment
Step 2-4: Remove barriers scoring below the threshold
NS Delphi
PHASE 1
Finalizing and confirming the critical barriers
STEP 3
Evaluate the degree of mutual influence among the barriers
STEP 4
Determine the weight of each barrier
STEP 5
Establish the cause-and-effect relationships among barriers
NS DEMATEL
PHASE 2
Determining the cause-effect relationships and weights of each barrier
STEP 6
Assess strategies based on barrier mitigation capabilities
STEP 7
Determine the relative importance of each strategy
STEP 8
Assign relative weights to strategies
Sensitive analysis
Comparative analysis
NS COCOSO
PHASE 3
Ranking strategies
STEP 9
Ranking strategies
END

Figure 1: Research Flowchart

In Phase 1, a panel of experts comprising academics and practitioners specializing in supply chain and logistics management is assembled to assess the identified factors influencing WMP adoption in SRL. These experts evaluate the relevance and impact of the factors based on their practical and theoretical insights. The NS Delphi technique is then applied to synthesize expert opinions, ensuring the convergence of judgments and validating significant factors for further analysis. This phase establishes a robust foundation for identifying key barriers systematically and evidence-based.

Phase 2 employs the NS DEMATEL method to analyze the interrelationships among the validated factors identified in Phase 1. The expert panel assesses the levels of mutual influence between these factors, facilitating the identification of cause-and-effect relationships. This phase highlights the root causes and dependent factors,

comprehensively understanding their dynamics. The influence weights of each factor are calculated, which are critical for prioritizing barriers and devising effective strategies within the SRL context.

In Phase 3, the NS COCOSO method evaluates and ranks strategies proposed to address the identified barriers. This phase integrates the influence weights derived from Phase 2, ensuring that the prioritization of strategies aligns with the underlying causal dynamics. Experts assess the effectiveness of each strategy in mitigating the barriers, and the NS COCOSO method systematically ranks these strategies based on their ideality. This final phase provides actionable insights and supports informed decision-making, enabling stakeholders to effectively improve warehouse management practices in SRL.

3.3 NS Delphi Method

Assume that q experts assess n factors, with each expert assigning significance to each factor using a linguistic scale. These linguistic evaluations are then transformed into NS numbers using NS. Furthermore, the experts are assigned weights according to their educational background and professional experience. The detailed steps of the calculation process are as follows:

Step 1: Calculate the weight of the expert.

The weights assigned to experts will be determined using NS numbers, which account for two key factors: the evaluation of the expert's experience and educational background. These NS numbers, reflecting the expert's rating based on years of experience and education, are combined in Equation (2) and subsequently transformed into a precise score through Equation (7). **Table 2** provides an expert-level evaluation alongside the relevant linguistic scale [38].

Table 2: Expert rating scale

Education	**Experience**	**Linguistic scale**	**Code**	**NS Number**
Doctor	Over 20 years	Extremely High	EH	(0.8,0.15,0.2)
Master	From 10 to 20 years	High	H	(0.6,0.35,0.4)
Bachelor	From 5 to 10 years	Medium	M	(0.4,0.65,0.6)
Under Bachelor	Under five years	Low	L	(0.2,0.85,0.8)
		Extremely Low	EL	(0,1,1)

For example, Expert 1 has a doctoral degree and 5 to 10 years of professional experience. Given their advanced qualifications and work experience, the expert's weight is evaluated as highly high (EH) based on qualifications and medium (M) based on experience. These evaluations are expressed as NS numbers: (0.8, 0.15, 0.2) for qualifications and (0.5, 0.65, 0) for experience.

The two evaluations, represented by NS numbers, are combined using Equation (7). The result is then converted into a crisp value using Equation (13). In the given example, the evaluation for Expert 1 is:

$(0.8, 0.15, 0.2) \oplus (0.5, 0.65, 0) = (0.88, 0.0975, 0.12)$

By applying Equation (13) to convert the NS number (0.88, 0.0975, 0.12) into a crisp score, the resulting value is 0.8913.

To calculate the evaluation values for q experts, we obtain a set of q values, denoted as $SQ: sq_a = \{sq_1, sq_2, sq_3, \dots, sq_q\}$. The weight assigned to each expert, SW, is represented as $sw_a = \{sw_1, sw_2, sw_3, \dots, sw_a\}$ is determined using the formula in Equation (14):

$$sw_a = \frac{sq_a}{\sum_{a=1}^{q} sq_a} \tag{14}$$

This formula calculates the weight of each expert by dividing their evaluation score sq_a by the total of all experts' scores. The result provides the relative weight or importance of that expert compared to the others in the group.

Step 2: Build a weighted expert evaluation matrix.

Experts will participate in assessing the significance of n factors. The initial assessments, which are provided in linguistic terms, are subsequently converted into NS numbers and organized into a matrix $\otimes$ $FM = [f_{ia}]_{n\times q}$, where n represents the number of factors being evaluated, and q denotes the number of

experts participating. The linguistic evaluation scale and the corresponding NS values are presented in **Table 3** [38].

Each element f_{ia} in the matrix indicates the evaluation score given by the expert a for factor i. This process transforms qualitative judgments into a quantitative form, enabling a structured analysis of the factors' importance across the group of experts.

Table 3: Linguistic Importance Scale in NS Delphi

Linguistic scale	**Code**	**Membership function**		
		T	**I**	**F**
Extremely High	EH	0.8	0.15	0.2
High	H	0.6	0.35	0.4
Medium	M	0.4	0.65	0.6
Low	L	0.2	0.85	0.8
Extremely Low	EL	0	1	1

The weighted expert evaluation matrix denoted as $\otimes\ FMW = [fw_{ia}]_{n\times q}$ is calculated using Equation (15) below:

$$\mathrm{fw_{ia}} = \mathrm{fw_{ia}} \otimes sw_a \tag{15}$$

Here, $i = 1,2,\dots,n$ represents the evaluated factors, and $a = 1,2,\dots,q$ corresponds to the experts involved. The value sw_a refers to the weight assigned to expert a, where the weights are expressed as a set $\{sw_1, sw_2, sw_3, \dots, sw_a\}$. This process ensures that each expert's evaluation is adjusted according to weight.

Step 3: Calculate the threshold and validate factors

Each factor will be evaluated by q experts. First, the evaluations from these experts are combined using Equation (6), resulting in aggregated outcomes for n factors represented in the form of Normalized NS. Next, Equation (13) is applied to convert these evaluations into crisp scores, generating a set of n evaluation values, denoted as $av_i = \{av_1, av_2, \dots, av_n\}$

To determine the acceptance threshold for these factors, we calculate the threshold value ɣ using the Equation (16):

$$\gamma = \frac{\sum_{i=1}^{n} av_i}{n} \tag{16}$$

If the evaluation value av_i is greater than or equal to the threshold ɣ, then factor i is deemed acceptable. However, if av_i is less than ɣ, factor i is rejected.

3.4 NS DEMATEL Method

Suppose there are k experts, each assigned a specific weight ew, assessing the mutual influence of n factors. Initially, the ratings are expressed in linguistic terms and then converted into NS. The rating scale and the corresponding NS values are provided in **Table 4**.

Table 4: Linguistic Importance Scale in NS DEMATEL

Linguistic scale	**Code**	**Membership function**		
		T	**I**	**F**
Absolute influence	AI	0.8	0.15	0.2
Strong influence	SI	0.6	0.35	0.4
Fair influence	FI	0.4	0.65	0.6
Weak influence	WI	0.2	0.85	0.8
No influence	NI	0	1	1

Once the assessments are transformed into NS, the data will be processed through the DEMATEL method. The steps for the calculations are detailed below [39].

Step 1: Creating the direct relationship matrix $\otimes V$.

The evaluations of the mutual influence among n factors (where factor i affects factor j) from k experts, denoted as v_{ij}^k, are converted into NS with their corresponding expert weights ew_t. These evaluations are then

consolidated using Equation (17), resulting in the direct influence matrix $\otimes V = [\otimes v_{ij}]_{n\times n}$, where:

$$v = SVNWA\left(v_{ij}^{1}, v_{ij}^{2}, \dots, p_{ij}^{k}\right) = \sum_{t=1}^{k} ew_t v_{ij}^{k} \tag{17}$$

Here, $i = 1, 2, \dots, n$, $j = 1, 2, \dots, n$, and $t = 1, 2, \dots, k$. The notation $\otimes v_{ij}$ is defined as $(v_{ij}^{\alpha}, v_{ij}^{\beta}, v_{ij}^{\gamma})$. It is important to note that the diagonal elements of this matrix are 0, i.e., meaning $\otimes v = 0$, when $i = j$.

Then, Equation (13) is applied to convert the matrix $\otimes V$ into crisp scores.

Step 2: Calculate the normalized direct relationship matrix $\otimes U$.

The matrix $\otimes U = \left[\otimes u_{ij}\right]_{n\times n}$ will undergo normalization to produce the matrix using Equations (18)-(20):

$$\otimes \mathrm{U} = \left[\otimes u_{ij}\right]_{n\times n} = \begin{bmatrix} \otimes \theta \cdot \mathrm{v}_{11} & \otimes \theta \cdot \mathrm{v}_{12} & \cdots & \otimes \theta \cdot \mathrm{v}_{1\mathrm{j}} & \cdots & \otimes \theta \cdot \mathrm{v}_{1\mathrm{n}} \\ \otimes \theta \cdot \mathrm{v}_{21} & \otimes \theta \cdot \mathrm{v}_{22} & \cdots & \otimes \theta \cdot \mathrm{v}_{2\mathrm{j}} & \cdots & \otimes \theta \cdot \mathrm{v}_{2\mathrm{n}} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ \otimes \theta \cdot \mathrm{v}_{\mathrm{i}1} & \otimes \theta \cdot \mathrm{v}_{\mathrm{i}2} & \cdots & \otimes \theta \cdot \mathrm{v}_{\mathrm{ij}} & \cdots & \otimes \theta \cdot \mathrm{v}_{\mathrm{in}} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ \otimes \theta \cdot \mathrm{v}_{\mathrm{n}1} & \otimes \theta \cdot \mathrm{v}_{\mathrm{n}2} & \cdots & \otimes \theta \cdot \mathrm{v}_{\mathrm{nj}} & \cdots & \otimes \theta \cdot \mathrm{v}_{\mathrm{nn}} \end{bmatrix}_{n\times n} \tag{18}$$

$$\otimes u_{ij} = \theta . v_{ij} \tag{19}$$

With:

$$\theta = \frac{1}{\underset{1\le i\le n}{max}\left(\sum_{j=1}^{n} v_{ij}\right)} \tag{20}$$

Here, $i = j = 1, 2, \dots n$.

Step 3: Calculate the total influence matrix $\otimes T$.

To compute the total influence matrix $\otimes T$, the normalized direct relationship matrix $\otimes U$ is integrated using Equations (21)-(22), which summarize all direct and indirect influence interactions from the first to infinite power.

$$\otimes \mathrm{T} = \left[\otimes t_{ij}\right]_{n\times n} = \begin{bmatrix} \otimes \mathrm{t}_{11} & \otimes \mathrm{t}_{12} & \cdots & \otimes \mathrm{t}_{1\mathrm{j}} & \cdots & \otimes \mathrm{t}_{1\mathrm{n}} \\ \otimes \mathrm{t}_{21} & \otimes \mathrm{t}_{22} & \cdots & \otimes \mathrm{t}_{2\mathrm{j}} & \cdots & \otimes \mathrm{t}_{2\mathrm{n}} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ \otimes \mathrm{t}_{\mathrm{i}1} & \otimes \mathrm{t}_{\mathrm{i}2} & \cdots & \otimes \mathrm{t}_{\mathrm{ij}} & \cdots & \otimes \mathrm{t}_{\mathrm{in}} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ \otimes \mathrm{t}_{\mathrm{n}1} & \otimes \mathrm{t}_{\mathrm{n}2} & \cdots & \otimes \mathrm{t}_{\mathrm{nj}} & \cdots & \otimes \mathrm{t}_{\mathrm{nn}} \end{bmatrix}_{nxn} \tag{21}$$

where: $i = j = 1, 2, \dots n$

$$\begin{aligned} \otimes T &= \otimes U + \otimes U^2 + \cdots + \otimes U^{\infty} \\ &= \otimes U(I + \otimes U + \otimes U^2 + \cdots + \otimes U^{\infty-1}) \\ &= \otimes U(I - \otimes U^{\infty})(I - \otimes U)^{-1} = \otimes U(I - \otimes U)^{-1} \end{aligned} \tag{22}$$

Where $\otimes U^{\infty} = [0]_{n\times n}$ and I is the identity matrix

The elements of matrix ⊗T in the form of neutrosophic are converted to crisp neutrosophic using Equation (13), resulting in the matrix $\otimes T^{*} = [t_{ij}^{*}]_{n\times n}$.

Step 4: Formulating a Cause and Effect map.

The value $\otimes r$ is derived by summing the rows of the total influence matrix $\otimes T^{*}$, while $\otimes c$ is obtained by summing the columns of matrix $\otimes T^{*}$.

$$\otimes r = [\otimes r_i]_{n\times 1} = (\otimes r_1, \otimes r_2, \dots, \otimes r_{i,} \dots, \otimes r_n) \tag{23}$$

$$[\otimes r_i]_{n\times 1} = \left[\sum_{j=1}^{n} \otimes t_{ij}^{*}\right]_{n\times 1} \tag{24}$$

$$\otimes c = [\otimes c_i]_{1\times n} = (\otimes c_1, \otimes c_2, \dots, \otimes c_{j,}, \dots, \otimes c_n)^{T} \tag{25}$$

$$\left[\otimes c_j\right]_{1\times n} = \left[\sum_{i=1}^{n} \otimes t_{ij}^{*}\right]_{1\times n} = [\otimes c_i]_{n\times 1}^{T} \tag{26}$$

The combined influence index, represented by $\otimes r_i + \otimes c_i$, measures the total strength of influence given

and received. The difference $\otimes r_i - \otimes c_i$ signifies the net influence. A larger value of $\otimes r_i + \otimes c_i$ implies that factor i has a significant impact on the evaluation system. A positive $\otimes r_i - \otimes c_i$ indicates that indicator i exerts considerable influence on other indicators, while a negative $\otimes r_i - \otimes c_i$ value suggests that other indicators affect indicator i more.

The overall impact of an indicator on the system is reflected in $\otimes r_i - \otimes c_i$. Thus, Equation (27) will calculate the indicator's impact weight.

$$\sigma_i = \frac{(r_i + c_i)}{\sum_{i=1}^{n}(r_i + c_i)} \tag{27}$$

3.5 NS COCOSO Method

Suppose there are k experts responsible for assessing how n strategies affect and address m factors. The evaluation scale and its corresponding neutrosophic values are shown in **Table 5**.

Table 5: Linguistic Importance Scale in NS COCOSO

Linguistic scale	**Code**	**Membership function**		
		T	**I**	**F**
Very Good	VG	0.8	0.15	0.2
Good	G	0.6	0.35	0.4
Fair	F	0.4	0.65	0.6
Poor	P	0.2	0.85	0.8
Extremely Poor	EP	0	1	1

After transforming the assessments into neutrosophic values, the data will be processed through the COCOSO method. The calculation procedure is outlined in the steps below [40].

Step 1: Calculate the synthesized expert assessment matrix $\otimes X$.

The effectiveness ratings of n strategies in addressing m factors, provided by k experts, denoted as e_{ij}^k and weighted by ew_t, are combined into the matrix $\otimes X = [\otimes x_{ij}]_{n\times n}$ as illustrated using Equation (28).

$$x_{ij} = SVNWA\left(x_{ij}^1, x_{ij}^2, \dots, x_{ij}^k\right) = \sum_{t=1}^{k} ew_t x_{ij}^k \tag{28}$$

while, $i = 1, 2, \dots, n$; $j = 1, 2, \dots, m$; $t = 1, 2, \dots, k$; and $\otimes x_{ij} = (x_{ij}^T, x_{ij}^I, x_{ij}^F)$.

Subsequently, Equation (13) is used to matrix $\otimes X$ into crisp scores.

Step 2: Normalizing matrix $\otimes X$ into matrix $\otimes X^*$.

Matrix $\otimes X = \left[\otimes x_{ij}\right]_{n\times n}$ is transformed into the normalized matrix $\otimes X^* = \left[\otimes x_{ij}^*\right]_{n\times n}$ by using Equations (29)-(30):

$$x_{ij}^* = \frac{x_{ij} - \min(x_{ij})}{\max(x_{ij}) - \min(x_{ij})} \tag{29}$$

$$x_{ij}^* = \frac{\max(x_{ij}) - x_{ij}}{\max(x_{ij}) - \min(x_{ij})} \tag{30}$$

Step 3: Assess the weight of importance assigned to each strategy.

Determine the sum of the weighted comparability sequence (S_i) and the cumulative power weight of the comparability sequence (P_i) for each alternative, using Equations (31)-(32).

$$S_i = \sum_{j}^{m} \sigma_i \, x_{ij}^* \tag{31}$$

$$P_i = \sum_{j}^{m} (x_{ij}^*)^{\sigma_i} \tag{32}$$

Here, $i = 1, 2, \dots, n$; $j = 1, 2, \dots, m$; $\sigma_i = (\sigma_1, \sigma_2, \dots, \sigma_m)$ is the weight of factor j.

Step 4: Computing the relative weight of each strategy.

Three evaluation scores k_{iI}, k_{iII}, and k_{iIII}, are used to establish the relative weights of other options, calculated using Equations (33)-(35).

Calculate k_{iI}

$$k_{iI} = \frac{S_i + P_i}{\sum_{i=1}^{n}(S_i + P_i)} \tag{33}$$

Calculate k_{iII}

$$k_{iI} = \frac{S_i}{\min C_i} + \frac{P_i}{\min P_i} \tag{34}$$

Calculate k_{iIII}

$$k_{iI} = \frac{\gamma S_i + (1-\gamma)P_i}{\gamma \max S_i + (1-\gamma)\max P_i} \tag{35}$$

Where $0 \leq \gamma \leq 1$; ω is usually taken as 0.5.

Calculate k_i and ranking

By applying Equation (36) to compute k_i, the resulting value is utilized to establish the ranking of the corresponding strategies.

$$k_i = (k_{iI}k_{iII}k_{iIII})^{\frac{1}{3}} + \frac{1}{3}(k_{iI} + k_{iII} + k_{iIII}) \tag{36}$$

4 WMPs Problem

WMPs, often called WMSs, are essential tools in the logistics and supply chain industries, particularly in supporting SRL in Vietnam. Implementing these platforms enables efficient warehouse operations management, including inventory control, order fulfillment, labor management, and resource optimization. They can increase customer satisfaction and more efficient operations while enabling supply chain traceability, promoting transparency, and supporting sustainability efforts [41]. With e-commerce, growing global supply chains, and technological advancements, WMPs have evolved into highly sophisticated systems increasingly integrated with AI, IoT, and automation.

Adopting AI and IoT in WMPs allows businesses to optimize warehouse layouts, predict demand, and automate inventory tracking. This is especially beneficial in RL, where returned products must be sorted, stored, and possibly re-entered into the supply chain. Smart WMPs can dynamically allocate resources, reduce human error, and streamline operations, helping companies minimize costs and maximize the recovery of valuable materials compared to traditional systems that rely on human handling.

WMPs connected with other business systems like ERP and delivery management tools can enable smooth data exchange across the supply chain, enhancing operational efficiency. Real-time inventory tracking and automated space management reduce reliance on manual processes, cut operational costs, and improve accuracy. For RL, this means quicker returns processing, lower storage costs, and better coordination between manufacturers and suppliers.

In Vietnam, the adoption of SRL is particularly relevant as the country's e-commerce and manufacturing sectors experience rapid growth. Sustainable practices in SRL can lead to cost reductions of up to 30% while also improving customer loyalty [6]. WMPs thus form the backbone of effective SRL systems, which are increasingly essential in maintaining competitiveness in a global market that values sustainability and efficient resource use. Despite the challenges associated with implementing WMPs, the benefits they bring make them an invaluable asset. For Vietnam, where reverse logistics is gaining prominence, WMPs represent a pathway toward a more competitive, sustainable, and customer-focused logistics industry.

4.1 Decision-making Criteria

Implementing WMPs in Vietnam's smart reverse logistics requires addressing numerous barriers across political, economic, social, technological, legal, and management-organization factors. A total of 55 decision-making criteria, organized into these six categories, are proposed for practical evaluation.

(i) Political aspects:

Consumer education *(OB1)* - Customers should be priced appropriately and fully informed that the efficiency of refurbished equipment may be lower than that of new products. Consumer education, awareness campaigns, and commercials should be implemented to help people improve their perception of remanufactured items by teaching them to be environmentally conscious [42].

Lack of knowledge on taxation *(OB2)* - There are various sectors where particular tax knowledge

requirements must be adopted to guarantee tax compliance in the digital economy. For those working in the digital economy, gaps in these knowledge areas put them in danger of non-compliance [43].

Lack of regulations and laws *(OB3)* - There are various sectors where particular tax knowledge requirements must be adopted to guarantee tax compliance in the digital economy. For those working in the digital economy, gaps in these knowledge areas put them in danger of non-compliance [11].

Bureaucracy and red tape *(OB4)* - Together with improving the nation's infrastructure, there needs to be a reduction in the excessive levels of bureaucracy and taxes [42].

Complex customs regulations *(OB5)* - Reverse logistics involving the cross-border movement of goods face hurdles due to stringent and sometimes unclear customs regulations. Exporting used or returned goods, even for recycling, may be delayed or prohibited [11].

Fragmented policy across provinces *(OB6)* - The inefficient collection and classification of scrap plastic, pointing out that local policies are not uniformly implemented across regions, which creates logistical hurdles for businesses operating nationwide [44].

Lack of cross-ministry coordination *(OB7)* - Poor coordination among government agencies leads to regulatory bottlenecks that inhibit the adoption of reverse logistics in developing countries [45].

Lack of integration with global environmental *commitments (OB8)* - Compliance with environmental policies and the need for governmental support play critical roles in achieving sustainability through reverse logistics [44].

Overemphasis on economic growth *(OB9)* - Economic globalization pressures in Vietnam contribute to environmental degradation. This has implications for reverse logistics, as economic objectives often overshadow sustainability-focused policies [11].

Limited role of civil society organizations *(OB10)* - The restricted impact of civil society on policymaking limits its ability to democratize global sustainable governance [42].

(ii) Economic aspect:

Lack of Economies of Scale *(OB11)* means that smaller or less efficient operations may face higher costs for handling, storing, and processing returned goods because they cannot leverage bulk purchasing, automation, or streamlined processes. As a result, the costs associated with reverse logistics, such as transportation, labor, and storage, remain high, reducing overall profitability [11].

Lack of Initial Capital *(OB12)* - Establishing collecting stations and machinery and linking them to the present forward logistics network incur significant upfront costs [46].

Less Return on Investment *(OB13)* - The barrier stems from uncertainty about investment returns and a lack of verified advantages [45].

Financial Constraints *(OB14)*- *R*efer to the limited availability of funds or resources that restrict a company's ability to invest in necessary technologies, infrastructure, or personnel for efficient warehouse management and reverse logistics [45].

High Training Cost *(OB15*) - This includes the costs of training programs, hiring experts or trainers, and the time spent away from regular duties [11].

Tax-related Financial Strain *(OB16)* refers to the financial pressure businesses face due to taxes, including high tax rates and complex tax regulations, which can increase overall costs and reduce profitability [44].

Uncertainty regarding economic issues *(OB17)* refers to the unpredictable nature of economic conditions, such as inflation, currency fluctuations, or market downturns. This can lead to fluctuating storage, transportation, and labor costs, making it difficult for businesses to plan and invest in efficient reverse logistics systems, potentially leading to delays or higher costs in processing returns [11].

The cost associated with inventory management of return products *(OB18)* includes additional storage space, labor for inspecting and restocking items, and system resources needed to track and manage returned inventory. These expenses can increase operational complexity and reduce profitability if not efficiently managed [47].

Lack of funds for product return monitoring systems *(OB19)* refers to insufficient financial resources to invest in technology that tracks and manages returned goods. This can result in inefficiencies such as delays in processing returns, inaccurate inventory tracking, and difficulty analyzing return patterns [11].

Expenditure of collecting used products *(OB20)* - These expenses include transportation, labor, and logistical coordination needed to retrieve products from various locations. These collection costs can significantly impact reverse logistics operations' overall efficiency and profitability, especially if the process is

not optimized for scale or cost-effectiveness [42].

(iii) Social aspect

Reluctance to modify existing models to integrate reverse logistics *(OB21)* - It refers to the unwillingness of organizations to adapt their current systems to incorporate reverse logistics practices. In warehouse management, this resistance can result in missed opportunities for improving efficiency and sustainability, as outdated processes hinder the effective handling of returned goods [45].

Customers' distrust of recovered products due to perceived lower quality (OB22) refers to consumers' hesitation in purchasing refurbished or returned items. In warehouse management and reverse logistics, this lack of trust can lead to slower resale of recovered goods, increased storage costs, and reduced profitability [46].

Cooperation Among Chain Participants *(OB23)* - Reuse and recycling efforts are not well supported by other supply chain participants, such as third-party logistics providers, clients, or intermediaries [45].

Negative Perceptions of Used or Returned Products *(OB24)* - This perception can lead to challenges in efficiently reselling returned goods, resulting in excess inventory and increased storage costs, ultimately reducing the profitability of reverse logistics operations [47].

Lack of credible information communicated to the public on reverse logistics *(OB25)* - It refers to inadequate or unclear information about the return and recycling process. This can lead to consumer confusion, reducing anticipation and complicating warehouse management and reverse logistics operations [45].

Demand uncertainty for return products *(OB26)* refers to the unpredictability in the volume or need for returned goods. This uncertainty can complicate inventory planning and allocation, leading to excess storage costs or insufficient capacity to handle returns efficiently [45].

Lack of support, encouragement, and motivation to adopt reverse logistics practices *(OB27*) refers to insufficient stakeholder incentives or backing to implement reverse logistics strategies. This can result in the slow adoption of efficient return-handling practices, leading to increased operational costs and reduced effectiveness in managing returned goods [47].

Lack of industry involvement in ethical and moral practice *(OB28)* refers to insufficient engagement from businesses in adopting responsible and sustainable practices for managing returned goods. This can lead to poor handling of returns, increased waste, and negative environmental impact [45].

Lack of population education regarding reverse logistics (OB29) means insufficient public awareness about return processes. This can lead to warehouse management and reverse logistics inefficiencies as people may not handle returns properly or understand the benefits [11].

(iv) Technical aspect:

Lack of technical skills *(OB30)* - There is a shortage of skilled personnel and insufficient capabilities to carry out reverse logistics activities [11].

Difficulty deciding on 3PL to partner with *(OB31)* - The company is finding it challenging to design a supply chain network for reverse logistics. Because most logistic follows forward logistics and poor service in reverse logistics [11].

Complexity in operation *(OB32)* - Companies find it challenging to make strategic decisions regarding recovery options and to operate reverse logistics systems efficiently and effectively [46].

Lack of available technological infrastructure to adopt RL practices *(OB33)* - The insufficient development of essential systems, such as digital platforms, automation, and connectivity, needed to implement efficient reverse logistics processes, leading to operational inefficiencies and delays [11].

Lack of flexibility to change from traditional system to new system *(OB34)* - Organizations face difficulty adapting their existing processes and infrastructure to modern, technology-driven systems, resulting in resistance to innovation, operational disruptions, and inefficiencies in managing reverse logistics [11].

Lack of effective environmental measure *(OB35)* - Inadequacy of strategies and actions to minimize environmental impact, resulting in inefficient resource management, waste handling, and sustainability efforts within warehouse operations [46].

Lack of efficient and effective systems to monitor returns and recalls *(OB36)* - The absence of robust tools and processes for tracking and managing product returns and recalls, leading to delays, inaccuracies, and inefficiencies in handling returned goods and addressing customer concerns or extent the cycle of the product [46].

Lack of technology for waste management and recycling *(OB37)* - Insufficient or outdated systems for efficiently handling and processing waste in reverse logistics. This barrier leads to inefficiencies, higher costs,

limited recycling capabilities, and increased environmental impact [46].

Low security of data and information within the supply chain *(OB38)* - Low security of data and information within the supply chain in warehouse management platforms refers to inadequate protection of sensitive data, leading to risks of breaches, fraud, and disruptions in reverse logistics operations [45].

(v) Legal aspect

Data protection and cybersecurity concerns *(OB39)* - Information safety, privacy, and network security issues are constantly severe. According to information from the Ministry, Vietnam is also among the top 10 nations experiencing cyberattacks and harmful malware infections. It ranks 7th in terms of cyberattack victims and 2nd in terms of nations most infected with malware that mines cryptocurrency [46]

Environmental regulations *(OB40)* - Inconsistent and unclear waste disposal regulations complicate compliance in reverse logistics operations [44].

Intellectual Property (IP) Laws *(OB41)* - Finding obstacles to Vietnam's intellectual property rights is crucial to preventing long-term dangers, making the necessary modifications, seizing the chances presented by the EVFTA for sustainable economic growth, and maximizing Vietnam's comparative advantages [44].

Data Sovereignty Laws *(OB42)* - The phrase "data sovereignty" is used to characterize government attempts to impose control over online activity; these efforts are frequently manifested through actions directed towards Internet service providers [43].

Lack of Standardization in Reverse Logistics Practices *(OB43)* - A lack of coding standards makes it challenging to utilize internationally. RFID standards are insufficient, even though specific extra initiatives must be implemented to create regional and global synergies and improve cooperation [44].

(vi) Management - Organization aspect

Company policies against reverse logistics *(OB44)* - A company may be concerned about cannibalizing its high-quality items [46].

Low importance of reverse logistics relative to other issues *(OB45)* - Product return activities are viewed as incompatible with the company's core business [46].

Lack of suitable indicators and metrics to measure performance *(OB46)* - There is a lack of performance evaluation to assess warehouse management efficiency in smart reverse logistics setups and product tracking systems [44].

Higher priority given to other issues in the supply chain *(OB47)* - The company focuses on forward logistics setup and sales of its regular product portfolio [46].

Lack of a business plan for returns *(OB48)* - Companies find it challenging to design methods for ensuring that returned items are sold and that the product portfolio is utilized correctly [44].

Variable quality and quantity of returned products *(OB49)* refers to the inconsistent condition and volume of goods returned by customers. This variability can complicate inventory management and processing, leading to challenges in handling and reselling returned items effectively [47].

Insufficient expertise in reverse logistics *(OB50)* refers to a lack of knowledge and skills in managing the return and processing of goods. This gap in expertise can lead to ineffective handling of returns, higher operational costs, and challenges in optimizing return processes [47].

Lack of commitment from top management *(OB51)* refers to the lack of active involvement and backing from top executives in implementing reverse logistics practices. This can result in inadequate resources and attention for optimizing return processes, leading to inefficiencies and missed opportunities for improving return handling and sustainability [11].

Unfavorable organizational culture *(OB52)* refers to a work environment lacking support for efficient reverse logistics practices. In warehouse management, this can result in low employee morale, resistance to change, and ineffective handling of returns, which impedes reverse logistics operations' overall effectiveness and efficiency[11].

Lack of specialized staff *(OB53)* refers to a shortage of trained reverse logistics and warehouse management experts. This can hinder the efficient handling and processing of returns, leading to operational inefficiencies and increased costs due to a lack of expertise in managing complex reverse logistics tasks [11].

Limited Training Programs *(OB54) r*efer to insufficient educational resources or structured training opportunities for employees to develop the skills to manage advanced warehouse systems and reverse logistics processes [48].

Poor Strategic Planning (OB55) may involve misestimating demand for returned goods, improper

warehouse space allocation, or lack of coordination in handling returned inventory. This results in higher operational costs, delayed processing of returns, and reduced overall efficiency in the reverse logistics system [49].

4.2 Alternatives

Drawing from the identified barriers and comprehensive research on solutions applied in various domains and countries, twelve strategic solutions have been formulated to tackle warehouse management challenges in Vietnam's SRL.

Investment in Infrastructure *(S1)* - This approach incorporates IoT, cloud computing, and AI to enable real-time tracking and data analysis for returned goods, remanufacturing and recycling processes, reduce waste, and enhance overall operational efficiency [3], [50].

Training and Educational Programs *(S2)* - Ensuring proper training and education for human resources is crucial to mitigate the dynamic and unpredictable risks that arise from technological transition challenges [51]. This proactive approach minimizes disruption and effectively empowers teams to leverage new technologies, ensuring seamless integration and long-term success. Furthermore, a well-trained workforce can anticipate potential obstacles, troubleshoot emerging issues, and contribute to a smoother technological evolution, ultimately enhancing overall productivity and competitiveness in a rapidly evolving digital landscape.

Improved Data Protection *(S3)* - The strategy involves utilizing blockchain technology as a reliable and strong solution for addressing data security and cybersecurity challenges [52].

Current State Assessment and Vision Formulation *(S4)* - Organizational culture significantly influences the transformation of business models, which is essential for evolving into a more innovative supply chain [52]. By assessing the current business model, organizations can identify gaps, inefficiencies, and bottlenecks in their supply chain. This helps pinpoint improvement areas and uncover opportunities to implement new technologies and processes. Formulating a future vision based on understanding the current state ensures that the transformation is purpose-driven and aligned with the organization's long-term objectives. It allows for a clear roadmap that all stakeholders can follow.

Apply Blockchain Technology *(S5)* - Blockchain can enhance traceability and transparency along the chain [53], [54]. With blockchain, every return can be traced back to its origin, ensuring that the entire lifecycle of products is documented. This is crucial for quality control and compliance with regulations. Additionally, a shared and unified blockchain for all stakeholders in the supply chain, including manufacturers, freight providers, and consumers, guarantees that the exchanged and altered goods are authenticated [55]. This approach helps prevent fraud and improves the availability of information.

Adopt Sustainability Practices *(S6)* - Sustainability practices aimed at reducing carbon footprints and adopting green strategies in logistics are crucial for enhancing value and competitiveness [56]. Investing in energy-efficient material handling equipment was suggested to minimize the environmental impact of warehouse operations.

Role of Government *(S7)* - The government can implement policies to encourage the adoption of Circular Economy (CE) and RL as part of Corporate Social Responsibility (CSR) obligations, requiring companies to support cleaner production and environmentally sustainable practices. Additionally, governments can incentivize consumers to reuse products, enhancing sustainability efforts. Providing tax rebates for remanufactured goods or imposing taxes on products that cannot be remanufactured would motivate organizations to incorporate CE-friendly designs into their products [57].

Raise Awareness for Customers *(S8)* - Customers must understand that refurbished products may be less efficient than new ones and should be priced accordingly. To foster this understanding, awareness programs and advertisements should be implemented to educate consumers about environmental sensitivity and encourage a positive attitude toward remanufactured products [58].

Outsourcing *(S9)* - Outsourcing recycling and remanufacturing tasks to a third-party logistics (3PL) provider can optimize production strategies, benefiting both manufacturers and the supply chain by maximizing profits. Identifying the right conditions regarding cost factors and the proportion of returned products is crucial to ensure the feasibility of an optimal threshold policy for outsourcing in warehouse management and reverse logistics [58].

Selective Approach to Reverse Logistics *(S10)* - Market demand must be carefully tracked to ensure RL is economically viable and maximizes returns from remanufactured products. Not all products are suitable for RL, so selecting the right items is crucial to avoid inefficiencies and increased storage costs. Collaborating with the sales and marketing teams to consider leasing products rather than selling them outright can create a circular product lifecycle, optimize warehouse operations, and reduce

Develop a Single Window System for Licenses and Financial Support *(S11)* - This simplifies the process for businesses to obtain necessary permits and funding. In warehouse management and reverse logistics, this approach can reduce administrative delays and provide easier access to financial resources, enabling smoother implementation of reverse logistics operations and fostering growth [47].

Develop a Single Window System for Licenses and Financial Support *(S11)* - This simplifies the process for businesses to obtain necessary permits and funding. In warehouse management and reverse logistics, this approach can reduce administrative delays and provide easier access to financial resources, enabling smoother implementation of reverse logistics operations and fostering growth [47].

Create Electronic Collaboration Among Supply Chain Partners *(S12)* - This strategy involves utilizing digital platforms to enhance communication and coordination between all supply chain members. In warehouse management and reverse logistics, this approach can improve the flow of information regarding returns, streamline processes, and increase efficiency in handling returned products, ultimately leading to better overall performance [59].

5 Result and Discussion

5.1 Expert Selection and Expert Weight

Delving into the participants' demographic details adds credibility to the research findings. The panel includes a broad range of experience levels, with 77.78% of experts having over 10 years of professional experience. This experience range emphasizes the substantial expertise the participants bring to the study, particularly in fields pertinent to logistics and supply chain management. Additionally, the diversity of age, with the majority (48.89%) falling within the 40-60 age group and 13.33% being over 60, suggests that the study benefits from the insights of seasoned professionals.

Educational qualifications further underline the panel's expertise; over 77% hold advanced degrees, with 51.11% having master's degrees and 26.67% possessing doctorates. This advanced education level highlights the academic rigor and specialization participants bring to the research. The diverse employment backgrounds—26.67% in teaching, 35.56% in research, and 40.00% actively working in industry—ensure a well-rounded perspective, blending theoretical knowledge with practical application. Such diversity in age, experience, education, and professional roles strengthens the research, offering insights grounded in various expert perspectives. The experts' qualifications, experience, and skills make them ideally suited to contribute to surveys in this field, as outlined in Table 6, and their corresponding weights are shown in Table 7.

Table 6: Experts' Profiles

Demographic	Sematic	Count	%
Age	Under 25	0	0.00%
	From 25 - 40	17	37.78%
	From 40 - 60	22	48.89%
	Over 60	6	13.33%
Gender	Male	33	73.33%
	Female	11	24.44%
	Other	1	2.22%
Education	Bachelor	10	22.22%
	Master	23	51.11%
	Doctor	12	26.67%
	Other	0	0.00%
Work	Teaching	12	26.67%
	Researching	16	35.56%
	Working	18	40.00%
Experience	Less than 5 years	0	0.00%

	From 5 - 10 years	10	22.22%
	From 10 - 20 years	22	48.89%
	Over 20 years	13	28.89%

Table 7: Experts' Weights

Experts	**Education**	**Experience**	**Evaluation Value (NS)**	**Crips Value**	**Weight**
Expert 1	Master	From 5 - 10 years	(0.76; 0.227; 0.24)	0.7665	0.0201
Expert 2	Master	From 10 - 20 years	(0.84; 0.122; 0.16)	0.8590	0.0225
Expert 3	Bachelor	From 5 - 10 years	(0.64; 0.423; 0.36)	0.6085	0.0160
Expert 4	Bachelor	From 10 - 20 years	(0.76; 0.227; 0.24)	0.7665	0.0201
Expert 5	Master	From 10 - 20 years	(0.84; 0.122; 0.16)	0.8590	0.0225
Expert 6	Doctor	Over 20 years	(0.96; 0.022; 0.04)	0.9690	0.0254
Expert 7	Doctor	Over 20 years	(0.96; 0.022; 0.04)	0.9690	0.0254
Expert 8	Master	From 10 - 20 years	(0.84; 0.122; 0.16)	0.8590	0.0225
Expert 9	Doctor	Over 20 years	(0.96; 0.022; 0.04)	0.9690	0.0254
Expert 10	Master	From 5 - 10 years	(0.76; 0.227; 0.24)	0.7665	0.0201
Expert 11	Master	From 10 - 20 years	(0.84; 0.122; 0.16)	0.8590	0.0225
Expert 12	Master	From 10 - 20 years	(0.84; 0.122; 0.16)	0.8590	0.0225
Expert 13	Master	Over 20 years	(0.92; 0.052; 0.08)	0.9340	0.0245
Expert 14	Bachelor	Over 20 years	(0.88; 0.098; 0.12)	0.8910	0.0234
Expert 15	Bachelor	From 10 - 20 years	(0.76; 0.227; 0.24)	0.7665	0.0201
Expert 16	Master	From 10 - 20 years	(0.84; 0.122; 0.16)	0.8590	0.0225
Expert 17	Master	From 10 - 20 years	(0.84; 0.122; 0.16)	0.8590	0.0225
Expert 18	Master	From 10 - 20 years	(0.84; 0.122; 0.16)	0.8590	0.0225
Expert 19	Doctor	From 10 - 20 years	(0.92; 0.052; 0.08)	0.9340	0.0245
Expert 20	Master	From 5 - 10 years	(0.76; 0.227; 0.24)	0.7665	0.0201
Expert 21	Doctor	Over 20 years	(0.96; 0.022; 0.04)	0.9690	0.0254
Expert 22	Master	From 5 - 10 years	(0.76; 0.227; 0.24)	0.7665	0.0201
Expert 23	Bachelor	From 5 - 10 years	(0.64; 0.423; 0.36)	0.6085	0.0160
Expert 24	Bachelor	From 10 - 20 years	(0.76; 0.227; 0.24)	0.7665	0.0201
Expert 25	Master	From 5 - 10 years	(0.76; 0.227; 0.24)	0.7665	0.0201
Expert 26	Bachelor	From 5 - 10 years	(0.64; 0.423; 0.36)	0.6085	0.0160
Expert 27	Doctor	From 10 - 20 years	(0.92; 0.052; 0.08)	0.9340	0.0245
Expert 28	Master	From 10 - 20 years	(0.84; 0.122; 0.16)	0.8590	0.0225
Expert 29	Master	From 10 - 20 years	(0.84; 0.122; 0.16)	0.8590	0.0225
Expert 30	Master	Over 20 years	(0.92; 0.052; 0.08)	0.9340	0.0245
Expert 31	Master	Over 20 years	(0.92; 0.052; 0.08)	0.9340	0.0245
Expert 32	Doctor	Over 20 years	(0.96; 0.022; 0.04)	0.9690	0.0254
Expert 33	Master	Over 20 years	(0.92; 0.052; 0.08)	0.9340	0.0245

Expert 34	Bachelor	From 5 - 10 years	(0.64; 0.423; 0.36)	0.6085	0.0160
Expert 35	Bachelor	From 10 - 20 years	(0.76; 0.227; 0.24)	0.7665	0.0201
Expert 36	Doctor	Over 20 years	(0.96; 0.022; 0.04)	0.9690	0.0254
Expert 37	Doctor	From 10 - 20 years	(0.92; 0.052; 0.08)	0.9340	0.0245
Expert 38	Doctor	From 10 - 20 years	(0.92; 0.052; 0.08)	0.9340	0.0245
Expert 39	Bachelor	From 10 - 20 years	(0.76; 0.227; 0.24)	0.7665	0.0201
Expert 40	Master	From 5 - 10 years	(0.76; 0.227; 0.24)	0.7665	0.0201
Expert 41	Master	From 10 - 20 years	(0.84; 0.122; 0.16)	0.8590	0.0225
Expert 42	Master	From 10 - 20 years	(0.84; 0.122; 0.16)	0.8590	0.0225
Expert 43	Doctor	Over 20 years	(0.96; 0.022; 0.04)	0.9690	0.0254
Expert 44	Master	From 10 - 20 years	(0.84; 0.122; 0.16)	0.8590	0.0225
Expert 45	Doctor	Over 20 years	(0.96; 0.022; 0.04)	0.9690	0.0254

5.2 NS Delphi Results

This study began by identifying a comprehensive list of 55 barriers through an extensive literature review, capturing a broad spectrum of challenges relevant to the field. To refine this list, experts were invited to assess the suitability and significance of these barriers using linguistic scales, which allow for nuanced evaluations based on expertise and subjective interpretation. Each barrier was then analyzed by comparing its normalized Crisp value to a predetermined threshold, a critical step in filtering out less impactful barriers. Through this rigorous comparison, five barriers, labeled explicitly as OB2 – Lack of knowledge on taxation, OB10 – Limited role of civil society organizations, OB28 – Lack of industry involvement in ethical and moral practice, OB41 – Intellectual Property (IP) Laws, and OB52 - Unfavorable organizational culture, did not meet the required criteria and were consequently eliminated from further consideration. This initial evaluation leaves a refined set of barriers, which will advance to Phase 2 for deeper analysis. The detailed outcomes of this assessment process, including the barriers retained and eliminated, are presented in Table 8, which summarizes the NS Delphi analysis results.

Table 8: The NS Delphi results

Barriers	Aggregate	Crips	Validate	Barriers	Aggregate	Crips	Validate
OB1	(0.6885; 0.2620; 0.3115)	0.0207	Accepted	**OB29**	(0.6892; 0.2606; 0.3108)	0.0208	Accepted
OB2	(0.2200; 0.8090; 0.7800)	0.0060	Rejected	**OB30**	(0.7067; 0.2420; 0.2933)	0.0213	Accepted
OB3	(0.6449; 0.3114; 0.3551)	0.0194	Accepted	**OB31**	(0.6816; 0.2667; 0.3184)	0.0206	Accepted
OB4	(0.6329; 0.3261; 0.3671)	0.0190	Accepted	**OB32**	(0.6193; 0.3397; 0.3807)	0.0186	Accepted
OB5	(0.6648; 0.2871; 0.3352)	0.0200	Accepted	**OB33**	(0.6713; 0.2801; 0.3287)	0.0202	Accepted
OB6	(0.6470; 0.3089; 0.3530)	0.0195	Accepted	**OB34**	(0.6769; 0.2717; 0.3231)	0.0204	Accepted
OB7	(0.6075; 0.3552; 0.3925)	0.0182	Accepted	**OB35**	(0.6485; 0.3067; 0.3515)	0.0195	Accepted
OB8	(0.6157; 0.3445; 0.3843)	0.0185	Accepted	**OB36**	(0.6592; 0.2965; 0.3408)	0.0198	Accepted
OB9	(0.6181; 0.3423; 0.3819)	0.0185	Accepted	**OB37**	(0.7006; 0.2469; 0.2994)	0.0211	Accepted
OB10	(0.1940; 0.8236; 0.8060)	0.0054	Rejected	**OB38**	(0.6115; 0.3475; 0.3885)	0.0184	Accepted
OB11	(0.6440; 0.3109; 0.3560)	0.0194	Accepted	**OB39**	(0.6075; 0.3541; 0.3925)	0.0182	Accepted
OB12	(0.7061; 0.2444; 0.2939)	0.0213	Accepted	**OB40**	(0.6199; 0.3358; 0.3801)	0.0187	Accepted

OB13	(0.6606; 0.2945; 0.3394)	0.0199	Accepted	**OB41**	(0.0767; 0.9435; 0.9233)	0.0019	Rejected
OB14	(0.6498; 0.3074; 0.3502)	0.0195	Accepted	**OB42**	(0.6503; 0.3023; 0.3497)	0.0196	Accepted
OB15	(0.6729; 0.2807; 0.3271)	0.0202	Accepted	**OB43**	(0.6491; 0.3066; 0.3509)	0.0195	Accepted
OB16	(0.6136; 0.3443; 0.3864)	0.0185	Accepted	**OB44**	(0.6302; 0.3303; 0.3698)	0.0189	Accepted
OB17	(0.6478; 0.3103; 0.3522)	0.0194	Accepted	**OB45**	(0.6746; 0.2779; 0.3254)	0.0203	Accepted
OB18	(0.6299; 0.3272; 0.3701)	0.0189	Accepted	**OB46**	(0.6111; 0.3484; 0.3889)	0.0184	Accepted
OB19	(0.6075; 0.3556; 0.3925)	0.0182	Accepted	**OB47**	(0.6335; 0.3259; 0.3665)	0.0190	Accepted
OB20	(0.6754; 0.2763; 0.3246)	0.0203	Accepted	**OB48**	(0.6271; 0.3345; 0.3729)	0.0188	Accepted
OB21	(0.6326; 0.3263; 0.3674)	0.0190	Accepted	**OB49**	(0.6469; 0.3086; 0.3531)	0.0195	Accepted
OB22	(0.6179; 0.3430; 0.3821)	0.0185	Accepted	**OB50**	(0.7073; 0.2415; 0.2927)	0.0213	Accepted
OB23	(0.6432; 0.3115; 0.3568)	0.0194	Accepted	**OB51**	(0.6275; 0.3323; 0.3725)	0.0188	Accepted
OB24	(0.6479; 0.3098; 0.3521)	0.0195	Accepted	**OB52**	(0.0830; 0.9388; 0.9170)	0.0021	Rejected
OB25	(0.6198; 0.3428; 0.3802)	0.0186	Accepted	**OB53**	(0.7362; 0.2133; 0.2638)	0.0221	Accepted
OB26	(0.6325; 0.3217; 0.3675)	0.0191	Accepted	**OB54**	(0.6363; 0.3175; 0.3637)	0.0192	Accepted
OB27	(0.6749; 0.2753; 0.3251)	0.0203	Accepted	**OB55**	(0.6894; 0.2612; 0.3106)	0.0208	Accepted
OB28	(0.2549; 0.7738; 0.7451)	0.0070	Rejected				
					Threshold	0.0182	

5.3 NS DEMATEL Results

Following the NS Delphi phase, fifty barriers were confirmed and integrated to assess their impact on the warehouse management platform within Vietnam's smart reverse logistics and the cause-and-effect relationships among these factors. Initially, to examine the mutual impact relationships of barriers in a pairwise setup, the impact scores ranged from "No influence" to "Absolutely influence," evaluating each barrier's influence relative to others, as illustrated in Table 6.

According to research experts' responses, the direct-relation matrix was initially organized using linguistic terms. This initial matrix was converted into the NS direct-relation matrix by translating the linguistic terms into corresponding Neutrosophic fuzzy algebraic values. Expert weights were integrated with each evaluation to ensure accuracy during this transformation. Utilizing forty-five expert responses arranged in the NS DEMATEL framework, an aggregated direct-influence matrix

$\otimes V$ matrix was created using Eq. (17), with results presented in Table 9. Next, elements within the Neutrosophic Sets were converted to crisp scores through Eq. (13). Subsequently, the total direct-influence matrix was normalized using Eq. (19) and (20), resulting in the normalized matrix $\otimes U$. The total direct-influence matrix was then formed using Eq. (21) and (22), with the total relation matrix provided in Table 10. Finally, Eq. (23)–(27) were applied to calculate the values of $\otimes r_i,\ \otimes c_i,\ \otimes r_i + \otimes c_i,\ \otimes r_i - \otimes c_i$, along with the corresponding risk weights, is shown in Table 11.

Table 9: Aggregated-direct-influence-matrix.

Barriers	**OB1**	**OB3**	**OB4**	**OB5**	**OB6**	⋯	**OB50**	**OB51**	**OB53**	**OB54**	**OB55**

OB1	(0,0,0)	(.656,.298,.344)	(.475,.481,.525)	(.646,.311,.354)	(.585,.379,.415)	⋯	(.655,.299,.345)	(.72,.227,.28)	(.532,.419,.468)	(.652,.3,.348)	(.587,.36,.413)
OB3	(.251,.753,.749)	(0,0,0)	(.211,.827,.789)	(.216,.81,.784)	(.22,.819,.78)	⋯	(.224,.824,.776)	(.196,.843,.804)	(.175,.85,.825)	(.164,.871,.836)	(.191,.844,.809)
OB4	(.277,.724,.723)	(.241,.78,.759)	(0,0,0)	(.265,.755,.735)	(.286,.718,.714)	⋯	(.223,.82,.777)	(.241,.768,.759)	(.139,.894,.861)	(.207,.84,.793)	(.163,.858,.837)
OB5	(.672,.28,.328)	(.655,.299,.345)	(.614,.347,.386)	(0,0,0)	(.63,.328,.37)		(.733,.216,.267)	(.643,.323,.357)	(.676,.277,.324)	(.694,.253,.306)	(.747,.201,.253)
OB6	(.665,.291,.335)	(.615,.341,.385)	(.648,.308,.352)	(.641,.316,.359)	(0,0,0)		(.689,.264,.311)	(.657,.298,.343)	(.63,.333,.37)	(.712,.242,.288)	(.718,.23,.282)
⋮	⋮	⋮	⋮			⋱	⋮	⋮	⋮		
OB50	(.655,.297,.345)	(.62,.341,.38)	(.662,.288,.338)	(.577,.385,.423)	(.642,.318,.358)		(0,0,0)	(.619,.344,.381)	(.693,.257,.307)	(.595,.369,.405)	(.55,.425,.45)
OB51	(.438,.531,.562)	(.435,.54,.565)	(.533,.434,.467)	(.453,.536,.547)	(.469,.517,.531)		(.566,.395,.434)	(0,0,0)	(.482,.491,.518)	(.422,.56,.578)	(.507,.459,.493)
OB53	(.638,.316,.362)	(.679,.269,.321)	(.568,.396,.432)	(.616,.348,.384)	(.649,.308,.351)	⋯	(.657,.297,.343)	(.658,.291,.342)	(0,0,0)	(.659,.293,.341)	(.679,.271,.321)
OB54	(.613,.344,.387)	(.617,.343,.383)	(.509,.459,.491)	(.586,.381,.414)	(.595,.379,.405)	⋯	(.631,.332,.369)	(.603,.37,.397)	(.697,.25,.303)	(0,0,0)	(.719,.227,.281)
OB55	(.645,.31,.355)	(.648,.304,.352)	(.607,.36,.393)	(.607,.359,.393)	(.679,.272,.321)	⋯	(.724,.223,.276)	(.724,.227,.276)	(.664,.288,.336)	(.757,.191,.243)	(0,0,0)

Table 10: Direct-influence-matrix after defuzzification of NS

Barriers	OB1	OB3	OB4	OB5	OB6	⋯	OB50	OB51	OB53	OB54	OB55
OB1	0.75	0.679	0.497	0.668	0.603	⋯	0.678	0.746	0.557	0.676	0.614
OB3	0.249	0.75	0.192	0.203	0.201	⋯	0.2	0.176	0.162	0.147	0.173
OB4	0.276	0.23	0.75	0.255	0.284	⋯	0.202	0.237	0.122	0.184	0.152
OB5	0.696	0.678	0.634	0.75	0.651		0.758	0.66	0.7	0.72	0.773
OB6	0.687	0.637	0.67	0.662	0.75		0.712	0.679	0.648	0.735	0.744
⋮	⋮	⋮	⋮			⋱	⋮	⋮	⋮		
OB50	0.679	0.639	0.687	0.596	0.662		0.75	0.637	0.718	0.613	0.563
OB51	0.454	0.448	0.55	0.459	0.476		0.586	0.75	0.495	0.431	0.524
OB53	0.661	0.705	0.586	0.634	0.67	⋯	0.68	0.683	0.75	0.683	0.704
OB54	0.635	0.637	0.525	0.603	0.608	⋯	0.649	0.617	0.723	0.75	0.746
OB55	0.667	0.672	0.623	0.624	0.703	⋯	0.75	0.748	0.688	0.783	0.75

Considering the top five barriers with the highest impact weights presented in **Table 11**, these findings identify Poor strategic planning (OB55) within the Management - Organization domain as the most critical barrier to implementing warehouse management platforms in Vietnam's smart reverse logistics. Strategic planning for reverse logistics is crucial for companies to cut costs, minimize waste, and comply with environmental standards. Key actions include monitoring returned product flows, offering clear return instructions to customers, and coordinating with retailers and distributors to ensure proper processing of returned items. This finding aligns with the research by Sonar et al. [60], emphasizing the substantial impact of strategic planning in the warehouse management process within reverse supply chains. Without well-structured strategic planning, companies may face challenges such as inconsistent processes, inefficient use of resources, and a lack of adaptability to new technologies. In the case of smart reverse logistics, a company might attempt to deploy an automated warehouse system but fail to effectively integrate it with its existing supply chain network due to inadequate planning. This could result in misaligned goals, causing delays, higher costs, and resource wastage, ultimately lowering the platform's efficiency. Strategic planning also ensures that investments in technology and training are prioritized correctly, enabling a smoother transition to innovative solutions. In this context, weak planning might prevent a company from recognizing critical issues such as demand forecasting for returns, capacity management, and regulatory compliance, all essential to smart reverse logistics. Therefore, "poor strategic planning" directly impacts the ability of organizations to implement and leverage warehouse management platforms effectively.

Insufficient expertise in reverse logistics (OB50), categorized under Management - Organization, ranks as the second most critical barrier. The significance of this barrier can be understood through the lens of operational efficiency and the effective implementation of reverse logistics practices. Insufficient expertise in this area can hinder a company's ability to manage returns, remanufacturing, and recycling processes effectively. Organizations may face challenges in designing efficient reverse logistics systems without knowledgeable personnel, leading to increased operational costs and reduced customer satisfaction. For instance, according to research by Govindan et al.[61], a lack of skilled professionals can limit the capacity to optimize logistics processes, negatively impacting sustainability and overall performance. This aligns with the findings in our study, highlighting the need for organizations to invest in training and development to build expertise in reverse logistics. By addressing this barrier, companies can enhance their capabilities, reduce inefficiencies, and better comply with environmental regulations, ultimately leading to improved competitiveness in the market. Research by Sonar et al. [60] has emphasized that without adequate expertise, companies struggle to handle the complexity of reverse logistics, from returns processing to resource allocation effectively.

The third ranking is Insufficiently suitable indicators and metrics for measuring performance (OB46). The

lack of appropriate indicators and metrics to assess performance can severely hinder the effectiveness of reverse logistics systems. Without proper metrics, organizations struggle to evaluate the efficiency of their processes, leading to poor decision-making and resource allocation. Effective performance measurement is essential for identifying areas that require improvement and assessing the impact of implemented strategies. As per previous research in the field, performance metrics are critical in supply chain management. Dominic and Orji [62] emphasize the importance of performance measurement in achieving competitive advantage and improving operational efficiency. In reverse logistics, inadequate performance metrics can lead to inefficiencies in product returns, waste management, and overall customer satisfaction. Therefore, addressing this barrier by developing suitable indicators and metrics is vital for enhancing performance and ensuring the success of reverse logistics initiatives.

Lack of specialized staff (OB53) was identified as the fourth most critical barrier. This factor reflects the challenge of finding personnel with specific expertise in reverse logistics - a field that requires warehouse management skills and specialized knowledge in handling returns, quality checks, recycling, and disposal. For example, without skilled staff, a company might experience inefficient handling of returned goods, increasing the time and cost associated with reverse logistics processes. In Vietnam, according to research by Tran and Ngo [63], a lack of knowledge in management, development, and staff specialization leads to inefficiencies in execution.

Ranked fifth among the top barriers with the highest influence is OB44 - Company policies against reverse logistics. In the study by Bouzon et al. [64], experts noted that companies often hesitate to adopt reverse logistics due to concerns about cannibalizing sales of their new products. They fear recovering and reselling used items could diminish the demand for their latest offerings. For example, if a company promotes the resale of older products, consumers might choose the cheaper used option instead of buying the new version, leading to reduced sales and profitability.

The five factors can be grouped into two clusters according to the values of $\otimes r_i - \otimes c_i$ in **Figure 2**. If the values of $\otimes r_i - \otimes c_i$ are positive, these factors belong to the cause group. These cause factors serve as primary influences on the other factors, with the highest values of $\otimes r_i - \otimes c_i$ having a direct impact on them. Fig 2 reveals that barriers OB55, OB50, OB46, OB53, OB44, OB31, OB27, OB39, OB19, OB54, OB36, OB30, OB33, OB43, OB15, OB18, OB6, OB45, OB17, OB37, OB21, OB42, OB8, OB29, OB12, OB13, OB20, OB34, OB1, and OB5 exhibit positive causality degree values $\otimes r_i - \otimes c_i$, placing them within the cause group. Among these, the barrier Lack of Cross-Ministry Coordination (OB7) ranks highest in causality degree $\otimes r_i - \otimes c_i$ with a value of 1.0596. In second place is the barrier Reluctance to modify existing models to integrate reverse logistics (OB21) with a value of 1.0270. This positioning indicates that they significantly influence other factors rather than being influenced by them and are direct contributors to reduced efficiency in reverse supply chain warehouse management. Fig. 2 and the $\otimes r_i - \otimes c_i$ data in **Table 13** illustrate the effect of group elements criteria with less influential characteristics. Based on the $\otimes r_i - \otimes c_i$ values, these barrier elements are classified into the effect group: OB49, OB51, OB48, OB40, OB25, OB26, OB32, OB47, OB23, OB22, OB38, OB24, OB35, OB14, OB4, OB16, OB11, OB9, and OB3. Criteria with lower $\otimes r_i - \otimes c_i$ values indicate lower influence and require less attention than the cause group criteria due to their relatively minor impact. Many effect elements will be removed by reducing the cause elements, benefiting the effect group without special consideration. This study identified Tax-related financial strain (OB16) and Overemphasis on Economic Growth (OB9) as the most susceptible factors.

Table 11: The NS DEMATEL results

Barriers	$\otimes r_i$	$\otimes c_i$	$\otimes r_i$	σ_i	Rank	$\otimes r_i - \otimes c_i$	Relations
OB1	5.1668	4.5421	9.7089	0.0210	30	0.6247	**Cause**
OB3	1.7197	4.4531	6.1727	0.0133	50	-2.7334	Effect
OB4	1.7730	4.5482	6.3212	0.0137	46	-2.7752	Effect
OB5	5.1191	4.5255	9.6446	0.0209	31	0.5936	**Cause**
OB6	5.4492	4.5293	9.9785	0.0216	21	0.9199	**Cause**
OB7	5.4553	4.3379	9.7932	0.0212	28	1.1174	**Cause**
OB8	5.4876	4.4575	9.9452	0.0215	23	1.0301	**Cause**
OB9	1.7914	4.4514	6.2428	0.0135	49	-2.6600	Effect

OB11	1.7064	4.5820	6.2884	0.0136	48	-2.8755	Effect
OB12	5.4169	4.4662	9.8831	0.0214	25	0.9507	**Cause**
OB13	5.3544	4.5190	9.8734	0.0214	26	0.8354	**Cause**
OB14	1.9825	4.7067	6.6893	0.0145	45	-2.7242	Effect
OB15	5.3650	4.7050	10.0700	0.0218	15	0.6600	**Cause**
OB16	1.8045	4.4889	6.2935	0.0136	47	-2.6844	Effect
OB17	5.4498	4.6030	10.0528	0.0217	17	0.8469	**Cause**
OB18	5.5123	4.5743	10.0866	0.0218	12	0.9380	**Cause**
OB19	5.5454	4.6277	10.1731	0.0220	8	0.9178	**Cause**
OB20	5.2904	4.5228	9.8132	0.0212	27	0.7676	**Cause**
OB21	5.5394	4.4614	10.0008	0.0216	20	1.0780	**Cause**
OB22	4.1489	4.5387	8.6877	0.0188	43	-0.3898	Effect
OB23	4.0851	4.6646	8.7498	0.0189	40	-0.5795	Effect
OB24	4.0440	4.6481	8.6920	0.0188	42	-0.6041	Effect
OB25	4.1030	4.7668	8.8698	0.0192	37	-0.6638	Effect
OB26	4.1907	4.7301	8.9208	0.0193	35	-0.5394	Effect
OB27	5.4784	4.6370	10.1154	0.0219	11	0.8414	**Cause**
OB29	5.1537	4.7772	9.9309	0.0215	24	0.3766	**Cause**
OB30	5.5483	4.6083	10.1566	0.0220	10	0.9399	**Cause**
OB31	5.5078	4.5713	10.0791	0.0218	14	0.9365	**Cause**
OB32	4.2203	4.5510	8.7713	0.0190	39	-0.3307	Effect
OB33	5.4949	4.7635	10.2584	0.0222	6	0.7314	**Cause**
OB34	5.2207	4.5377	9.7584	0.0211	29	0.6831	**Cause**
OB35	2.6029	4.5354	7.1383	0.0154	44	-1.9324	Effect
OB36	5.3307	4.7487	10.0794	0.0218	13	0.5820	**Cause**
OB37	5.2522	4.7605	10.0126	0.0217	19	0.4917	**Cause**
OB38	4.1027	4.7035	8.8062	0.0190	38	-0.6008	Effect
OB39	5.4946	4.7456	10.2402	0.0221	7	0.7490	**Cause**
OB40	4.2375	4.7114	8.9489	0.0194	33	-0.4739	Effect
OB42	5.3083	4.7530	10.0612	0.0218	16	0.5553	**Cause**
OB43	5.4380	4.6142	10.0522	0.0217	18	0.8238	**Cause**
OB44	5.4900	4.7756	10.2656	0.0222	5	0.7143	**Cause**
OB45	5.2714	4.6820	9.9534	0.0215	22	0.5893	**Cause**
OB46	5.5743	4.6965	10.2708	0.0222	3	0.8779	**Cause**
OB47	4.0890	4.6061	8.6951	0.0188	41	-0.5171	Effect
OB48	4.2757	4.6433	8.9190	0.0193	36	-0.3676	Effect
OB49	4.1947	4.7778	8.9724	0.0194	32	-0.5831	Effect
OB50	5.4927	4.7848	10.2775	0.0222	2	0.7080	**Cause**

OB51	4.2381	4.7064	8.9445	0.0193	34	-0.4683	Effect
OB53	5.5853	4.6853	10.2706	0.0222	4	0.9001	**Cause**
OB54	5.4751	4.6949	10.1700	0.0220	9	0.7802	**Cause**
OB55	5.6482	4.7056	10.3538	0.0224	1	0.9426	**Cause**

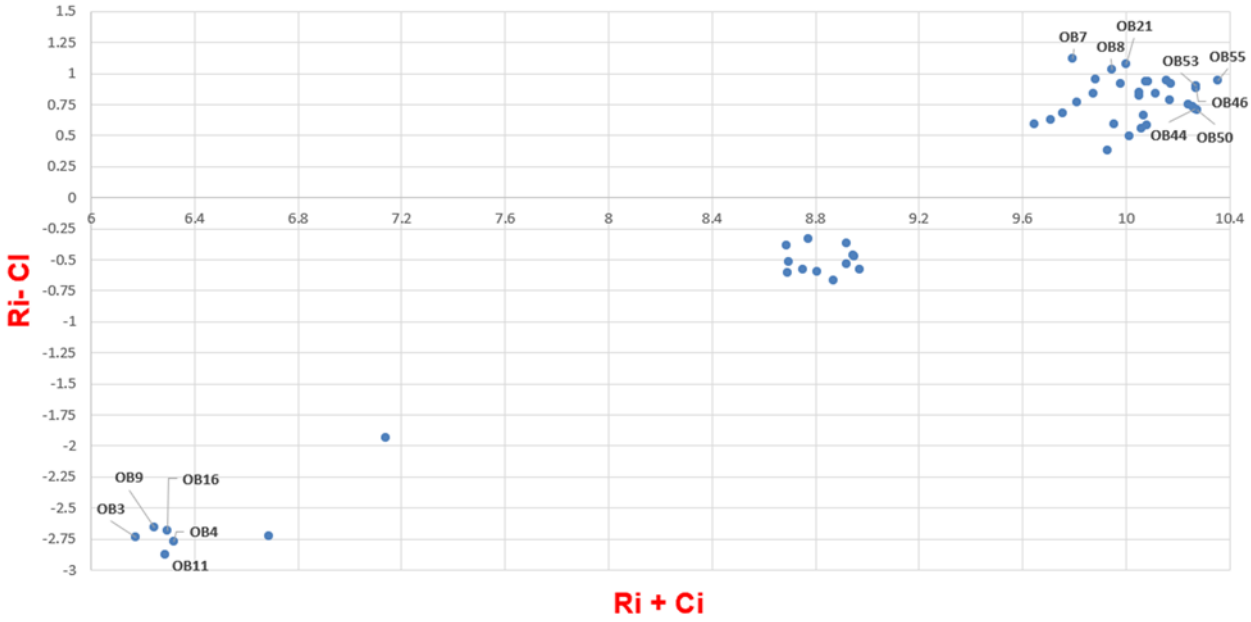

Figure 2: Cause-effect diagram

5.4 NS COCOSO Results

To evaluate the effectiveness of the twelve strategies to reduce and manage barriers, experts will utilize the scale detailed in Table 7. Table 12 presents the aggregated, standardized assessment results from 45 experts derived using Eq. (28).

Table 12: Aggregated assessment matrix

	OB1	**OB3**	**OB4**	**OB5**	**OB6**	⋯	**OB50**	**OB51**	**OB53**	**OB54**	**OB55**
S1	(.514,. 462, .486)	(.538,. 438, .462)	(.536,. 434, .464)	(.48,.5, .52)	(.503,. 479, .497)	⋯	(.62,.3 34, .38)	(.498,. 472, .502)	(.422,. 576, .578)	(.558,. 4, .442)	(.494,. 484, .506)
S2	(.569,. 397, .431)	(.445,. 553, .555)	(.459,. 535, .541)	(.49,.5 04, .51)	(.465,. 518, .535)	⋯	(.512,. 456, .488)	(.502,. 471, .498)	(.692,. 254, .308)	(.728,. 219, .272)	(.543,. 422, .457)
S3	(.355,. 659, .645)	(.311,. 704, .689)	(.299,. 717, .701)	(.242,. 783, .758)	(.291,. 749, .709)	⋯	(.359,. 666, .641)	(.345,. 66, .655)	(.318,. 703, .682)	(.324,. 684, .676)	(.31,.6 91, .69)

S4	(.415,.58,.585)	(.461,.531,.539)	(.414,.586,.586)	(.405,.6,.595)	(.42,.576,.58)	⋯	(.482,.51,.518)	(.474,.517,.526)	(.478,.51,.522)	(.495,.489,.505)	(.691,.258,.309)
S5	(.282,.746,.718)	(.327,.69,.673)	(.323,.685,.677)	(.263,.774,.737)	(.267,.766,.733)	⋯	(.302,.728,.698)	(.307,.726,.693)	(.311,.718,.689)	(.36,.673,.64)	(.388,.618,.612)
S6	(.37,.645,.63)	(.308,.72,.692)	(.407,.587,.593)	(.273,.752,.727)	(.383,.644,.617)	⋯	(.455,.538,.545)	(.37,.659,.63)	(.41,.578,.59)	(.375,.636,.625)	(.285,.737,.715)
S7	(.698,.252,.302)	(.599,.359,.401)	(.586,.369,.414)	(.7,.251,.3)	(.685,.268,.315)	⋯	(.443,.545,.557)	(.474,.524,.526)	(.506,.47,.494)	(.419,.575,.581)	(.529,.444,.471)
S8	(.718,.232,.282)	(.368,.643,.632)	(.339,.667,.661)	(.346,.682,.654)	(.441,.553,.559)	⋯	(.378,.643,.622)	(.343,.684,.657)	(.309,.724,.691)	(.384,.6,.616)	(.355,.651,.645)
S9	(.365,.63,.635)	(.342,.648,.658)	(.349,.648,.651)	(.356,.637,.644)	(.349,.645,.651)	⋯	(.369,.635,.631)	(.347,.639,.653)	(.355,.653,.645)	(.415,.578,.585)	(.418,.573,.582)
S10	(.309,.684,.691)	(.339,.671,.661)	(.312,.695,.688)	(.366,.629,.634)	(.389,.625,.611)	⋯	(.376,.624,.624)	(.411,.584,.589)	(.385,.633,.615)	(.413,.58,.587)	(.489,.478,.511)
S11	(.446,.541,.554)	(.403,.594,.597)	(.445,.556,.555)	(.384,.614,.616)	(.368,.636,.632)	⋯	(.407,.586,.593)	(.506,.466,.494)	(.459,.527,.541)	(.411,.596,.589)	(.5,.47,.5)
S12	(.223,.803,.777)	(.279,.759,.721)	(.198,.829,.802)	(.32,.708,.68)	(.398,.622,.602)	⋯	(.338,.687,.662)	(.454,.54,.546)	(.398,.627,.602)	(.363,.655,.637)	(.433,.569,.567)

Table 13 displays data processed through Equations (29)-(36) to calculate the values of S_i, P_i and, $k_{iI}, k_{iII}, k_{iIII}$ along with the ranking order of the strategies.

Table 13: The COCOSO results

STT	S_i	P_i	$S_i + P_i$	k_{iI}	k_{iII}	k_{iIII}	k_i	Rank
S1	0.7716	49.708	50.48	0.093	6.9113	1	3.531	1
S2	0.7367	49.6611	50.398	0.0928	6.6704	0.9984	3.4391	3
S3	0.1458	36.401	36.547	0.0673	2.185	0.724	1.4661	11
S4	0.4936	49.231	49.725	0.0916	4.9884	0.985	2.788	5
S5	0.1475	30.719	30.866	0.0568	2.0119	0.6115	1.3054	12
S6	0.2351	43.528	43.763	0.0806	3.0295	0.8669	1.9217	10
S7	0.7387	49.657	50.395	0.0928	6.684	0.9983	3.4441	2
S8	0.3127	45.62	45.932	0.0846	3.63	0.9099	2.1953	7

S9	0.2537	45.566	45.819	0.0844	3.2235	0.9077	2.0326	8
S10	0.2716	43.72	43.991	0.081	3.2863	0.8715	2.0274	9
S11	0.529	49.273	49.803	0.0917	5.2331	0.9866	2.8833	4
S12	0.3481	44.885	45.233	0.0833	3.8494	0.8961	2.2695	6

Based on the NS COCOSO method, the strategies have been ranked in order to mitigate the impact of barriers on WMS in SRL in Vietnam, including Investment in Infrastructure, Role of Government, Training and Educational Programs, Developing a Single Window System for Licenses and Financial Support, Current State Assessment and Vision Formulation, Create Electronic Collaboration Among Supply Chain Partners, Raise Awareness for Customers, Outsourcing, Selective Approach to Reverse Logistics, Apply Blockchain Technology, Improved Data Protection, and Adopt Sustainability Practices.

Based on the NS COCOSO method results, S1 - Investment in Infrastructure is rated the most effective strategy, ranking highest (1st place) in mitigating barriers to WMS in reverse logistics in Vietnam. According to the data, this strategy is particularly effective in addressing barriers related to the lack of available digital infrastructure and technology (OB33). Investing in infrastructure not only enhances companies' resilience to disruptions but also promotes more efficient implementation of reverse logistics processes. This, in turn, contributes to improving expertise in reverse logistics (OB50) and boosting enterprises' competitive capabilities in the market. High-quality infrastructure is critical in supporting logistics activities, especially in deeper economic integration and increasing digitalization.

Based on the NS COCOSO method, S2 - Training and Educational Programs ranks third among the key strategies to mitigate barriers to WMS in reverse logistics in Vietnam. This strategy is particularly effective in addressing barriers related to the Lack of specialized staff (OB53), Limited training programs (OB54), Technical skill shortages (OB30), and Lack of awareness about reverse logistics in the community (OB29). Investing in training and educational programs enhances employees' skills and enables businesses to adapt to evolving logistics and supply chain management requirements quickly. According to the Vietnam Logistics Report 2021 published by the Ministry of Industry and Trade, developing logistics human resources is a key factor in improving logistics efficiency and warehouse management. The report highlights that a shortage of technical skills and limited awareness of reverse logistics is hindering the sector's growth. Therefore, investing in training and educational programs will help enhance staff capabilities, improve the efficiency of warehouse management systems, and boost the competitive edge of businesses (Bo Cong Thuong 2021).

Strategy S7 - The role of the Government is ranked second among Vietnam's most crucial strategies to mitigate barriers to WMS in reverse logistics. This strategy addresses issues such as the lack of coordination between ministries (OB7) and integration with international environmental commitments (OB8). Currently, logistics policies in Vietnam face difficulties harmonizing across different ministries, leading to fragmentation in regulations and enforcement. The inconsistency in legal regulations among various ministries not only hampers businesses in complying with regulations but also increases logistics costs and operational time. Particularly in international integration, the lack of alignment with environmental commitments could result in losing competitive advantages when exporting to demanding markets like the EU and the United States.

Finally, S4 - Current State Assessment and Vision Formulation considerably addresses critical barriers such as Poor Strategic Planning (OB55) and Lack of Funds for Product Return Monitoring Systems (OB19). Conducting a thorough assessment of the current state enables businesses to develop strategic plans and allocate resources more efficiently, optimizing costs and enhancing operational effectiveness. Assessing the current situation helps businesses identify gaps in their logistics processes and implement strategic improvements. This proactive approach optimizes costs and enhances the efficiency of WMSs in reverse logistics, effectively addressing challenges related to strategy and budgeting. By setting a clear vision, companies can align their objectives with industry best practices, making their supply chains more agile and responsive. This strategy not only improves return on investment but also ensures long-term sustainability in a competitive market environment.

5.5 Sensitive Analysis

The sensitivity analysis assesses the stability of the strategy rankings generated by the NS COCOSO method by varying the γ parameter between 0.1 and 0.9. This approach aims to mitigate potential biases in human judgment that could affect decision-making outcomes. **Table 14** presents the computed k_i values for each strategy (S1 to S12) under different γ values.

Table 14: k_i under different γ values

γ	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
S1	3.531	3.531	3.531	3.531	3.531	3.531	3.531	3.531	3.531
S2	3.4394	3.4394	3.4393	3.4392	3.4391	3.4389	3.4385	3.4379	3.4361
S3	1.4701	1.4695	1.4687	1.4676	1.4661	1.4638	1.4602	1.4531	1.4336
S4	2.7908	2.7904	2.7898	2.7891	2.788	2.7865	2.7839	2.779	2.7656
S5	1.3086	1.3081	1.3075	1.3066	1.3054	1.3036	1.3007	1.2951	1.2796
S6	1.926	1.9253	1.9245	1.9233	1.9217	1.9193	1.9153	1.9078	1.8869
S7	3.4445	3.4444	3.4443	3.4442	3.4441	3.4439	3.4436	3.443	3.4414
S8	2.1993	2.1986	2.1978	2.1968	2.1953	2.1931	2.1895	2.1826	2.1635
S9	2.0371	2.0363	2.0354	2.0342	2.0326	2.0301	2.026	2.0182	1.9967
S10	2.0315	2.0308	2.03	2.0289	2.0274	2.0252	2.0215	2.0144	1.9949
S11	2.8857	2.8853	2.8848	2.8842	2.8833	2.8819	2.8797	2.8754	2.8636
S12	2.273	2.2725	2.2718	2.2708	2.2695	2.2676	2.2644	2.2582	2.2412

Figure 3 provides a visualization of the stability of the strategy rankings across varying γ values, ranging from 0.1 to 0.9, in the NS COCOSO method. Despite these adjustments, the rankings remain consistent, demonstrating the robustness of the proposed model in handling parameter variations. This consistency highlights the framework's reliability for prioritizing warehouse management strategies for reverse logistics. A deeper examination of the results reveals several key observations. Strategy S1 consistently maintains the top rank across all γ values, underscoring its effectiveness as the most impactful priority for mitigating barriers in warehouse management. This finding highlights the critical role of investment in infrastructure (S1) as a foundational strategy. Similarly, strategies S7 (The Role of Government) and S2 (Training and Educational Programs) exhibit high rankings across all scenarios, reaffirming their importance in addressing systemic challenges such as policy alignment, regulatory support, and workforce development.

Conversely, lower-ranking strategies, such as S3 (Outsourcing) and S5 (Adopting Advanced Technologies), maintain relatively stable positions toward the bottom of the rankings. This stability indicates their lesser direct impact on overcoming immediate barriers, though they may still hold long-term relevance under specific operational contexts. The consistent positioning of all strategies, regardless of γ adjustments, reinforces the method's ability to produce reliable outcomes.

The sensitivity analysis results provide robust evidence for the adaptability and resilience of the NS COCOSO method in prioritizing strategies under varying parameter settings. The stability observed across all γ values mitigates concerns about human judgment biases and demonstrates that the decision-making framework can maintain reliable rankings even under fluctuating conditions. This is particularly critical in volatile environments, such as reverse logistics in developing economies like Vietnam, where parameters and priorities may shift due to operational, regulatory, or market changes. From a practical perspective, the stability of the rankings gives stakeholders confidence that the recommendations are consistent and actionable. For instance, infrastructure investments (S1) emerge as the most critical strategy regardless of parameter adjustments, ensuring its prioritization aligns with enhancing operational efficiency.

Furthermore, the high rankings of strategies S7 and S2 emphasize the need for collaboration between private and public sectors and the importance of capacity-building initiatives. The sensitivity analysis also opens avenues for further exploration. While the stable rankings demonstrate the robustness of the model, future studies could incorporate additional dynamic factors, such as real-time data inputs, to further refine the adaptability of the framework. Moreover, exploring how these rankings change under varying economic conditions or across different industry sectors could provide a more nuanced understanding of strategy prioritization in diverse contexts.

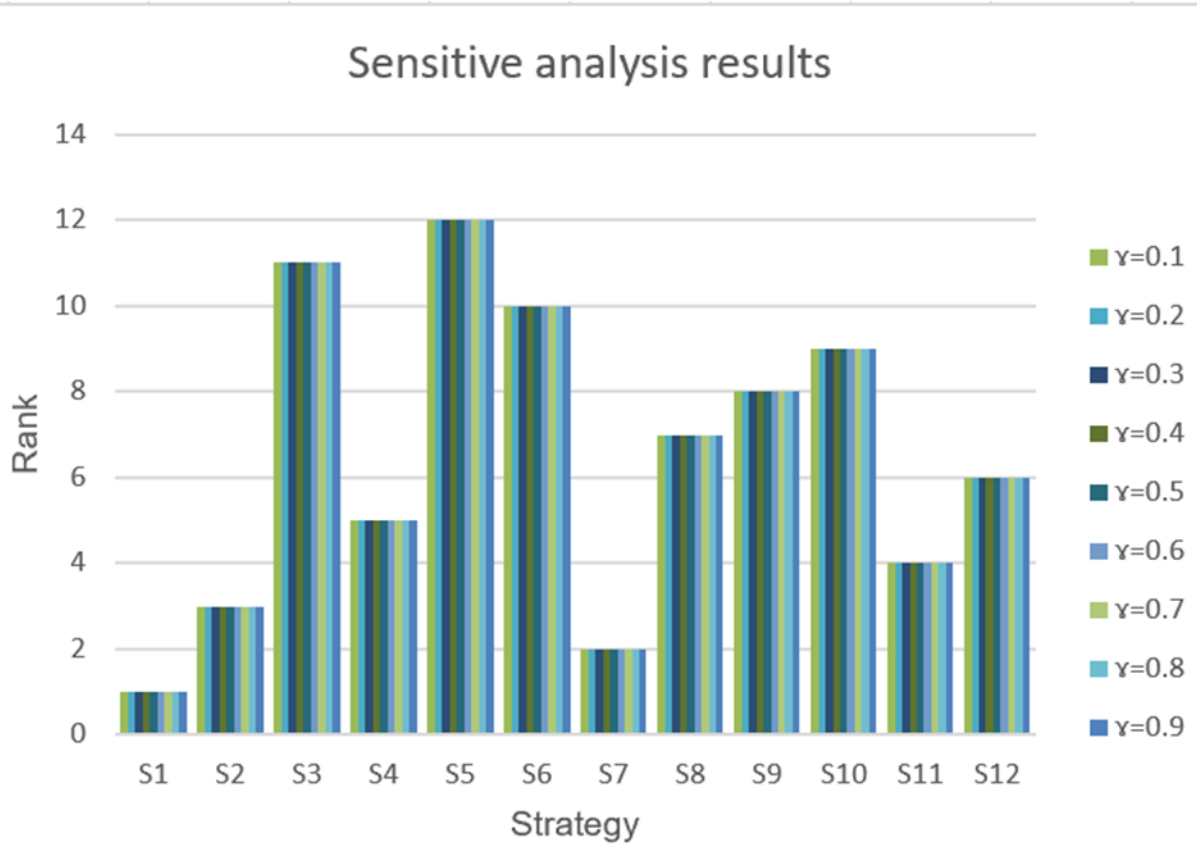

Figure 3: Sensitive analysis results

The sensitivity analysis validates the robustness of the NS COCOSO method, demonstrating its ability to produce consistent and reliable outcomes under varying parameter settings. This resilience enhances the credibility of the strategy rankings and highlights the framework's adaptability to complex and dynamic logistics ecosystems. For stakeholders, this ensures that the recommendations remain practical and effective, providing a solid foundation for decision-making in optimizing warehouse management for reverse logistics. Such reliability is essential for aligning theoretical frameworks with real-world applications, particularly in environments characterized by uncertainty and rapid change.

5.6 Comparative analysis

Figure 4 presents a comparison of rankings for 12 different strategies (S1 through S12) using three decision-making methods: WASPAS, EDAS, and COCOSO. The rankings are displayed in a bar chart, where the y-axis represents rank values (0 to 14), with lower numbers indicating better performance.

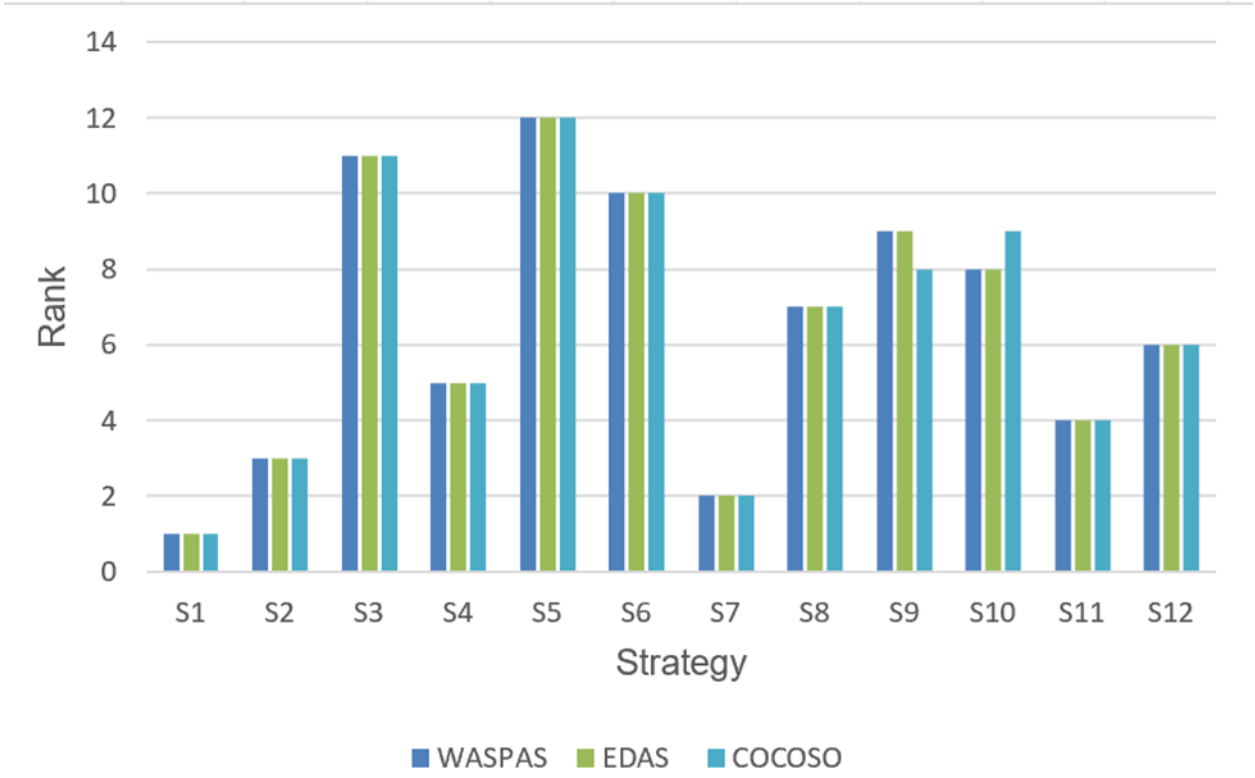

Figure 4: Comparative Analysis

The results reveal a strong alignment in ranking patterns across the three methods, indicating consistency and reliability in the evaluation process. Strategy S1 consistently achieves the top rank (approximately 1) across all methods, followed closely by S2, further reinforcing the robustness of these strategies. In contrast, S5 ranks the lowest (approximately 12) in all methods, indicating its limited effectiveness compared to other strategies. For middle-ranking strategies, such as S8-S12, minor variations in rankings are observed across the methods. For example, S9 exhibits consistent rankings near position 9, with COCOSO showing a slightly better performance (rank ~8) than WASPAS and EDAS. Similarly, for S10, COCOSO ranks slightly lower (rank ~9) than WASPAS and EDAS, which show similar results (~8).

Overall, the close alignment of rankings among WASPAS, EDAS, and COCOSO demonstrates a high degree of agreement in their assessment methodologies, validating the reliability of the evaluation framework. However, this study's neutrosophic decision support framework offers unique advantages by incorporating uncertainty, indeterminacy, and expert judgments more effectively than conventional methods. This capability provides additional depth and nuance in decision-making, ensuring more precise and actionable insights for strategy prioritization. By combining consistency across methods with the advanced capabilities of neutrosophic techniques, this study provides a comprehensive and reliable framework for strategy evaluation and decision support in reverse logistics.

6 Conclusion, Implications, Limitations, and Future Work

6.1 Conclusions

This study identifies and prioritizes the critical barriers to WMP implementation in Vietnam's SRL using the Neutrosophic DEMATEL and NS COCOSO methods. Key barriers, including poor strategic planning (OB55), insufficient expertise (OB50), and lack of suitable performance metrics (OB46), were identified as having the most significant impact on SRL efficiency. To address these barriers, the study highlights the top-ranked strategies: Investment in Infrastructure (S1), The Role of Government (S7), and Training and Educational Programs (S2). The findings emphasize the importance of comprehensive strategic planning, workforce development, and policy alignment for successfully adopting WMPs. Robust sensitivity analyses and comparative assessments confirm the reliability of these results, providing actionable and context-specific recommendations for stakeholders. This research contributes to both theory and practice by offering a structured framework for overcoming barriers and enhancing SRL systems, paving the way for more efficient and sustainable logistics operations in Vietnam.

6.2 Implications

6.2.1 Theoretical Implications

This study contributes significantly to reverse logistics and warehouse management literature by integrating NS with MCDM techniques. The research advances theoretical understanding by uncovering causal relationships among barriers, allowing scholars to examine these challenges' systemic and interdependent nature. Integrating NS with the NS COCOSO method offers a novel framework for addressing logistics systems' uncertainty, indeterminacy, and complexity. This study demonstrates the robustness of these methodologies, providing a foundation for future research in domains such as sustainability, green logistics, and supply chain resilience.

Additionally, the study bridges gaps in understanding how barriers in SRL differ between developing economies, like Vietnam, and developed markets. Offering a contextual perspective paves the way for cross-country comparative studies. The study also underscores the importance of aligning academic research with real-world challenges by providing strategy recommendations that are both theoretically robust and practically relevant. Future research could extend this work by incorporating blockchain, artificial intelligence (AI), Internet of Things (IoT), and machine learning to refine the framework and enhance its adaptability to dynamic logistics ecosystems.

6.2.2 Managerial Implications

For practitioners, this study provides actionable insights for optimizing WMP adoption within SRL systems. Managers should prioritize addressing poor strategic planning (OB55) by developing comprehensive strategies that align reverse logistics goals with resources and technological investments. This includes allocating budgets for infrastructure, advanced technologies, and workforce development.

Investment in human capital is another critical area. Managers should implement robust training and educational programs (S2) to enhance staff expertise in reverse logistics and build cross-functional teams capable of bridging knowledge gaps. Specialized training initiatives can significantly improve operational efficiency and foster innovation.

Policy alignment and collaboration with government entities (S7) are essential to overcome regulatory fragmentation and ensure compliance with environmental standards. Managers should actively engage policymakers to create supportive frameworks for smoother WMP implementation. Lastly, performance measurement systems must be adapted to track efficiency, optimize resource allocation, and evaluate the effectiveness of reverse logistics strategies. Developing metrics aligned with organizational objectives will enable managers to drive continuous improvement and align logistics operations with long-term business goals.

6.3 Limitations and Future Work

This research focuses on selecting WMPs for general warehousing environments in Vietnam, which may limit its applicability to warehouses with specialized operational needs, such as cold storage facilities or warehouses handling hazardous materials. These specialized contexts often have unique criteria and operational priorities that could yield different insights. Additionally, the criterion weights used in this study are derived from input provided by a select panel of experts. While these weights represent informed perspectives, they may not fully capture the diversity of viewpoints from other stakeholders, potentially leading to variations in outcomes. This limitation underscores the importance of contextual understanding, as stakeholders may prioritize criteria differently, influencing the results.

Moreover, the study emphasizes prominent WMP software currently used in Vietnam. Consequently, the findings may not be directly generalizable to other regions or countries with distinct market conditions, regulatory frameworks, and logistical challenges. Caution is advised when applying these findings in different geographical contexts, as WMP requirements and priorities often vary significantly across international markets.

To address these limitations, future research could focus on extending the NS-Delphi-DEMATEL-COCOSO model to accommodate real-time data, enabling dynamic adaptation to changing criteria and evolving demands in WMP selection. Incorporating multi-objective optimization into the framework could help balance competing priorities, such as cost minimization, operational efficiency maximization, and sustainability enhancement. Such adaptive decision-support systems would enable real-time adjustments that reflect operational conditions, inventory levels, and supply chain fluctuations, thereby improving the robustness of WMP selection processes.

Machine learning (ML) and AI technologies could also play a transformative role in refining decision-making processes. By analyzing historical data and user feedback, these technologies could predict WMP performance regarding reliability, functionality, and user satisfaction. Integrating AI-driven predictive insights into the decision framework would ensure that selected platforms align with practical operational needs. Additionally, iterative feedback mechanisms could facilitate continuous improvement in WMP selection, allowing adjustments based on real-time user experiences and changing requirements.

Comparative studies across industries could yield valuable insights into sector-specific WMP requirements, enabling the development of tailored decision models for unique operational demands. Future research should also emphasize sustainability metrics in WMP selection to reduce the environmental impact of warehouse operations. Platforms that support energy-efficient practices, waste reduction, and resource optimization could be prioritized, contributing to the broader goals of sustainable supply chain management.

Collaborations with industry experts, practitioners, and WMP vendors could help establish standardized evaluation criteria, ensuring consistency and reliability in WMP assessments across diverse contexts. Future studies could significantly enhance decision-making by addressing these research gaps, delivering actionable, industry-specific insights, and fostering innovation in WMP selection and implementation.

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- **Ethics Declaration Statement:** The authors state that this is their original work, which is neither
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- **Ethical and informed consent for data used:** No data are used.
- **Data availability and access:** The data that support the findings of this study are available from the corresponding author, Phi-Hung Nguyen: hungnp30@fe.edu.vn, upon reasonable request.

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Supplementary Materials

QUESTIONNAIRE

The influence level of barriers to warehouse management platforms in Vietnam's smart reverse logistics.

We are researching "Evaluating Critical Barriers to Warehouse Management Platforms in Vietnam's Smart Reverse Logistics." Our study aims to explore the key challenges affecting the implementation and efficiency of warehouse management systems (WMS) within reverse logistics operations. By identifying these barriers, we hope to contribute valuable insights to support the development of more innovative and more sustainable logistics solutions in Vietnam. This survey will take approximately 30 minutes to complete. We assure you that all your information will remain confidential and be used strictly for research purposes.

Your participation is crucial to the success of our research, and we greatly appreciate your time and thoughtful responses.

Section 1: General Information

1. Please indicate what age you are in:

☐ Under 25

☐ From 25 - 40

☐ From 40 - 60

☐ Over 60

2. Please indicate your gender:

☐ Male

☐ Female

☐ Other

3. Please indicate your current level of education qualification:

☐ Bachelor

☐ Master

☐ Doctor

☐ Other

4. Please indicate what industry you are currently working in:

☐ Teaching on Supply Chain Management/Logistics Management

☐ Researching on Supply Chain Management/Logistics Management

☐ Working on Supply Chain Management/Logistics Management

☐ Other

5. Please indicate your experience in this field:

☐ Less than 5 years

☐ From 5 to 10 years

☐ From 10 to 20 years

☐ Over 20 years

Section 2: Evaluate the important level of the factors

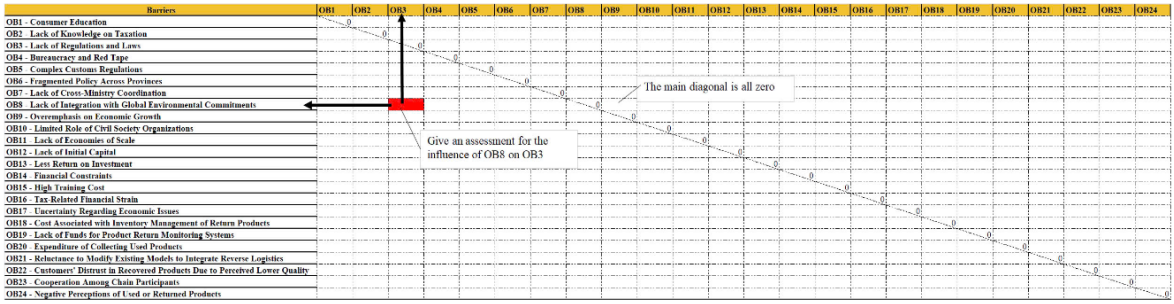
Please assess the level of importance of the factor in the row to the factor in the column as instructed above.

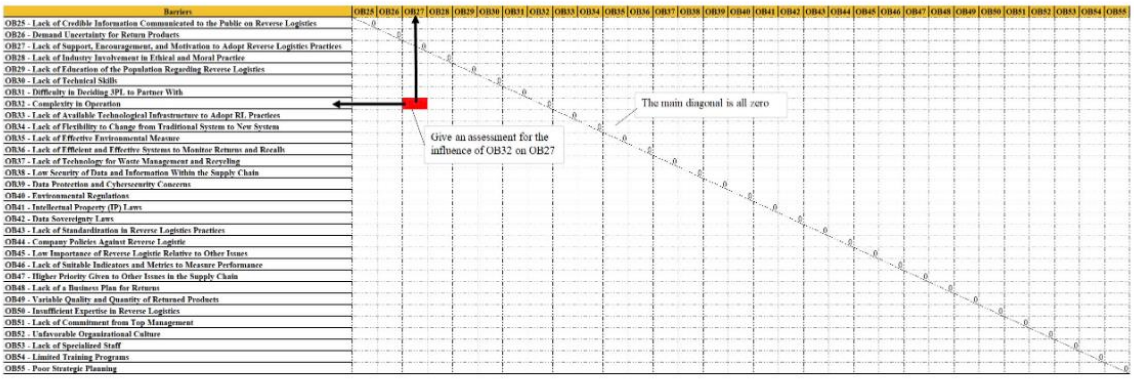
Importance Scale	
Linguistic scale	Code
Extremely High	EH
High	H
Medium	M
Low	L
Extremely Low	EL

BARRIERS	Importance Scale
OB1 - Consumer Education	
OB2 - Lack of Knowledge on Taxation	
OB3 - Lack of Regulations and Laws	
OB4 - Bureaucracy and Red Tape	
OB5 - Complex Customs Regulations	
OB6 - Fragmented Policy Across Provinces	
OB7 - Lack of Cross-Ministry Coordination	
OB8 - Lack of Integration with Global Environmental Commitments	
OB9 - Overemphasis on Economic Growth	
OB10 - Limited Role of Civil Society Organizations	
OB11 - Lack of Economies of Scale	
OB12 - Lack of Initial Capital	
OB13 - Low Return on Investment	
OB14 - Financial Constraints	
OB15 - High Training Cost	
OB16 - Tax-Related Financial Strain	
OB17 - Uncertainty Regarding Economic Issues	
OB18 - Cost Associated with Inventory Management of Return Products	

OB19 - Lack of Funds for Product Return Monitoring Systems	
OB20 - Expenditure of Collecting Used Products	
OB21 - Reluctance to Modify Existing Models to Integrate Reverse Logistics	
OB22 - Customer Distrust in Recovered Products Due to Perceived Lower Quality	
OB23 - Cooperation Among Chain Participants	
OB24 - Negative Perceptions of Used or Returned Products	
OB25 - Lack of Credible Information Communicated to the Public on Reverse Logistics	
OB26 - Demand Uncertainty for Return Products	
OB27 - Lack of Support, Encouragement, and Motivation to Adopt Reverse Logistics Practices	
OB28 - Lack of Industry Involvement in Ethical and Moral Practices	
OB29 - Lack of Education of the Population Regarding Reverse Logistics	
OB30 - Lack of Technical Skills	
OB31 - Difficulty in Deciding 3PL to Partner With	
OB32 - Complexity in Operation	
OB33 - Lack of Available Technological Infrastructure to Adopt RL Practices	
OB34 - Lack of Flexibility to Change from Traditional System to New System	
OB35 - Lack of Effective Environmental Measures	
OB36 - Lack of Efficient and Effective Systems to Monitor Returns and Recalls	
OB37 - Lack of Technology for Waste Management and Recycling	
OB38 - Low Security of Data and Information within the Supply Chain	
OB39 - Data Protection and Cybersecurity Concerns	
OB40 - Environmental Regulations	
OB41 - Intellectual Property (IP) Laws	
OB42 - Data Sovereignty Laws	
OB43 - Lack of Standardization in Reverse Logistics Practices	
OB44 - Company Policies Against Reverse Logistics	
OB45 - Low Importance of Reverse Logistics Relative to Other Issues	
OB46 - Lack of Suitable Indicators and Metrics to Measure Performance	
OB47 - Higher Priority Given to Other Issues in the Supply Chain	
OB48 - Lack of a Business Plan for Returns	
OB49 - Variable Quality and Quantity of Returned Products	
OB50 - Insufficient Expertise in Reverse Logistics	
OB51 - Lack of Commitment from Top Management	
OB52 - Unfavorable Organizational Culture	
OB53 - Lack of Specialized Staff	
OB54 - Limited Training Programs	
OB55 - Poor Strategic Planning	

Section 3: Evaluate the influence level of the factors





Influence Scale	
Linguistic scale	Code
Absolute influence	AI
Strong influence	SI
Fair influence	FI
Weak influence	WI
No influence	NI

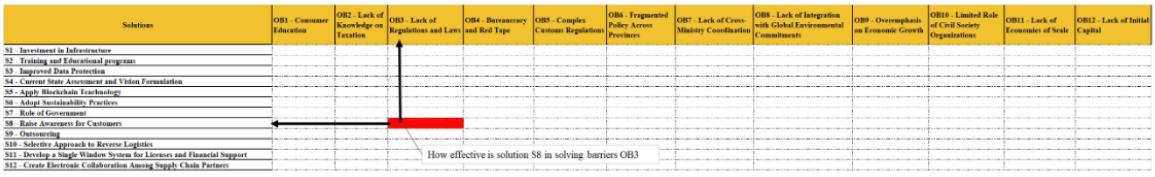
Please assess the factor's influence level in the row on the factor in the column as instructed above.

BARRIERS	**OB1**	**OB2**	**OB3**	**OB4**	**OB5**	**OB6**	**OB7**	**OB8**	**…**	**OB 50**	**OB 51**	**OB 52**	**OB 53**	**OB 54**	**OB 55**
OB1 - Consumer Education	0														
OB2 - Lack of Knowledge on Taxation		0													
OB3 - Lack of Regulations and Laws			0												
OB4 - Bureaucracy and Red Tape				0											
OB5 - Complex Customs Regulations					0										
OB6 - Fragmented Policy Across Provinces						0									
OB7 - Lack of Cross-Ministry Coordination							0								
OB8 - Lack of Integration with Global Environmental Commitments								0							
...									0						

OB50 - Insufficient Expertise in Reverse Logistics										0					
OB51 - Lack of Commitment from Top Management											0				
OB52 - Unfavorable Organizational Culture												0			
OB53 - Lack of Specialized Staff													0		
OB54 - Limited Training Programs														0	
OB55 - Poor Strategic Planning															0

Section 4: Evaluation solutions

Solutions
S1 - Investment in Infrastructure
S2 - Training and Educational programs
S3 - Improved Data Protection
S4 - Current State Assessment and Vision Formulation
S5 - Apply Blockchain Teachnology
S6 - Adopt Sustainability Practices
S7 - Role of Government
S8 - Raise Awareness for Customers
S9 - Outsourcing
S10 - Selective Approach to Reverse Logistics
S11 - Develop a Single Window System for Licenses and Financial Support
S12 - Create Electronic Collaboration Among Supply Chain Partners


Solutions
OB1 - Consumer Education
OB2 - Lack of Knowledge on Taxation
OB3 - Lack of Regulations and Laws
OB4 - Bureaucracy and Red Tape
OB5 - Complex Customs Regulations
OB6 - Fragmented Policy Across Provinces
OB7 - Lack of Cross-Ministry Coordination
OB8 - Lack of Integration with Global Environmental Commitments
OB9 - Overemphasis on Economic Growth
OB10 - Limited Role of Civil Society Organizations
OB11 - Lack of Economies of Scale
OB12 - Lack of Initial Capital
S1 - Investment in Infrastructure
S2 - Training and Educational programs
S3 - Improved Data Protection
S4 - Current State Assessment and Vision Formulation
S5 - Apply Blockchain Teachnology
S6 - Adopt Sustainability Practices
S7 - Role of Government
S8 - Raise Awareness for Customers
S9 - Outsourcing
S10 - Selective Approach to Reverse Logistics
S11 - Develop a Single Window System for Licenses and Financial Support
S12 - Create Electronic Collaboration Among Supply Chain Partners
How effective is solution S8 in solving barriers OB3

Assessment Scale	
Linguistic scale	Code
Very Good	VG
Good	G
Fair	F
Poor	P
Extremely Poor	EP

SOLUTIONS	OB1	OB2	OB3	OB4	OB5	OB6	OB7	OB8	…	OB50	OB51	OB52	OB53	OB54	OB55
S1 - Investment in Infrastructure															

S2 - Training and Educational programs															
S3 - Improved Data Protection															
S4 - Current State Assessment and Vision Formulation															
S5 - Apply Blockchain Technology															
S6 - Adopt Sustainability Practices															
S7 - Role of Government															
S8 - Raise Awareness for Customers															
S9 - Outsourcing															
S10 - Selective Approach to Reverse Logistics															
S11 - Develop a Single Window System for Licenses and Financial Support															
S12 - Create Electronic Collaboration Among Supply Chain Partners															

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Interval-Valued Neutrosophic Framework for Improving the Decision-Making on the Quality Evaluation of English Classics Translation

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Abstract—English translations of literary classics are a subject of interest for many scholars and evaluators. However, the evaluation of the translations is a complex and multifaceted task, which necessitates a balance between linguistic accuracy, cultural fidelity, and readability. This makes traditional methods often fall short of catching the uncertainties, ambiguities, and subjective judgments. In an attempt to bridge these gaps between computational linguistics and translation studies, we introduce a mathematically grounded IVNS approach to evaluate the translation of English classics by flexible modeling of the ambiguities inherent in translation decisions. Then, we introduced an extended version of the MULTIMOORA method, called MULTIVNSMOORA, that can simultaneously account for multiple, often conflicting criteria to ensure well-adjusted decision-making about translation quality. Finally, we present a case study on Don Quixote to validate the practical applicability of the proposed IVNS approach, by which we offer a reproducible framework for assessing translations of other English literary works.

Keywords: Neutrosophic Logic, Interval-Valued Neutrosophic (IVN), English Classics, Corpus-based evaluation.

1. Introduction

The translation of literary classics into English is a complex and nuanced endeavor, requiring not only linguistic proficiency but also a deep understanding of cultural, historical, and contextual subtleties [1]. As these works traverse linguistic boundaries, the fidelity, readability, and aesthetic quality of their translations often become subjects of intense scrutiny. Traditional methods of translation evaluation, while useful, frequently struggle to capture the inherent uncertainties, ambiguities, and subjective judgments

involved in the process [2]. This always applies when dealing with corpus-based translations, in which large-scale textual analysis adds additional layers of complexity [3].

In recent years, the development of computational and mathematical approaches has gained traction in translation studies, offering new ways to assess and compare translated texts. Among these, neutrosophic logic—a generalization of fuzzy logic—has emerged as a powerful tool for handling uncertainty, indeterminacy, and imprecision in decision-making processes [4], [5]. The design of NS presented a neutrality degree as a new component that is not predefined fuzzy logic [6], [7]. Thus, neutrosophic logic has three membership components which are falsity, indeterminacy, and truth, allowing more valuable application in different decision-making tasks [8]. Lately, researchers have introduced different types of NS have been proposed in different publications including Intuitionistic NS [9], multi-valued NSs (MVNs) [10], single-valued NSs (SVNSs) [11], [12], and Trapezoidal NS (TNS) [13]. Interval-valued neutrosophic sets (IVNS), in particular, provide a robust framework for modeling multi-dimensional and uncertain data, making them well-suited for the evaluation of translation quality [14].

In this article, we propose a novel IVNSs approach for the evaluation of corpus-based English translations of literary classics through the integration of linguistic, stylistic, and contextual parameters. Our IVNSs approach aims to address the limits that appear in dealing with traditional evaluations of assessment of translation quality. To guarantee well-adjusted decision-making about translation quality, we then presented MULTIVNSMOORA, an expanded version of the MULTIMOORA approach [15] that may concurrently account for several, frequently competing criteria. Lastly, to verify the usefulness of the suggested IVNS technique, we provide a replicable framework for evaluating translations of other English literary works through a case study on Don Quixote.

In subsequent sections, we outline the theoretical foundations of IVNSs, describe the methodology for applying this framework to translation evaluation, and present a case study involving the analysis of English translations of selected literary classics. By doing so, we shed light on the potential of neutrosophic logic as a transformative tool in translation studies.

2. Preliminaries

Definition 1. Neutrosophic set [4] to extend the fuzzy set for modeling uncertainty, and is defined as:

$$\mathcal{P} = \{\langle c, \mathfrak{t}_{\mathcal{P}}(c), \mathfrak{i}_{\mathcal{P}}(c), \mathfrak{f}_{\mathcal{P}}(c)\rangle : c \in U\} \tag{1}$$

$$0^- \leq \mathfrak{t}_A(x) + \mathfrak{i}_A(x) + \mathfrak{f}_A(x) \leq 3^+ \tag{2}$$

$$\mathfrak{t}_{\mathcal{P}} : X \to]^-0, 1[^+, \quad \mathfrak{i}_{\mathcal{P}} : X \to]^-0, 1[^+, \quad \mathfrak{f}_{\mathcal{P}} : X \to]^-0, 1[^+ \tag{3}$$

Definition 2. An IVNS redefines the membership of each component to be interval instead of scalars, which can be expressed as:

$$\mathcal{P} = \{\langle c, [\mathfrak{t}_{\mathfrak{L}}(c), \mathfrak{t}_{\mathfrak{U}}(c)], [\mathfrak{i}_{\mathfrak{L}}(c), \mathfrak{i}_{\mathfrak{U}}(c)], [\mathfrak{f}_{\mathfrak{L}}(c), \mathfrak{f}_{\mathfrak{U}}(c)]\rangle : c \in \mathbf{U}\} \tag{4}$$

where the constituting intervals satisfy the following conditions:

$$\mathfrak{t}_{\mathfrak{L}}(c) + \mathfrak{i}_{\mathfrak{L}}(c) + \mathfrak{f}_{\mathfrak{L}}(c) \leq 3, \ \mathfrak{t}_{\mathfrak{U}}(c) + \mathfrak{i}_{\mathfrak{U}}(c) + \mathfrak{f}_{\mathfrak{U}}(c) \leq 3 \tag{5}$$

In the above formula, $\mathfrak{t}_{\mathfrak{L}}(c)$ and $\mathfrak{t}_{\mathfrak{U}}(c)$ Denote the lower and upper bounds of the truth membership, and similarly for $\mathfrak{i}$ and $\mathfrak{f}$.

Definition 3. Given two IVNSs $\mathcal{P}$ and $\mathcal{Q}$, the union $\mathcal{P} \cup \mathcal{Q}$ is defined as:

$$\mathcal{P} \cup \mathcal{Q} = \left\{ \left\langle c, \begin{bmatrix} max\left(\mathfrak{t}_{\mathfrak{L}}^{\mathcal{P}}(c), \mathfrak{t}_{\mathfrak{L}}^{\mathcal{Q}}(c)\right), max\left(\mathfrak{t}_{\mathfrak{U}}^{\mathcal{P}}(c), \mathfrak{t}_{\mathfrak{U}}^{\mathcal{Q}}(c)\right) \end{bmatrix}, \begin{bmatrix} min\left(\mathfrak{i}_{\mathfrak{L}}^{\mathcal{P}}(c), \mathfrak{i}_{\mathfrak{L}}^{\mathcal{Q}}(c)\right), min\left(\mathfrak{i}_{\mathfrak{U}}^{\mathcal{P}}(c), \mathfrak{i}_{\mathfrak{U}}^{\mathcal{Q}}(c)\right) \end{bmatrix}, \begin{bmatrix} min\left(\mathfrak{f}_{\mathfrak{L}}^{\mathcal{P}}(c), \mathfrak{f}_{\mathfrak{L}}^{\mathcal{Q}}(c)\right), min\left(\mathfrak{f}_{\mathfrak{U}}^{\mathcal{P}}(c), \mathfrak{f}_{\mathfrak{U}}^{\mathcal{Q}}(c)\right) \end{bmatrix} \right\rangle : c \in U \right\} \tag{6}$$

Definition 4. Given two IVNSs $\mathcal{P}$ and $\mathcal{Q}$, the intersection $\mathcal{P} \cap \mathcal{Q}$ is defined as:

$$\mathcal{P} \cap \mathcal{Q} = \left\{ \left\langle c, \begin{bmatrix} min\left(\mathfrak{t}_{\mathfrak{L}}^{\mathcal{P}}(c), \mathfrak{t}_{\mathfrak{L}}^{\mathcal{Q}}(c)\right), min\left(\mathfrak{t}_{\mathfrak{U}}^{\mathcal{P}}(c), \mathfrak{t}_{\mathfrak{U}}^{\mathcal{Q}}(c)\right) \end{bmatrix}, \begin{bmatrix} max\left(\mathfrak{i}_{\mathfrak{L}}^{\mathcal{P}}(c), \mathfrak{i}_{\mathfrak{L}}^{\mathcal{Q}}(c)\right), max\left(\mathfrak{i}_{\mathfrak{U}}^{\mathcal{P}}(c), \mathfrak{i}_{\mathfrak{U}}^{\mathcal{Q}}(c)\right) \end{bmatrix}, \begin{bmatrix} max\left(\mathfrak{f}_{\mathfrak{L}}^{\mathcal{P}}(c), \mathfrak{f}_{\mathfrak{L}}^{\mathcal{Q}}(c)\right), max\left(\mathfrak{f}_{\mathfrak{U}}^{\mathcal{P}}(c), \mathfrak{f}_{\mathfrak{U}}^{\mathcal{Q}}(c)\right) \end{bmatrix} \right\rangle : c \in U \right\} \tag{7}$$

Definition 5. Given an IVNSs $\mathcal{P} = \{\langle c, [\mathfrak{t}_{\mathcal{P}}^{\mathfrak{L}}, \mathfrak{t}_{\mathcal{P}}^{\mathfrak{U}}], [\mathfrak{i}_{\mathcal{P}}^{\mathfrak{L}}, \mathfrak{i}_{\mathcal{P}}^{\mathfrak{U}}], [\mathfrak{f}_{\mathcal{P}}^{\mathfrak{L}}, \mathfrak{f}_{\mathcal{P}}^{\mathfrak{U}}]\rangle : c \in U\}$, then the complement $\mathcal{P}^c$ can be expressed as:

$$\mathcal{P}^c = \{\langle c, [\mathfrak{f}_{\mathcal{P}}^{\mathfrak{L}}, \mathfrak{f}_{\mathcal{P}}^{\mathfrak{U}}], [1 - \mathfrak{i}_{\mathcal{P}}^{\mathfrak{L}}, 1 - \mathfrak{i}_{\mathcal{P}}^{\mathfrak{U}}], [\mathfrak{t}_{\mathcal{P}}^{\mathfrak{L}}, \mathfrak{t}_{\mathcal{P}}^{\mathfrak{U}}]\rangle : c \in U\} \tag{8}$$

Definition 6. Given two IVNSs $\mathcal{P} = \{\langle c, [\mathfrak{t}_{\mathcal{P}}^{\mathfrak{L}}, \mathfrak{t}_{\mathcal{P}}^{\mathfrak{U}}], [\mathfrak{i}_{\mathcal{P}}^{\mathfrak{L}}, \mathfrak{i}_{\mathcal{P}}^{\mathfrak{U}}], [\mathfrak{f}_{\mathcal{P}}^{\mathfrak{L}}, \mathfrak{f}_{\mathcal{P}}^{\mathfrak{U}}]\rangle : c \in \mathfrak{U}\}$, and $\mathcal{Q} = \left\{\left\langle c, \left[\mathfrak{t}_{\mathcal{Q}}^{\mathfrak{L}}, \mathfrak{t}_{\mathcal{Q}}^{\mathfrak{U}}\right], \left[\mathfrak{i}_{\mathcal{Q}}^{\mathfrak{L}}, \mathfrak{i}_{\mathcal{Q}}^{\mathfrak{U}}\right], \left[\mathfrak{f}_{\mathcal{Q}}^{\mathfrak{L}}, \mathfrak{f}_{\mathcal{Q}}^{\mathfrak{U}}\right]\right\rangle : c \in U\right\}$, the following relations apply as follows:

(9)

$$\mathcal{P} \subseteq \mathcal{Q} \; iff \; \mathfrak{t}_{\mathcal{P}}^{\mathfrak{L}} \leq \mathfrak{t}_{\mathcal{Q}}^{\mathfrak{L}}, \mathfrak{t}_{\mathcal{P}}^{\mathfrak{U}} \leq \mathfrak{t}_{\mathcal{Q}}^{\mathfrak{U}}; \mathfrak{i}_{\mathcal{P}}^{\mathfrak{L}} \geq \mathfrak{i}_{\mathcal{Q}}^{\mathfrak{L}}, \mathfrak{i}_{\mathcal{P}}^{\mathfrak{U}} \geq \mathfrak{i}_{\mathcal{Q}}^{\mathfrak{U}}; \; \mathfrak{f}_{\mathcal{P}}^{\mathfrak{L}} \geq \mathfrak{f}_{\mathcal{Q}}^{\mathfrak{L}}, \mathfrak{f}_{\mathcal{P}}^{\mathfrak{U}} \geq \mathfrak{f}_{\mathcal{Q}}^{\mathfrak{U}}$$

$$\mathcal{P} = \mathcal{Q} \; if \; and \; only \; if \; \mathcal{P} \subseteq \mathcal{Q} \; and \; \mathcal{Q} \subseteq \mathcal{P}. \tag{10}$$

Definition 7. Given an IVNS $\mathcal{P} = \{\langle c, [\mathfrak{t}_{\mathcal{P}}^{\mathfrak{L}}, \mathfrak{t}_{\mathcal{P}}^{\mathfrak{U}}], [\mathfrak{i}_{\mathcal{P}}^{\mathfrak{L}}, \mathfrak{i}_{\mathcal{P}}^{\mathfrak{U}}], [\mathfrak{f}_{\mathcal{P}}^{\mathfrak{L}}, \mathfrak{f}_{\mathcal{P}}^{\mathfrak{U}}]\rangle : c \in \mathfrak{U}\}$, we can compute the score, accuracy, and certainty functions as follows:

$$S_{\mathcal{P}}(c) = \frac{\left(\mathfrak{t}_{\mathcal{P}}^{\mathfrak{L}} + \mathfrak{t}_{\mathcal{P}}^{\mathfrak{U}} + (1 - \mathfrak{i}_{\mathcal{P}}^{\mathfrak{U}}) + (1 - \mathfrak{f}_{\mathcal{P}}^{\mathfrak{U}}) + (1 - \mathfrak{i}_{\mathcal{P}}^{\mathfrak{L}}) + (1 - \mathfrak{f}_{\mathcal{P}}^{\mathfrak{U}})\right)}{6}. \tag{11}$$

$$a_{\mathcal{P}}(c) = \frac{\left(\mathfrak{t}_{\mathcal{P}}^{\mathfrak{L}} + \mathfrak{t}_{\mathcal{P}}^{\mathfrak{U}} - \mathfrak{f}_{\mathcal{P}}^{\mathfrak{L}} - \mathfrak{f}_{\mathcal{P}}^{\mathfrak{U}}\right)}{2} \tag{12}$$

Definition 8. Given two IVNSs $\mathcal{P} = \{\langle c, [\mathfrak{t}_{\mathcal{P}}^{\mathfrak{L}}, \mathfrak{t}_{\mathcal{P}}^{\mathfrak{U}}], [\mathfrak{i}_{\mathcal{P}}^{\mathfrak{L}}, \mathfrak{i}_{\mathcal{P}}^{\mathfrak{U}}], [\mathfrak{f}_{\mathcal{P}}^{\mathfrak{L}}, \mathfrak{f}_{\mathcal{P}}^{\mathfrak{U}}]\rangle : c \in U\}$, and $\mathcal{Q} = \left\{\left\langle \mathbf{c}, \left[\mathfrak{t}_{\mathcal{Q}}^{\mathfrak{L}}, \mathfrak{t}_{\mathcal{Q}}^{\mathfrak{U}}\right], \left[\mathfrak{i}_{\mathcal{Q}}^{\mathfrak{L}}, \mathfrak{i}_{\mathcal{Q}}^{\mathfrak{U}}\right], \left[\mathfrak{f}_{\mathcal{Q}}^{\mathfrak{L}}, \mathfrak{f}_{\mathcal{Q}}^{\mathfrak{U}}\right]\right\rangle : \mathbf{c} \in \mathbf{U}\right\}$, there are many elementary operations to be applied:

- Scalar multiplications

$$\lambda\mathcal{P} = \left\{\left\langle \begin{matrix} c, \left[1 - (1 - \mathfrak{t}_{\mathcal{P}}^{\mathfrak{L}})^{\lambda}, 1 - (1 - \mathfrak{t}_{\mathcal{P}}^{\mathfrak{U}})^{\lambda}\right], \\ \left[(\mathfrak{i}_{\mathcal{P}}^{\mathfrak{U}})^{\lambda}, (\mathfrak{i}_{\mathcal{Q}}^{\mathfrak{U}})^{\lambda}\right], \\ \left[(\mathfrak{f}_{\mathcal{P}}^{\mathfrak{U}})^{\lambda}, (\mathfrak{f}_{\mathcal{Q}}^{\mathfrak{U}})^{\lambda}\right] \end{matrix} \right\rangle : c \in \mathfrak{U}\right\} \tag{13}$$

- Power

$$\mathcal{P}^{\lambda} = \left\{\left\langle \begin{matrix} c, \left[(\mathfrak{t}_{\mathcal{P}}^{\mathfrak{L}})^{\lambda}, (\mathfrak{t}_{\mathcal{P}}^{\mathfrak{U}})^{\lambda}\right], \\ \left[1 - (1 - \mathfrak{t}_{\mathcal{P}}^{\mathfrak{U}})^{\lambda}, 1 - (1 - \mathfrak{t}_{\mathcal{Q}}^{\mathfrak{U}})^{\lambda}\right], \\ \left[1 - (1 - \mathfrak{f}_{\mathcal{P}}^{\mathfrak{U}})^{\lambda}, 1 - (1 - \mathfrak{f}_{\mathcal{Q}}^{\mathfrak{U}})^{\lambda}\right] \end{matrix} \right\rangle : c \in \mathfrak{U}\right\} \tag{14}$$

- Addition

$$\mathcal{P} \oplus \mathcal{Q} = \left\{\left\langle \begin{matrix} c, \left[\mathfrak{t}_{\mathcal{P}}^{\mathfrak{L}} + \mathfrak{t}_{\mathcal{Q}}^{\mathfrak{L}} - \mathfrak{t}_{\mathcal{P}}^{\mathfrak{L}} \cdot \mathfrak{t}_{\mathcal{Q}}^{\mathfrak{L}}, \mathfrak{t}_{\mathcal{P}}^{\mathfrak{U}} + \mathfrak{t}_{\mathcal{Q}}^{\mathfrak{U}} - \mathfrak{t}_{\mathcal{P}}^{\mathfrak{U}} \cdot \mathfrak{t}_{\mathcal{Q}}^{\mathfrak{U}}\right], \\ \left[\mathfrak{i}_{\mathcal{P}}^{\mathfrak{L}} \cdot \mathfrak{i}_{\mathcal{Q}}^{\mathfrak{L}}, \mathfrak{i}_{\mathcal{P}}^{\mathfrak{U}} \cdot \mathfrak{i}_{\mathcal{Q}}^{\mathfrak{U}}\right], \\ \left[\mathfrak{f}_{\mathcal{P}}^{\mathfrak{L}} \cdot \mathfrak{f}_{\mathcal{Q}}^{\mathfrak{L}}, \mathfrak{f}_{\mathcal{P}}^{\mathfrak{U}} \cdot \mathfrak{f}_{\mathcal{Q}}^{\mathfrak{U}}\right] \end{matrix} \right\rangle : c \in \mathfrak{U}\right\} \tag{15}$$

- Subtraction

$$\mathcal{P} \ominus \mathcal{Q} = \left\{\left\langle \begin{matrix} c, \left[\mathfrak{t}_{\mathcal{P}}^{\mathfrak{L}} - \mathfrak{t}_{\mathcal{Q}}^{\mathfrak{L}}, \mathfrak{t}_{\mathcal{P}}^{\mathfrak{U}} - \mathfrak{t}_{\mathcal{Q}}^{\mathfrak{U}}\right], \\ \left[max(\mathfrak{i}_{\mathcal{P}}^{\mathfrak{L}}, \mathfrak{i}_{\mathcal{Q}}^{\mathfrak{L}}), max(\mathfrak{i}_{\mathcal{P}}^{\mathfrak{U}}, \mathfrak{i}_{\mathcal{Q}}^{\mathfrak{U}})\right], \\ \left[\mathfrak{f}_{\mathcal{P}}^{\mathfrak{L}} - \mathfrak{f}_{\mathcal{Q}}^{\mathfrak{L}}, \mathfrak{f}_{\mathcal{P}}^{\mathfrak{U}} - \mathfrak{f}_{\mathcal{Q}}^{\mathfrak{U}}\right] \end{matrix} \right\rangle : c \in \mathfrak{U}\right\} \tag{16}$$

- Multiplication

$$\mathcal{P} \otimes \mathcal{Q} = \left\{ \left\langle \begin{matrix} \mathbf{c}, \left[\mathfrak{t}_{\mathcal{P}}^{\mathfrak{L}} \cdot \mathfrak{t}_{\mathcal{Q}}^{\mathfrak{L}}, \mathfrak{t}_{\mathcal{P}}^{\mathfrak{U}} \cdot \mathfrak{t}_{\mathcal{Q}}^{\mathfrak{U}}\right], \\ \left[\mathfrak{i}_{\mathcal{P}}^{\mathfrak{L}} + \mathfrak{i}_{\mathcal{Q}}^{\mathfrak{L}} - \mathfrak{i}_{\mathcal{P}}^{\mathfrak{L}} \cdot \mathfrak{i}_{\mathcal{Q}}^{\mathfrak{L}}, \mathfrak{i}_{\mathcal{P}}^{\mathfrak{U}} + \mathfrak{i}_{\mathcal{Q}}^{\mathfrak{U}} - \mathfrak{i}_{\mathcal{P}}^{\mathfrak{U}} \cdot \mathfrak{i}_{\mathcal{Q}}^{\mathfrak{U}}\right], \\ \left[\mathfrak{f}_{\mathcal{P}}^{\mathfrak{L}} + \mathfrak{f}_{\mathcal{Q}}^{\mathfrak{L}} - \mathfrak{f}_{\mathcal{P}}^{\mathfrak{L}} \cdot \mathfrak{f}_{\mathcal{Q}}^{\mathfrak{L}}, \mathfrak{f}_{\mathcal{P}}^{\mathfrak{U}} + \mathfrak{f}_{\mathcal{Q}}^{\mathfrak{U}} - \mathfrak{f}_{\mathcal{P}}^{\mathfrak{U}} \cdot \mathfrak{f}_{\mathcal{Q}}^{\mathfrak{U}}\right] \end{matrix} \right\rangle : \mathbf{c} \in \mathfrak{U} \right\} \tag{17}$$

Definition 9. Given $\mathcal{P}$, $\mathcal{Q}$, $\mathcal{R}$ representing three IVNSs, where $\lambda, \lambda_1, \lambda_2 > 0$, then, the following condition apply

$$\begin{gathered} \mathcal{P} + \mathcal{Q} = \mathcal{Q} + \mathcal{P} \\ \mathcal{P} \cdot \mathcal{Q} = \mathcal{Q} \cdot \mathcal{P} \\ \lambda(\mathcal{P} + \mathcal{Q}) = \lambda\mathcal{Q} + \lambda\mathcal{P} \\ (\mathcal{P} \cdot \mathcal{Q})^{\lambda} = \mathcal{Q}^{\lambda} \cdot \mathcal{P}^{\lambda} \\ \lambda_1\mathcal{P} + \lambda_2\mathcal{P} = (\lambda_1 + \lambda_2)\mathcal{P} \\ \mathcal{P}^{\lambda_1} \cdot \mathcal{P}^{\lambda_1} = \mathcal{P}^{\lambda_1 + \lambda_2} \\ (\mathcal{P} + \mathcal{Q}) + \mathcal{R} = \mathcal{P} + (\mathcal{Q} + \mathcal{R}) \end{gathered} \tag{18}$$

Definition 9. Given an IVN $\mathcal{P} = \left[\mathfrak{t}_{\mathcal{P}}^{\mathfrak{L}}, \mathfrak{t}_{\mathcal{P}}^{\mathfrak{U}}\right], \left[\mathfrak{i}_{\mathcal{P}}^{\mathfrak{L}}, \mathfrak{i}_{\mathcal{P}}^{\mathfrak{U}}\right], \left[\mathfrak{f}_{\mathcal{P}}^{\mathfrak{L}}, \mathfrak{f}_{\mathcal{P}}^{\mathfrak{U}}\right]$, the deneutrosophication can be computed as follows:

$$\mathfrak{D}(\mathcal{P}) = \left(\frac{\left(\mathfrak{t}_{\mathcal{P}}^{\mathfrak{L}} + \mathfrak{t}_{\mathcal{P}}^{\mathfrak{U}}\right)}{2} + \left(1 - \frac{\left(\mathfrak{i}_{\mathcal{P}}^{\mathfrak{L}} + \mathfrak{i}_{\mathcal{P}}^{\mathfrak{U}}\right)}{2}\right)\left(\mathfrak{i}_{\mathcal{P}}^{\mathfrak{U}}\right) - \left(\frac{\left(\mathfrak{f}_{\mathcal{P}}^{\mathfrak{L}} + \mathfrak{f}_{\mathcal{P}}^{\mathfrak{U}}\right)}{2}\right)\left(1 - \mathfrak{f}_{\mathcal{P}}^{\mathfrak{U}}\right)\right) \tag{19}$$

Definition 10. Given a set of IVNSs $\mathcal{P}_j = \left\{\left\langle \boldsymbol{c}, \left[\mathfrak{t}_{\mathcal{P}_j}^{\mathfrak{L}}, \mathfrak{t}_{\mathcal{P}_j}^{\mathfrak{U}}\right], \left[\mathfrak{i}_{\mathcal{P}_j}^{\mathfrak{L}}, \mathfrak{i}_{\mathcal{P}_j}^{\mathfrak{U}}\right], \left[\mathfrak{f}_{\mathcal{P}_j}^{\mathfrak{L}}, \mathfrak{f}_{\mathcal{P}_j}^{\mathfrak{U}}\right] \right\rangle : \boldsymbol{c} \in \boldsymbol{U}\right\}$, and corresponding weight vector $W = (w_1, w_2, \ldots, w_n)^{\mathfrak{t}}$, where $w_j \geq 0$, and $\sum w_j = \mathbf{1}$. Their general weighted aggregation function is formulated as follows:

if $\lambda \to 0$

$$Z = \mathrm{IVNS}_{\mathrm{GWA}}(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n) = \prod_{j=1}^{n} A_j^{w_j} = \left\{ \begin{matrix} \left\langle \left[\prod_{j=1}^{n} (\mathfrak{t}_j^{\mathfrak{L}})^{w_j}, \prod_{j=1}^{n} (\mathfrak{t}_j^{\mathfrak{U}})^{w_j}\right], \right. \\ \left[1 - \prod_{j=1}^{n} (1 - \mathfrak{i}_j^{\mathfrak{L}})^{w_j}, 1 - \prod_{j=1}^{n} (1 - \mathfrak{i}_j^{\mathfrak{U}})^{w_j}\right], \\ \left. \left[1 - \prod_{j=1}^{n} (1 - \mathfrak{f}_j^{\mathfrak{L}})^{w_j}, 1 - \prod_{j=1}^{n} (1 - \mathfrak{f}_j^{\mathfrak{U}})^{w_j}\right] \right\} \end{matrix} \tag{20}$$

if $\lambda \to 1$

$$Z = \mathrm{IVNS}_{\mathrm{GWA}}(\mathcal{P}_1, \mathcal{P}_2, \dots, \mathcal{P}_n) = \sum_{j=1}^{n} w_j \mathcal{P}_j =$$
$$\left\{ \left\langle \begin{array}{c} \left[1 - \prod_{j=1}^{n} (1 - \mathfrak{t}_j^{\mathfrak{L}})^{w_j}, 1 - \prod_{j=1}^{n} (1 - \mathfrak{t}_j^{\mathfrak{U}})^{w_j}\right], \\ \left[\prod_{j=1}^{n} (\mathfrak{i}_j^{\mathfrak{L}})^{w_j}, \prod_{j=1}^{n} (\mathfrak{i}_j^{\mathfrak{U}})^{w_j}\right], \left[\prod_{j=1}^{n} (\mathfrak{f}_j^{\mathfrak{L}})^{w_j}, \prod_{j=1}^{n} (\mathfrak{f}_j^{\mathfrak{U}})^{w_j}\right] \end{array} \right\rangle \right\}$$
$$= \left\{ \left\langle \begin{array}{c} \left[1 - \prod_{j=1}^{n} (1 - \mathfrak{t}_j^{\mathfrak{L}})^{w_j}, 1 - \prod_{j=1}^{n} (1 - \mathfrak{t}_j^{\mathfrak{U}})^{w_j}\right], \\ \left[\prod_{j=1}^{n} (\mathfrak{i}_j^{\mathfrak{L}})^{w_j}, \prod_{j=1}^{n} (\mathfrak{i}_j^{\mathfrak{U}})^{w_j}\right], \left[\prod_{j=1}^{n} (\mathfrak{f}_j^{\mathfrak{L}})^{w_j}, \prod_{j=1}^{n} (\mathfrak{f}_j^{\mathfrak{U}})^{w_j}\right] \end{array} \right\rangle \right\} \quad (21)$$

3. Research Method

This section provides a holistic explanation of the proposed IVNSMCDM approach for assessing and uncertain process of evaluating the translations of English. Our explanation of the proposed method will go through sequence of steps that are detailed as follows:

Step 1: we construct the Decision Matrix in IVNS form, in which each alternative A_i is evaluated against each criterion C_j using IVNSs:

$$\boldsymbol{D_{n\times m}} = \begin{bmatrix} \left\langle [\mathfrak{t}_{11}^{\mathfrak{L}}, \mathfrak{t}_{11}^{\mathfrak{U}}], [\mathfrak{i}_{11}^{\mathfrak{L}}, \mathfrak{i}_{11}^{\mathfrak{U}}], [\mathfrak{f}_{11}^{\mathfrak{L}}, \mathfrak{f}_{11}^{\mathfrak{U}}] \right\rangle & \cdots & \left\langle [\mathfrak{t}_{1m}^{\mathfrak{L}}, \mathfrak{t}_{1m}^{\mathfrak{U}}], [\mathfrak{i}_{1m}^{\mathfrak{L}}, \mathfrak{i}_{1m}^{\mathfrak{U}}], [\mathfrak{f}_{1m}^{\mathfrak{U}}, \mathfrak{f}_{1m}^{\mathfrak{U}}] \right\rangle \\ \vdots & \ddots & \vdots \\ \left\langle [\mathfrak{t}_{n1}^{\mathfrak{L}}, \mathfrak{t}_{n1}^{\mathfrak{U}}], [\mathfrak{i}_{n1}^{\mathfrak{L}}, \mathfrak{i}_{n1}^{\mathfrak{U}}], [\mathfrak{f}_{n1}^{\mathfrak{L}}, \mathfrak{f}_{n1}^{\mathfrak{U}}] \right\rangle & \cdots & \left\langle [\mathfrak{t}_{nm}^{\mathfrak{L}}, \mathfrak{t}_{nm}^{\mathfrak{U}}], [\mathfrak{i}_{nm}^{\mathfrak{L}}, \mathfrak{i}_{nm}^{\mathfrak{U}}], [\mathfrak{f}_{nm}^{\mathfrak{U}}, \mathfrak{f}_{nm}^{\mathfrak{U}}] \right\rangle \end{bmatrix}, \quad (22)$$

where n = number of alternatives, m = number of criteria, and $\left\langle [\mathfrak{t}_{1j}^{\mathfrak{L}}, \mathfrak{t}_{1j}^{\mathfrak{U}}], [\mathfrak{i}_{1j}^{\mathfrak{L}}, \mathfrak{i}_{1j}^{\mathfrak{U}}], [\mathfrak{f}_{1j}^{\mathfrak{U}}, \mathfrak{f}_{1j}^{\mathfrak{U}}] \right\rangle$ represent IVNS evaluation of alternative A_i for criterion C_j.

Step 2: we conduct normalization of the $\boldsymbol{D_{n\times m}}$ to ensure all criteria are comparable. For IVNSs, the normalized decision matrix $\boldsymbol{R}$ is calculated as:

$$\mathbf{R_{ij}} = \left\langle [\mathfrak{t}_{ij}^{\mathfrak{L}}, \mathfrak{t}_{ij}^{\mathfrak{U}}], [\mathfrak{i}_{ij}^{\mathfrak{L}}, \mathfrak{i}_{ij}^{\mathfrak{U}}], [\mathfrak{f}_{ij}^{\mathfrak{U}}, \mathfrak{f}_{ij}^{\mathfrak{U}}] \right\rangle \quad (23)$$

Step 3: we apply a ratio system to calculate the overall score for each alternative by summing the normalized IVNSs for benefit criteria and subtracting those for cost criteria.

Step 3.1. Aggregate the IVNSs for each alternative $\boldsymbol{A_i}$:

$$\boldsymbol{RS_i} = \sum_{j=1}^{k} \boldsymbol{w_j} \cdot \boldsymbol{R_{ij}} - \sum_{j=k+1}^{m} \boldsymbol{w_j} \cdot \boldsymbol{R_{ij}} \quad (24)$$

where w_j = weight of criterion C_j Obtained using the CRITIC method [16], k = number of benefit criteria, m = total number of criteria.

Step 3.2. calculate the score function for

$$\boldsymbol{S_{(RS_i)}} = \frac{\left\langle \mathfrak{t}_{\mathbf{ij}}^{\mathfrak{L}} + \mathfrak{t}_{\mathbf{ij}}^{\mathfrak{U}} - \left(\mathfrak{i}_{\mathbf{ij}}^{\mathfrak{L}} + \mathfrak{i}_{\mathbf{ij}}^{\mathfrak{U}}\right) - \left(\mathfrak{f}_{\mathbf{ij}}^{\mathfrak{U}}, \mathfrak{f}_{\mathbf{ij}}^{\mathfrak{U}}\right)\right\rangle}{\mathbf{2}} \tag{25}$$

Step 3.3. Rank the alternatives based on $\boldsymbol{S_{(RS_i)}}$ in descending order.

Step 4: Apply the Reference Point (RP) Approach

RP identifies the best alternative by minimizing the distance from an ideal reference point.

Step 4.1. Determine the reference point $\boldsymbol{RP_j}$ for each criterion $\boldsymbol{C_j}$:

$$\boldsymbol{RP_j} = \left\langle \left[\mathbf{max}\left(\mathfrak{t}_{\mathbf{ij}}^{\mathfrak{L}}\right), \mathbf{max}\left(\mathfrak{t}_{\mathbf{ij}}^{\mathfrak{U}}\right)\right], \left[\mathbf{min}\left(\mathfrak{i}_{\mathbf{ij}}^{\mathfrak{L}}\right), \mathbf{min}\left(\mathfrak{i}_{\mathbf{ij}}^{\mathfrak{U}}\right)\right], \left[\mathbf{min}\left(\mathfrak{f}_{\mathbf{ij}}^{\mathfrak{L}}\right), \mathbf{min}\left(\mathfrak{f}_{\mathbf{ij}}^{\mathfrak{U}}\right)\right]\right\rangle \tag{26}$$

Step 4.2. Calculate the distance between each alternative $\boldsymbol{A_i}$ and the reference point $\boldsymbol{RP_j}$ Using a distance measure for IVNSs, such as:

$$\boldsymbol{d}\left(\boldsymbol{R_{ij}}, \boldsymbol{RP_j}\right) = \sqrt{\sum_{k=1}^{n} \boldsymbol{R_{ij}^k} - \boldsymbol{RP_{ij}^k}} \tag{27}$$

Step 4.3. Compute the overall distance for each alternative:

$$\boldsymbol{RP_i} = \max_{j}\left(\boldsymbol{w_j} \cdot \boldsymbol{d}\left(\boldsymbol{R_{ij}}, \boldsymbol{RP_j}\right)\right) \tag{28}$$

Step 4.4. Rank the alternatives based on RP_i in ascending order.

Step 5: Apply the Full Multiplicative Form (FMF) by taking the overall score for each alternative by multiplying the normalized IVNSs for benefit criteria and dividing by those for cost criteria.

Step 5.1. Aggregate the IVNSs for each alternative A_i :

$$\boldsymbol{FMF_i} = \frac{\prod_{j=1}^{k}\left(\boldsymbol{R_{ij}}\right)^{\boldsymbol{w_j}}}{\prod_{j=k+1}^{m}\left(\boldsymbol{R_{ij}}\right)^{\boldsymbol{w_j}}} \tag{29}$$

Step 5.2. Calculate the score function for FMF_i :

$$\boldsymbol{S_{(FMF_i)}} = \frac{\left\langle \mathfrak{t}_{\mathbf{ij}}^{\mathfrak{L}} + \mathfrak{t}_{\mathbf{ij}}^{\mathfrak{U}} - \left(\mathfrak{i}_{\mathbf{ij}}^{\mathfrak{L}} + \mathfrak{i}_{\mathbf{ij}}^{\mathfrak{U}}\right) - \left(\mathfrak{f}_{\mathbf{ij}}^{\mathfrak{U}}, \mathfrak{f}_{\mathbf{ij}}^{\mathfrak{U}}\right)\right\rangle}{\mathbf{2}} \tag{30}$$

Step 5.3. Rank the alternatives based on $\boldsymbol{S_{(FMF_i)}}$ in descending order.

Step 6: we aggregate the ranking of alternatives from the RS, RP, and FMF based on a common aggregation method called the Borda Rule.

4. Case Study & Quantitative Results

Herein, we introduce our case study for English translation of classics and analyze the results obtained by the proposed method.

4.1. Case Study: English Translations of Don Quixote

Translating literary classics into English involves balancing fidelity to the original text, readability, cultural context, and accessibility. In our study, we used “Don Quixote" by Miguel de Cervantes, as one of the most translated and analyzed classics in literature. Various translations of "Don Quixote" by Miguel de Cervantes have been published over time, each with its strengths and weaknesses. Selecting the most suitable translation requires evaluating different criteria. The assessment of each of the above translations is conducted based on the combination of benefit criteria (higher values are better) as well as cost criteria to assess the translations. First, readability (C1) refers to the degree to which the translation can be easily understood by a contemporary reader. It is always captured by sentence complexity, vocabulary, and flow. Second, cultural Fidelity (C2) measures how well the translation preserves the cultural and historical context of the original. It is always captured by the retention of idioms, cultural references, and tone. Third, literary Style (C3) measures how effectively the translation captures the literary essence of the original. It is always captured by narrative voice, humor, and stylistic devices. Forth, complexity (C4), is the level of linguistic and structural difficulty in the translation. It is always captured by sentence length, archaic language, and syntactic complexity. Fifth, deviation from Source Text (C5) measures the extent to which the translation diverges from the original text. After an internet investigation, we found Below, we found five distinct English translations of the same classic: Translation A1: John Ormsby (1885), Translation A2: Samuel Putnam (1949), Translation A3: Edith Grossman (2003), Translation A4: Tobias Smollett (1755), and Translation A5: Tom Lathrop (2005).

Table 1. Experts-aggregated IVNS-decision matrix for our case study.

Alternative	C1	C2	C3	C4	C5
A1 (Ormsby)	{[0.834, 0.955], [0.062, 0.112], [0.144, 0.223]}	{[0.741, 0.846], [0.144, 0.170], [0.244, 0.328]}	{[0.763, 0.860], [0.149, 0.232], [0.167, 0.278]}	{[0.623, 0.737], [0.136, 0.209], [0.353, 0.418]}	{[0.611, 0.724], [0.132, 0.214], [0.222, 0.349]}

A2 (Putnam)	{[0.780, 0.900], [0.116, 0.219], [0.156, 0.262]}	{[0.810, 0.902], [0.051, 0.153], [0.122, 0.203]}	{[0.718, 0.807], [0.128, 0.196], [0.203, 0.354]}	{[0.502, 0.619], [0.153, 0.238], [0.437, 0.543]}	{[0.424, 0.738], [0.107, 0.216], [0.348, 0.428]}
A3 (Grossman)	{[0.929, 1.000], [0.023, 0.053], [0.035, 0.124]}	{[0.872, 0.963], [0.091, 0.149], [0.076, 0.159]}	{[0.831, 0.919], [0.070, 0.141], [0.139, 0.226]}	{[0.726, 0.824], [0.124, 0.204], [0.232, 0.503]}	{[0.402, 0.634], [0.066, 0.139], [0.119, 0.218]}
A4 (Smollett)	{[0.532, 0.627], [0.239, 0.346], [0.441, 0.504]}	{[0.614, 0.718], [0.154, 0.285], [0.330, 0.421]}	{[0.521, 0.652], [0.210, 0.311], [0.413, 0.540]}	{[0.514, 0.825], [0.125, 0.251], [0.413, 0.742]}	{[0.454, 0.622], [0.130, 0.229], [0.445, 0.525]}
A5 (Lathrop)	{[0.745, 0.824], [0.145, 0.201], [0.236, 0.342]}	{[0.770, 0.885], [0.142, 0.207], [0.161, 0.274]}	{[0.673, 0.751], [0.200, 0.274], [0.260, 0.377]}	{[0.619, 0.713], [0.105, 0.234], [0.341, 0.525]}	{[0.515, 0.842], [0.129, 0.212], [0.224, 0.339]}

The weights associated with the evaluation criteria were calculated using the CRITIC method (as mentioned earlier), which determines the importance of each criterion according to the contrast intensity of the data as well as the struggle between criteria. This process was conducted in two distinct ways to ensure robustness and comprehensiveness. First, the weights were derived based on the scores of the IVNS decision matrix, which were computed using the interval components for each alternative-criterion pair (refer to Figure 1). Second, the weights were calculated based on the accuracies of the IVNS

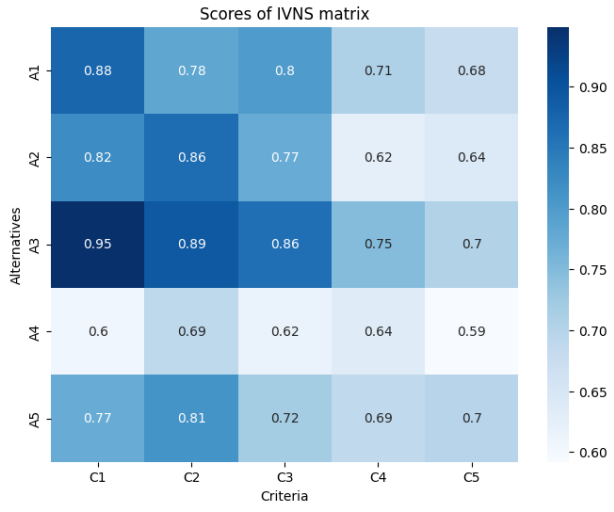

Figure 1. IVNS score matrix driven from the aggregated decision matrix

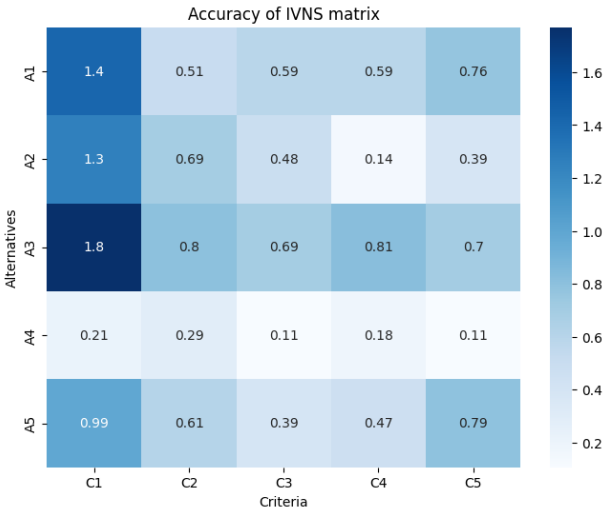

Figure 2. IVNS accuracy matrix driven from the aggregated decision matrix.

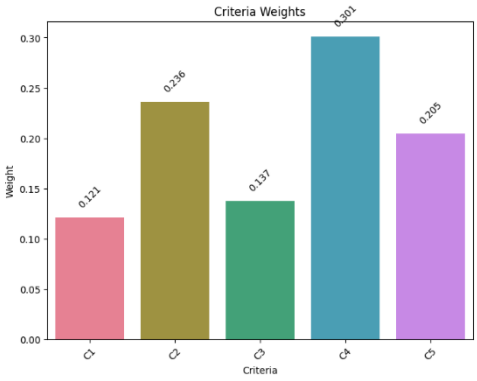

Figure 3. Criterion weights based on IVNS scores matrix.

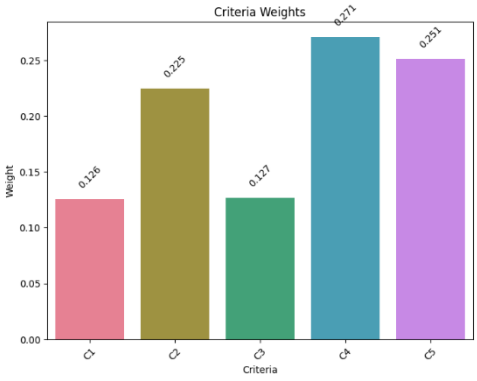

Figure 4. Criterion weights based on IVNS accuracy matrix.

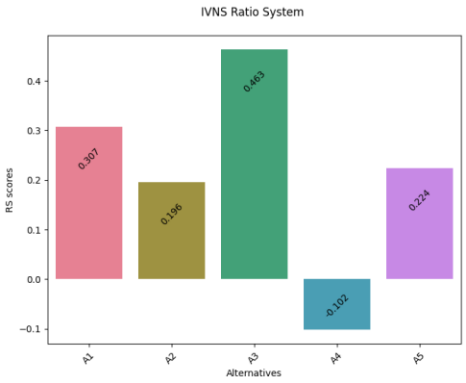

Figure 5. Ratio System Scores for alternatives based on scored weighting scheme.

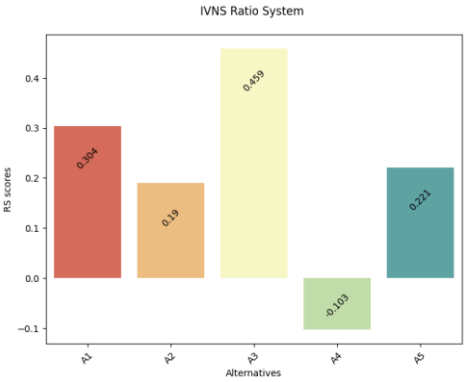

Figure 6: Ratio System Scores for alternatives using the accuracy weighting scheme.

decision matrix, which reflects the degree of certainty or precision in the evaluations (refer to Figure 2). The CRITIC method was applied to both approaches to capture the relative importance of each criterion in the context of the decision-making problem. The resulting weights from the first approach are visualized in Figure 3, which illustrates the distribution of weights across the criteria. Similarly, the weights derived from the second approach are presented in Figure 4, providing a comparative perspective on how the criteria weights vary when calculated based on accuracies. This dual application of the CRITIC method ensures a more nuanced and reliable weighting scheme, enhancing the validity of the evaluation framework.

To visualize the impact of different weighting schemes on the evaluation of alternatives, we plot the Ratio System Scores for both weighting schemes in Figure 5 and Figure 6. In Figure 5, the scores are plotted using the score-based weighting scheme, where all criteria are assigned equal importance. This provides a baseline understanding of how the alternatives perform when no criterion is prioritized over another. In Figure 6, the scores are plotted using the accuracy weighting scheme, where criteria are weighted based on their relative importance as determined by expert judgment or prior analysis.

This allows us to observe how the prioritization of specific criteria influences the overall ranking of alternatives.

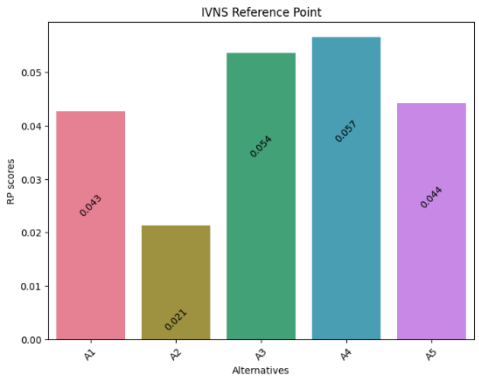

Figure 7: RP Scores for alternatives using the scored weighting scheme.

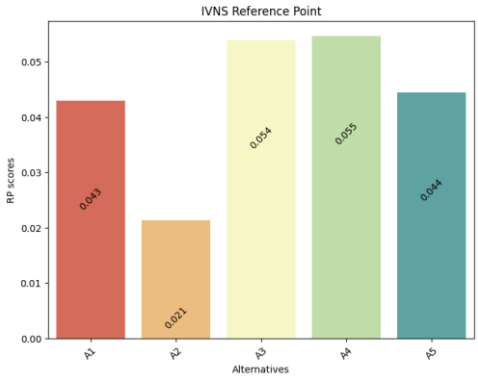

Figure 8: RP Scores for alternatives using the accuracy weighting scheme.

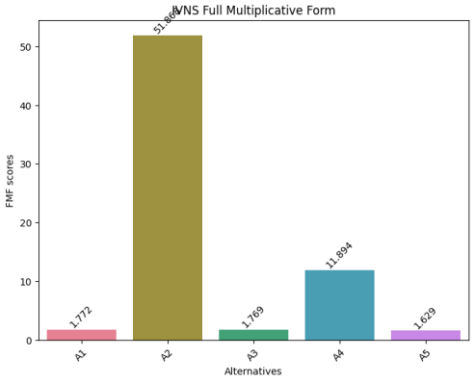

Figure 9: FMF Scores for alternatives using the scored weighting scheme.

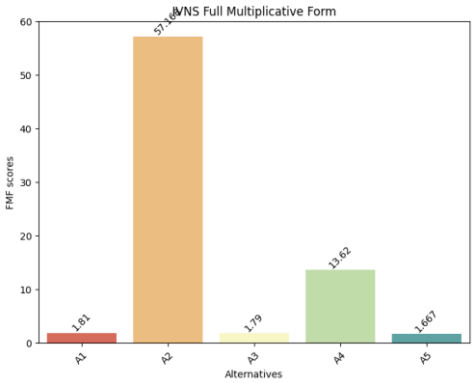

Figure 10: FMF Scores for alternatives using the accuracy weighting scheme.

To further analyzes the robustness of the evaluations processes, we plot the Reference Point Scores for both weighting schemes in Figure 7 and Figure 8. This allow readers to perceive how the arrangement of explicit criteria stimuli the distance of alternatives from the ideal reference point. To ample our analysis, again, we plot the FMF Scores for both weighting schemes in Figure 9 and Figure 10. This enables observing how the ordering of definite criteria can affect the overall multiplicative scores and the subsequent ranking of alternatives.

To validate the effectiveness of the proposed approach, we conduct a comparative analysis against other IVNS-MCDM methods, as summarized in Table 2. The comparison includes methods such as IVNS-TOPSIS [17], MULTIMOORA [15], and IVNS-EDAS [18], which are widely utilized in previous studies for handling uncertainty and imprecision in decision-making. The results demonstrate that the proposed approach consistently outperforms or aligns with these methods in terms of ranking accuracy, robustness, and

computational efficiency. Specifically, the proposed approach provides a more balanced evaluation by integrating the strengths of the Ratio System, Reference Point, and Full Multiplicative Form methods, while effectively handling the indeterminacy and ambiguity inherent in IVNSs.

Table 2. Comparative analysis of alternative ranking from different IVNS-MCDM methods.

Rank	IVNS-TOPSIS	MULTIMOORA	IVNS-EDAS	Proposed
1	A2	A2	A2	A2
2	A4	A3	A4	A1
3	A3	A1	A3	A3
4	A1	A5	A1	A5
5	A5	A4	A5	A4

5. Conclusion

This research article proposes a novel IVNS approach that integrates an extended version of the MULTIMOORA method to deal with uncertainty challenges inherent in the translation of English classics. Our work introduces a proof-of-concept case study based on a corpus of English translations of Don Quixote, to be used to assess the evaluation of our approach for evaluating and ranking translations according to different translation criteria, including readability, cultural fidelity, literary style, complexity, and deviation from the source text. The quantitative and qualitative results demonstrate the versatility and reliability of the MULTIVNSMOORA method in handling the strengths and weaknesses of each translation, meanwhile offering valued intuitions for translators, scholars, and publishers.

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High-Quality Development Evaluation of Cultural and Tourism Integration Empowered by the Digital Economy under Double Valued Neutrosophic Sets

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Abstract: This study proposed a decision-making approach to evaluate the cultural and tourism integration empowered by the digital economy. Two methods are used in this study, such as Entropy method to compute the criteria weights and the MABAC method to rank the alternatives. These methods are used under the neutrosophic sets. Neutrosophic sets (NSs) are used to represent ambiguous, inconsistent, and unpredictable data that arise in practical issues. Indeterminacy leaning toward truth membership and indeterminacy leaning toward falsity membership are two separate components of double-valued neutrosophic sets (DVNSs), a variant of NSs. Three experts have evaluated the criteria and alternatives. Ten criteria and six alternatives are used in this study. We show the Digital Infrastructure criterion has the highest weight.

Keywords: Neutrosophic Sets; Double Valued Neutrosophic sets; Entropy Method; MABAC Method; Cultural and Tourism; Digital Economy.

1. Introduction

Zadeh proposed the fuzzy set (FS) theory, which has been successfully used in many different domains. A single number within the closed interval [0,1] represents an element's degree of membership in FSs[1], [2]. Nevertheless, in practical scenarios, it could not always be certain that an element's degree of non-membership in the FS is equivalent to one minus degree of membership. In other words, there might be some hesitancy. For this reason, Atanassov proposed the idea of intuitionistic fuzzy sets (IFSs), which are a generalization of FSs. The inability to describe the degree of hesitation individually is the only drawback of IFSs. To get beyond this flaw, as a generalization of IFSs and FSs, Smarandache introduced the idea of neutrosophic sets (NSs). Subclasses of NSs, including simplified neutrosophic sets (SNS), interval neutrosophic sets (INSs), and single-valued neutrosophic sets (SVNSs), were then defined by various scholars[3], [4]. Let's look at an example where we ask someone about a statement, and that person may be

certain that the statement has a 0.8 chance of being true and a 0.4 chance of being false. Furthermore, 0.3 indicates that he or she is unsure but believes it to be true, and 0.4 indicates that he or she is unsure but believes it to be untrue. Kandasamy developed the idea of double-valued neutrosophic sets (DVNSs) as a substitute for NSs to handle this type of data, giving indeterminacy greater clarity and dependability. DVNSs distinguish between two types of indeterminacy: indeterminacy that leans toward truth membership and indeterminacy that leans toward falsity membership[5], [6]. Smarandache performed the first refinement of neutrosophic sets, splitting the truth value (T) into different sub-truths (T1, T2, etc.), splitting the indeterminacy (I) into different sub-indeterminacies (I1, I2, etc.), and splitting the sub-falsehood (F) into F1, F2, etc. A particular instance of n-valued neutrosophic sets is DVNSs[7], [8]. To give indeterminacy in this work greater accuracy and precision, the indeterminacy found in the neutrosophic set has been divided into two membership-based categories: indeterminacy leaning toward false membership and indeterminacy leaning toward truth membership. Indeterminacy leaning towards truth is the word used to describe indeterminacy that can be recognized as having a greater proportion of the truth value than the false value but cannot be categorized as truth. It is deemed to be indeterminacy leaning towards false when it is possible to identify indeterminacy that is more of a false value than the truth value but cannot be categorized as false[9], [10].

Since it is unclear whether indeterminacy favors truth or false membership, the single value of indeterminacy is ambiguous. The results or outcome will undoubtedly be better than when a single value is employed when the indeterminacy is favoring or leaning toward truth and when it is favoring or leaning toward false. Both truth-leaning and falsity-leaning indeterminacy make the scenario's indeterminacy more exact and accurate. A Double-Valued Neutrosophic Set (DVNS) is the definition of this modified neutrosophic set[11], [12].

Since there is currently little research on DVNSs, it is important to examine certain fundamental theories. Accordingly, this article's objectives were to:

(1) Proposed two decision-making methods to evaluate the decision-making issue.

(2) The entropy method is used to compute the criteria weights and the MABAC method is used to rank the alternatives.

(3) Ten criteria and six alternatives are used in this study.

(4) Sensitivity analysis is conducted to show the rank of alternatives under different criteria weights.

The rest of this article is organized as follows: Section 2 reviews some fundamental ideas about DVNSs; Section 3 proposes the two methods such as entropy to compute the criteria weights and the MABAC method to rank the alternatives; Section 4 shows the results of two methods; and Section 6 shows the conclusions of this study.

2. Double Valued Neutrosophic Set (DVNS)

We can define the DVNS with four membership functions such as truth $T_A(x)$, indeterminacy toward truth function $I_{TA}(x)$, indeterminacy toward falsity function $I_{FA}(x)$, and falsity function $F_A(x)$[8], [9].

$$\big(T_A(x), I_{TA}(x), I_{FA}(x), F_A(x)\big) \in [0,1] \tag{1}$$

$$0 \leq T_A(x), I_{TA}(x), I_{FA}(x), F_A(x) \leq 4 \tag{2}$$

$$A = \{(x, T_A(x), I_{TA}(x), I_{FA}(x), F_A(x) | x \in X)\} \tag{3}$$

Definition 1.

The complement of the DVNS can be defined as:

$$T_{c(A)}(x) = F_A(x) \tag{4}$$

$$I_{Tc(A)}(x) = 1 - I_{TA}(x) \tag{5}$$

$$I_{Fc(A)}(x) = 1 - I_{FA}(x) \tag{6}$$

$$F_{c(A)}(x) = T_A(x) \tag{7}$$

Definition 2.

The DVNS A is contained in DVNS B $A \sqsubseteq B$ if and only if

$$T_A(x) \leq T_B(x) \tag{8}$$

$$I_{TA}(x) \leq I_{TB}(x) \tag{9}$$

$$I_{FA}(x) \leq I_{FB}(x) \tag{10}$$

$$F_A(x) \geq F_B(x) \tag{11}$$

Definition 3.

The union of two DVNSs can be defined as:

$$T_c(x) = \max\big(T_A(x), T_B(x)\big) \tag{12}$$

$$I_{Tc}(x) = \max\big(I_{TA}(x), I_{TB}(x)\big) \tag{13}$$

$$I_{Fc}(x) = \max\big(I_{FA}(x), I_{FB}(x)\big) \tag{14}$$

$$F_c(x) = \min\big(F_A(x), F_B(x)\big) \tag{15}$$

Definition 4.

The intersection of two DVNSs can be defined as:

$$T_c(x) = \min\big(T_A(x), T_B(x)\big) \tag{16}$$

$$I_{Tc}(x) = \min\left(I_{TA}(x), I_{TB}(x)\right) \tag{17}$$

$$I_{Fc}(x) = \min\left(I_{FA}(x), I_{FB}(x)\right) \tag{18}$$

$$F_c(x) = \min\left(F_A(x), F_B(x)\right) \tag{19}$$

Definition 5.

The difference between the two DVNSs can be defined as:

$$T_c(x) = \min\left(T_A(x), F_B(x)\right) \tag{20}$$

$$I_{Tc}(x) = \min\left(I_{TA}(x), 1 - I_{TB}(x)\right) \tag{21}$$

$$I_{Fc}(x) = \min\left(I_{FA}(x), 1 - I_{FB}(x)\right) \tag{22}$$

$$F_c(x) = \min\left(F_A(x), T_B(x)\right) \tag{23}$$

Definition 6.

The falsity favorite of DVNS can be defined as:

$$T_B(x) = T_A(x) \tag{24}$$

$$I_{TB}(x) = 0 \tag{25}$$

$$I_{FB}(x) = 0 \tag{26}$$

$$F_B(x) = \min(F_A(x) + I_{FA}(x), 1) \tag{28}$$

Definition 7.

The truth favorite of DVNS can be defined as:

$$T_B(x) = \min(T_A(x) + I_{TA}(x), 1) \tag{29}$$

$$I_{TB}(x) = 0 \tag{30}$$

$$I_{FB}(x) = 0 \tag{31}$$

$$F_B(x) = F_A(x) \tag{32}$$

Definition 8.

The indeterminate nature of DVNS can be defined as:

$$T_B(x) = T_A(x) \tag{33}$$

$$I_{TB}(x) = \min(I_{TA}(x) + I_{TB}(x), 1) \tag{34}$$

$$I_{FB}(x) = 0 \tag{35}$$

$$F_B(x) = F_A(x) \tag{36}$$

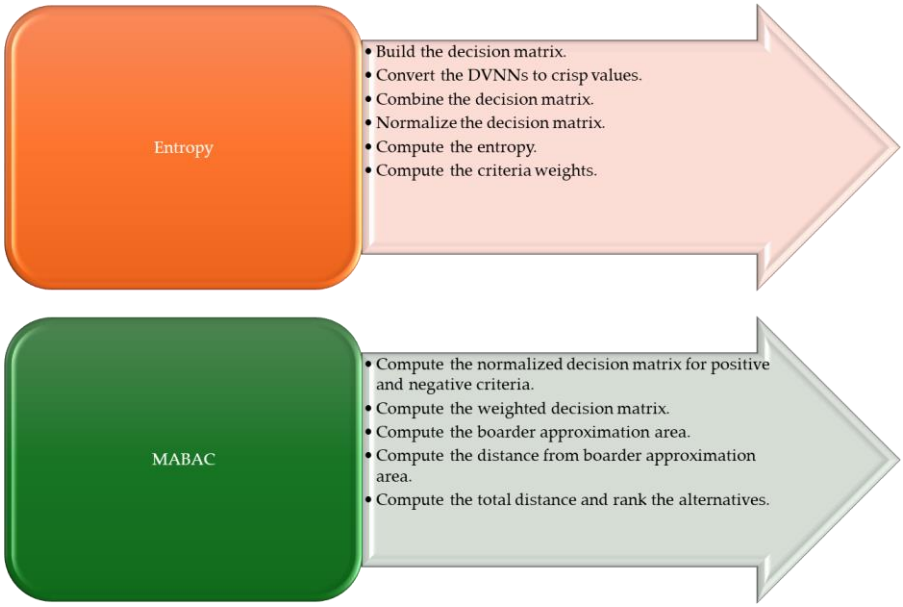
Entropy
• Build the decision matrix.
• Convert the DVNNs to crisp values.
• Combine the decision matrix.
• Normalize the decision matrix.
• Compute the entropy.
• Compute the criteria weights.
MABAC
• Compute the normalized decision matrix for positive and negative criteria.
• Compute the weighted decision matrix.
• Compute the boarder approximation area.
• Compute the distance from boarder approximation area.
• Compute the total distance and rank the alternatives.

Figure 1. The DVNS-Entropy-MABAC Method.

3. DVNS-Entropy-MABAC Approach

This section shows the steps of the DVNS-Entropy-MABAC approach to the MCDM problem. The entropy method is used to compute the criteria weights. The MABAC method is used to rank alternatives[13], [14]. Fig 1 shows the steps of the proposed approach.

Step 1. Build the decision matrix.

We used the double-value neutrosophic numbers (DVNNs) to evaluate the criteria and alternatives to build the decision matrix.

Step 2. Convert the DVNNs to crisp values.

Step 3. Combine the decision matrix.

The decision matrix is combined into a single matrix.

Step 4. Normalize the decision matrix.

$$q_{ij} = \frac{r_{ij}}{\sum_{i=1}^{m} q_{ij}} \tag{37}$$

Step 5. Compute the entropy

$$e_j = -h \sum_{i=1}^{m} q_{ij} \ln q_{ij} \ \ and \ h = \frac{1}{\ln(m)} \tag{38}$$

Step 6. Compute the criteria weights.

$$w_j = \frac{1-e_j}{\sum_{j=1}^{n}(1-e_j)} \tag{39}$$

Step 7. Apply the steps of the MABAC method to rank the alternatives. Compute the normalized decision matrix for positive and negative criteria such as:

$$d_{ij} = \frac{r_{ij} - \min r_i}{\max r_i - \min r_i} \tag{40}$$

$$d_{ij} = \frac{r_{ij} - \max r_i}{\min r_i - \max r_i} \tag{41}$$

Step 8. Compute the weighted decision matrix

$$O_{ij} = w_j + w_j d_{ij} \tag{42}$$

Step 9. Compute the boarder approximation area

$$k_j = \left(\prod_{i=1}^{m} O_{ij}\right)^{\frac{1}{m}} \tag{43}$$

Step 10. Compute the distance from k_i

$$T_{ij} = Q_{ij} - k_i \tag{44}$$

Step 11. Compute the total distance and rank the alternatives

$$S_i = \sum_{j=1}^{n} T_{ij} \tag{45}$$

4. Illustrative Example

The digital economy plays a vital role in enabling the high-quality development of cultural and tourism integration. By leveraging cutting-edge technologies such as big data, artificial intelligence, and blockchain, it enhances the efficiency, innovation, and sustainability of cultural and tourism industries. Digital platforms enable better resource allocation, providing tourists with personalized experiences and seamless services. Smart tourism applications, such as virtual reality (VR) and augmented reality (AR), allow the immersive exploration of cultural heritage, enriching the visitor experience while preserving historical assets. The digital economy also fosters new business models, such as online travel services, digital marketing, and e-commerce for cultural products, expanding market reach and boosting revenue. Data-driven insights improve decision-making for destinations, aiding in the optimization of tourism capacity and environmental sustainability. Moreover, digital tools promote cultural exchange by making intangible heritage more accessible worldwide. Evaluating this integration focuses on indicators such as technological innovation, cultural preservation, economic impact, and consumer satisfaction. High-quality development is reflected in the sustainability of tourism ecosystems, the depth of cultural engagement, and the balance between economic growth and cultural heritage protection. Ultimately, the digital economy empowers the cultural and tourism sectors to align with the demands of modern consumers, driving growth and innovation. This section shows the results of the two MCDM methods to compute the criteria weights and rank the alternatives. Three experts and decision-makers who have experience in the field of the MCDM issue evaluated the criteria and alternatives. We gathered ten criteria and six alternatives from the

previous studies as shown in Fig 2. Then we evaluated the decision matrix by the DVNNs as shown in Tables 1-3. Then we obtained crisp values. Then we combined these values into a single matrix.

Table 1. The first DVNNs

	A_1	A_2	A_3	A_4	A_5	A_6
C_1	(0.9, 0.2, 0.3, 0.0)	(0.2, 0.1, 0.2, 0.7)	(0.9, 0.2, 0.3, 0.0)	(0.5, 0.1, 0.2, 0.4)	(0.2, 0.4, 0.3, 0.4)	(0.4, 0.2, 0.3, 0.4)
C_2	(0.8, 0.1, 0.2, 0.1)	(0.9, 0.2, 0.3, 0.0)	(0.2, 0.1, 0.2, 0.7)	(0.9, 0.2, 0.3, 0.0)	(0.3, 0.2, 0.1, 0.5)	(0.8, 0.1, 0.2, 0.1)
C_3	(0.6, 0.3, 0.2, 0.3)	(0.2, 0.4, 0.3, 0.4)	(0.3, 0.2, 0.1, 0.5)	(0.9, 0.2, 0.3, 0.0)	(0.2, 0.1, 0.2, 0.7)	(0.6, 0.3, 0.2, 0.3)
C_4	(0.7, 0.2, 0.3, 0.1)	(0.4, 0.2, 0.3, 0.4)	(0.8, 0.1, 0.2, 0.1)	(0.9, 0.2, 0.3, 0.0)	(0.2, 0.1, 0.2, 0.7)	(0.7, 0.2, 0.3, 0.1)
C_5	(0.4, 0.2, 0.3, 0.4)	(0.7, 0.2, 0.3, 0.1)	(0.9, 0.2, 0.3, 0.0)	(0.2, 0.1, 0.2, 0.7)	(0.5, 0.1, 0.2, 0.4)	(0.4, 0.2, 0.3, 0.4)
C_6	(0.2, 0.4, 0.3, 0.4)	(0.6, 0.3, 0.2, 0.3)	(0.9, 0.2, 0.3, 0.0)	(0.2, 0.1, 0.2, 0.7)	(0.9, 0.2, 0.3, 0.0)	(0.2, 0.4, 0.3, 0.4)
C_7	(0.5, 0.1, 0.2, 0.4)	(0.8, 0.1, 0.2, 0.1)	(0.4, 0.2, 0.3, 0.4)	(0.9, 0.2, 0.3, 0.0)	(0.9, 0.2, 0.3, 0.0)	(0.2, 0.1, 0.2, 0.7)
C_8	(0.9, 0.2, 0.3, 0.0)	(0.3, 0.2, 0.1, 0.5)	(0.2, 0.4, 0.3, 0.4)	(0.9, 0.2, 0.3, 0.0)	(0.2, 0.1, 0.2, 0.7)	(0.9, 0.2, 0.3, 0.0)
C_9	(0.9, 0.2, 0.3, 0.0)	(0.2, 0.1, 0.2, 0.7)	(0.5, 0.1, 0.2, 0.4)	(0.2, 0.4, 0.3, 0.4)	(0.6, 0.3, 0.2, 0.3)	(0.3, 0.2, 0.1, 0.5)
C_{10}	(0.8, 0.1, 0.2, 0.1)	(0.5, 0.1, 0.2, 0.4)	(0.9, 0.2, 0.3, 0.0)	(0.2, 0.1, 0.2, 0.7)	(0.7, 0.2, 0.3, 0.1)	(0.8, 0.1, 0.2, 0.1)

Table 2. The second DVNNs

	A_1	A_2	A_3	A_4	A_5	A_6
C_1	(0.9, 0.2, 0.3, 0.0)	(0.2, 0.1, 0.2, 0.7)	(0.5, 0.1, 0.2, 0.4)	(0.9, 0.2, 0.3, 0.0)	(0.3, 0.2, 0.1, 0.5)	(0.4, 0.2, 0.3, 0.4)
C_2	(0.8, 0.1, 0.2, 0.1)	(0.9, 0.2, 0.3, 0.0)	(0.2, 0.1, 0.2, 0.7)	(0.2, 0.4, 0.3, 0.4)	(0.5, 0.1, 0.2, 0.4)	(0.9, 0.2, 0.3, 0.0)
C_3	(0.6, 0.3, 0.2, 0.3)	(0.2, 0.4, 0.3, 0.4)	(0.9, 0.2, 0.3, 0.0)	(0.2, 0.1, 0.2, 0.7)	(0.9, 0.2, 0.3, 0.0)	(0.3, 0.2, 0.1, 0.5)
C_4	(0.7, 0.2, 0.3, 0.1)	(0.2, 0.4, 0.3, 0.4)	(0.5, 0.1, 0.2, 0.4)	(0.9, 0.2, 0.3, 0.0)	(0.2, 0.1, 0.2, 0.7)	(0.7, 0.2, 0.3, 0.1)
C_5	(0.4, 0.2, 0.3, 0.4)	(0.2, 0.4, 0.3, 0.4)	(0.5, 0.1, 0.2, 0.4)	(0.9, 0.2, 0.3, 0.0)	(0.3, 0.2, 0.1, 0.5)	(0.4, 0.2, 0.3, 0.4)
C_6	(0.2, 0.4, 0.3, 0.4)	(0.5, 0.1, 0.2, 0.4)	(0.9, 0.2, 0.3, 0.0)	(0.2, 0.1, 0.2, 0.7)	(0.9, 0.2, 0.3, 0.0)	(0.2, 0.4, 0.3, 0.4)
C_7	(0.5, 0.1, 0.2, 0.4)	(0.2, 0.4, 0.3, 0.4)	(0.5, 0.1, 0.2, 0.4)	(0.9, 0.2, 0.3, 0.0)	(0.3, 0.2, 0.1, 0.5)	(0.5, 0.1, 0.2, 0.4)
C_8	(0.9, 0.2, 0.3, 0.0)	(0.3, 0.2, 0.1, 0.5)	(0.2, 0.4, 0.3, 0.4)	(0.5, 0.1, 0.2, 0.4)	(0.9, 0.2, 0.3, 0.0)	(0.3, 0.2, 0.1, 0.5)
C_9	(0.3, 0.2, 0.1, 0.5)	(0.9, 0.2, 0.3, 0.0)	(0.9, 0.2, 0.3, 0.0)	(0.2, 0.1, 0.2, 0.7)	(0.5, 0.1, 0.2, 0.4)	(0.9, 0.2, 0.3, 0.0)
C_{10}	(0.8, 0.1, 0.2, 0.1)	(0.9, 0.2, 0.3, 0.0)	(0.2, 0.1, 0.2, 0.7)	(0.5, 0.1, 0.2, 0.4)	(0.9, 0.2, 0.3, 0.0)	(0.3, 0.2, 0.1, 0.5)

Table 3. The third DVNNs

	A_1	A_2	A_3	A_4	A_5	A_6
C_1	(0.2, 0.4, 0.3, 0.4)	(0.6, 0.3, 0.2, 0.3)	(0.9, 0.2, 0.3, 0.0)	(0.2, 0.1, 0.2, 0.7)	(0.2, 0.4, 0.3, 0.4)	(0.4, 0.2, 0.3, 0.4)
C_2	(0.8, 0.1, 0.2, 0.1)	(0.9, 0.2, 0.3, 0.0)	(0.2, 0.1, 0.2, 0.7)	(0.7, 0.2, 0.3, 0.1)	(0.3, 0.2, 0.1, 0.5)	(0.8, 0.1, 0.2, 0.1)
C_3	(0.6, 0.3, 0.2, 0.3)	(0.2, 0.4, 0.3, 0.4)	(0.9, 0.2, 0.3, 0.0)	(0.2, 0.1, 0.2, 0.7)	(0.7, 0.2, 0.3, 0.1)	(0.3, 0.2, 0.1, 0.5)
C_4	(0.7, 0.2, 0.3, 0.1)	(0.9, 0.2, 0.3, 0.0)	(0.2, 0.1, 0.2, 0.7)	(0.6, 0.3, 0.2, 0.3)	(0.7, 0.2, 0.3, 0.1)	(0.3, 0.2, 0.1, 0.5)
C_5	(0.9, 0.2, 0.3, 0.0)	(0.2, 0.1, 0.2, 0.7)	(0.6, 0.3, 0.2, 0.3)	(0.7, 0.2, 0.3, 0.1)	(0.3, 0.2, 0.1, 0.5)	(0.4, 0.2, 0.3, 0.4)
C_6	(0.2, 0.4, 0.3, 0.4)	(0.6, 0.3, 0.2, 0.3)	(0.7, 0.2, 0.3, 0.1)	(0.3, 0.2, 0.1, 0.5)	(0.9, 0.2, 0.3, 0.0)	(0.2, 0.4, 0.3, 0.4)
C_7	(0.5, 0.1, 0.2, 0.4)	(0.9, 0.2, 0.3, 0.0)	(0.2, 0.1, 0.2, 0.7)	(0.6, 0.3, 0.2, 0.3)	(0.7, 0.2, 0.3, 0.1)	(0.3, 0.2, 0.1, 0.5)
C_8	(0.9, 0.2, 0.3, 0.0)	(0.2, 0.1, 0.2, 0.7)	(0.7, 0.2, 0.3, 0.1)	(0.3, 0.2, 0.1, 0.5)	(0.8, 0.1, 0.2, 0.1)	(0.9, 0.2, 0.3, 0.0)
C_9	(0.3, 0.2, 0.1, 0.5)	(0.9, 0.2, 0.3, 0.0)	(0.2, 0.1, 0.2, 0.7)	(0.7, 0.2, 0.3, 0.1)	(0.3, 0.2, 0.1, 0.5)	(0.3, 0.2, 0.1, 0.5)
C_{10}	(0.8, 0.1, 0.2, 0.1)	(0.5, 0.1, 0.2, 0.4)	(0.2, 0.4, 0.3, 0.4)	(0.6, 0.3, 0.2, 0.3)	(0.7, 0.2, 0.3, 0.1)	(0.3, 0.2, 0.1, 0.5)

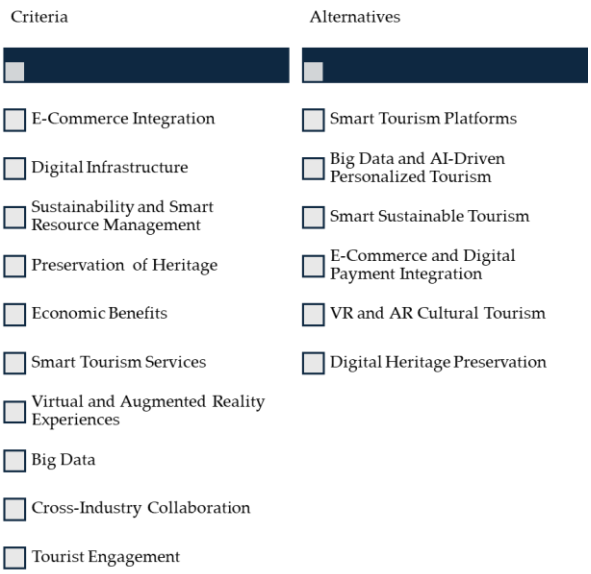

Figure 2. The MCDM criteria and alternatives.

Eq. (37) is used to normalize the decision matrix by the entropy method as shown in Table 4. Then we used Eq. (38) to compute the entropy value. Then we compute the criteria weights by using Eq. (39) as shown in Fig 3.

Table 4. The normalized DVNNs using the Entropy method.

	A_1	A_2	A_3	A_4	A_5	A_6
C_1	0.197889	0.137203	0.200528	0.163588	0.150396	0.150396
C_2	0.188406	0.202899	0.101449	0.173913	0.140097	0.193237
C_3	0.18509	0.14653	0.192802	0.143959	0.172237	0.159383
C_4	0.186104	0.163772	0.148883	0.198511	0.131514	0.171216
C_5	0.174603	0.153439	0.190476	0.177249	0.153439	0.150794
C_6	0.14467	0.172589	0.205584	0.119289	0.213198	0.14467
C_7	0.153453	0.186701	0.13555	0.204604	0.184143	0.13555
C_8	0.205379	0.127139	0.154034	0.163814	0.166259	0.183374
C_9	0.171429	0.181818	0.161039	0.150649	0.163636	0.171429
C_{10}	0.191646	0.167076	0.149877	0.142506	0.191646	0.157248

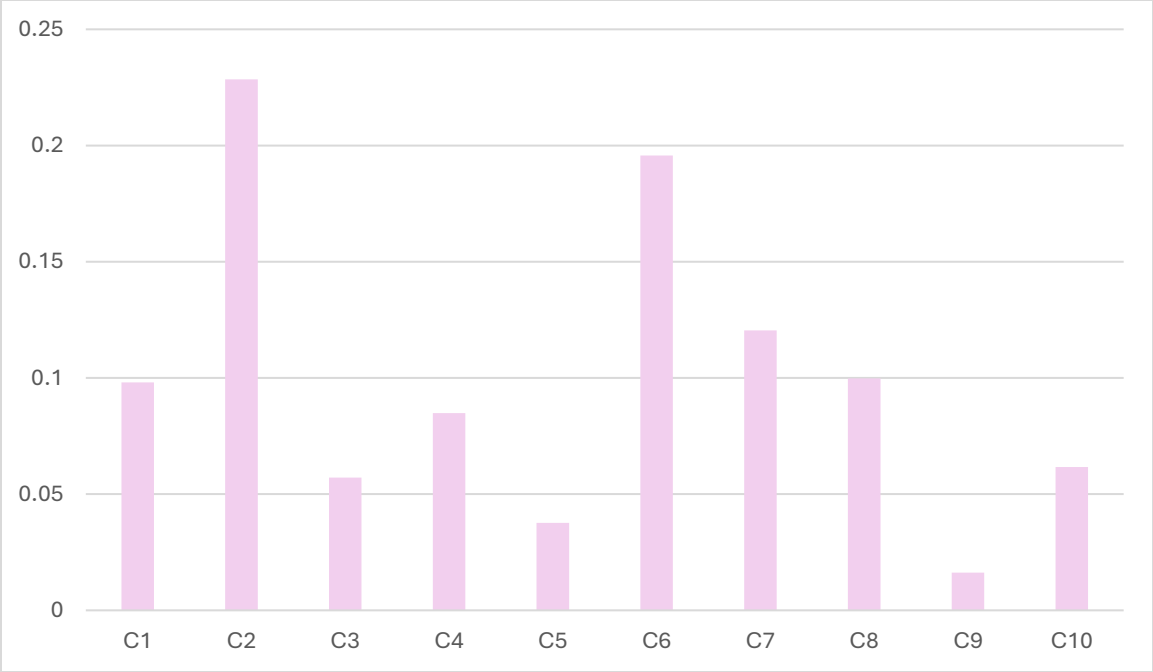

Figure 3. The criteria weights.

Then we used Eq. (40) to normalize the decision matrix by the MABAC method as shown in Table 5. Then we used Eq. (42) to compute the weighted decision matrix as shown in Table 6. Then we used Eq. (43) to compute the values of k_j. Then we used Eq. (44) to compute the distance from k_j as shown in Table 7. Then we compute the total distance as shown in Table 8. Then we ranked the alternatives as shown in Fig 4.

Table 5. The normalized DVNNs using the MABAC method.

	A_1	A_2	A_3	A_4	A_5	A_6
C_1	0.958333	0	1	0.416667	0.208333	0.208333
C_2	0.857143	1	0	0.714286	0.380952	0.904762
C_3	0.842105	0.052632	1	0	0.578947	0.315789
C_4	0.814815	0.481481	0.259259	1	0	0.592593
C_5	0.6	0.066667	1	0.666667	0.066667	0
C_6	0.27027	0.567568	0.918919	0	1	0.27027
C_7	0.259259	0.740741	0	1	0.703704	0
C_8	1	0	0.34375	0.46875	0.5	0.71875
C_9	0.666667	1	0.333333	0	0.416667	0.666667
C_{10}	1	0.5	0.15	0	1	0.3

Table 6. The weighted DVNNs using the MABAC method.

	A_1	A_2	A_3	A_4	A_5	A_6
C_1	0.192015	0.09805	0.1961	0.138904	0.118477	0.118477
C_2	0.424324	0.456965	0.228482	0.391684	0.315523	0.435204

C_3	0.105243	0.060139	0.114263	0.057132	0.090208	0.075173
C_4	0.154021	0.125731	0.106872	0.169738	0.084869	0.135161
C_5	0.060166	0.040111	0.075207	0.062673	0.040111	0.037604
C_6	0.248578	0.306755	0.375511	0.195689	0.391378	0.248578
C_7	0.151706	0.209712	0.120473	0.240946	0.20525	0.120473
C_8	0.199674	0.099837	0.134156	0.146635	0.149755	0.171595
C_9	0.026996	0.032395	0.021596	0.016197	0.022946	0.026996
C_{10}	0.123335	0.092502	0.070918	0.061668	0.123335	0.080168

Table 7. The distance from k_j.

	A_1	A_2	A_3	A_4	A_5	A_6
C_1	0.05306	-0.0409	0.057145	-5.1E-05	-0.02048	-0.02048
C_2	0.058857	0.091497	-0.13699	0.026216	-0.04994	0.069737
C_3	0.02435	-0.02075	0.033371	-0.02376	0.009316	-0.00572
C_4	0.027874	-0.00042	-0.01928	0.04359	-0.04128	0.009014
C_5	0.00938	-0.01067	0.024422	0.011887	-0.01067	-0.01318
C_6	-0.03724	0.020941	0.089697	-0.09013	0.105564	-0.03724
C_7	-0.01673	0.041276	-0.04796	0.07251	0.036814	-0.04796
C_8	0.052705	-0.04713	-0.01281	-0.00033	0.002787	0.024626
C_9	0.003029	0.008428	-0.00237	-0.00777	-0.00102	0.003029
C_{10}	0.034445	0.003611	-0.01797	-0.02722	0.034445	-0.00872

Table 8. The distance from k_j.

	Total distance
A_1	0.209735
A_2	0.045873
A_3	-0.03274
A_4	0.004942
A_5	0.065529
A_6	-0.02689

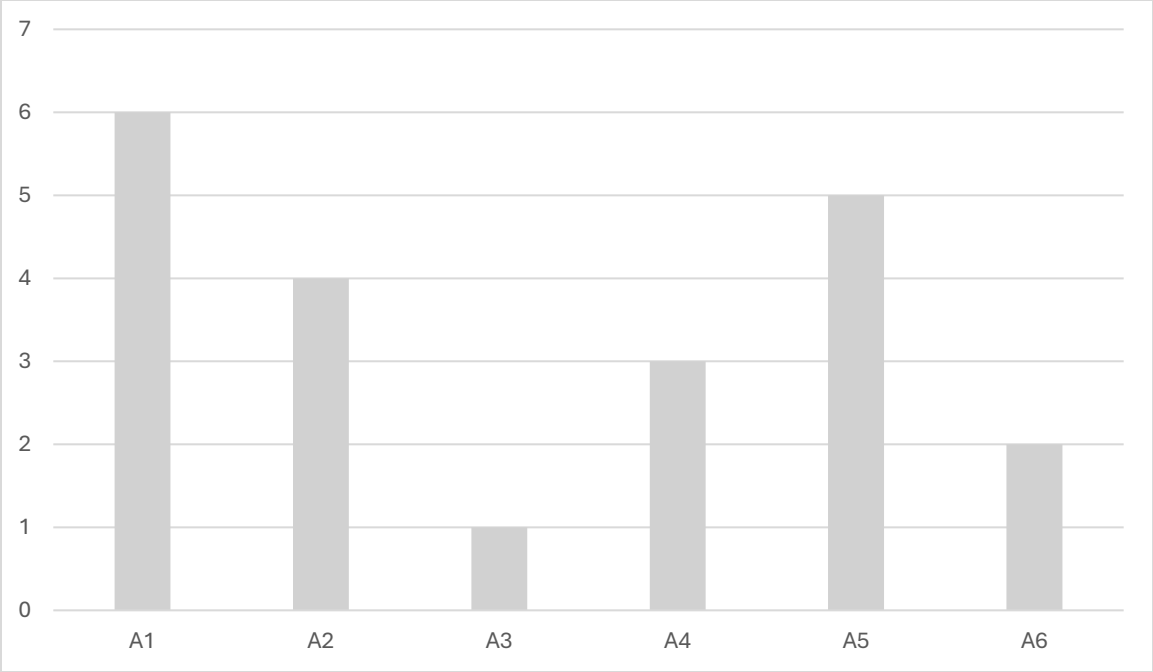

Fig 4. The rank of alternatives.

Sensitivity analysis

This part shows the sensitivity analysis between the rank of alternatives by changing the criteria weights under different cases. We change the criteria weights by the 11 cases as shown in Fig 5. In the first case, we put all criteria weights by the same weights. Then in the second case, we increase the first criterion by 22% and other criteria with the same weights. Then in the third case, we increase the second criterion by 22% and other criteria with the same weights. Then in the fourth case, we increase the third criterion by 22% and other criteria with the same weights.

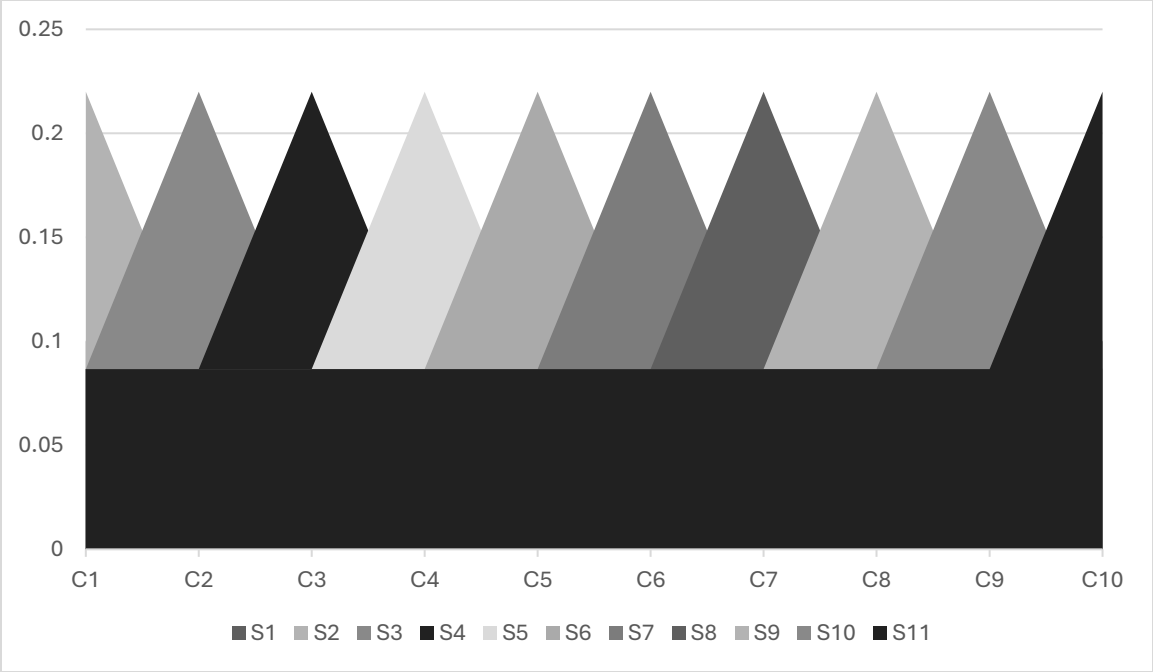

Figure 5. The different criteria weights.

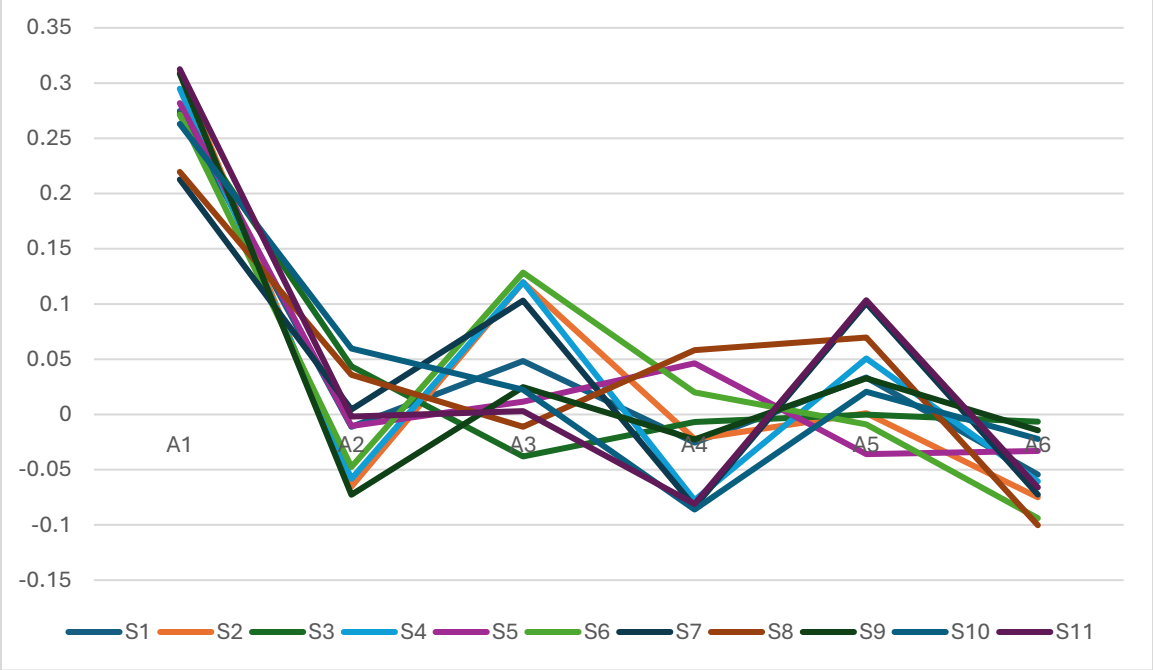

Figure 6. The total distance with different weights.

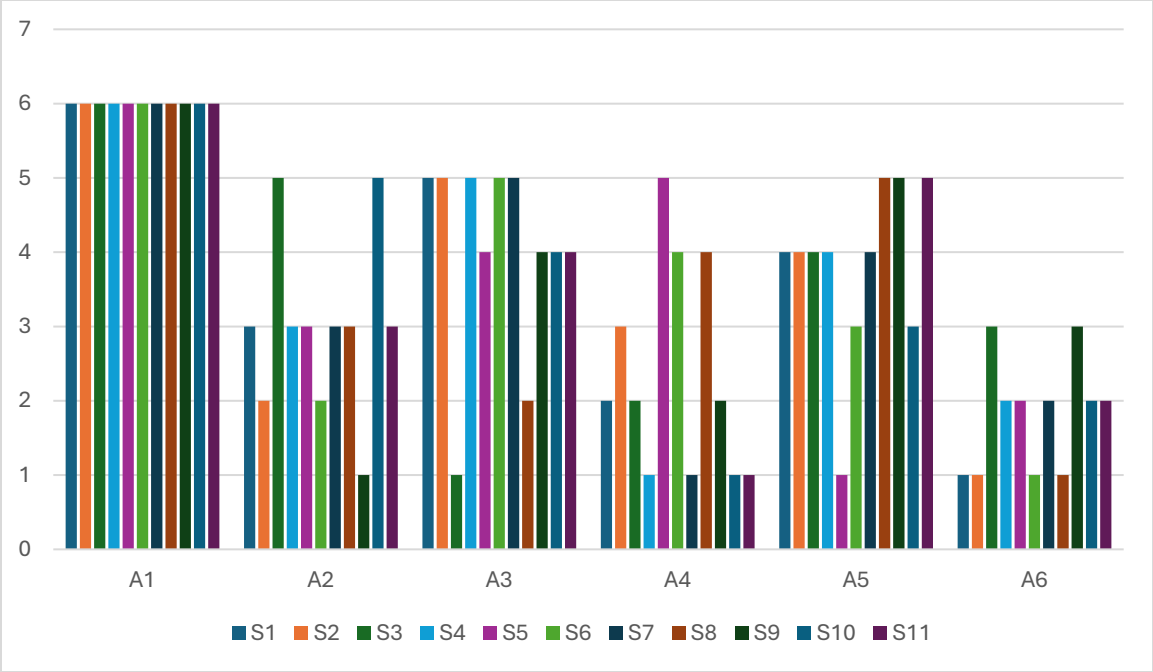

Figure 7. The rank of alternatives under different weights.

Then we rank the alternatives by 11 cases by obtaining the total distance as shown in Fig 6. The results show alternative 1 is the best and alternative 3 is the worst. We show that alternative 1 is the best in all cases. So, the ranks of alternatives are stale under different cases as Fig 7.

5. Conclusions

Uncertain, inconsistent, and indeterminate information that arises in real-world problems is illustrated via neutrophilic sets (NSs). A kind of neutrosophic sets known as double-valued neutrosophic sets (DVNSs) include indeterminacy that is composed of two different components: indeterminacy that leans toward truth membership and indeterminacy that leans toward falsehood membership. This article's objective was to evaluate the cultural and Tourism Integration Empowered by the Digital Economy. This study proposed two methods such as the Entropy method to compute the criteria weights and rank the alternatives. The two methods are used under the DVNSs. The results show that alternative 1 is the best and alternative 3 is the worst.

Acknowledgments

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Isotopic Properties of Neutrosophic Soft Quasigroup and Its Application in Decision-making

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Abstract. A Q-neutrosophic soft quasigroup $(\phi^Q, \mathfrak{A})$ represents a novel mathematical framework designed to address scenarios characterized by indeterminate occurrences. This paper seeks to extend the classical examination of quasigroups to include Q-neutrosophic soft quasigroups. It provides a thorough investigation into isotopism, homomorphism, isomorphism, and the direct product concerning the hybrid concept of neutrosophic soft sets under a non-associative structure. This research expands the concept of the Q-set to a groupoid, which facilitates the development of an algorithm for Q-neutrosophic soft quasigroups. The definitions of isotopism and homomorphism for Q-neutrosophic soft quasigroups are introduced, revealing that every isotope of a Q-neutrosophic soft quasigroup is indeed a Q-neutrosophic soft quasigroup. Furthermore, it is established that the homomorphic image of a Q-neutrosophic soft quasigroup under a quasigroup is not necessarily a Q-neutrosophic soft quasigroup. The findings indicate that the direct product of any two Q-neutrosophic soft quasigroups is also a Q-neutrosophic soft quasigroup. Generalized properties regarding the direct product of Q-neutrosophic soft quasigroups of order n have been established. Additionally, the study presents the necessary and sufficient conditions under which the direct product of two Q-neutrosophic soft subquasigroups, $\psi^Q(a)\times\psi^Q(b)$, contains at least one Q-neutrosophic soft subquasigroup, $\pi_Q(c)$, that is isomorphic to the Q-neutrosophic soft subquasigroup $\pi_Q(a)$. Specifically, $\psi^Q(a) \cong \pi_Q(c) \leq \psi^Q(a)\times\psi^Q(b)$ if and only if a homomorphism exists from $\psi^Q(a)$ to $\psi^Q(b)$. To validate some of these results, examples have been constructed, and a practical application utilizing real-life data has been demonstrated.

Keywords: Isotopism; homomorphism; Isomorphism; Soft set; Neutrosophic set; Quasigroup

1. Introduction

Suppose that $\hat{G}$ is a non-empty set and the binary operation $(\circ)$ is define on 'Q such that $d \circ w \in \hat{G}$ for all $d, w \in \hat{G}$, the pair $(\hat{G}, \circ)$ is called a *groupoid*. If the equations:

$$\alpha \circ d = \beta \text{ and } w \circ \alpha = \beta$$

has unique solutions $d, w \in \hat{G}$ for all $\alpha, \beta \in \hat{G}$, then $(\hat{G}, \circ)$ is called quasigroup. Suppose there is a unique element $1 \in \hat{G}$ called the identity element such that $1 \circ d = d \circ 1 = d$ for all $d \in \hat{G}$, then $(\hat{G}, \circ)$ is a loop. In this research, we write dw instead of $d \circ w$, and stipulate that $\circ$ has lower priority than juxtaposition among factors to be multiplied. Suppose that d is a fixed element in a quasigroup $(\hat{G}, \circ)$. Then, the left and right translation maps for all $d \in \hat{G}$, written as L_d and R_d respectively are defined by

$$wL_d = d \circ w \qquad \text{and} \qquad wR_d = w \circ d.$$

It is shown that a groupoid $(\hat{G}, \circ)$ is a quasigroup if the left and right translation maps are bijective. Hence, the inverse mappings L_d^{-1} and R_d^{-1} also exist. Thus,

$$d\backslash w = wL_d^{-1} \qquad \text{and} \qquad d/w = dR_w^{-1}.$$

In his publication [20], Molodtsov introduced soft set theory as a mathematical tool for managing uncertainty. Algebra experts have subsequently applied this concept to analyze the structural characteristics of algebraic structures. For example, Oyem delved into the algebraic properties of soft sets within the non-associative structure known as quasigroup [24,25]. Soft set theory is esteemed for its capacity to accommodate a broad spectrum of information for decision-making in real-world scenarios. Nonetheless, the characterization of membership degrees in neutrosophic sets does not extend to the realm of soft set theory. Consequently, soft set theory may not be suitable for resolving issues involving indeterminate data.

A neutrosophic set, introduced by Smarandache in 1998, is a mathematical framework designed to handle real-life data with indeterminate characteristics. Unlike classical set theory, which deals with precise data, neutrosophic sets are well-suited for addressing problems involving uncertainty and indeterminate. The structure of a neutrosophic set is defined by three distinct membership degrees: true, indeterminate, and false, denoted as T, I, and F, respectively.

In recent times, there has been a surge of interest in the study of neutrosophic sets combined with soft set theory in the field of mathematics [2–6,19,21,26,31]. This combined approach offers a generalized framework for addressing uncertainties and indeterminacy in real-world problems. For instance, the $Q-$neutrosophic soft set extends the concept of the neutrosophic soft set by accommodating two universal sets, allowing for the simultaneous handling of indeterminate memberships from both sets.

Furthermore, researchers in [16] introduced a novel concept of neutrosophic logic in the development of an inspection assignment form for product quality control. Additionally, [14] explored characterizations of separation axioms in neutrosophic topological spaces, while the study in [15] presented the application of neutrosophic methods in operations research for managing corporate

The concept research on a neutrosophic soft set of two universal sets was first introduced in [9]. Various researchers have explored its applications in associative structures such as fields, groups, rings, and modules [7,9]. Oyebo et al. extended the $Q-$neutrosophic soft group to a $Q-$neutrosophic soft quasigroup, which is a generalization of the former. Additionally, Osoba et al. conducted a further investigation into the properties of a $Q-$neutrosophic soft quasigroup and loop [22, 23]. For more on recent works in neutrosophic structure, the reader should check [17, 18]. It is particularly noteworthy that this research marks the first extension of the Q-set to groupoids and uses it to construct an algorithm for decision-making, which inherently lacks the associative property.

The motivation behind this study is as follows: The work of Abu-Qamar and Hassan [2, 7] reports that the homomorphic images of certain classes of associative algebra—specifically, $Q-$neutrosophic soft fields, $Q-$neutrosophic soft rings, and $Q-$neutrosophic soft groups—remain within their respective categories. This assertion also holds true within classical studies on fields, rings, and groups. Moreover, in classical quasigroup studies, it is well established that the homomorphic image of a quasigroup is not necessarily a quasigroup itself (Pflugfelder, [29]). Additionally, while it is known that the direct product of two groups (or loops) P and H must incorporate a subgroup (or subloop) isomorphic to P—specifically, $P \times \{1\}$—this is not the case in classical quasigroup studies (see Fuval Foguel [12], 2008).

In line with the above motivation, the objectives of the research are as follows:

(1) to carry out the homomorphic characterization of $Q-$NS quasigroups and by extension study their homomorphic images and pre-image.
(2) to examine the direct product of two $Q-$NS quasigroups.
(3) to construct an example of $Q-$NS quasigroups useable to make decision in finance, particularity in stock-exchange related problems

Therefore, with the above objective, this study aims to close some of the existing knowledge gaps by investigating the following research questions:

- is the direct product of two $Q-$neutrosophic soft quasigroups also a $Q-$neutrosophic soft quasigroup?
- does the direct product $(\Psi^Q, A) \times (\Lambda^Q, B)$ of two $Q-$neutrosophic soft quasigroups isomorphic (Ψ^Q, A)?

- is the isotope of $Q-$neutrosophic soft quasigroups also a $Q-$neutrosophic soft quasigroup?
- are the isomorphic images and homomorphic images of $Q-$neutrosophic soft quasigroup also $Q-$neutrosophic soft quasigroup?
- is it possible to create a practical example that can be used to address real-life financial problems requiring decision-making?

This paper introduces a new study area by exploring $Q-$neutrosophic soft set under a non-associative algebra known as quasigroup, expanding on previous research in the field. While many authors have investigated the homomorphism of $Q-$neutrosophic soft sets under associative structures such as groups, rings, modules, and fields, it is worth noting that the homomorphic images of $Q-$neutrosophic soft groups, rings, and fields are indeed $Q-$neutrosophic soft groups, rings, and fields, as demonstrated by the authors in [7]. However, this research shall provide the solution to each research question mentioned above.

Definition 1.1. [13] Let $(\hat{G}, \circ)$ be a groupoid (quasigroup) and $\emptyset \neq H \subseteq \hat{G}$. Then, H is called subgroupoid (subquasigroup) of $\hat{G}$ if $(H, \circ)$ is a groupoid(quasigroup) and it is denoted as $H \leq \hat{G}$. Let S and T be non empty subsets of $\hat{G}$, then $S \circ T = \{s \circ t \mid s \in S, t \in T\}, S/T = \{s/t \mid s \in S, t \in T\}$ and $S\backslash T = \{s\backslash t \mid s \in S, t \in T\}$.

Definition 1.2. [20] Let W be set, a pair $(F, \mathfrak{A})$ is called a soft set if $F : \mathfrak{A} \to P(W)$, where $P(W)$ is power set of W and $\mathfrak{A}$ is a set of parameters.

Definition 1.3. [28] Let W be a set. A neutrosophic set (NS) is described as $\phi = \{\langle w, (T_\phi(w), I_\phi(w), F_\phi(w))\rangle : w \in W\}$ such that $T_\phi, I_\phi, F_\phi : W \to]^-0, 1^+[$. where T, I, F are the truth, indeterminate and falsity membership degrees respectively.

Definition 1.4. [2] Let Q be a non-empty set, a $Q-$neutrosophic set ϕ^Q in W and Q is described in the form $\phi^Q = \{\langle (w, u), T_{\phi^Q}(w, u), I_{\phi^Q}(w, u), F_{\phi^Q}(w, u)\rangle : w \in W, u \in Q\}$, where $T_{\phi^Q}, I_{\phi^Q}, F_{\phi^Q} : W \times Q \to]^-0, 1^+[$ are the membership degrees.

Definition 1.5. [21] Let W be a set and $\mathfrak{A}$ be a parameter sets. A neutrosophic soft set $(\phi, \mathfrak{A})$ described as $(\phi, \mathfrak{A}) = \{\langle w, (T_\phi(w), I_\phi(w), F_\phi(w))\rangle : w \in W\}$.

Definition 1.6. [2] Let $\alpha : M \times Q \mapsto N \times Q$ and $\rho : \mathfrak{A} \mapsto \mathfrak{B}$ be two functions, where $\mathfrak{A}$ and $\mathfrak{B}$ are parametric sets for $M \times Q$ and $N \times Q$ respectively. Then, (α, ρ) is called a $Q-$neutrosophic soft mapping or function from the set $M \times Q$ to the set $N \times Q$.

Definition 1.7. [13] A triple (A, B, C) of bijection from a set M onto a set N is called an isotopism of a groupoid $(M, \circ)$ onto a groupiod $(N, \odot)$ if for all $x, y \in M$, $xA \circ yB = (x \odot y)C$.

2. Properties of Quasigroup under Neutrosophic Soft set

Definition 2.1. Let $(\phi^Q, \mathfrak{A})$ be a $Q-$neutrosophic soft set defined over a quasigroup $(\hat{G}, *)$. Then, $(\phi^Q, \mathfrak{A})$ is called a $Q-$neutrosophic soft subquasigroup over a quasigroup $\hat{G}$ if $\phi^Q(a)$ is a $Q-$neutrosophic soft quasigroup given a mapping $\phi^Q(a) : \hat{G} \times Q \to [0,1]^3$ for all $a \in \mathfrak{A}, v \in Q$.

Definition 2.2. Let $(\phi^Q, \mathfrak{A})$ be a $Q-$neutrosophic soft set defined under a quasigroup $(\hat{G}, \circ, /, \backslash)$. Then, $(\phi^Q, \mathfrak{A})$ is called a $Q-$neutrosphic soft subquasigroup of quasigroup $\hat{G}$ if for all $a \in \mathfrak{A}, r_1, r_2 \in \hat{G}, v \in Q$ satisfies the following conditions

(1) $T_{\phi^Q(a)}(r_1 * r_2, v) \geq \min\{T_{\phi^Q(a)}(r_1, v), T_{\phi^Q(a)}(r_2, v)\}$
(2) $I_{\phi^Q(a)}(r_1 * r_2, v) \leq \max\{I_{\phi^Q(a)}(r_1, v), I_{\phi^Q(a)}(r_1, v)\}$
(3) $F_{\phi^Q(a)}(r_1 * r_2, v) \leq \max\{F_{\phi^Q(a)}(r_1, v), F_{\phi^Q(a)}(r_2, v)\}$

where $* \in \{\circ, /, \backslash\}$.

Example 2.3. Let $\hat{G} = \{i, j, k, l, m, n, o\}$ be quasigroup of order 7 and $A \subseteq E = \hat{G}$ be the parametric set. Given the quasigeoup in Cayley table below.

Define a $Q-$ neutrosophic soft set $(\phi^Q, \mathfrak{A})$ as follows, for all $u \in Q$ and $w, t, z \in \hat{G}, A = \hat{G}$ a

TABLE 1. Quasigroup of order 7

$\odot$	i	j	k	l	m	n	o
i	i	m	o	n	j	l	k
j	m	j	n	o	i	k	l
k	o	n	k	m	l	j	i
l	n	o	m	k	l	i	j
m	j	i	l	k	m	o	n
n	l	k	j	i	o	n	m
o	k	l	i	j	n	m	o

parameters set with $z = w * t$ and $n \in \mathbb{N}$.

$$T_{\phi^Q(a)}(w \odot t, u) = \begin{cases} 1 - \frac{1}{2n}, & \text{if } z = \{j, k, l, m, n, o\} \\ 1, & \text{otherwise.} \end{cases}$$

$$I_{\phi^Q(a)}(w \odot t, u) = \begin{cases} 0, & \text{if } z = \{j, k, l, m, n, o\} \\ 1 - \frac{1}{2n}, & \text{otherwise.} \end{cases}$$

$$F_{\phi^Q(a)}(w \odot t, u) = \begin{cases} 0, & \text{if } z = \{j, k, l, m, n, o\} \\ 1 - \frac{1}{2n}, & \text{otherwise.} \end{cases}$$

$$T_{\phi^{Q}(a)}(w/t,u)=\begin{cases}1-\frac{1}{3n}, & \text{if } z=\{j,k,l,m,n,o\}\\ 0, & \text{otherwise.}\end{cases}$$

$$I_{\phi^{Q}(a)}(w/t,u)=\begin{cases}0, & \text{if } z=\{j,k,l,m,n,o\}\\ 1-\frac{1}{2n}, & \text{otherwise.}\end{cases}$$

$$F_{\phi^{Q}(a)}(w/t,u)=\begin{cases}0, & \text{if } z=\{j,k,l,m,n,o\}\\ 1-\frac{1}{2n}, & \text{otherwise.}\end{cases}$$

$$T_{\phi^{Q}(a)}(w\backslash t,u)=\begin{cases}1-\frac{1}{2n}, & \text{if } w=\{j,k,l,m,n,o\}\\ 0, & \text{otherwise.}\end{cases}$$

$$I_{\phi^{Q}(a)}(w\backslash t,u)=\begin{cases}0, & \text{if } z=\{j,k,l,m,n,o\}\\ 1-\frac{1}{2n}, & \text{otherwise.}\end{cases}$$

$$F_{\phi^{Q}(a)}(w\backslash t,u)=\begin{cases}0, & \text{if } z=\{j,k,l,m,n,o\}\\ 1-\frac{1}{2n}, & \text{otherwise.}\end{cases}$$

Let consider the operation $'\odot'$ then, $T_{\phi^{Q}(a)}(w\odot t,u)\geq\min\{T_{\phi^{Q}(a)}(w,u),T_{\phi^{Q}(a)}(t,u)\}$. Put $w=j,t=n$, then we have

$$T_{\phi^{Q}(a)}(j\odot n,u)\geq\min\{T_{\phi^{Q}(a)}(m,u),T_{\phi^{Q}(a)}(m,u)\}=$$
$$=1-\frac{1}{2n}=0.5\in[0,1]\text{ where n=1 }\in\mathbb{N}. \tag{1}$$

On the other hand,

$$\min\{T_{\phi^{Q}(a)}(j,u),T_{\phi^{Q}(a)}(n,u)\}=$$
$$\min\{0,1-\frac{1}{2n}\}=0\in[0,1]\text{ for all }n\in\mathbb{N} \tag{2}$$

Hence, $1-\frac{1}{2n}\geq 0$ for all $n\in\mathbb{N}$. This is true for membership degree.

3. Direct Product of $Q-$Neutrosophic Soft quasigroups

Let (ψ^{Q},A) and (ψ^{Q},B) be two $Q-$neutrosophic soft quasigroups, then their direct product $(\psi^{Q},A)\times(\lambda^{Q},B)$ can be constructed in the same way like in $Q-$neutrosophic soft groups, and it will be a $Q-$neutrosophic soft quasigroup. Also if (ψ^{Q},A) is a $Q-$neutrosophic soft quasigroup of order n, any direct product of (ψ^{Q},A) with another $Q-$neutrosophic soft quasigroup (ψ^{Q},B) of order m can form different range of $Q-$neutrosophic soft quasigroups of order nm, which are generalized products variously called extensions, quasidirect products or twisted products Chein et al. [11], which is an attempt to generalize the work in Brucks [10].

It has been established in Theorem 3.2 that if (ψ^Q, A) and (ψ^Q, B) are two $Q-$neutrosophic soft quasigroups over quasigroups $\hat{G}_1$ and $\hat{G}_2$ respectively, then the direct product $(\psi^Q, A) \times (\psi^Q, B)$ is a $Q-$ neutrosophic soft quasigroup.

Definition 3.1. Let (ψ^Q, A) and (ψ, B) be two $Q-$neutrosophic soft sets over a quasigroups $\hat{G}_1$ and $\hat{G}_2$ respectively. The direct product donated by $(\psi^Q, A) \times (\psi^Q, B)$ over $\hat{G}_1 \times \hat{G}_2$ such that $(\phi^Q, A) \times (\psi^Q, B) = (\pi^Q, C)$ is given by $\pi^Q(a,b) = \Big\{\langle((x_1,x_2),q), T_{\phi_Q(a,b)}((x_1,x_2),q), I_{\phi_Q(a,b)}((x_1,x_2),q), F_{\phi_Q(a,b)}((x_1,x_2),q)\rangle : (x_1,x_2) \in \hat{G}_1 \times \hat{G}_2, q \in Q\Big\}$. $T_{\pi_Q(a,b)}((x_1,x_2),q) = \min\{T_{\phi_Q(a)}(x_1,q), T_{\psi_Q(b)}(x_2,q)\}$,
$I_{\pi_Q(a,b)}((x_1,x_2),q) = \max\{I_{\phi_Q(a)}(x_1,q), I_{\psi_Q(b)}(x_2,q)\}$,
$F_{\pi_Q(a,b)}((x_1,x_2),q) = \max\{F_{\phi_Q(a)}(x_1,q), F_{\psi_Q(b)}(x_2,q))\}$.

Theorem 3.2. *Let (ψ^Q, A) and (ψ^Q, B) be two $Q-$neutrosophic soft quasigroups over quasigroups $\hat{G}_1$ and $\hat{G}_2$ respectively. Then $(\psi^Q, A) \times (\psi^Q, B) = (\pi^Q, C)$ is a $Q-$neutrosophic soft quasigroup over $(\hat{G}_1 \times \hat{G}_2)$.*

Proof. Let $(\psi^Q, A) \times (\psi^Q, B) = (\pi_Q, C)$ such that $\psi^Q(a) \times \psi^Q(b) = \pi_Q(a,b)$, for all $a \in A, b \in B$, $(a,b) \in A \times B, (x,y), (z,t) \in \hat{G}_1 \times \hat{G}_2$. The True membership:

$$
\begin{aligned}
&T_{\pi_Q(a,b)}\big((x,y) \times (z,t)), q\big) \\
&= T_{\pi_Q(a,b)}\big((x \odot z, y \odot t), q\big) \\
&= \min\Big\{T_{\phi_Q(a)}\big((x \odot z), q\big), T_{\psi_Q(b)}\big((y \odot t), q\big)\Big\} \\
&\geq \min\Big\{\min\underbrace{\{T_{\phi_Q(a)}(x,q), T_{\phi_Q(a)}(z,q)\}}, \min\underbrace{\{T_{\psi_Q(b)}(y,q), T_{\psi_Q(b)}(t,q)\}}\Big\} \\
&\geq \min\Big\{\min\underbrace{\{T_{\phi_Q(a)}(x,q), T_{\psi_Q(b)}(y,q)\}}, \min\underbrace{\{T_{\phi_Q(a)}(z,q), T_{\psi_Q(b)}(t,q)\}}\Big\} \\
&= \min\Big\{\{T_{\pi_Q(a,b)}((x,y),q), T_{\pi_Q(a,b)}((z,t),q)\Big\}
\end{aligned}
$$

Next:

$$
\begin{aligned}
&I_{\pi_Q(a,b)}(x,y)\times((z,t),q)\\
&I_{\pi_Q(a,b)}(x\odot z),((y\odot t),q)\\
&=\max\left\{I_{\phi_Q(a)}(x\odot z),q),I_{\psi_Q(b)}((y\odot t),q)\right\}\\
&\leq\max\left\{\max\underbrace{\{I_{\phi_Q(a)}(x),q),I_{\phi_Q(a)}(z,q)\}},\max\{\underbrace{I_{\psi_Q(b)}(y,q),I_{\psi_Q(b)}(t,q)\}}\right\}\\
&\leq\max\left\{\max\underbrace{\{I_{\phi_Q(a)}(x,q),I_{\psi_Q(a)}(y,q)\}},\max\{\underbrace{I_{\phi_Q(b)}(z,q),I_{\psi_Q(b)}(t,q)\}}\right\}\\
&=\max\left\{I_{\pi_Q(a,b)}((x,y),q),I_{\pi_Q(a,b)}((z,t),q)\right\}
\end{aligned}
$$

Next:

$$
\begin{aligned}
&F_{\pi_Q(a,b)}(x,y)\times((z,t),q)\\
&=F_{\pi_Q(a,b)}(x\odot z),((y\odot t),q)\\
&=\max\{F_{\phi_Q(a)}(x\odot z),q),F_{\psi_Q(b)}((y\odot t),q)\}\\
&\leq\max\left\{\max\underbrace{\{F_{\phi_Q(a)}(x,q),F_{\phi_Q(a)}(z,q)\}},\max\{\underbrace{F_{\psi_Q(b)}(y,q),F_{\psi_Q(b)}(t,q)\}}\right\}\\
&\leq\max\left\{\max\underbrace{\{F_{\phi_Q(a)}(x,q),F_{\psi_Q(b)}(y,q)\}},\max\{\underbrace{F_{\phi_Q(a)}(z,q),F_{\psi_Q(b)}(t,q)\}}\right\}\\
&=\max\left\{F_{\pi_Q(a,b)}((x,y),q),F_{\pi_Q(a,b)}((z,t),q)\right\}
\end{aligned}
$$

□

Corollary 3.3. *Let (ψ^Q,A_1) and (ψ^Q,A_2) be two $Q-$ neutrosophis soft quasigroups. Then, $(\psi^Q,A_1)\times(\psi^Q,A_2)$ is a $Q-$ neutrosophis soft quasigroup over $\hat{G_1}\times\hat{G_2}$.*

Proof. It's easy from Theorem 3.2. □

Definition 3.4. Let (ψ^Q,A_i) be set of $Q-$ neutrosophic soft quasigroups under the quasigroups $\hat{Q_i}$, then $(\psi^Q,A_i)^n=(\psi^Q,A_i^n)$ is called $Q-$ neutrosophis power soft quasigroup $(Q-NPS\hat{G})$ for all $n\in\mathbb{Z}$.

Theorem 3.5. *Let (ψ^Q,A_1) and (ψ^Q,A_2) be two $Q-$ neutrosophis soft quasigroups. Then, $((\psi^Q,A_1)\times(\psi^Q,A_2))^n=(\psi^Q,A_1)^n\times(\psi^Q,A_2)^n$ any natural number $n\in\mathbb{Z}$.*

Proof. Let $n=2$, then we show that $((\psi^Q,A_1)\times(\psi^Q,A_2))^2=(\psi^Q,A_1)^2\times(\psi^Q,A_2)^2$

Let $((\psi^Q,A_1)\times(\psi^Q,A_2))^2=((\psi^Q,A_1\times A_2))^2$ and let $x_i,y_i\in\hat{Q_i}$ where $i=1,2$ and for $(a_1,a_1)=a_1^2\in A_1\times A_1)$ and $(a_2,a_2)=a_2^2\in A_2\times A_2)$. Then, the True membership degree is

given as $T_{\psi^Q{}_{(a_1,a_2)^2}}((x_1,x_2)(y_1,y_2))$

$$
\begin{aligned}
&= T_{\psi^Q{}_{(a_1,a_2)^2}}((x_1y_1,q)(x_2y_2,q)\\
&= \min\{T_{\psi^Q{}_{(a_1,a_2)}}((x_1y_1,q),T_{\psi^Q{}_{(a_1,a_2)}}(x_2y_2,q))\}\\
&= \min\left\{\min\left\{T_{\psi^Q{}_{(a_1)}}((x_1y_1,q),T_{\psi^Q{}_{(a_2)}}((x_1y_1,q)\right\},\min\left\{T_{\psi^Q{}_{(a_1)}}((x_2y_2,q),T_{\psi^Q{}_{(a_2)}}((x_2y_2,q)\right\}\right\}\\
&= \min\left\{\min\left\{T_{\psi^Q{}_{(a_1)}}((x_1y_1,q),T_{\psi^Q{}_{(a_1)}}((x_2y_2,q)\right\},\min\left\{T_{\psi^Q{}_{(a_2)}}((x_1y_1q),T_{\psi^Q{}_{(a_2)}}((x_2y_2,q)\right\}\right\}\\
&\geq \min\left\{\min\left[\left\{\min\{T_{\psi^Q{}_{(a_1)}}((x_1,q),T_{\psi^Q{}_{(a_1)}}((y_1,q)\},\min\{T_{\psi^Q{}_{(a_1)}}((x_2,q),T_{\psi^Q{}_{(a_1)}}((y_2,q)\}\right\}\right],\right.\\
&\quad \left.\min\left[\left\{\min\{T_{\psi^Q{}_{(a_2)}}((x_1,q),T_{\psi^Q{}_{(a_2)}}((y_1,q)\},\min\{T_{\psi^Q{}_{(a_2)}}((x_2,q),T_{\psi^Q{}_{(a_2)}}((y_2,q)\}\right\}\right]\right\}\\
&= \min\left\{\left\{\min\{T_{\psi^Q{}_{(a_1,a_1)}}((x_1y_1,q),T_{\psi^Q{}_{(a_1,a_1)}}((x_2y_2,q)\},\min\{T_{\psi^Q{}_{(a_2,a_2)}}(x_1y_1,q),T_{\psi^Q{}_{(a_2,a_2)}}((x_2y_2,q)\}\right\}\right\}\\
&= \min\left\{T_{\psi^Q{}_{(a_1,a_1)^2}}((x_1y_1),(x_2y_2)q),T_{\psi^Q{}_{(a_2,a_2)^2}}((x_1y_1),(x_2y_2)q)\right\}
\end{aligned}
$$

Hence, $((\psi^Q,A_1)\times(\psi^Q,A_2))^2=(\psi^Q,A_1)^2\times(\psi^Q,A_2)^2$.

Similarly, we can show that

$$
\begin{aligned}
((\psi^Q,A_1)\times(\psi^Q,A_2))^3 &= ((\psi^Q,A_1)\times(\psi^Q,A_2))((\psi^Q,A_1)\times(\psi^Q,A_2))^2\\
&= ((\psi^Q,A_1)\times(\psi^Q,A_2))((\psi^Q,A_1)^2\times(\psi^Q,A_2)^2
\end{aligned}
$$

Since, Mann's multiplication is associative, this mean, it can be show that $((\psi^Q,A_1)\times(\psi^Q,A_2))^3=(\psi^Q,A_1)^3\times(\psi^Q,A_2)^3$. Hence, by induction on a positive integer n, Theorem 3.5 is true for all $n\in\mathbb{Z}$. □

Corollary 3.6. *Let $(\psi^Q,A_1),(\psi^Q,A_2),...,(\psi^Q,A_n)$ be $Q-$ neutrosophis soft quasigroups. Then, $((\psi^Q,A_1)^m\times(\psi^Q,A_2)^m\times...\times(\psi^Q,A_n)^m=(\psi^Q,A_1)^m\times(\psi^Q,A_2)^m\times...\times(\psi^Q,A_n)^m$.*

Proof. It follows from Theorem 3.5. □

Proposition 3.7. *If (ψ^Q,A) has a proper $Q-$neutrosophic soft subquasigroup (ψ^Q,B), then $(\psi^Q,A)\times(\pi_Q,C)$ has the proper $Q-$neutrosophic soft quasigroup $(\psi^Q,B)\times(\pi_Q,C)$.*

Proof. Let $(\psi^Q,A)\times(\pi_Q,C)=(\lambda^Q,D)$ and $(\psi^Q,B)\times(\pi_Q,C)=(\Upsilon_Q,E)$. Then, by Theorem 3.2, (λ^Q,D) and (Υ_Q,E) are $Q-$neutrosophic soft quasigroups. Since (ψ^Q,B) is a proper $Q-$neutrosophic soft subquasigroup of (ψ^Q,A), then (Υ_Q,E) is a proper $Q-$neutrosophic soft subquasigroup of (λ^Q,D). The membership degrees are characterize as $T_{\Upsilon_Q}(x)\leq$

$T_{\lambda^Q}(x), I_{\lambda^Q}(x) \geq I_{\Upsilon_Q}(x), F_{\lambda^Q}(x) \geq F_{\Upsilon_Q}(x)$, are respectively truth membership function, indeterminacy membership function and falsity membership function for all $x \in \hat{G}$ and $q \in Q$.

□

4. Isotopism of $Q-$Neutrosophic Soft Quasigroup

Definition 4.1. Let (ψ^Q, A) and (λ^Q, B) be two $Q-$neutrosophic soft sets and under the operation of the sets $M \times Q$ and $N \times Q$ respectively and $\alpha : M \times Q \mapsto N \times Q$ and $\rho : A \mapsto B$ be two functions. Then, the image $(\alpha, \rho)(\psi^Q, A)$ of (ψ^Q, A) under (α, ρ) is a $Q-$ neutrosophic soft set over $N \times Q$. That is,

$$(\alpha, \rho)(\psi^Q, A) = \big(\alpha(\psi^Q), \rho(A)\big) = \big\{b, \alpha(\psi^Q)(b) : b \in \rho(A)\big\} \tag{3}$$

Then, membership degrees are given as for all $n \in N, q \in Q$

$$T_{\alpha_{(\psi^Q)(b)}}(n,q) = \begin{cases} \max_{\alpha(m,q)=(n,q)} \max_{\rho(a)=b}[T_{\phi_{Q(a)}}(m,q)], & \text{if } (m,q) \in \alpha^{-1}(n,q) \\ 0, & \text{otherwise.} \end{cases}$$

$$I_{\alpha_{(\psi^Q)(b)}}(n,q) = \begin{cases} \min_{\alpha(m,q)=(n,q)} \min_{\rho(a)=b}[I_{\phi_{Q(a)}}(m,q)], & \text{if } (m,q) \in \alpha^{-1}(n,q) \\ 1, & \text{otherwise.} \end{cases}$$

$$F_{\alpha_{(\psi^Q)(b)}}(n,q) = \begin{cases} \min_{\alpha(m,q)=(n,q)} \min_{\rho(a)=b}[F_{\phi_{Q(a)}}(m,q)], & \text{if } (m,q) \in \alpha^{-1}(n,q) \\ 1, & \text{otherwise.} \end{cases}$$

for all $n \in N$, $q \in Q$.

The preimage $(\alpha, \rho)^{-1}(\lambda^Q, B)$ of (λ^Q, B) under (α, ρ) is a $Q-$ neutrosophic soft set over $M \times Q$ define as

$$(\alpha, \rho)^{-1}(\lambda^Q, B) = \big(\alpha^{-1}(\lambda^Q), \rho(B)\big) = \big\{a, \alpha^{-1}(\psi^Q)(a) : a \in \alpha^{-1}(B)\big\} \tag{4}$$

for all $m \in M$,

$$\begin{aligned} T_{\alpha^{-1}(\lambda^Q)(a)}(m,q) &= T_{\psi^Q[\rho(a)]}(\alpha(n,q)) \\ , I_{\alpha^{-1}(\lambda^Q)(a)}(m,q) &= I_{\psi^Q[\rho(a)]}(\alpha(n,q)), \\ F_{\alpha^{-1}(\lambda^Q)(a)}(m,q) &= F_{\psi^Q[\rho(a)]}(\alpha(n,q)). \end{aligned}$$

Definition 4.2. Let the triple $((\alpha, \beta, \gamma)\rho)$ of bijections of a $Q-$neutrosophic soft sets $M \times Q$ onto the set $N \times Q$. If $((\alpha, \beta, \gamma)\rho)$ is an isotopism of the set $(M \times Q, \odot)$ onto a set $(N \times Q, \circ)$,

then $((\alpha,\beta,\gamma)\rho)$ is said to be isotope between the two $Q-$ neutrosophic soft quasigroups. The membership degrees are defined by

$$T_{\alpha(\psi^Q)(b)}(n,q)=\begin{cases}\max_{\beta,\gamma(m,q)=(n,q)}\max_{\rho(a)=b}[T_{\phi_{Q(a)}}(m,q)], & \text{if } (m,q)\in\beta^{-1},\gamma^{-1}(n,q)\\ 0, & \text{otherwise.}\end{cases}$$

$$I_{\alpha(\psi^Q)(b)}(n,q)=\begin{cases}\min_{\beta,\gamma(m,q)=(n,q)}\min_{\rho(a)=b}[I_{\phi_{Q(a)}}(m,q)], & \text{if } (m,q)\in\beta^{-1},\gamma^{-1}(n,q)\\ 1, & \text{otherwise.}\end{cases}$$

$$F_{\alpha(\psi^Q)(b)}(n,q)=\begin{cases}\min_{\beta,\gamma(m,q)=(n,q)}\min_{\rho(a)=b}[F_{\phi_{Q(a)}}(m,q)], & \text{if } (m,q)\in\beta^{-1},\gamma^{-1}(n,q)\\ 1, & \text{otherwise.}\end{cases}$$

Definition 4.3. Suppose that the triple $((\alpha,\beta,\gamma)\rho)=(\alpha,\rho)$, where $\alpha=\beta=\gamma$ then (α,ρ) is said to be homomorphism from $(M\times Q,\odot)$ to $(N\times Q,\circ)$. The set

$$(\alpha,\rho)(\psi^Q,A)=\big(\alpha(\psi^Q),\rho(A)\big)=\big\{b,\alpha(\psi^Q)(b):b\in\rho(A)\big\} \tag{5}$$

is called the homomorphic image of $(M\times Q,\odot)$ under (α,ρ).

Suppose that $\alpha: M\times Q\mapsto M\times Q$ and $\rho: A\mapsto A$, then the homomorphism is called endomorphism. It is called isomorphism if (α,ρ) is a bijection. Suppose that $\alpha: M\times Q\mapsto M\times Q$ and $\rho: A\mapsto A$ is a homomorphism, then it called isomorphism.

Theorem 4.4. *Let (ψ^Q,A) be a $Q-$neutrosophis soft quasigroup over a quasigroup $\hat{G}_1$ and $T=\alpha,\beta,\gamma:(\hat{G}_1\times Q,\odot)\to(\hat{G}_2\times Q,\circ)$ be isotopism and $\rho:A\to B$. Then $(T,\rho)(\psi^Q,A)$ is a $Q-$neutrosophic soft quasigroup over $\hat{G}_2$.*

Proof. Let $b\in\rho(E)$ the parametric set $s,t\in\hat{G}_2$ and $q\in Q$. Suppose that there exist $x,y\in\hat{G}_1$ such that $(s,q)=\alpha(x,q)$ and $(t,q)=\beta(y,q)$. Let $z\in\hat{G}_1$ such that $(z,q)=((x\odot y),q)$. Then,

$$\begin{aligned}T_{\gamma(\psi^Q)(b)}(s\circ t,q)&=\max_{\alpha(x,q)=(s,q),\beta(y,q)=(t,q)}\max_{\rho(a)=b}[T_{(\psi^Q)(a)}(z,q)]\\&\geq\max_{\rho(a)=b}[T_{(\psi^Q)(a)}(x\odot y,q)]\\&\geq\max_{\rho(a)=b}\Big[\min\big\{T_{(\psi^Q)(a)}(x,q),T_{(\psi^Q)(a)}(y,q)\big\}\Big]\\&\min\big\{[\max_{\rho(a)=b}T_{(\psi^Q)(a)}(x,q)],\max_{\rho(a)=b}[T_{(\psi^Q)(a)}(y,q)]\big\}.\end{aligned}$$

For each $x,y\in\hat{G}_1$, and $q\in Q$, the equality $(s,q)=\alpha(x,q)$ and $(t,q)=\beta(y,q)$ hold for all $s,t\in\hat{G}_2$. Then, this follows the last equality

$$\begin{aligned}&T_{\gamma(\psi^Q)(b)}(s\circ t,q)\\&\geq\min\big\{\max_{\alpha(x,q)=(s,q)}\max_{\rho(a)=b}[T_{(\psi^Q)(a)}(x,q)],\max_{\beta(y,q)=(t,q)}\max_{\rho(a)=b}[T_{(\psi^Q)(a)}(y,q)]\big\}\\&=\min\{T_{\alpha(\psi^Q)(b)}(s,q),T_{\beta(\psi^Q)(b)}(t,q).\end{aligned}$$

Let (ψ^Q, B) be a $Q-$neutrosophis soft quasigroup over a quasigroup $\hat{G}_2$ such that $T^{-1}(\psi^Q, B) = (\psi^Q, A)$ and $\rho^{-1}(B) = A$, where $T = (\alpha, \beta, \gamma)$. For all $s, t \in \hat{G}_1$ and $q \in Q$, we have

$$\begin{aligned} T_{\gamma^{-1}(\psi^Q)(a)}(s \odot t, q) &= T_{(\psi^Q)\rho^{-1}(b)}[\alpha(t,q) \circ \beta(s,q)] \\ &\geq \min\{T_{(\psi^Q)\rho^{-1}(b)}\alpha(t,q), T_{(\psi^Q)\rho^{-1}(b)}\beta(t,q)\} \\ &= \min\{T_{\alpha^{-1}(\psi^Q)(a)}(t,q), T_{\beta^{-1}(\psi^Q)(a)}(t,q)\}. \end{aligned}$$

Similarly, we show:
The indeterminate membership degree is given by

$$\begin{aligned} & I_{\gamma(\psi^Q)(b)}(s \circ t, q) \\ \geq \max\Big\{ & \min_{\alpha(x,q)=(s,q)} \min_{\rho(a)=b} [I_{(\psi^Q)(a)}(x,q)], \min_{\beta(y,q)=(t,q)} \min_{\rho(a)=b} [I_{(\psi^Q)(a)}(y,q)] \Big\} \\ &= \max\{I_{\alpha(\psi^Q)(b)}(s,q), I_{\beta(\psi^Q)(b)}(t,q) \end{aligned}$$

and

$$I_{\gamma^{-1}(\psi^Q)(a)}(s \odot t, q) = \max\{I_{\alpha^{-1}(\psi^Q)(a)}(t,q), I_{\beta^{-1}(\psi^Q)(a)}(t,q)\}.$$

And the falsity membership degree is given as

$$\begin{aligned} & F_{\gamma(\psi^Q)(b)}(s \circ t, q) \\ \geq \max\Big\{ & \min_{\alpha(x,q)=(s,q)} \min_{\rho(a)=b} [F_{(\psi^Q)(a)}(x,q)], \min_{\beta(y,q)=(t,q)} \min_{\rho(a)=b} [F_{(\psi^Q)(a)}(y,q)] \Big\} \\ &= \max\{F_{\alpha(\psi^Q)(b)}(s,q), F_{\beta(\psi^Q)(b)}(t,q) \end{aligned}$$

$$F_{\gamma^{-1}(\psi^Q)(a)}(s \odot t, q) = \max\{F_{\alpha^{-1}(\psi^Q)(a)}(t,q), F_{\beta^{-1}(\psi^Q)(a)}(t,q)\}$$

□

Example 4.5. Let (ψ^Q, A) be a $Q-$neutrosophis soft quasigroup over a quasigroup $\hat{G}_1$ and $(T, \rho) : (\hat{G}_1 \times Q, \odot) \to (\hat{G}_2 \times Q, \circ)$ be a $Q-$neutrosophic soft isotopism with the parameters $\rho : A \to B$. Let $\hat{G}_1 = \{1, 2, 3, 4\}$ and $\hat{G}_2 = \{a, b, c, d\}$ given in table below.

$\odot$	1	2	3	4
1	2	3	4	1
2	3	2	1	4
3	4	1	2	3
4	1	4	3	2

$\longrightarrow$

$\circ$	a	b	c	d
a	b	d	1	c
b	a	c	b	d
c	d	b	c	a
d	c	a	d	b

TABLE 2. Quasigroup of order 4

Next, construct $Q-$neutrosophic soft set as follows

$$T_{(\psi^Q)(b)}(s\circ t,q)=\begin{cases}0.90, & \text{if } z\in\{a,d,e\}\\ 0, & \text{otherwise.}\end{cases}$$

$$I_{(\psi^Q)(b)}(s\circ t,q)=\begin{cases}0.90, & \text{if } z\in\{a,d,e\}\\ 1, & \text{otherwise.}\end{cases}$$

$$F_{(\psi^Q)(b)}(s\circ t,q)=\begin{cases}0.90, & \text{if } z\in\{a,d,e\}\\ 1, & \text{otherwise.}\end{cases}$$

Let $(T,\rho):(\hat{G}_1\times Q,\odot)\to(\hat{G}_2\times Q,\circ)$ be isotopism of the $Q-$neutrosophic soft quasigroups defined as

$$\alpha(\psi^Q)(b)=\begin{pmatrix}1&2&3&4\\a&c&b&d\end{pmatrix},\beta(\psi^Q)(b)=\begin{pmatrix}1&2&3&4\\b&a&d&c\end{pmatrix}$$

and

$$\gamma(\psi^Q)(b)=\begin{pmatrix}1&2&3&4\\a&c&b&d\end{pmatrix}$$

where $T=\alpha,\beta,\gamma$ and $\rho(a)=b$. The membership degree is as follows: In particular, let $s=1$, and $t=3$, then the true membership degree is given as

$$\begin{aligned}T_{\gamma(\psi^Q)(b)}(1\odot 3,q) &= \max_{\rho(a)=b}[\{\min T_{\alpha(\psi^Q)(a)}(1,q),T_{\beta(\psi^Q)(a)}(3,q)\}]\\ T_{\gamma(\psi^Q)(b)}(4,q) &\geq \min\{[\max_{\rho(a)=b}T_{\alpha(\psi^Q)(a)}(1,q)],\max_{\rho(a)=b}[T_{\beta(\psi^Q)(a)}(3,q)]\}\\ T_{(\psi^Q)(b)}(d,q) &\geq \min\{T_{(\psi^Q)(b)}(a,q),T_{(\psi^Q)(b)}(d,q)\}.\end{aligned}$$

Consider the LHS, we have $T_{(\psi^Q)(b)}(d,q)=0.90$ since $d\in\{a,d,e\}$. Observing the RHS, we have $\min\{T_{(\psi^Q)(b)}(a,q),T_{(\psi^Q)(b)}(d,q)\}=\min\{0.90,0,90\}=0.90$. Hence, $RHS=LHS$. Indeed, $(T,\rho):(\hat{G}_1\times Q,\odot)\to(\hat{G}_2\times Q,\circ)$ is a $Q-$neutrosophic soft isotopic quasigroups and $(T,\rho)(\psi^Q,A)$ is a $Q-$neutrosophic soft quasigroup over $\hat{G}_2$.

The indeterminate membership degree is by

$$\begin{aligned}I_{\gamma(\psi^Q)(b)}(1\odot 3,q) &= \min_{\rho(a)=b}[\{\max I_{\alpha(\psi^Q)(a)}(1,q),I_{\beta(\psi^Q)(a)}(3,q)\}]\\ I_{\gamma(\psi^Q)(b)}(4,q) &\leq \max\{[\min_{\rho(a)=b}I_{\alpha(\psi^Q)(a)}(1,q)],\min_{\rho(a)=b}[I_{\beta(\psi^Q)(a)}(3,q)]\}\\ I_{(\psi^Q)(b)}(d,q) &\leq \max\{I_{(\psi^Q)(b)}(a,q),I_{(\psi^Q)(b)}(d,q)\}\end{aligned}$$

Consider the LHS of the last equality, $T_{(\psi^Q)(b)}(d,q)=1.0$ since $d\in\{a,d,e\}$ and the RHS, gives $\max\{T_{(\psi^Q)(b)}(a,q),T_{(\psi^Q)(b)}(d,q)\}=\max\{1.0,1.0\}=1.0$. Thus, $RHS=LHS$.

Theorem 4.6. *Let (ψ^Q,A) be a $Q-$neutrosophis soft quasigroup over a finite quasigroup $\hat{G}_1$ and let $\alpha:(\hat{G}_1\times Q,\odot)\to(\hat{G}_2\times Q,\circ)$ be a homomorphism such that $\rho:A\to B$. Then $(\alpha,\rho)(\psi^Q,A)$ is a $Q-$neutrosophic soft quasigroup over $\hat{G}_2$.*

Proof. Let $\alpha : (\hat{G}_1 \times Q, \odot) \to (\hat{G}_2 \times Q, \circ)$ be a $Q-$neutrosophic soft homomorphism and $\rho : A \to B$ by setting $T = \alpha = \beta = \gamma$ in theorem (4.4), we have $(\alpha, \rho)(\psi^Q, A)$ is a $Q-$neutrosophic soft quasigroup over $\hat{G}_2$. □

Theorem 4.7. *Let (ψ^Q, A) be a $Q-$neutrosophis soft quasigroup over a finite quasigroup $\hat{G}_1$. If $\alpha : \hat{G}_1 \to \hat{G}_2$ is a $Q-$neutrosophic soft isomorphism. Then, $(\alpha, \rho)((\psi^Q, A))$ is a $Q-$neutrosophis soft quasigroup over $\hat{G}_2$.*

Proof. From theorem (4.4), (α, β, γ) is a $Q-$neutrosophis soft quasigroup isotopism. If $\alpha = \beta = \gamma$, is a $Q-$neutrosophis soft isotopism, then $\alpha(\phi_{Q(a)})$ is a $Q-$neutrosophic soft quasigroup over $\hat{G}_2$. □

Theorem 4.8. *Let (ψ^Q, B) be a $Q-$neutrosophic soft quasigroup over a finite quasigroup $\hat{G}_1$ and $\alpha : (\hat{G}_1 \times Q, \odot) \to (\hat{G}_2 \times Q, \circ)$ be a $Q-$neutrosophic soft homomorphism and $\rho : A \to B$. Then, $(\alpha, \rho)^{-1}(\psi^Q, B)$ is a $Q-$neutrosophic soft quasigroup over $\hat{G}_1$.*

Proof. Let $s, t \in \hat{G}_1, q \in Q$ and $\rho : A \to B$. For any parameter $a \in \rho^{-1}(B)$, we have

$$\begin{aligned} T_{\alpha^{-1}(\psi^Q)(a)}(s \circ t, q) &= T_{(\psi^Q)[\rho(a)]}(\alpha(s \odot t, q)) \\ &= T_{(\psi^Q)[\rho(a)]}(\alpha(s, q)\alpha(t, q)) \\ &\geq \min\{T_{(\psi^Q)[\rho(a)]}(\alpha(s, q)), T_{(\psi^Q)[\rho(a)]}(\alpha(t, q))\} \\ &= \min\{T_{\alpha^{-1}(\psi^Q)(a)}(s, q), T_{\alpha^{-1}(\psi^Q)(a)}(t, q)\}. \end{aligned}$$

Similarly, we shows that

$$I_{\alpha^{-1}(\psi^Q)(a)}(s \circ t, q) = \max\{I_{\alpha^{-1}(\psi^Q)(a)}(s, q), I_{\alpha^{-1}(\psi^Q)(a)}(t, q)\}$$
$$\text{and } F_{\alpha^{-1}(\psi^Q)(a)}(s \circ t, q) = \max\{F_{\alpha^{-1}(\psi^Q)(a)}(s, q), F_{\alpha^{-1}(\psi^Q)(a)}(t, q)\}.$$

□

Definition 4.9. Let (ψ^Q, A) be a $Q-$neutrosophic soft quasigroup and (U, V, W) be an autotopism of a finite quasigroup $(\hat{G}, \odot)$. Then the following hold.

(1) $T_{W(\psi^Q)(a)}(st, q) = \max_{\rho(a)=a}\{T_{U(\psi^Q)(a)}(s, q), T_{V(\psi^Q)(a)}(t, q)\}$.
(2) $I_{W(\psi^Q)(a)}(st, q) = \min_{\rho(a)=a}\{I_{U(\psi^Q)(a)}(s, q), I_{V(\psi^Q)(a)}(t, q)\}$.
(3) $F_{W(\psi^Q)(a)}(st, q) = \min_{\rho(a)=a}\{F_{U(\psi^Q)(a)}(s, q), F_{V(\psi^Q)(a)}(t, q)\}$.

for all $s, t \in \hat{G}, q \in Q$

Lemma 4.10. *Let (ψ^Q, A) be a $Q-$neutrosophic soft quasigroup and $\Gamma = (U, V, W)$ be an autotopism of a finite quasigroup $(\hat{G}, \odot)$. Then*

(1) $(T_{U(\psi^Q)(a)}, T_{V(\psi^Q)(a)}, T_{W(\psi^Q)(a)}) \in AUT(\psi^Q, A)^{(\hat{G},\odot)}$ *for true membership.*
(2) $(I_{U(\psi^Q)(a)}, I_{V(\psi^Q)(a)}, I_{W(\psi^Q)(a)}) \in AUT(\psi^Q, A)^{(\hat{G},\odot)}$ *for unterminate membership.*
(3) $(F_{U(\psi^Q)(a)}, F_{V(\psi^Q)(a)}, F_{W(\psi^Q)(a)}) \in AUT(\psi^Q, A)^{(\hat{G},\odot)}$ *for unterminate membership.*

Proof. It follows from Definition 4.9. □

Theorem 4.11. *Let (ψ^Q, A) be a $Q-$neutrosophis soft quasigroup over a quasigroup $\hat{G}_1$. Let $T_1 = (\alpha_1, \beta_1, \gamma_1)$ and $T_2 = (\alpha_2, \beta_2, \gamma_2)$ be autotopism of $(\hat{G}, \odot)$ and $\rho : A \to A$. Then $(T_1 \circ T_2, \rho)$ is an autotopism of $(\psi^Q, A)^{(\hat{G},\odot)}$.*

Proof. Let Let $T_1 = (\alpha_1, \beta_1, \gamma_1)$ and $T_2 = (\alpha_2, \beta_2, \gamma_2)$ be isotopisms of $(\hat{G}, \odot)$ and $\rho : A \to A$. Then $(\alpha_1, \beta_1, \gamma_1) \circ (\alpha_2, \beta_2, \gamma_2) = (\alpha_1\alpha_2, \beta_1\beta_2, \gamma_1\gamma_2)$ is also an isotopism of $(\hat{G}, \odot)$. To prof this Theorem, it is suffix to show that

$$T_{(\gamma_1\circ\gamma_2)(\psi^Q)(a)}(x \odot y, q) = \min\{T_{(\alpha_1\circ\alpha_2)(\psi^Q)(a)}(y, q), T_{(\beta_1\circ\beta_2)(\psi^Q)(a)}(x, q)\}$$

Let $a \in \rho(E)$ the parametric set, for all $x, y \in \hat{G}$ and $q \in Q$.

$$\begin{aligned}
&\min\{T_{(\psi^Q)(a)}(\alpha_1 \circ \alpha_2)(x, q), T_{(\psi^Q)(a)}(\beta_1 \circ \beta_2)(y, q))\} \\
&\geq \max_{\rho(a)=a}\left[\min\left\{T_{(\psi^Q)(a)}(\alpha_1 \circ \alpha_2)(x, q), T_{(\psi^Q)(a)(\beta_1\circ\beta_2)}(y, q)\right\}\right] \\
&= \max_{\rho(a)=a}\left[\min\left\{T_{\alpha_1(\psi^Q)(a)}[\alpha_2(x, q)], T_{\beta_1(\psi^Q)(a)}[\beta_2(y, q)]\right\}\right] \\
&= \max_{\rho(a)=a}\left[\min\left\{T_{\alpha_1[\alpha_2(\psi^Q)(a)]}(x, q), T_{\beta_1[\beta_2(\psi^Q)(a)]}(y, q)]\right\}\right] \\
&= \min\left[\left\{\max_{\rho(a)=a}\left[T_{\alpha_1[\alpha_2(\psi^Q)(a)]}(x, q)\right], \max_{\rho(a)=a}\left[T_{\beta_1[\beta_2(\psi^Q)(a)]}(y, q)\right]\right]\right\}\right]. \qquad (6)
\end{aligned}$$

Let $T_{\gamma_2(\psi^Q)(a)}(x \odot y, q)] = \min\{T_{\alpha_2(\psi^Q)(a)}(x, q), T_{\beta_2(\psi^Q)(a)}(x, q)\}$.

It follows from equation (6),

$$\begin{aligned}
&\min\left[\left\{\max_{\rho(a)=a}\left[T_{\alpha_1[\alpha_2(\psi^Q)(a)]}(x, q)\right], \max_{\rho(a)=a}\left[T_{\beta_1[\beta_2(\psi^Q)(a)]}(y, q)\right]\right]\right\}\right] \\
&= \max_{\rho(a)=a}[T_{\gamma_1(\gamma_2(\psi^Q)}(a))(x \odot y, q)]. \qquad (7)
\end{aligned}$$

Similarly, it is shows for indeterminate and falsity membership degrees. □

Corollary 4.12. (1) *Let (ψ^Q, A) be a $Q-$neutrosophis soft quasigroup over a quasigroup $\hat{G}_1$. Let $T = (\alpha, \beta, \gamma)$ be an autotopism of $(\hat{G}, \odot)$ and $\rho : A \to A$. Then,*

$$(T_{\alpha^{-1}(\psi^Q)(a)}, T_{\beta^{-1}(\psi^Q)(a)}, T_{\gamma^{-1}\psi^Q)(a)})$$
$$(I_{\alpha^{-1}(\psi^Q)(a)}, I_{\beta^{-1}(\psi^Q)(a)}, I_{\gamma^{-1}(\psi^Q)(a)})$$
$$(F_{\alpha^{-1}(\psi^Q)(a)}, F_{\beta^{-1}(\psi^Q)(a)}, F_{\gamma^{-1}(\psi^Q)(a)})$$

are autotopisms of $(\psi^Q, A)^{(\hat{G}, \odot)}$.

Proof. It has been reported in [11] that an autotopism is an isotopism of a quasigroup $(\hat{G}, \odot)$ onto itself. In Theorem 4.4, it was shown that the isotopic image of $Q-$neutrosophic soft quasigroup is also a $Q-$neutrosophic soft quasigroup. So, if the isotopic image has the same operation with the domain under (T, ρ), then second part of Theorem 4.4 gives the desire result. □

Lemma 4.13. *If $\psi^*_Q(b)$ is a homomorphic image of $Q-$ neutrosophic soft quasigroup $\psi^Q(a)$ and $\psi^*_Q(b) \subseteq \psi^Q(b)$ is a $Q-$ neutrosophic soft quasigroup, then $\psi^*_Q(b)$ is a $Q-$ neutrosophic soft quasigroup.*

Proof. We established Theorem 4.8 that homomorphic image of a finite $Q-$ neutrosophic soft quasigroup is also a $Q-$ neutrosophic soft quasigroup. Since $\psi^*_Q(b) \leq \psi^Q(b)$ a $Q-$ neutrosophic soft quasigroup, then $\psi^*_Q(b)$ is a $Q-$ neutrosophic soft quasigroup □

Theorem 4.14. *Let $\psi^Q(a)$ and $\psi^Q(b)$ be $Q-$ neutrosophic soft quasigroups. Then, $\psi^Q(a) \cong \pi_Q(c) \leq \psi^Q(a) \times \psi^Q(b)$ if and only if there exist a homomorphism $T : \psi^Q(a) \to \psi^Q(b)$.*

Proof. Suppose that $\psi^Q(a) \cong \pi_Q(c) \leq \psi^Q(a) \times \psi^Q(b)$. Then, π is a homomorphism from $\pi_Q(c) \to \psi^Q(a)$ and since $\psi^Q(a) \cong \pi_Q(c)$ there exist a homomorphism $T : \psi^Q(a) \to \psi^Q(b)$.

Conversely, suppose that there exist a homomorphism $T : \psi^Q(a) \to \psi^Q(b)$, then $\psi^Q(a) \cong \{((x, q), T(\psi^Q(a)(x, q)) | x \in \psi^Q(a), q \in Q\} \leq \psi^Q(a) \times \psi^Q(b)$ □

Corollary 4.15. *$\phi_{Q(a)} \cong \pi_{Q(c)} \leq \phi_{Q(a)} \times \psi_{Q(b)}$ where $\phi_{Q(a)}$ and $\psi_{Q(b)}$ are $Q-$ neutrosophic soft quasigroups if and only if there $\psi_{Q(b)}$ contain a $Q-$ neutrosophic soft subquasigroup that is a homomorphic image of $\phi_{Q(a)}$.*

Proof. Using Lemma 4.13 and Theorem 4.14. □

Definition 4.16. Let $\phi_{Q(a)} \times \psi_{Q(b)}$ be a direct product of two $Q-$ neutrosophic soft quasigroups $\phi_{Q(a)}$ and $\psi_{Q(b)}$. A $Q-$ neutrosophic soft subquasigroup $\pi_{Q(c)}$ is call a generalised diagonal subquasigroup if

$$\pi_{Q(c)} = \{(x,q), T(\phi_{Q(c)}(x,q)) | x \in \phi_{Q(a)}, q \in Q\} \leq \phi_{Q(a)} \times \psi_{Q(b)}$$

where T is a homomorphism from $\phi_{Q(a)}$ to $\psi_{Q(b)}$.

Theorem 4.17. *$\phi_{Q(a)} \cong \pi_{Q(c)} \leq \phi_{Q(a)} \times \psi_{Q(b)}$ where $\phi_{Q(a)}$ and $\psi_{Q(b)}$ are $Q-$ neutrosophic soft quasigroups if and only if $\phi_{Q(a)} \times \psi_{Q(b)}$ contain a generalised diagonal subquasigroup.*

Proof. By Definition 4.16, $\pi_{Q(c)} \cong \phi_{Q(a)}$. If $\phi_{Q(a)} \cong \pi_{Q(c)} \leq \phi_{Q(a)} \times \psi_{Q(b)}$, then by Lemma 4.13 there exist a homomorphism $T : \phi_{Q(a)} \to \psi_{Q(b)}$. That is

$$\{(x,q), T(\phi_{Q(a)}(x,q)) | x \in \phi_{Q(a)}, q \in Q\} \leq \phi_{Q(a)} \times \psi_{Q(b)}$$

is a generalised diagonal subquasigroup $\phi_{Q(a)} \times \psi_{Q(b)}$. □

5. An Application of $Q-$neutrosophic Soft Quasigroups

. In the works of Abu-Qumar et al. [9,30], algorithms based on Q-neutrosophic soft sets were developed to address real-life problems. Their approach was formulated using arbitrary sets without relying on a binary operation. Additionally, Oyem et al. [25] in 2022 explored the application of soft quasigroups and created an algorithm focused on uniformity and equity. While this method effectively tackles certain real-world issues related to property distribution, it falls short in managing challenges involving indeterminate data. Specifically, it cannot ascertain the degree of membership functions concerning truth, falsity, and indeterminate data.

We propose a generalized method capable of addressing uncertainties and indeterminacies pertinent to two universal sets: Q-set (groupoid) and quasigroup, in relation to real-life scenarios. It is widely acknowledged that decisions in areas such as politics, health, marriage, careers, and stock market investments are crucial, particularly due to the presence of indeterminate factors. Notably, stock market investments are fraught with uncertainties and indeterminacies, as they occur within a virtual market where buyers and sellers trade existing securities. This marketplace, typically facilitated by a government body or institution, involves the trading of shares, stocks, debentures, bonds, futures, options, among others. Given the inherent uncertainties and indeterminacies in this virtual environment, it becomes essential for investors to employ rigorous and analytical mathematical approaches, such as neutrosophic soft quasigroups, to better navigate their choices and determine the most suitable stock exchange companies for investment.

Some results on finite $Q-$neutrosophic soft quasigroup was adopted as guides for setting up an algorithm to describe truth, falsity and indeterminate membership degree. For a finite $Q-$neutrosophic soft quasigroup $(\Psi^Q, \mathfrak{A})$ over $(\hat{G}, \odot)$, the following was adopted:

$$\begin{aligned}\Psi^Q(a) &= \text{ Truth, indeterminate and falsity membership degrees}\\ \mathfrak{A} &= \text{Parameters (Beneficiaries)}\\ Q &= \text{ Company's policy}\\ \hat{G} &= \text{Stock Exchange Companies}\end{aligned}$$

Then, $T_{\Phi Q}, I_{\Phi Q}, F_{\Phi Q} : X \times Q \to [0,1]$ defines how the parameters sets $\mathfrak{A}$ will assign its choices to subset in $[0,1]$ such that condition $0 \leq T_{\Phi Q} + I_{\Phi Q} + F_{\Phi Q} \leq 3^+$ hold. Knowing that $\hat{G}$ has a structure (called quasigroup) which determines the interaction of choices of $Q-$neutrosophic soft set $\Psi^Q(a)$ for all $a \in \mathfrak{A}$. To determine the truth, falsity and indeterminate membership degrees, Definition 2.1 and Definition 2.2 are used as guide for constructing an algorithm below.

Step 1 Construct a qroupoid of finite order.

Step 2 Construct a quasigroup of finite order.

Step 3 Construct two $Q-$neutrosophic soft quasigroups $(\Phi^Q, \mathfrak{A})$ and $(\Psi^Q, \mathfrak{B})$ under a finite quasigroup $\hat{G}$.

Step 4 Use Definition 2.2 and Definition 2.1to compute $(\Pi, \mathfrak{C}) = (\Psi^Q, \mathfrak{A}) \odot (\Psi^Q, \mathfrak{B})$ such that $a_i \odot b_j = c_k$ where c_k is unique in $\hat{G}$.

Step 5 From 4, compute $\Pi_{Q(e_i \odot e_j)}$ for all $i \neq j$.

Step 6 Compute the comparison table using the formula: $T_{\Pi^Q(c)} + I_{\Pi^Q(c)} - F_{\Phi^Q(c)}$.

Step 7 Compute the sum of these numerical grades score $S_{(x,q)} \in (\hat{G} \times Q)$ of each object (x, q).

Step 8 Find highest numerical grades corresponding to every pair of parameters on the score table.

Step 9 The best decision is any one of the elements of $V = \max_{(x,q) \in \hat{G} \times Q}(a)(x, q)$.

Example 5.1. Suppose that a newly wedded couple $\mathfrak{A}$ and $\mathfrak{B}$ want to invest their money in stock exchange company. Let $\hat{G} = \{a, b, c, d, e, f, g, h\}$ be the set alternative companies (elements of quasigroup) and $Q = \{q_1, q_2, q_3, q_4, q_5\}$ the company's policies and $E = \{e_1, e_2, e_3, e_4, e_5, e_6\}$ set of parameters where e_1= trend in earning growth, e_2 = company strength relative to its peers, e_3 = effectiveness of execution leadership skill, e_4 = debt-to-equity ratio in line with industry norms, e_5 = long-term strength and stability and e_6 = rate of dividend. If the couple has to invest according to their choice of parameters $\mathfrak{A} = \{e_1, e_2, e_5\}$ and $\mathfrak{B} = \{e_3, e_4, e_6\}$. The problem is the selection of best stock exchange company which

satisfies their criteria. Below are choice of degrees for $(\Psi^Q, \mathfrak{A})$ and $(\Psi^Q, \mathfrak{B})$:

$$(\Psi^Q, \mathfrak{A}) = \Big\{ \begin{aligned} \Phi^Q(e_1) &= \{[(a,q_1)0.9,0.7,0.4],[(b,q_2)0.8,0.5,0.7],[(e,q_3)0.9,0.7,0.2], \\ &\quad [(h,q_4)0.9,0.7,0.2]\} \\ \Phi^Q(e_2) &= \{[(a,q_2)0.3,0.5,0.2],[(b,q_3)0.9,0.6,0.4],[(f,q_4)0.1,0.5,0.7], \\ &\quad [(g,q_5)0.6,0.7,0.9]\} \\ \Phi^Q(e_5) &= \{[(b,q_1)0.9,0.6,0.3],[(e,q_3)0.6,0.3,0.7],[(f,q_4)0.5,0.4,0.9], \\ &\quad [(h,5_4)0.5,0.4,0.9]\} \Big\} \end{aligned}$$

$$(\Psi^Q, \mathfrak{B}) = \Big\{ \begin{aligned} \Psi^Q(e_3) &= \{[(c,q_1)0.1,0.4,0.5],[(e,q_3)0.7,0.3,0.2],[(f,q_4)0.8,0.3,0.6], \\ &\quad [(g,q_5)0.8,0.3,0.6]]\} \\ \Psi^Q(e_4) &= \{[(a,q_2)0.1,0.6,0.3],[(b,q_3)0.1,0.3,0.5],[(c,q_4)0.7,0.1,0.9], \\ &\quad [(d,q_5)0.7,0.5,0.9]\} \\ \Psi^Q(e_6) &= \{[(c,q_1)0.3,0.4,0.6],[(d,q_2)0.9,0.1,0.6],[(f,q_4)0.6,0.2,0.3], \\ &\quad [(g,q_5)0.5,0.2,0.6]\} \Big\}. \end{aligned}$$

Suppose that the $Q-$NSS $(\Psi^Q, \mathfrak{A})$ describes the relations: "trend in earning growth", "company strength relative to its peers" and "long-term strength and stability" with some rate of dividends and $Q-$NSS $(\Psi^Q, \mathfrak{B})$ describes the relations: "effectiviness of execution leadership skill", "debt-to-equity ratio in line with industry norms" and "price-earning ratio as an indicator of valuation".

Next, compute step 2 to get a $Q-$neutrosophic soft quasigroup $(\Pi^Q, \mathfrak{C})$ such that $(\Psi^Q, \mathfrak{A}) \odot (\Psi^Q, \mathfrak{B}) = (\Pi^Q, \mathfrak{C})$ under a quasigroup of order 8. Then, $\Pi^Q(e_i \odot e_j) = \Psi^Q(e_i) \odot \Psi^Q(e_j)$, for all $e_i \in \mathfrak{A}, e_j \in \mathfrak{B}$ we have $\mathfrak{A} \odot \mathfrak{B} = \mathfrak{C}$

$$\begin{aligned} \Pi^Q(e_1 \odot e_3) = \Big\{ &\langle [(c,q_1)0.1,0.7,0.5],[(e,q_3)0.7,0.7,0.4],[(f,q_4)0.8,0.7,0.6],[(g,q_5)0.9,0.7,0.6] \\ &[(d,q_2)0.1,0.5,0.7],[(f,q_3)0.7,0.5,0.7],[(e,q_5)0.8,0.5,0.7],[(h,q_2)0.8,0.5,0.7] \\ &[(g,q_3)0.1,0.7,0.4],[(a,q_4)0.9,0.7,0.2],[(b,q_2)0.8,0.7,0.7],[(c,q_1)0.8,0.7,0.6] \\ &[(f,q_4)0.1,0.7,0.5],[(c,q_1)0.9,0.7,0.2],[(d,q_3)0.9,0.7,0.6],[(b,q_5)0.8,0.7,0.6]\rangle \Big\} \end{aligned}$$

TABLE 3. Given a quasigroup $(\hat{G}) = \{a, b, c, d, e, f, g, h, \}$ of order 8 representing 8 companies with the method of relative operation of $(\hat{G}, \odot)$ shown in the Cayley table below such that $(ge)h \neq g(eh)$.

$\odot$	a	b	c	d	e	f	g	h
a	a	b	c	d	e	f	g	h
b	b	a	d	c	f	e	h	g
c	c	d	a	b	h	g	e	f
d	d	c	b	a	g	h	f	e
e	e	f	g	h	a	b	c	d
f	f	e	h	g	b	a	d	c
g	g	h	e	f	d	c	a	b
h	h	g	f	e	c	d	b	a

TABLE 4. Given a groupoid $(Q) = \{q_1, q_2, q_3, q_4, q_5\}$ of order 5 representing 5 different policies with their mode of operations $(Q, +)$ is shown in Cayley table below.

$+$	q_1	q_2	q_3	q_4	q_5
q_1	q_1	q_2	q_3	q_4	q_5
q_2	q_2	q_1	q_3	q_5	q_4
q_3	q_3	q_5	q_4	q_2	q_1
q_4	q_4	q_2	q_1	q_3	q_5
q_5	q_5	q_2	q_3	q_1	q_4

$$\Pi^Q(e_1 \odot e_4) = \Big\{\langle[(a,q_5)0.1,0.7,0.4],[(b,q_3)0.1,0.7,0.5],[(c,q_4)0.7,0.7,0.9],[(d,q_1)0.7,0.7,0.4]$$
$$[(b,q_1)0.1,0.6,0.7],[(a,q_3)0.1,0.5,0.7],[(d,q_5)0.7,0.5,0.9],[(c,q_4)0.7,0.5,0.9]$$
$$[(e,q_5)0.1,0.7,0.3],[(f,q_4)0.1,0.7,0.5],[(g,q_2)0.7,0.7,0.9],[(h,q_1)0.7,0.7,0.9]$$
$$[(h,q_2)0.1,0.7,0.3],[(g,q_1)0.1,0.7,0.5],[(f,q_3)0.7,0.7,0.9],[(e,q_5)0.7,0.7,0.9]\rangle\Big\}$$

$$\Pi^Q(e_1 \odot e_6) = \Big\{\langle[(c,q_1)0.3,0.7,0.5],[(d,q_2)0.9,0.7,0.6],[(f,q_4)0.6,0.7,0.4],[(g,q_5)0.5,0.7,0.6]$$
$$[(d,q_2)0.3,0.5,0.7],[(e,q_1)0.8,0.5,0.7],[(e,q_5)0.6,0.5,0.7],[(h,q_2)0.5,0.5,0.7]$$
$$[(g,q_3)0.3,0.7,0.6],[(h,q_5)0.9,0.7,0.6],[(b,q_1)0.6,0.7,0.3],[(c,q_3)0.5,0.7,0.6]$$
$$[(f,q_4)0.3,0.7,0.6],[(e,q_2)0.9,0.7,0.6],[(d,q_3)0.6,0.7,0.3],[(b,q_5)0.5,0.7,0.6]\rangle\Big\}$$

$$\Pi^Q(e_2 \odot e_3) = \Big\{\langle[(c,q_2)0.1,0.5,0.5],[(e,q_3)0.3,0.5,0.2],[(f,q_4)0.8,0.7,0.6],[(g,q_5)0.9,0.7,0.6]$$
$$[(d,q_2)0.1,0.5,0.7],[(f,q_3)0.7,0.5,0.7],[(e,q_5)0.8,0.5,0.7],[(h,q_2)0.8,0.5,0.7]$$
$$[(g,q_3)0.1,0.7,0.4],[(a,q_4)0.9,0.7,0.2],[(b,q_2)0.8,0.7,0.7],[(c,q_1)0.8,0.7,0.6]$$
$$[(f,q_4)0.1,0.7,0.5],[(c,q_1)0.9,0.7,0.2],[(d,q_3)0.9,0.7,0.6],[(b,q_5)0.8,0.7,0.6]\rangle\Big\}$$

$$\Pi^Q(e_2 \odot e_4) = \Big\{\langle[(a,q_1)0.1,0.6,0.3],[(b,q_3)0.1,0.5,0.5],[(c,q_5)0.3,0.5,0.9],[(d,q_4)0.3,0.5,0.9]$$
$$[(b,q_5)0.1,0.6,0.4],[(a,q_4)0.1,0.3,0.5],[(d,q_2)0.7,0.6,0.9],[(c,q_1)0.7,0.6,0.9]$$
$$[(f,q_2)0.1,0.6,0.7],[(e,q_1)0.1,0.5,0.7],[(h,q_3)0.1,0.5,0.9],[(g,q_5)0.1,0.5,0.9]$$
$$[(g,q_2)0.1,0.7,0.9],[(h,q_1)0.1,0.7,0.9],[(e,q_3)0.6,0.7,0.9],[(f,q_5)0.6,0.7,0.9]\rangle\Big\}$$

$$\Pi^Q(e_1 \odot e_6) = \Big\{\langle[(c,q_1)0.3,0.5,0.6],[(d,q_2)0.3,0.5,0.6],[(f,q_5)0.3,0.5,0.3],[(g,q_4)0.3,0.5,0.6]$$
$$[(d,q_1)0.3,0.6,0.6],[(c,q_5)0.9,0.6,0.6],[(e,q_2)0.6,0.6,0.4],[(h,q_1)0.5,0.6,0.6]$$
$$[(h,q_1)0.1,0.5,0.7],[(g,q_2)0.1,0.5,0.7],[a,q_3)0.1,0.5,0.7],[(d,q_5)0.1,0.5,0.7]$$
$$[e,q_5)0.3,0.7,0.9],[(f,q_2)0.6,0.7,0.9],[(c,q_1)0.6,0.7,0.9],[(a,q_4)0.5,0.7,0.9]\rangle\Big\}$$

$$\Pi^Q(e_5 \odot e_3) = \Big\{\langle[(d,q_1)0.1,0.4,0.3],[(f,q_3)0.7,0.6,0.3],[(e,q_4)0.8,0.6,0.6],[(h,q_5)0.8,0.5,0.6]$$
$$[(g,q_3)0.1,0.4,0.7],[(a,q_4)0.6,0.3,0.7],[(b,q_2)0.6,0.3,0.7],[(c,q_4)0.6,0.3,0.7]$$
$$[(h,q_4)0.1,0.4,0.9],[(b,q_1)0.5,0.4,0.9],[a,q_3)0.5,0.9,0.4],[(c,q_5)0.5,0.4,0.9]$$
$$[f,q_5)0.1,0.4,0.9],[(c,q_3)0.5,0.4,0.9],[(d,q_1)0.5,0.4,0.9],[(b,q_4)0.5,0.4,0.9]\rangle\Big\}$$

$$\Pi^Q(e_5 \odot e_4) = \Big\{\langle[(b,q_2)0.1,0.6,0.6],[(a,q_3)0.1,0.6,0.5],[(d,q_4)0.7,0.6,0.9],[(c,q_5)0.8,0.6,0.6]$$
$$[(e,q_5)0.1,0.6,0.7],[(f,q_4)0.1,0.3,0.7],[(g,q_2)0.6,0.3,0.9],[(h,q_1)0.6,0.5,0.9]$$
$$[(a,q_2)0.1,0.6,0.9],[(e,q_1)0.1,0.4,0.9],[h,q_3)0.5,0.4,0.9],[(g,q_5)0.5,0.5,0.9]$$
$$[h,q_2)0.1,0.6,0.9],[(g,q_1)0.1,0.4,0.9],[(f,q_3)0.5,0.4,0.9],[(e,q_5)0.5,0.5,0.9]\rangle\Big\}$$

$$\Pi^Q(e_5 \odot e_6) = \Big\{\langle[(d,q_3)0.3,0.6,0.6],[(c,q_2)0.9,0.6,0.9],[(e,q_4)0.6,0.6,0.3],[(h,q_5)0.5,0.6,0.6]$$
$$[(g,q_3)0.3,0.4,0.7],[(h,q_5)0.6,0.3,0.7],[(b,q_2)0.6,0.3,0.3],[(c,q_1)0.5,0.3,0.7]$$
$$[(h,q_4)0.3,0.4,0.9],[(g,q_2)0.3,0.4,0.9],[a,q_3)0.5,0.4,0.9],[(d,q_5)0.5,0.4,0.9]$$
$$[f,q_5)0.3,0.4,0.9],[(e,q_2)0.5,0.4,0.9],[(d,q_1)0.5,0.4,0.9],[(b,q_5)0.5,0.5,0.9]\rangle\Big\}$$

Nest, We compute the comparison table to obtain the highest numerical grade for each column.

TABLE 5. Compute $T_{\Pi_Q}(c)) + I_{\Pi_Q}(c) - F_{\Pi_Q}(c)$ for company a

$(\hat{G} \times Q)$	(e_1e_3)	(e_1e_4)	(e_1e_6)	(e_2e_3)	(e_2e_4)	(e_2e_6)	(e_5e_3)	(e_5e_4)	(e_5e_6)
(a, q_1)	-	-	-	-	0.4	-	-	-	-
(a, q_2)	-	-	-	-	-	-	-	-0.2	-
(a, q_3)	-	-	-	-	-	-0.1	1.0	0.2	0.0
(a, q_4)	1.4	-0.1	-	1.4	-0.1	0.3	0.2	-	-
(a, q_5)	-	-	-	-	-	-	-	-	-

TABLE 6. Compute $T_{\Pi_Q}(c)) + I_{\Pi_Q}(c) - F_{\Pi_Q}(c)$ for company b

$(\hat{G} \times Q)$	(e_1e_3)	(e_1e_4)	(e_1e_6)	(e_2e_3)	(e_2e_4)	(e_2e_6)	(e_5e_3)	(e_5e_4)	(e_5e_6)
(b, q_1)	-	0.0	1.0	-	-	-	0.0	-	-
(b, q_2)	8.0	-	-	8.0	-	-	0.2	0.1	0.6
(b, q_3)	-	-	-	-	0.1	-	-	-	-
(b, q_4)	-	-	-	-	-	-	0.0	-	-
(b, q_5)	0.9	-	0.6	0.9	0.3	-	-	-	0.1

TABLE 7. Compute $T_{\Pi^Q}(c)) + I_{\Pi^Q}(c) - F_{\Pi^Q}(c)$ for company c

$(\hat{G} \times Q)$	(e_1e_3)	(e_1e_4)	(e_1e_6)	(e_2e_3)	(e_2e_4)	(e_2e_6)	(e_5e_3)	(e_5e_4)	(e_5e_6)
(c, q_1)	0.3,0.9,1.4	0.5	0.5	0.9	0.4	0.2, 0.4	-	-	0.1
(c, q_2)	-	-	-	0.1	-	-	-	-	0.6
(c, q_3)	-	-	0.6	1.4	-	-	0.0	-	-
(c, q_4)	-	0.3, 0.5	-	-	-	-	0.2	-	-
(c, q_5)	-	-	-	-	-0.1	0.9	0.0	0.8	-

TABLE 8. Compute $T_{\Pi^Q}(c)) + I_{\Pi^Q}(c) - F_{\Pi^Q}(c)$ for company d

$(\hat{G} \times Q)$	(e_1e_3)	(e_1e_4)	(e_1e_6)	(e_2e_3)	(e_2e_4)	(e_2e_6)	(e_5e_3)	(e_5e_4)	(e_5e_6)
(d, q_1)	-	1.0	-	-	-	0.3	0.0. 0.2	-	0.0
(d, q_2)	-0.1	1.0	1.0	-0.1	0.4	0.2	-	-	-
(d, q_3)	0.1 -	0.1, 1.0	1.0	-	-	-	-	-	0.3
(d, q_4)	-	-	-	-	-0.1	-	-	0.4	-
(d, q_5)	-	0.3	-	-	-	-0.1	-	-	0.0

TABLE 9. Compute $T_{\Pi^Q}(c)) + I_{\Pi^Q}(c) - F_{\Pi^Q}(c)$ for company e

$(\hat{G} \times Q)$	(e_1e_3)	(e_1e_4)	(e_1e_6)	(e_2e_3)	(e_2e_4)	(e_2e_6)	(e_5e_3)	(e_5e_4)	(e_5e_6)
(e, q_1)	-	-	0.6	-	-0.1	-	-	-0.4	0.0
(e, q_2)	-	-	1.0	-	-	0.8	-	-	0.0
(e, q_3)	1.0	-	-	0.3	-	-	-	-	-
(e, q_4)	-	-	-	-	-	-	0.8	-	0.8
(e, q_5)	0.6	0.5	0.4	0.6	0.5	0.1	-	-	0.1,0.0

TABLE 10. Compute $T_{\Pi^Q}(c)) + I_{\Pi^Q}(c) - F_{\Pi^Q}(c)$ for company f

$(\hat{G} \times Q)$	(e_1e_3)	(e_1e_4)	(e_1e_6)	(e_2e_3)	(e_2e_4)	(e_2e_6)	(e_5e_3)	(e_5e_4)	(e_5e_6)
(f, q_1)	-	-	0.6	-	-	-	-	-	-
(f, q_2)	-	-	-	-	0.0	0.4	-	-	-
(f, q_3)	0.5	-	-	0.5	-	-	0.4	0.0	-
(f, q_4)	0.9,0.3	0.3	0.9, 0.4	0.9, 0.3	-	-	-	-	-0.3
(f, q_5)	-	-	-	-	0.5	0.5	-0.4	0.1	-0.2

TABLE 11. Compute $T_{\Pi^Q}(c)) + I_{\Pi^Q}(c) - F_{\Pi^Q}(c)$ for company g

$(\hat{G} \times Q)$	(e_1e_3)	(e_1e_4)	(e_1e_6)	(e_2e_3)	(e_2e_4)	(e_2e_6)	(e_5e_3)	(e_5e_4)	(e_5e_6)
(g, q_1)	-	-	-	-	-	-	-	-0.7	-
(g, q_2)	-	0.5	-	-	-0.1	-0.1	-	0.0	-0.2
(g, q_3)	0.4	-	0.4	0.4	-	-	0.2	-	0.0
(g, q_4)	-	-	-	-	-	0.2	-	-	-
(g, q_5)	1.0	-	0.5	1.0	-0.3	-	-	-	-

TABLE 12. Compute $T_{\Pi^Q}(c)) + I_{\Pi^Q}(c) - F_{\Pi^Q}(c)$ for company h

$(\hat{G} \times Q)$	(e_1e_3)	(e_1e_4)	(e_1e_6)	(e_2e_3)	(e_2e_4)	(e_2e_6)	(e_5e_3)	(e_5e_4)	(e_5e_6)
(h, q_1)	-	0.5	-	-	-0.1	0.5, -0.1	-	0.2	-
(h, q_2)	0.6	-	0.3	0.6	-	-	-	-0.3	-
(h, q_3)	-	-	-	-	-0.3	-	0.0	-	
(h, q_4)	-	-	-	-	-	-	0.4	-	-0.2
(h, q_5)	-	-	1.0	-	-	-	0.7	-	-0.5, 0.2

Compute the sum of these numerical grades score $S_{(x,q)} \in (\hat{G} \times Q)$ of each object $\Pi_{Q(e_i \odot e_j)}(x, q_k)$ for all $k = 1, ..., 5)$.

TABLE 13. Compute $\sum_{\Pi^Q(e_i \odot e_j)}(x, q_k)$ for all k=1,..., 5

$\hat{G} \times Q$	Total
(a, q_1)	0.4
(a, q_2)	-0.2
(a, q_3)	1.1
(a, q_4)	3.1
(a, q_5)	-

$\hat{G} \times Q$	Total
(b, q_1)	1.0
(b, q_2)	2.5
(b, q_3)	0.1
(b, q_4)	0.0
(b, q_5)	2.6

$\hat{G} \times Q$	Total
(c, q_1)	7.6
(c, q_2)	o.5
(c, q_3)	2.0
(c, q_4)	1.0
(c, q_5)	1.6

$\hat{G} \times Q$	Total
(d, q_1)	1.5
(d, q_2)	1.4
(d, q_3)	2.5
(d, q_4)	0.3
(d, q_5)	0.2

$\hat{G} \times Q$	Total
(e, q_1)	0.1
(e, q_2)	1.8
(e, q_3)	1.3
(e, q_4)	1.6
(e, q_5)	2.8

$\hat{G} \times Q$	Total
(f, q_1)	0.6
(f, q_2)	0.4
(f, q_3)	1.4
(f, q_4)	3.7
(f, q_5)	0.5

$\hat{G} \times Q$	Total
(g, q_1)	-0.7
(g, q_2)	0.1
(g, q_3)	1.4
(g, q_4)	0.2
(g, q_5)	2.2

$\hat{G} \times Q$	Total
(h, q_1)	1.0
(h, q_2)	1.2
(h, q_3)	-0.3
(h, q_4)	0.4
(h, q_5)	0.4

Now, selecting the highest numerical value from company "a, b, c, d, e, f, g, h" with their respective policy

(1) $(a, q_4) = 3.1 > (a, q_3) = 1.1 > (a, q_1) = 0.4 > (a, q_2) = -0.2 > (a, q_5) - invalid$
(2) $(b, q_5) = 2.6 > (b, q_2) = 2.5 > (b, q_1) = 1.0 > (b, q_3) = 0.1 > (b, q_4) = 0.0$
(3) $(c, q_1) = 7.6 > (c, q_3) = 2.0 > (c, q_5) = 1.6 > (c, q_4) = 1.0 > (c, q_2) = 0.5$
(4) $(d, q_3) = 2.5 > (d, q_1) = 1.5 > (d, q_2) = 1.4 > (d, q_4) = 0.3 > (d, q_5) = 0.2$
(5) $(e, q_5) = 2.8 > (e, q_2) = 1.8 > (e, q_4) = 1.6 > (e, q_3) = 1.3 > (e, q_1) = 0.1$
(6) $(f, q_4) = 3.7 > (f, q_3) = 1.4 > (f, q_1) = 0.6 > (f, q_5) = 0.5 > (f, q_2) = 0.4$
(7) $(g, q_5) = 2.2 > (g, q_3) = 1.4 > (g, q_2) = 0.1 > (g, q_4) = 0.2 > (g, q_1) = -0.7$
(8) $(h, q_2) = 1.2 > (h, q_1) = 1.0 > (h, q_4) = 0.4 = (h, q_5) = 0.4 > (h, q_3) = -0.3$

Select the highest numerical value from each company with their respectively policies q_k for $k = 1, ..., 5$ in table 13
$(c, q_1) = 7.6 > (f, q_4) = 3.7 > (a, q_3) = 3.1 > (b, q_5) = 2.6 > (d, q_3) = 2.5 = (e, q_5) > (g, q_5) = 2.2 > (h, q_2) = 1.2$. Hence, company 'c' with the policy q_1 is the best to invest in.

6. **Discussion**

In contrast to previous findings on $Q-$neutrosophic soft sets, this work does not assume the associative law, thereby providing a new perspective. It has been demonstrated that the exploration of $Q-$neutrosophic soft quasigroups offers a more comprehensive understanding of the properties of $Q-$neutrosophic soft groups, compared to existing generalizations in $Q-$neutrosophic soft sets. This research also showcases the superiority of non-associativity as a better generalization in comparison to its counterpart in the existing literature. It has been

established that the discoveries in $Q-$NS group theory can be extended to $Q-$NS quasigroups, thus paving the way for further research in this area. These findings not only expand the scope of $Q-$NS sets theory and classical quasigroup theory but also contribute to the development of a new theoretical framework. Overall, this work has opened up new avenues of exploration and is a significant improvement upon existing.

7. Conclusions

This paper explores the concepts of isotopism, homomorphism, and isomorphism within the framework of $Q-$neutrosophic soft quasigroups, along with an examination of the direct product of these quasigroups. It addresses two relevant questions and provides appropriate solutions. An extension of the $Q-$set to a groupoid was employed to develop an algorithm for decision-making. However, it is important to note that the characteristics of the homomorphic image and preimage of a $Q-$neutrosophic soft quasigroup do not necessarily form a $Q-$neutrosophic soft quasigroup. This paper outlines the necessary and sufficient conditions for this situation and tackles a specific issue regarding whether the direct product of two $Q-$neutrosophic soft quasigroups contains a $Q-$neutrosophic soft subquasigroup isomorphic to (ψ^Q, A).

Future research may delve into the properties of isotopy and homomorphism in quasigroups, with the potential to extend these concepts to $Q-$neutrosophic soft n-ary quasigroups, thereby generalizing the notion of $Q-$neutrosophic soft quasigroups. Additionally, it may involve constructing an algorithm using the hybrid model of $Q-$neutrosophic soft n-ary quasigroups for decision-making purposes.

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Adaptability Evaluation of Vocational Education and Economic Development Under the Bipolar Neutrosophic Sets

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Abstract: This study suggests a decision-making methodology to evaluate the risks in vocational education and economic development. This problem has different criteria and alternatives which must be evaluated. This evaluation can include uncertainty in the evaluation process. So, we use the bipolar neutrosophic sets (BNSs) to deal with uncertainty and vague data. The BNSs are extensions of the neutrosophic sets. We used two decision-making methods for the evaluation of risks in vocational education and economic development such as BWM to compute the criteria weights and the COPRAS method to rank the alternatives based on a set of criteria. This study uses nine criteria and six alternatives. The results show that Public Vocational Training Institutions is the best alternative and the Corporate Training and Upskilling Programs is the worst alternative.

Keywords: Bipolar Neutrosophic Sets (BNSs); Vocational Education and Economic Development; Uncertainty; BWM; COPRAS.

1. Introduction

Zadeh was the first to introduce fuzzy set theory. Any value inside the unit closed interval [0,1] can be used to represent the degree of membership that an object has in the set according to fuzzy set theory. Neutosophy, a field of philosophy that studies genesis, character, and extent of neutralities as well as their relationships to various intellectual spectrums, was first presented by Smarandache. Smarandache and Wang et al. devised single-valued neutrosophic sets, which take the value from the subset of [0,1], to more easily apply neutrosophic sets to real-life issues[1], [2]. A single-valued neutrosophic set is hence an example of a neutrosophic set and can be practically applied to real-world issues, particularly in decision support. Bipolar neutrosophic sets, are an extension of bipolar fuzzy sets[3], [4]. The process of selecting the best option with the maximum degree of attainment from a group of options that are defined by several competing criteria is known as multi-criteria decision-making or MCDM. One of the most significant areas of decision-

making theory is MCDM. In terms of the problem's solution space, MCDM problems are typically classified as either continuous or discrete. Multi-objective decision-making (MODM) techniques are applied to continuous issues. In contrast, multi-attribute decision-making (MADM) techniques are used to handle discrete problems[5], [6]. However, MCDM is frequently used to describe the discrete MCDM in the literature that is currently available, which is why we use it in this paper as well.

This study proposes a novel approach to multi-criteria decision-making (MCDM) problems: the best-worst method (BWM). To choose the best alternative or alternatives in an MCDM situation, several options are assessed based on a variety of factors. BWM states that the decision-maker first determines the best (i.e., most desirable, most important) and worst (i.e., least desirable, least important) criteria[7], [8].

Then, each of these two criteria (best and worst) is compared pairwise with the other criteria. To ascertain the weights of various criteria, a maximin problem is then developed and resolved. The same procedure is used to determine the alternatives' weights according to other parameters. The best option is chosen based on the final scores of the alternatives, which are obtained by adding the weights from several sets of criteria and alternatives[9], [10].

The following is how the article is planned: The background theory, along with some initial thoughts on bipolar neutrosophic numbers and the suggested model, is explained in Section 2. A case study is presented in Section 3 to validate the BWM–COPRAS method's applicability. The sensitivity analysis is acknowledged in Section 4. Finally, we offer some observations to round up our investigation.

2. Bipolar Neutrosophic BWM Method and COPRAS Method

Definition 1. The bipolar neutrosophic sets (BNSs) can be defined as[11], [12], [13]:

$$B = \left\{c, \left(T_B^+(c), I_B^+(c), F_B^+(c), T_B^-(c), I_B^-(c), F_B^-(c)\right) c \in C\right\} \tag{1}$$

$$T_B^+(c), I_B^+(c), F_B^+(c): C \to [0,1] \tag{2}$$

$$T_B^-(c), I_B^-(c), F_B^-(c): C \to [-1,0] \tag{3}$$

Definition 2. Let two bipolar neutrosophic numbers (BNNs) such as:

$$A_1 = \{T_1^+(c), I_1^+(c), F_1^+(c), T_1^-(c), I_1^-(c), F_1^-(c)\}$$

$$A_2 = \{T_2^+(c), I_2^+(c), F_2^+(c), T_2^-(c), I_2^-(c), F_2^-(c)\}$$

$$A_1 \cup A_2 = \begin{pmatrix} \max\left(T_1^+(c), T_2^+(c)\right), \frac{I_1^+(c)+I_2^+(c)}{2}, \\ \min\left(F_1^+(c), F_2^+(c)\right), \min\left(T_1^-(c), T_2^-(c)\right), \frac{I_1^-(c)+I_2^-(c)}{2}, \\ \max\left(F_1^-(c), F_2^-(c)\right) \end{pmatrix} \tag{4}$$

Definition 3. Some operations of BNSs

$$a_1 + a_2 = \begin{pmatrix} T_1^+(c) + T_2^+(c) - T_1^+(c)T_2^+(c), \\ I_1^+(c)I_2^+(c), \\ F_1^+(c)F_2^+(c), \\ -T_1^-(c)T_2^-(c), \\ -\left(-I_1^-(c) - I_2^-(c) - I_1^-(c)I_2^-(c)\right), \\ -\left(-F_1^-(c) - F_2^-(c) - F_1^-(c)F_2^-(c)\right) \end{pmatrix} \tag{5}$$

$$a_1 a_2 = \begin{pmatrix} T_1^+(c)T_2^+(c), I_1^+(c) + I_2^+(c) - I_1^+(c)I_2^+(c) + \\ F_1^+(c) + F_2^+(c) - F_1^+(c)F_2^+(c), \\ -\left(-T_1^-(c) - T_2^-(c) - T_1^-(c)T_2^-(c)\right), \\ -I_1^-(c)I_2^-(c), \\ -F_1^-(c)F_2^-(c) \end{pmatrix} \tag{6}$$

$$\theta a_1 = \begin{pmatrix} \left(1 - \left(1 - T_1^+(c)\right)\right)^\theta, \\ \left(I_1^+(c)\right)^\theta, \\ \left(F_1^+(c)\right)^\theta, \\ -\left(-\left(T_1^-(c)\right)^\theta\right), \\ -\left(-\left(I_1^-(c)\right)^\theta\right), \\ -\left(1 - \left(1 - F_1^-(c)\right)\right)^\theta \end{pmatrix} \tag{7}$$

$$a_1^\theta = \begin{pmatrix} \left(T_1^+(c)\right)^\theta, \\ \left(1 - \left(1 - I_1^+(c)\right)\right)^\theta, \\ \left(1 - \left(1 - F_1^+(c)\right)\right)^\theta, \\ -\left(1 - \left(1 - T_1^-(c)\right)\right)^\theta, \\ -\left(-\left(I_1^-(c)\right)^\theta\right) \\ -\left(-\left(F_1^-(c)\right)^\theta\right) \end{pmatrix} \tag{8}$$

Definition 4. The score function of BNSs can be computed such as:

$$S(a_1) = \frac{\begin{pmatrix} T_1^+(c)+1-I_1^+(c)+1-F_1^+(c)+ \\ 1+T_1^-(c)-I_1^-(c)-F_1^-(c) \end{pmatrix}}{6} \tag{10}$$

Then the steps of the BWM and COPRAS methods under a neutrosophic environment are presented in Fig. 1.

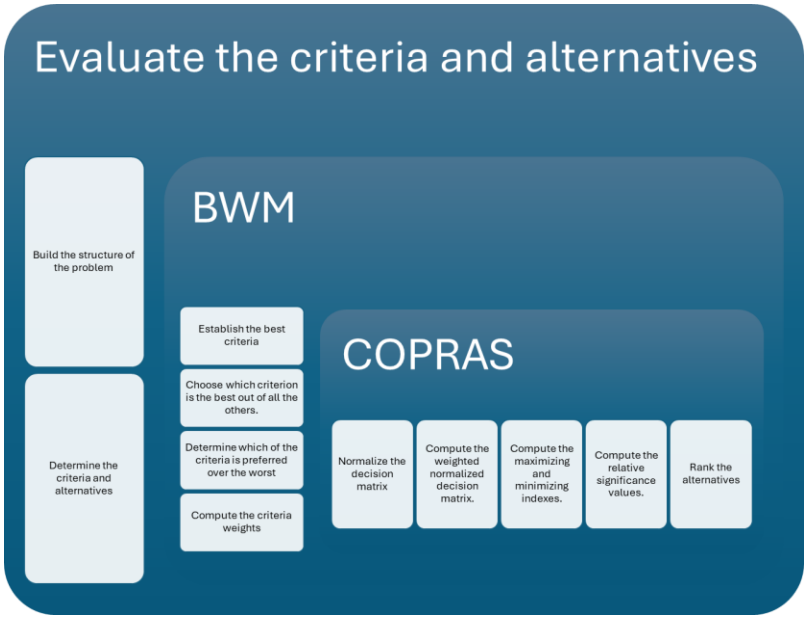
Evaluate the criteria and alternatives
Build the structure of the problem
Determine the criteria and alternatives
BWM
Establish the best criteria
Choose which criterion is the best out of all the others.
Determine which of the criteria is preferred over the worst
Compute the criteria weights
COPRAS
Normalize the decision matrix
Compute the weighted normalized decision matrix.
Compute the maximizing and minimizing indexes.
Compute the relative significance values.
Rank the alternatives

Figure 1. Suggested method.

We applied the steps of the BWM[14], [15].

A) Build the structure of the problem.

The structure of this problem is presented.

B) Determine the criteria and alternatives.

The list of experts and decision-makers determines the criteria and alternatives based on their experience.

C) Establish the best criteria (such as the most important or desirable) and the worst criteria (such as the least important or least attractive).

The decision-maker determines the best and worst criteria overall in this stage. At this point, no comparison is made.

D) Choose which criterion is the best out of all the others.

$A_B = (a_{B_1}, a_{B_2}, \ldots, a_{B_n})$

Where a_{B_j} refers the preference of the best criterion B over the criterion j.

$a_{B_B} = 1$

E) Determine which of the criteria is preferred over the worst. The outcome of the Vector of Others-to-Worst is obtained.

F) Compute the criteria weights.

$$\min \max_j \left\{ \left| \frac{w_B}{w_j} - a_{B_j} \right| \right\}, \left| \frac{w_j}{w_W} - a_{j_W} \right| \quad (11)$$

$\sum_j w_j = 1$

$w_j \geq 0 \; for \; all \; j$

G) This phase applies the COPRAS method to rank the alternatives[16], [17]. Build the decision matrix.

H) Normalize the decision matrix.

$$r_{ij} = \frac{x_{ij}}{\sum_{i=1}^{m} x_{ij}} \quad (12)$$

I) Compute the weighted normalized decision matrix.

$$q_{ij} = r_j r_{ij} \quad (13)$$

J) Compute the maximizing and minimizing indexes.

For positive criteria.

$$D_{+i} = \sum_{j=1}^{g} q_{ij} \quad (14)$$

For negative criteria.

$$D_{-i} = \sum_{j=g+1}^{n} q_{ij} \quad (15)$$

K) Compute the relative significance values.

$$Z_i = D_{+i} + \frac{\sum_{i=1}^{m} D_{-i}}{D_{-i} \sum_{i=1}^{m} 1/D_{-i}} \quad (16)$$

L) Rank the alternatives

3. Case Study

This study proposed the DM approach for the evaluation of Risk Areas in Vocational Education and Economic Development. Four experts have evaluated the criteria and alternatives. This study gathers nine criteria and six alternatives to be evaluated. Fig. 2. shows the criteria and alternatives in this study.

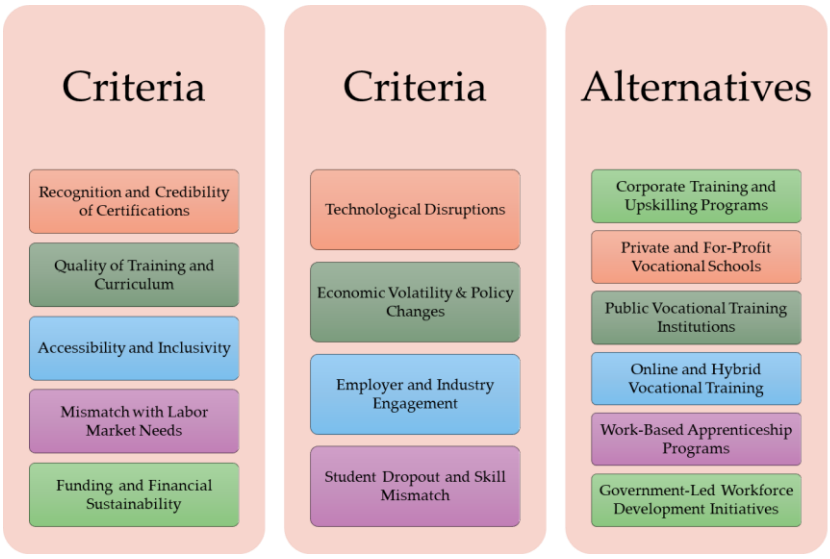

Figure .2. The criteria and alternatives.

A) We invited four experts to evaluate the criteria and alternatives.

B) We collected nine criteria and six alternatives.

C) Then we establish the best criteria and the worst criteria.

D) Then we choose which criterion is the best out of all the others.

E) Then we determine which of the criteria is preferred over the worst.

F) Then we computer the criteria weights as shown in Fig. 3.

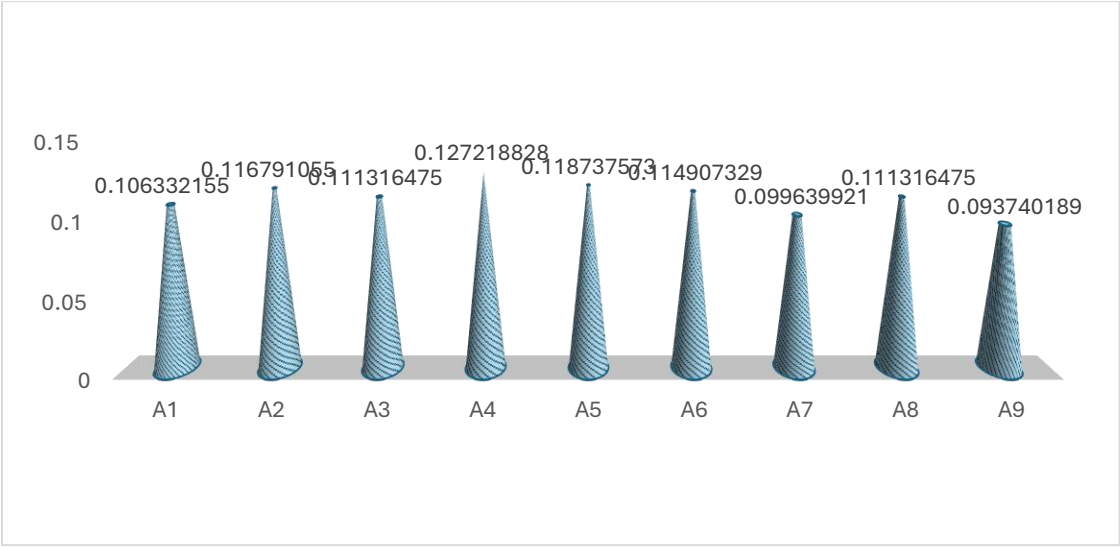

Figure 3. Criteria weights.

G) We build the decision matrix between the criteria and alternatives. We used the BNNs to evaluate the decision matrix as shown in Tables 1-4. Then we apply the score function to obtain one number. Then we combined these numbers into a single matrix.

Table 1. The first BNNs

	A_1	A_2	A_3	A_4	A_5	A_6
C_1	(0.8,0.2,0.1,-0.3,-0.2,-0.4)	(0.4,0.1,0.4,-0.1,-0.2,-0.5)	(0.2,0.4,0.4,-0.1,-0.4,-0.5)	(0.7,0.3,0.2,-0.4,-0.2,-0.1)	(0.6,0.4,0.4,-0.3,-0.2,-0.3)	(0.6,0.4,0.4,-0.3,-0.2,-0.3)
C_2	(0.5,0.4,0.3,-0.4,-0.3,-0.3)	(0.8,0.2,0.1,-0.3,-0.2,-0.4)	(0.4,0.1,0.4,-0.1,-0.2,-0.5)	(0.2,0.4,0.4,-0.1,-0.4,-0.5)	(0.1,0.4,0.3,-0.1,-0.2,-0.3)	(0.7,0.3,0.2,-0.4,-0.2,-0.1)
C_3	(0.8,0.2,0.1,-0.3,-0.2,-0.4)	(0.4,0.1,0.4,-0.1,-0.2,-0.5)	(0.2,0.4,0.4,-0.1,-0.4,-0.5)	(0.7,0.3,0.2,-0.4,-0.2,-0.1)	(0.1,0.4,0.4,-0.1,-0.3,-0.5)	(0.4,0.1,0.4,-0.1,-0.2,-0.5)
C_4	(0.1,0.4,0.3,-0.1,-0.2,-0.3)	(0.8,0.2,0.1,-0.3,-0.2,-0.4)	(0.4,0.1,0.4,-0.1,-0.2,-0.5)	(0.2,0.4,0.4,-0.1,-0.4,-0.5)	(0.5,0.4,0.3,-0.4,-0.3,-0.3)	(0.4,0.3,0.3,-0.1,-0.2,-0.3)
C_5	(0.6,0.4,0.4,-0.3,-0.2,-0.3)	(0.1,0.4,0.4,-0.1,-0.3,-0.5)	(0.8,0.2,0.1,-0.3,-0.2,-0.4)	(0.4,0.1,0.4,-0.1,-0.2,-0.5)	(0.2,0.4,0.4,-0.1,-0.4,-0.5)	(0.5,0.4,0.3,-0.4,-0.3,-0.3)
C_6	(0.7,0.3,0.2,-0.4,-0.2,-0.1)	(0.8,0.2,0.1,-0.3,-0.2,-0.4)	(0.4,0.1,0.4,-0.1,-0.2,-0.5)	(0.2,0.4,0.4,-0.1,-0.4,-0.5)	(0.8,0.2,0.1,-0.3,-0.2,-0.4)	(0.7,0.3,0.2,-0.4,-0.2,-0.1)
C_7	(0.8,0.2,0.1,-0.3,-0.2,-0.4)	(0.4,0.1,0.4,-0.1,-0.2,-0.5)	(0.2,0.4,0.4,-0.1,-0.4,-0.5)	(0.5,0.4,0.3,-0.4,-0.3,-0.3)	(0.2,0.4,0.4,-0.1,-0.4,-0.5)	(0.1,0.4,0.3,-0.1,-0.2,-0.3)
C_8	(0.2,0.4,0.4,-0.1,-0.4,-0.5)	(0.8,0.2,0.1,-0.3,-0.2,-0.4)	(0.4,0.1,0.4,-0.1,-0.2,-0.5)	(0.2,0.4,0.4,-0.1,-0.4,-0.5)	(0.4,0.1,0.4,-0.1,-0.2,-0.5)	(0.6,0.4,0.4,-0.3,-0.2,-0.3)
C_9	(0.8,0.2,0.1,-0.3,-0.2,-0.4)	(0.4,0.1,0.4,-0.1,-0.2,-0.5)	(0.2,0.4,0.4,-0.1,-0.4,-0.5)	(0.8,0.2,0.1,-0.3,-0.2,-0.4)	(0.7,0.3,0.2,-0.4,-0.2,-0.1)	(0.4,0.3,0.3,-0.1,-0.2,-0.3)

Table 2. The second BNNs

	A_1	A_2	A_3	A_4	A_5	A_6
C_1	(0.2,0.4,0.4,-0.1,-0.4,-0.5)	(0.7,0.3,0.2,-0.4,-0.2,-0.1)	(0.1,0.4,0.3,-0.1,-0.2,-0.3)	(0.7,0.3,0.2,-0.4,-0.2,-0.1)	(0.6,0.4,0.4,-0.3,-0.2,-0.3)	(0.4,0.3,0.3,-0.1,-0.2,-0.3)
C_2	(0.5,0.4,0.3,-0.4,-0.3,-0.3)	(0.2,0.4,0.4,-0.1,-0.4,-0.5)	(0.7,0.3,0.2,-0.4,-0.2,-0.1)	(0.1,0.4,0.3,-0.1,-0.2,-0.3)	(0.1,0.4,0.3,-0.1,-0.2,-0.3)	(0.5,0.4,0.3,-0.4,-0.3,-0.3)
C_3	(0.1,0.4,0.4,-0.1,-0.3,-0.5)	(0.4,0.3,0.3,-0.1,-0.2,-0.3)	(0.2,0.4,0.4,-0.1,-0.4,-0.5)	(0.7,0.3,0.2,-0.4,-0.2,-0.1)	(0.1,0.4,0.3,-0.1,-0.2,-0.3)	(0.2,0.4,0.4,-0.1,-0.4,-0.5)
C_4	(0.1,0.4,0.3,-0.1,-0.2,-0.3)	(0.2,0.4,0.4,-0.1,-0.4,-0.5)	(0.7,0.3,0.2,-0.4,-0.2,-0.1)	(0.1,0.4,0.3,-0.1,-0.2,-0.3)	(0.5,0.4,0.3,-0.4,-0.3,-0.3)	(0.1,0.4,0.3,-0.1,-0.2,-0.3)
C_5	(0.6,0.4,0.4,-0.3,-0.2,-0.3)	(0.1,0.4,0.4,-0.1,-0.3,-0.5)	(0.2,0.4,0.4,-0.1,-0.4,-0.5)	(0.7,0.3,0.2,-0.4,-0.2,-0.1)	(0.1,0.4,0.3,-0.1,-0.2,-0.3)	(0.6,0.4,0.4,-0.3,-0.2,-0.3)
C_6	(0.7,0.3,0.2,-0.4,-0.2,-0.1)	(0.1,0.4,0.3,-0.1,-0.2,-0.3)	(0.2,0.4,0.4,-0.1,-0.4,-0.5)	(0.7,0.3,0.2,-0.4,-0.2,-0.1)	(0.1,0.4,0.3,-0.1,-0.2,-0.3)	(0.6,0.4,0.4,-0.3,-0.2,-0.3)
C_7	(0.4,0.1,0.4,-0.1,-0.2,-0.5)	(0.2,0.4,0.4,-0.1,-0.4,-0.5)	(0.7,0.3,0.2,-0.4,-0.2,-0.1)	(0.1,0.4,0.3,-0.1,-0.2,-0.3)	(0.2,0.4,0.4,-0.1,-0.4,-0.5)	(0.7,0.3,0.2,-0.4,-0.2,-0.1)
C_8	(0.2,0.4,0.4,-0.1,-0.4,-0.5)	(0.7,0.3,0.2,-0.4,-0.2,-0.1)	(0.1,0.4,0.3,-0.1,-0.2,-0.3)	(0.4,0.3,0.3,-0.1,-0.2,-0.3)	(0.4,0.1,0.4,-0.1,-0.2,-0.5)	(0.1,0.4,0.3,-0.1,-0.2,-0.3)
C_9	(0.8,0.2,0.1,-0.3,-0.2,-0.4)	(0.4,0.1,0.4,-0.1,-0.2,-0.5)	(0.2,0.4,0.4,-0.1,-0.4,-0.5)	(0.8,0.2,0.1,-0.3,-0.2,-0.4)	(0.7,0.3,0.2,-0.4,-0.2,-0.1)	(0.6,0.4,0.4,-0.3,-0.2,-0.3)

Table 3. The third BNNs

	A_1	A_2	A_3	A_4	A_5	A_6
C_1	(0.4,0.3,0.3,-0.1,-0.2,-0.3)	(0.2,0.4,0.4,-0.1,-0.4,-0.5)	(0.4,0.1,0.4,-0.1,-0.2,-0.5)	(0.7,0.3,0.2,-0.4,-0.2,-0.1)	(0.6,0.4,0.4,-0.3,-0.2,-0.3)	(0.4,0.3,0.3,-0.1,-0.2,-0.3)
C_2	(0.4,0.3,0.3,-0.1,-0.2,-0.3)	(0.5,0.4,0.3,-0.4,-0.3,-0.3)	(0.6,0.4,0.4,-0.3,-0.2,-0.3)	(0.4,0.1,0.4,-0.1,-0.2,-0.5)	(0.1,0.4,0.3,-0.1,-0.2,-0.3)	(0.5,0.4,0.3,-0.4,-0.3,-0.3)
C_3	(0.1,0.4,0.4,-0.1,-0.3,-0.5)	(0.4,0.3,0.3,-0.1,-0.2,-0.3)	(0.5,0.4,0.3,-0.4,-0.3,-0.3)	(0.6,0.4,0.4,-0.3,-0.2,-0.3)	(0.1,0.4,0.4,-0.1,-0.3,-0.5)	(0.1,0.4,0.4,-0.1,-0.3,-0.5)
C_4	(0.1,0.4,0.3,-0.1,-0.2,-0.3)	(0.5,0.4,0.3,-0.4,-0.3,-0.3)	(0.4,0.3,0.3,-0.1,-0.2,-0.3)	(0.5,0.4,0.3,-0.4,-0.3,-0.3)	(0.6,0.4,0.4,-0.3,-0.2,-0.3)	(0.1,0.4,0.3,-0.1,-0.2,-0.3)

C_5	(0.6,0.4,0.4,-0.3,-0.2,-0.3)	(0.4,0.3,0.3,-0.1,-0.2,-0.3)	(0.5,0.4,0.3,-0.4,-0.3,-0.3)	(0.6,0.4,0.4,-0.3,-0.2,-0.3)	(0.4,0.3,0.3,-0.1,-0.2,-0.3)	(0.4,0.3,0.3,-0.1,-0.2,-0.3)
C_6	(0.7,0.3,0.2,-0.4,-0.2,-0.1)	(0.1,0.4,0.3,-0.1,-0.2,-0.3)	(0.4,0.3,0.3,-0.1,-0.2,-0.3)	(0.5,0.4,0.3,-0.4,-0.3,-0.3)	(0.6,0.4,0.4,-0.3,-0.2,-0.3)	(0.5,0.4,0.3,-0.4,-0.3,-0.3)
C_7	(0.4,0.1,0.4,-0.1,-0.2,-0.5)	(0.4,0.3,0.3,-0.1,-0.2,-0.3)	(0.5,0.4,0.3,-0.4,-0.3,-0.3)	(0.6,0.4,0.4,-0.3,-0.2,-0.3)	(0.2,0.4,0.4,-0.1,-0.4,-0.5)	(0.6,0.4,0.4,-0.3,-0.2,-0.3)
C_8	(0.4,0.3,0.3,-0.1,-0.2,-0.3)	(0.5,0.4,0.3,-0.4,-0.3,-0.3)	(0.6,0.4,0.4,-0.3,-0.2,-0.3)	(0.4,0.3,0.3,-0.1,-0.2,-0.3)	(0.4,0.1,0.4,-0.1,-0.2,-0.5)	(0.1,0.4,0.3,-0.1,-0.2,-0.3)
C_9	(0.8,0.2,0.1,-0.3,-0.2,-0.4)	(0.4,0.3,0.3,-0.1,-0.2,-0.3)	(0.5,0.4,0.3,-0.4,-0.3,-0.3)	(0.6,0.4,0.4,-0.3,-0.2,-0.3)	(0.7,0.3,0.2,-0.4,-0.2,-0.1)	(0.6,0.4,0.4,-0.3,-0.2,-0.3)

Table 4. The fourth BNNs

	A_1	A_2	A_3	A_4	A_5	A_6
C_1	(0.4,0.3,0.3,-0.1,-0.2,-0.3)	(0.2,0.4,0.4,-0.1,-0.4,-0.5)	(0.4,0.1,0.4,-0.1,-0.2,-0.5)	(0.7,0.3,0.2,-0.4,-0.2,-0.1)	(0.6,0.4,0.4,-0.3,-0.2,-0.3)	(0.4,0.3,0.3,-0.1,-0.2,-0.3)
C_2	(0.5,0.4,0.3,-0.4,-0.3,-0.3)	(0.8,0.2,0.1,-0.3,-0.2,-0.4)	(0.2,0.4,0.4,-0.1,-0.4,-0.5)	(0.4,0.1,0.4,-0.1,-0.2,-0.5)	(0.1,0.4,0.3,-0.1,-0.2,-0.3)	(0.5,0.4,0.3,-0.4,-0.3,-0.3)
C_3	(0.1,0.4,0.4,-0.1,-0.3,-0.5)	(0.4,0.3,0.3,-0.1,-0.2,-0.3)	(0.4,0.1,0.4,-0.1,-0.2,-0.5)	(0.7,0.3,0.2,-0.4,-0.2,-0.1)	(0.1,0.4,0.4,-0.1,-0.3,-0.5)	(0.2,0.4,0.4,-0.1,-0.4,-0.5)
C_4	(0.1,0.4,0.3,-0.1,-0.2,-0.3)	(0.5,0.4,0.3,-0.4,-0.3,-0.3)	(0.8,0.2,0.1,-0.3,-0.2,-0.4)	(0.6,0.4,0.4,-0.3,-0.2,-0.3)	(0.5,0.4,0.3,-0.4,-0.3,-0.3)	(0.7,0.3,0.2,-0.4,-0.2,-0.1)
C_5	(0.6,0.4,0.4,-0.3,-0.2,-0.3)	(0.1,0.4,0.4,-0.1,-0.3,-0.5)	(0.4,0.3,0.3,-0.1,-0.2,-0.3)	(0.1,0.4,0.3,-0.1,-0.2,-0.3)	(0.4,0.3,0.3,-0.1,-0.2,-0.3)	(0.1,0.4,0.3,-0.1,-0.2,-0.3)
C_6	(0.7,0.3,0.2,-0.4,-0.2,-0.1)	(0.1,0.4,0.3,-0.1,-0.2,-0.3)	(0.5,0.4,0.3,-0.4,-0.3,-0.3)	(0.1,0.4,0.4,-0.1,-0.3,-0.5)	(0.8,0.2,0.1,-0.3,-0.2,-0.4)	(0.1,0.4,0.4,-0.1,-0.3,-0.5)
C_7	(0.4,0.1,0.4,-0.1,-0.2,-0.5)	(0.6,0.4,0.4,-0.3,-0.2,-0.3)	(0.1,0.4,0.4,-0.1,-0.3,-0.5)	(0.5,0.4,0.3,-0.4,-0.3,-0.3)	(0.2,0.4,0.4,-0.1,-0.4,-0.5)	(0.5,0.4,0.3,-0.4,-0.3,-0.3)
C_8	(0.2,0.4,0.4,-0.1,-0.4,-0.5)	(0.7,0.3,0.2,-0.4,-0.2,-0.1)	(0.1,0.4,0.3,-0.1,-0.2,-0.3)	(0.4,0.3,0.3,-0.1,-0.2,-0.3)	(0.4,0.1,0.4,-0.1,-0.2,-0.5)	(0.4,0.3,0.3,-0.1,-0.2,-0.3)
C_9	(0.8,0.2,0.1,-0.3,-0.2,-0.4)	(0.4,0.1,0.4,-0.1,-0.2,-0.5)	(0.2,0.4,0.4,-0.1,-0.4,-0.5)	(0.8,0.2,0.1,-0.3,-0.2,-0.4)	(0.7,0.3,0.2,-0.4,-0.2,-0.1)	(0.8,0.2,0.1,-0.3,-0.2,-0.4)

H) Then we used Eq. (12) to normalize the decision matrix as shown in Table 5.

I) Then we used Eq. (13) to compute the weighted normalized decision matrix as shown in Table 6.

Table 5. The normalized BNNs

	A_1	A_2	A_3	A_4	A_5	A_6
C_1	0.175393	0.170157	0.170157	0.162304	0.157068	0.164921
C_2	0.16245	0.183755	0.170439	0.173103	0.149134	0.161119
C_3	0.168865	0.172823	0.170185	0.162269	0.155673	0.170185
C_4	0.151967	0.176391	0.184532	0.162822	0.162822	0.161465
C_5	0.161725	0.16442	0.177898	0.167116	0.167116	0.161725
C_6	0.164675	0.162019	0.171315	0.163347	0.177955	0.160691
C_7	0.188158	0.169737	0.161842	0.155263	0.168421	0.156579
C_8	0.16732	0.169935	0.15817	0.16732	0.183007	0.154248
C_9	0.186962	0.168512	0.154982	0.177122	0.152522	0.159902

Table 6. The weighted normalized BNNs

	A_1	A_2	A_3	A_4	A_5	A_6

C_1	0.01865	0.018093	0.018093	0.017258	0.016701	0.017536
C_2	0.018973	0.021461	0.019906	0.020217	0.017418	0.018817
C_3	0.018798	0.019238	0.018944	0.018063	0.017329	0.018944
C_4	0.019333	0.02244	0.023476	0.020714	0.020714	0.020541
C_5	0.019203	0.019523	0.021123	0.019843	0.019843	0.019203
C_6	0.018922	0.018617	0.019685	0.01877	0.020448	0.018465
C_7	0.018748	0.016913	0.016126	0.01547	0.016781	0.015602
C_8	0.018626	0.018917	0.017607	0.018626	0.020372	0.01717
C_9	0.017526	0.015796	0.014528	0.016603	0.014297	0.014989

J) Then we used Eq. (14) to compute the maximizing and minimizing indexes.

K) Then we used Eq. (16) to compute the relative significance values as shown in Table 7.

L) Then we ranked the alternatives as shown in Table 7 and Fig. 4.

Table 7. The rank of alternatives.

	D_{+i}	D_{-i}	Z_i	RANKS
A_1	0.113301	0.017071	0.122083	1
A_2	0.118766	0.013186	0.130135	5
A_3	0.117711	0.009531	0.133441	6
A_4	0.113647	0.012231	0.125904	3
A_5	0.112982	0.011236	0.126324	4
A_6	0.111474	0.011388	0.124638	2

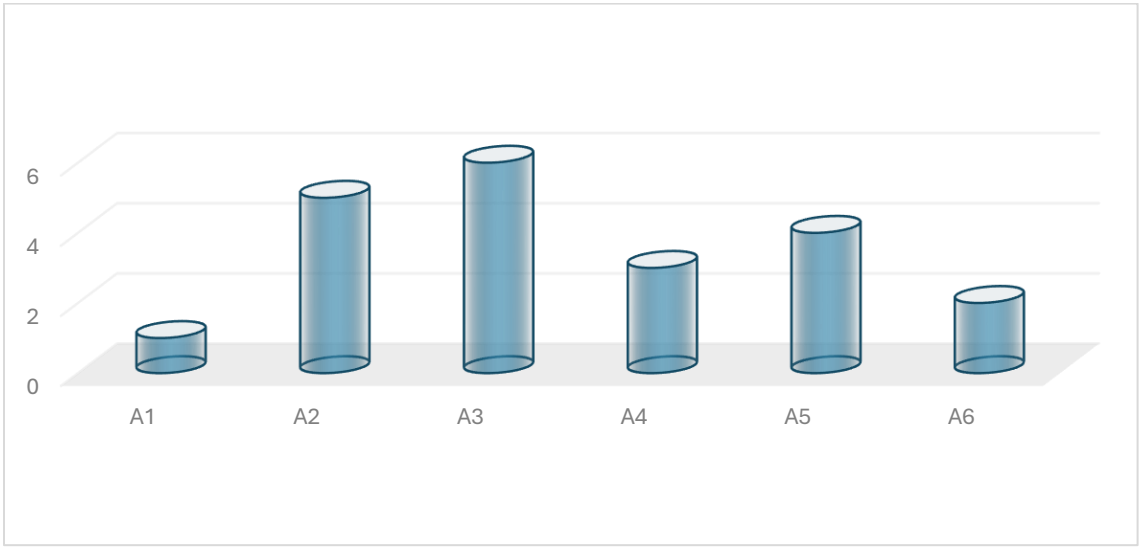

Fig. 4. Ranks of alternatives.

4. Analysis

This section shows the sensitivity analysis for changing the criteria weights based on the set of cases. Fig. 5. shows the criteria weights under different cases. In the first case, we put all criteria with the same weights. In each case, we increase the criteria weights by 15%. In the second case, we increase the first criterion by 15%. In the third case, we increase the second criterion with 15% weights.

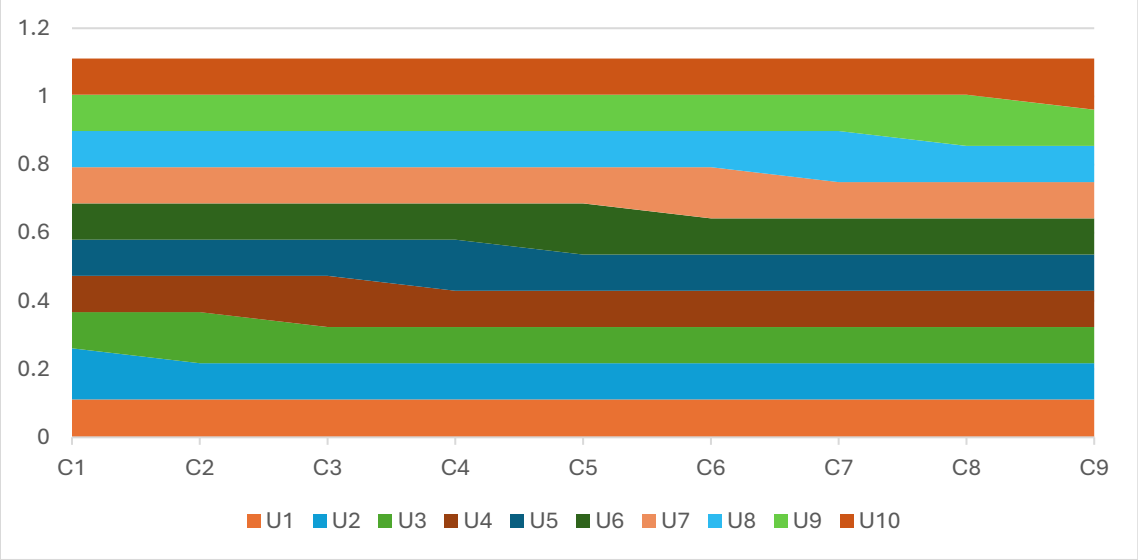

Figure 5. The various criteria weights.

Then we applied the COPRAS method under different criteria weights. We obtained the normalized decision matrix. Then we computed the weighted normalized decision matrix. Then we compute the relative significant values as shown in Table 8. Then ranked the alternatives as shown in Fig. 6. In all cases, we show the alternative 3 is the best and the alternative 1 is the worst.

Table 8. The relative significance values under different weights.

	U_1	U_2	U_3	U_4	U_5	U_6	U_7	U_8	U_9	U_{10}
A_1	0.124281	0.126517	0.125951	0.126232	0.125493	0.111815	0.126048	0.125238	0.126164	0.125253
A_2	0.132449	0.134099	0.134694	0.134216	0.134372	0.119047	0.133743	0.133822	0.134089	0.133849
A_3	0.135688	0.137196	0.137209	0.137198	0.137825	0.123541	0.137247	0.137155	0.136672	0.137522
A_4	0.128473	0.129953	0.130426	0.129952	0.129976	0.11529	0.129999	0.130667	0.130173	0.129686
A_5	0.129226	0.130444	0.130097	0.130383	0.130696	0.116048	0.131358	0.130766	0.131579	0.131528
A_6	0.12766	0.12929	0.129124	0.12952	0.129139	0.114259	0.129105	0.129851	0.128823	0.129661

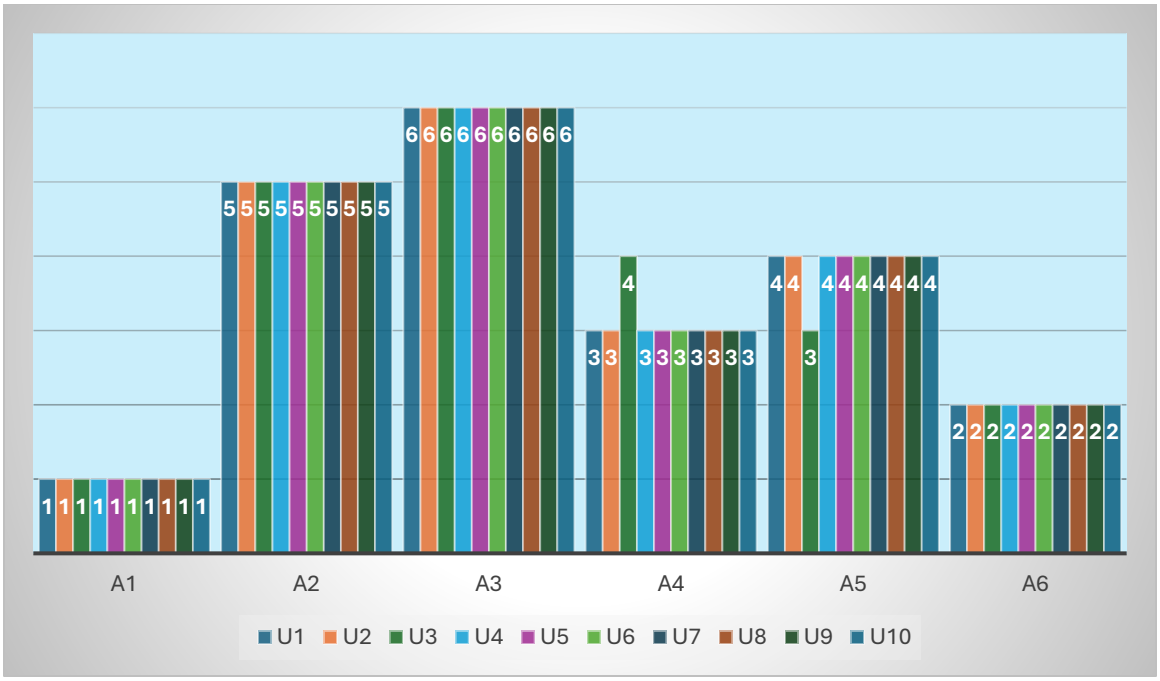

Figure 6. The rank of alternatives is based on a set of weights.

5. Conclusions

Two decision-making methods are used in this study to evaluate the risks of Vocational Education and Economic Development. BWM is used to compute the criteria weights. The COPRAS method is used to rank the alternatives. Two decision-making methods are used under the bipolar neutrosophic sets to deal with uncertainty in the evaluation. Four experts have evaluated the criteria and alternatives using the bipolar neutrosophic numbers to build the decision matrix. The results show the Mismatch with Labor Market Needs has the highest weights and the Student Dropout and Skill Mismatch has the lowest weights. Then we applied the COPRAS method to rank the alternatives. The results show alternative 3 is the best and alternative 1 is the worst. Then we applied the sensitivity analysis to show the stability of the ranks. We proposed ten cases in criteria weights. The results show the ranks of the alternatives are stable under different cases.

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Neutrosophic Linguistic Rhotrix in Managerial Decision Making

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Abstract: The developments in rhotrix theory and their applications are unveiled at recent times. This research work proposes the notion of linguistic rhotrix conjoining the theories of linguistic matrix and rhotrix. The theory of linguistic rhotrix comprising fundamental definitions, operators and classifications is presented in this study. The algorithmic framework of applying linguistic rhotrix to decisioning problem is presented in this work. A managerial decision-making illustration is putforth to validate the proposed decisioning framework based on neutrosophic quantifications of linguistic rhotrix. The results obtained are compared with fuzzy and intuitionistic fuzzy quantifications to demonstrate the efficacy of neutrosophic quantifications of linguistic variables. The results obtained using neutrosophic quantifications are more optimal and reliable in comparisons with the other two quantifications. The proposed decisioning approach based on neutrosophic linguistic rhotrix is more robust and competent in handling intricate decisioning environment.

Keywords: Linguistic Rhotrix, Neutrosophy Linguistic Rhotrix, Decisioning

1. Introduction

Rhotrix is basically termed as rhombus matrix comprising the rhomboidal arrangements of the elements. Ajibade [1] conceptualized the novel theory of rhotrix and researchers are attributing to both the theoretical developments and practical implications of this evolving concept. The properties and theoretical aspects of rhotrix are more in align with matrix theory and some of the significant contributions in this context are discoursed. Sani [2-3] introduced an alternate method for rhotrix multiplication and generalized the row-column multiplication for higher order matrices contributing to the development of rhotrix algebra. Absalom et al [4] discussed the concept of heart-oriented multiplication of rhotrices and its generalization. Mohammed and Sani [5] described the graphical representation of rhotrices as rhomtrees. Mohammed and Balarade [6] presented a revie of rhotrix theory highlighting its inceptions, theoretical concepts and applications. Sharma and Kumar [7] explained the applications of Hadamard rhotrices in designing balanced incomplete block. Usaini and Mohammed [8] focused on rhotrix based eigenvalues and eigen vectors which is more associated with matrices. Mohammed [9] introduced a new expression of rhotrix. Sharma et al [10] discussed the construction of Hadamard codes using Hadamard rhotrices which is a transformation form of Hadamard matrix. The hill cipher is basically a polygraphic substitution cipher based on linear algebra and Kumar [11] extended with rhotrices. Sharma et al [12] discussed a special type of vandermonde rhotrix contributing to algebraic studies. Kaurangini and Aminu [13] discussed Hermitian and skew -Hermitian rhotrices stating the explicit transformation of matrix theory. Anuradha et al [14] applied the theory of rhotrix in solving a coupled linear programming problem.

Isere [15] discussed the natural rhotrix and Mohammed attributed to non-commutative full rhotrix ring. Kavitha et al [16] developed a prediction model for weather forecasting integrating rhotrix optimization based genetic algorithm with interval valued rhotrix representations. Verma et al [17] explored the applications of rhotrix in the construction of balance incomplete block design. Ananda Priya et al [18] contributed to the classification of commutative general rhotrix group. Aminu [19] introduced the theory of fuzzy rhotrix with the integration of fuzzy representations. Ananda Priya et al [20] developed a diagnostic decision model based on neutrosophic based rhotrices. The above contributions present a vivid picture of developments and applications of rhotrix theory based on matrix and its properties. Similar to the aforementioned contributions, this research work extends the theory of linguistic matrix to linguistic rhotrix. Smarandache et al [21] conceptualized the theory of linguistic matrices highlighting the significance of its applications in determining optimal solutions to the realistic problems. Inspired by the works of Smarandache with special reference to linguistic matrix and neutrosophy, this research work contributes to the theoretical development of linguistic rhotrix and extends the same to neutrosophic rhotrix to develop a comprehensive decision-making model. At recent times, the researchers are evolving neutrosophic based decision making models to cater to the dynamic demands of the managerial decision makers. Neutrosophic theory is integrated with multi-criteria decision-making methods [22-23], soft sets[24-25], hypersoft sets [26-27], matrices [28], rough sets [29], graph [30] and algebraic structures [31-32] to formulate robust and reliable decisioning models. The neutrosophic blended decision models are gaining more momentum as the neutrosophic representations facilitate in handling intricate data sets. This has also motivated the authors to exclusively explore the applications of neutrosophic linguistic matrix

The remaining contents of the paper are described as follows, section 2 discusses the theoretical developments of linguistic rhotrix comprising the conceptualization of linguistic rhotrix, operators and classifications of linguistic rhotrix. Section 3 explains the algorithmic framework of linguistic rhotrix applied in decisioning process. Section 4 presents an illustration to the proposed algorithm framework with neutrosophic representations. Section 5 compares the same with both fuzzy and intuitionistic fuzzy based representations. Section 6 concludes the work.

2 Conceptualization of Linguistic Rhotrix

This section explains the theoretical developments of linguistic rhotrix based on the contributions of Smarandache to the theory of linguistic matrix and based on the contributions of Ajibade [1] to the theory of rhotrix. The concept of linguistic matrices and the operators applied are extended to rhotrix theory facilitating the theoretical developments of linguistic rhotrix.

Rhotrix is basically characterized as the rhomboidal arrangement of matrix elements. It is also termed as mathematical arrays comprising square matrix of two and three dimensions. The smallest rhotrix R of order 3 can be defined as.

$$\boldsymbol{R} = \left\langle \begin{matrix} & \mathbf{a} & \\ \mathbf{b} & \mathbf{c} & \mathbf{d} \\ & \mathbf{e} & \end{matrix} \right\rangle \quad \text{where a, b, c, d and e} \in \boldsymbol{R}$$

2.1 Linguistic Rhotrix

Linguistic rhotrix is a kind of rhotrix in which each entry are linguistic variables. The general form of linguistic rhotrix is

$$\mathbf{R_L} = \left\langle \begin{matrix} & & \mathbf{r_{L11}} & \\ & \mathbf{r_{L21}} \; \mathbf{r_{L11}} \; \mathbf{r_{L12}} & & \\ \mathbf{r_{L31}} & \mathbf{r_{L21}} \; \mathbf{r_{L22}} \; \mathbf{r_{L12}} & & \mathbf{r_{L13}} \\ & \mathbf{r_{L32}} \; \mathbf{r_{L21}} \; \mathbf{r_{L23}} & & \\ & & \mathbf{r_{L33}} & \end{matrix} \right\rangle \text{ where } \mathbf{r_{Lij}} \text{ is a linguistic variable.}$$

2.2 Operators on Linguistic Matrix

The operators of MAX and MIN are applied to a linguistic rhotrix. Let $\mathbf{R_L}$ and $\mathbf{S_L}$ be two linguistic matrices with entries as linguistic variables.

2.2.1 MAX Operator

$$\mathbf{R_L} = \left\langle \begin{matrix} & & \mathbf{r_{L11}} & \\ & \mathbf{r_{L21}} \; \mathbf{r_{L11}} \; \mathbf{r_{L12}} & & \\ \mathbf{r_{L31}} & \mathbf{r_{L21}} \; \mathbf{r_{L22}} \; \mathbf{r_{L12}} & & \mathbf{r_{L13}} \\ & \mathbf{r_{L32}} \; \mathbf{r_{L22}} \; \mathbf{r_{L23}} & & \\ & & \mathbf{r_{L33}} & \end{matrix} \right\rangle \text{ and } \mathbf{S_L} = \left\langle \begin{matrix} & & \mathbf{s_{L11}} & \\ & \mathbf{s_{L21}} \; \mathbf{s_{L11}} \; \mathbf{s_{L12}} & & \\ \mathbf{s_{L31}} & \mathbf{s_{L21}} \; \mathbf{s_{L22}} \; \mathbf{s_{L12}} & & \mathbf{s_{L13}} \\ & \mathbf{s_{L32}} \; \mathbf{s_{L22}} \; \mathbf{s_{L23}} & & \\ & & \mathbf{s_{L33}} & \end{matrix} \right\rangle$$

MAX(R_L,S_L)=

$$\left\langle \begin{matrix} & & \mathbf{max}(\mathbf{r_{L11}}, \mathbf{s_{L11}}) & \\ & \mathbf{max}(\mathbf{r_{L21}}, \mathbf{s_{L21}}) \; \mathbf{max}(\mathbf{r_{L11}}, \mathbf{s_{L11}}) \; \mathbf{max}(\mathbf{r_{L12}}, \mathbf{s_{L12}}) & & \\ \mathbf{max}(\mathbf{r_{L31}}, \mathbf{s_{L31}}) & \mathbf{max}(\mathbf{r_{L21}}, \mathbf{s_{L21}}) \; \mathbf{max}(\mathbf{r_{L22}}, \mathbf{s_{L22}}) \; \mathbf{max}(\mathbf{r_{L12}}, \mathbf{s_{L12}}) & & \mathbf{max}(\mathbf{r_{L13}}, \mathbf{s_{L13}}) \\ & \mathbf{max}(\mathbf{r_{L32}}, \mathbf{s_{L32}}) \; \mathbf{max}(\mathbf{r_{L22}}, \mathbf{s_{L22}}) \; \mathbf{max}(\mathbf{r_{L23}}, \mathbf{s_{L23}}) & & \\ & & \mathbf{max}(\mathbf{r_{L33}}, \mathbf{s_{L33}}) & \end{matrix} \right\rangle$$

Example

Let us consider two linguistic rhotrix with entries as linguistic variables from the set {Low, Medium, High}

$$\mathbf{R_L} = \left\langle \begin{matrix} & & \mathbf{r_{L11}} & \\ & \mathbf{r_{L21}} \; \mathbf{r_{L11}} \; \mathbf{r_{L12}} & & \\ \mathbf{r_{L31}} & \mathbf{r_{L21}} \; \mathbf{r_{L22}} \; \mathbf{r_{L12}} & & \mathbf{r_{L13}} \\ & \mathbf{r_{L32}} \; \mathbf{r_{L22}} \; \mathbf{r_{L23}} & & \\ & & \mathbf{r_{L33}} & \end{matrix} \right\rangle \text{ and } \mathbf{S_L} = \left\langle \begin{matrix} & & \mathbf{s_{L11}} & \\ & \mathbf{s_{L21}} \; \mathbf{s_{L11}} \; \mathbf{s_{L12}} & & \\ \mathbf{s_{L31}} & \mathbf{s_{L21}} \; \mathbf{s_{L22}} \; \mathbf{s_{L12}} & & \mathbf{s_{L13}} \\ & \mathbf{s_{L32}} \; \mathbf{s_{L22}} \; \mathbf{s_{L23}} & & \\ & & \mathbf{s_{L33}} & \end{matrix} \right\rangle$$

MAX(R_L,S_L)= $\left\langle \begin{matrix} & & \boldsymbol{H} & \\ & \mathbf{H} \; \mathbf{M} \; \mathbf{H} & & \\ \mathbf{H} & \mathbf{M} \; \mathbf{H} \; \mathbf{M} & & \mathbf{M} \\ & \mathbf{M} \; \mathbf{H} \; \mathbf{H} & & \\ & & \mathbf{H} & \end{matrix} \right\rangle$

2.1.2 MIN Operator

MIN(R_L,S_L)=

$$\left\langle \begin{matrix} & & \mathbf{min}(\mathbf{r_{L11}}, \mathbf{s_{L11}}) & \\ & \mathbf{min}(\mathbf{r_{L21}}, \mathbf{s_{L21}}) \; \mathbf{min}(\mathbf{r_{L11}}, \mathbf{s_{L11}}) \; \mathbf{min}(\mathbf{r_{L12}}, \mathbf{s_{L12}}) & & \\ \mathbf{min}(\mathbf{r_{L31}}, \mathbf{s_{L31}}) & \mathbf{min}(\mathbf{r_{L21}}, \mathbf{s_{L21}}) \; \mathbf{min}(\mathbf{r_{L22}}, \mathbf{s_{L22}}) \; \mathbf{min}(\mathbf{r_{L12}}, \mathbf{s_{L12}}) & & \mathbf{min}(\mathbf{r_{L13}}, \mathbf{s_{L13}}) \\ & \mathbf{min}(\mathbf{r_{L32}}, \mathbf{s_{L32}}) \; \mathbf{min}(\mathbf{r_{L22}}, \mathbf{s_{L22}}) \; \mathbf{min}(\mathbf{r_{L23}}, \mathbf{s_{L23}}) & & \\ & & \mathbf{min}(\mathbf{r_{L33}}, \mathbf{s_{L33}}) & \end{matrix} \right\rangle$$

Example

Let us consider two linguistic rhotrix with entries as linguistic variables from the set {Low, Medium, High}

$$\mathbf{R_L} = \left\langle \begin{matrix} & & \mathbf{H} & & \\ & \mathbf{H} & \mathbf{M} & \mathbf{M} & \\ \mathbf{M} & \mathbf{H} & \mathbf{L} & \mathbf{L} & \mathbf{H} \\ & \mathbf{L} & \mathbf{H} & \mathbf{M} & \\ & & \mathbf{M} & & \end{matrix} \right\rangle \quad \text{and} \quad \mathbf{S_L} = \left\langle \begin{matrix} & & \mathbf{M} & & \\ & \mathbf{M} & \mathbf{H} & \mathbf{L} & \\ \mathbf{L} & \mathbf{M} & \mathbf{M} & \mathbf{H} & \mathbf{L} \\ & \mathbf{M} & \mathbf{H} & \mathbf{L} & \\ & & \mathbf{H} & & \end{matrix} \right\rangle$$

$$\text{MIN}\,(R_L, S_L) = \left\langle \begin{matrix} & & \mathbf{M} & & \\ & \mathbf{M} & \mathbf{M} & \mathbf{L} & \\ \mathbf{L} & \mathbf{M} & \mathbf{L} & \mathbf{L} & \mathbf{L} \\ & \mathbf{L} & \mathbf{H} & \mathbf{L} & \\ & & \mathbf{M} & & \end{matrix} \right\rangle$$

2.3 Classification of Linguistic Rhotrix

The Linguistic rhotrix is classified based on both the representations of linguistic variable and quantifications of the linguistic variable.

2.3.1 Classification based on Representations of Linguistic Variable

The linguistic rhotrix shall be classified primarily into three types based on the representations of linguistic variable.

(i) Fuzzy Linguistic Rhotrix

A linguistic rhotrix is said to be fuzzy linguistic rhotrix if the linguistic variable is represented in fuzzy forms.

$$\mathbf{R_{L_{RF}}} = \left\langle \begin{matrix} & & \mathbf{r_{L_F11}} & & \\ & \mathbf{r_{L_F21}} & \mathbf{r_{L_F11}} & \mathbf{r_{L_F12}} & \\ \mathbf{r_{L_F31}} & \mathbf{r_{L_F21}} & \mathbf{r_{L_F22}} & \mathbf{r_{L_F12}} & \mathbf{r_{L_F13}} \\ & \mathbf{r_{L_F32}} & \mathbf{r_{L_F22}} & \mathbf{r_{L_F23}} & \\ & & \mathbf{r_{L_F33}} & & \end{matrix} \right\rangle$$

Example:

$$\mathbf{R_{L_{RF}}} = \left\langle \begin{matrix} & & \boldsymbol{VH} & & \\ & \mathbf{H} & \mathbf{L} & \mathbf{VL} & \\ \mathbf{VH} & \mathbf{VL} & \mathbf{M} & \mathbf{L} & \mathbf{VL} \\ & \mathbf{L} & \mathbf{H} & \mathbf{VH} & \\ & & \mathbf{VL} & & \end{matrix} \right\rangle$$

(ii) Intuitionistic Linguistic Rhotrix

A linguistic rhotrix is said to be intuitionistic linguistic rhotrix if the linguistic variable is represented in intuitionistic forms.

$$\mathbf{R}_{\mathbf{L}_{RIF}} = \left\langle \begin{array}{ccccc} & & \mathbf{r_{LIF11}} & & \\ & \mathbf{r_{LIF21}} & \mathbf{r_{LIF11}} & \mathbf{r_{LIF12}} & \\ \mathbf{r_{LIF31}} & \mathbf{r_{LIF21}} & \mathbf{r_{LIF22}} & \mathbf{r_{LIF12}} & \mathbf{r_{LIF13}} \\ & \mathbf{r_{LIF32}} & \mathbf{r_{LIF22}} & \mathbf{r_{LIF23}} & \\ & & \mathbf{r_{LIF33}} & & \end{array} \right\rangle$$

Example:

$$\mathbf{R}_{\mathbf{L}_{RIF}} = \left\langle \begin{array}{ccccc} & & \mathbf{(VH, VL)} & & \\ & \mathbf{(H, VL)} & \mathbf{(H, L)} & \mathbf{(VH, VH)} & \\ \mathbf{(L, H)} & \mathbf{(VH, L)} & \mathbf{(VH, VH)} & \mathbf{(VL, VL)} & \mathbf{(VH, VL)} \\ & \mathbf{(VL, VL)} & \mathbf{(M, VL)} & \mathbf{(H, M)} & \\ & & \mathbf{(VL, VH)} & & \end{array} \right\rangle$$

(iii) Neutrosophic Linguistic Rhotrix

A linguistic rhotrix is said to be neutrosophic linguistic rhotrix if the linguistic variable is represented in neutrosophic forms.

$$\mathbf{R}_{\mathbf{L}_{RNF}} = \left\langle \begin{array}{ccccc} & & \mathbf{r_{LNF11}} & & \\ & \mathbf{r_{LNF21}} & \mathbf{r_{LNF11}} & \mathbf{r_{LNF12}} & \\ \mathbf{r_{LNF31}} & \mathbf{r_{LNF21}} & \mathbf{r_{LNF22}} & \mathbf{r_{LNF12}} & \mathbf{r_{LNF13}} \\ & \mathbf{r_{LNF32}} & \mathbf{r_{LNF22}} & \mathbf{r_{LNF23}} & \\ & & \mathbf{r_{LNF33}} & & \end{array} \right\rangle$$

Example:

$$\mathbf{R}_{\mathbf{L}_{RNF}} = \left\langle \begin{array}{ccccc} & & \mathbf{(VH, VL, VL)} & & \\ & \mathbf{(VH, L, M)} & \mathbf{(M, H, VL)} & \mathbf{(VH, VL, L)} & \\ \mathbf{(VH, VH, L)} & \mathbf{(L, M, VL)} & \mathbf{(H, VL, L)} & \mathbf{(VH, M, VL)} & \mathbf{(H, M, VL)} \\ & \mathbf{(L, VH, VH)} & \mathbf{(H, VL, M)} & \mathbf{(VL, H, VH)} & \\ & & \mathbf{(VH, L, M)} & & \end{array} \right\rangle$$

2.3.2 Classification based on Quantifications of Linguistic Variable

The linguistic rhotrix shall be classified based on the quantifications of linguistic variable. If the linguistic variable is quantified using fuzzy sets, it shall be also referred as fuzzy linguistic rhotrix. Similarly based on the quantifications of linguistic variable, a linguistic rhotrix shall be categorized respectively.

Let us consider a linguistic rhotrix

$$\mathbf{R}_{\mathbf{L}} = \left\langle \begin{array}{ccccc} & & \mathbf{VH} & & \\ & \mathbf{H} & \mathbf{L} & \mathbf{VL} & \\ \mathbf{VH} & \mathbf{VL} & \mathbf{M} & \mathbf{L} & \mathbf{VL} \\ & \mathbf{L} & \mathbf{H} & \mathbf{VH} & \\ & & \mathbf{VL} & & \end{array} \right\rangle$$

(i) **Linguistic Rhotrix with fuzzy quantifications**

A linguistic rhotrix with fuzzy values to quantify the linguistic variables is termed as fuzzy quantified linguistic rhotrix.

$$\mathbf{R}_{\mathbf{L}_{QF}} = \left\langle \begin{matrix} & & \mathbf{0.94} & \\ & \mathbf{0.75} & \mathbf{0.25} & \mathbf{0.08} & \\ \mathbf{0.94} & \mathbf{0.08} & \mathbf{0.50} & \mathbf{0.25} & \mathbf{0.08} \\ & \mathbf{0.25} & \mathbf{0.75} & \mathbf{0.94} & \\ & & \mathbf{0.08} & & \end{matrix} \right\rangle$$

(ii) **Linguistic Rhotrix with intuitionistic quantifications**

A linguistic rhotrix with intuitionistic values to quantify the linguistic variables is termed as intuitionistic quantified linguistic rhotrix.

$$\mathbf{R}_{\mathbf{L}_{QF}} = \left\langle \begin{matrix} & & (\mathbf{0.75,1.00}) & & \\ & (\mathbf{0.50,0.75}) & (\mathbf{0.25,0.50}) & (\mathbf{0.00,0.25}) & \\ (\mathbf{0.75,1.00}) & (\mathbf{0.00,0.25}) & (\mathbf{0.50,0.50}) & (\mathbf{0.25,0.50}) & (\mathbf{0.00,0.25}) \\ & (\mathbf{0.25,0.50}) & (\mathbf{0.50,0.75}) & (\mathbf{0.75,1.00}) & \\ & & (\mathbf{0.75,1.00}) & & \end{matrix} \right\rangle$$

(iii) **Linguistic Rhotrix with neutrosophic quantifications**

A linguistic rhotrix with neutrosophic values to quantify the linguistic variables is termed as neutrosophic quantified linguistic rhotrix.

$$\mathbf{R}_{\mathbf{L}_{QF}} =$$

$$\left\langle \begin{matrix} & & (\mathbf{0.90,0.50,0.50}) & & \\ & (\mathbf{0.75,0.25,0.25}) & (\mathbf{0.25,0.75,0.75}) & (\mathbf{0.50,0.95,0.95}) & \\ (\mathbf{0.90,0.50,0.50}) & (\mathbf{0.50,0.95,0.95}) & (\mathbf{0.50,0.50,0.50}) & (\mathbf{0.25,0.75,0.75}) & (\mathbf{0.50,0.95,0.95}) \\ & (\mathbf{0.25,0.75,0.75}) & (\mathbf{0.75,0.25,0.25}) & (\mathbf{0.90,0.50,0.50}) & \\ & & (\mathbf{0.50,0.95,0.95}) & & \end{matrix} \right\rangle$$

2.4 Multiplication of Rhotrix

The Heart Based Multiplication of rhotrix is defined as

$$\mathrm{R} = \begin{matrix} & \mathbf{a} & \\ \mathbf{b} & \mathbf{h(R)} & \mathbf{d} \\ & \mathbf{e} & \end{matrix}$$

Then, rows of R can be viewed as (a d), (b e) whereas the columns can be viewed as (a b), (d e). Rhotrix multiplication is then same as matrix row-column multiplication applied in both the coupled matrices. For example,

$$\mathrm{R} \circ \mathrm{Q} = \left\langle \begin{matrix} \boldsymbol{a} \\ \boldsymbol{b} \quad \boldsymbol{h(R)} \quad \boldsymbol{d} \\ \boldsymbol{e} \end{matrix} \right\rangle \circ \left\langle \begin{matrix} \boldsymbol{f} \\ \boldsymbol{g} \quad \boldsymbol{h(Q)} \quad \boldsymbol{j} \\ \boldsymbol{k} \end{matrix} \right\rangle = \left\langle \begin{matrix} \boldsymbol{af+dg} \\ \boldsymbol{bf+eg} \quad \boldsymbol{h(R)h(Q)} \quad \boldsymbol{aj+dk} \\ \boldsymbol{bj+ek} \end{matrix} \right\rangle$$

2.5 Heart Based Multiplication of Linguistic rhotrix with Neutrosophic representations

The heart based multiplication of linguistic rhotrices with neutrosophic quantifications is defined as

The ith entry of $A_N \circ B_N$ = (max {min($\mathbf{a_i^T}$, $\mathbf{e_b^T}$), min($\mathbf{b_i^T}$, $\mathbf{e_a^T}$)}, max {min($\mathbf{a_i^I}$, $\mathbf{e_b^I}$), min($\mathbf{b_i^I}$, $\mathbf{e_a^I}$)}, min {max($\mathbf{a_i^F}$, $\mathbf{e_b^F}$), max($\mathbf{b_i^F}$, $\mathbf{e_a^F}$)}), for all I, except the heart

Heart of $A_N \circ B_N$ = (min{e_a, e_b}, min{e_a, e_b}, max{e_a, e_b})

2.6 Trace of a rhotrix

The trace of a rhotrix R_n is the sum of its diagonal entries and also the sum of all eigen values of that rhotrix. The trace is denoted by T r and defined as

$$\boldsymbol{Tr(R_n)}=\sum_{i=1}^{n} \boldsymbol{r_{ii}}$$

2.7 Trace of Linguistic rhotrix with Neutrosophic representations

The trace of a linguistic rhotrix of size '*n*' is denoted by adding the entries in major vertical axis and it is denoted as *Tr*()

That is for a rhotrix $\boldsymbol{A_N}$= $\left\langle \begin{matrix} & (\boldsymbol{a_1^T}, \boldsymbol{a_1^I}, \boldsymbol{a_1^F}) & \\ (\boldsymbol{a_2^T}, \boldsymbol{a_2^I}, \boldsymbol{a_2^F}) & (\boldsymbol{a_3^T}, \boldsymbol{a_3^I}, \boldsymbol{a_3^F}) & (\boldsymbol{a_4^T}, \boldsymbol{a_4^I}, \boldsymbol{a_4^F}) \\ & (\boldsymbol{a_5^T}, \boldsymbol{a_5^I}, \boldsymbol{a_5^F}) & \end{matrix} \right\rangle$

Tr($\boldsymbol{A_N}$) = (max{$\boldsymbol{a_1^T}$, $\boldsymbol{a_3^T}$, $\boldsymbol{a_5^T}$},max{$\boldsymbol{a_1^I}$, $\boldsymbol{a_3^I}$, $\boldsymbol{a_5^I}$},min{$\boldsymbol{a_1^F}$, $\boldsymbol{a_3^F}$, $\boldsymbol{a_5^F}$})

3. Algorithmic Framework of Linguistic Rhotrix in Managerial Decisioning

This section presents the steps involved in applying linguistic rhotrix in making optimal managerial decisions using the relations between the decision entities.

Step 1: In this step, the managerial decisioning problem is well defined and the associated entities are selected. Let us assume that the manufacturing companies are preferring sustainable production methods with preference to certain sets of criteria. A linguistic rhotrix CC_L = [cc_{Lij}] of order n, representing the relation between companies and the criteria is formulated with each cc_{Lij} is a linguistic variable indicating the degree of preference of the criteria by the companies.

$$\mathbf{CC_L} = \left\langle \begin{matrix} & & \mathbf{cc_{L11}} & & \\ & \mathbf{cc_{L21}} & \mathbf{cc_{L11}} & \mathbf{cc_{L12}} & \\ \mathbf{cc_{L31}} & \mathbf{cc_{L21}} & \mathbf{cc_{L22}} & \mathbf{cc_{L12}} & \mathbf{cc_{L13}} \\ & \mathbf{cc_{L32}} & \mathbf{cc_{L22}} & \mathbf{cc_{L23}} & \\ & & \mathbf{cc_{L33}} & & \end{matrix} \right\rangle$$

Step 2 : Formulate another linguistic rhotrix CM_L = [cm_{Lij}] of order n relating criteria and the sustainable methods chosen for decision. Each of the cells with linguistic variable represents the criteria possessed by the methods opted.

$$\mathbf{CM_L} = \left\langle \begin{matrix} & & \mathbf{cm_{L11}} & & \\ & \mathbf{cm_{L21}} & \mathbf{cm_{L11}} & \mathbf{cm_{L12}} & \\ \mathbf{cm_{L31}} & \mathbf{cm_{L21}} & \mathbf{cm_{L22}} & \mathbf{cm_{L12}} & \mathbf{cm_{L13}} \\ & \mathbf{cm_{L32}} & \mathbf{cm_{L22}} & \mathbf{cm_{L23}} & \\ & & \mathbf{cm_{L33}} & & \end{matrix} \right\rangle$$

Step 3: The linguistic variables presented in each of the cells cc_{Lij} and cm_{Lij} are quantified using neutrosophic representations.

$$\mathbf{CC_{NL}} = \left\langle \begin{matrix} & & (\mathbf{cc}^T_{L11}, \mathbf{cc}^I_{L11}, \mathbf{cc}^F_{L11}) & & \\ & (\mathbf{cc}^T_{L21}, \mathbf{cc}^I_{L21}, \mathbf{cc}^F_{L21}) & (\mathbf{cc}^T_{L11}, \mathbf{cc}^I_{L11}, \mathbf{cc}^F_{L11}) & (\mathbf{cc}^T_{12}, \mathbf{cc}^I_{12}, \mathbf{cc}^F_{12}) & \\ (\mathbf{cc}^T_{L31}, \mathbf{cc}^I_{L31}, \mathbf{cc}^F_{L31}) & (\mathbf{cc}^T_{L21}, \mathbf{cc}^I_{L21}, \mathbf{cc}^F_{L21}) & (\mathbf{cc}^T_{L22}, \mathbf{cc}^I_{L22}, \mathbf{cc}^F_{L22}) & (\mathbf{cc}^T_{L12}, \mathbf{cc}^I_{L12}, \mathbf{cc}^F_{L12}) & (\mathbf{cc}^T_{13}, \mathbf{cc}^I_{13}, \mathbf{cc}^F_{13}) \\ & (\mathbf{cc}^T_{L32}, \mathbf{cc}^I_{L32}, \mathbf{cc}^F_{L32}) & (\mathbf{cc}^T_{L22}, \mathbf{cc}^I_{L22}, \mathbf{cc}^F_{L22}) & (\mathbf{cc}^T_{L23}, \mathbf{cc}^I_{L23}, \mathbf{cc}^F_{L23}) & \\ & & (\mathbf{cc}^T_{L33}, \mathbf{cc}^I_{L33}, \mathbf{cc}^F_{L33}) & & \end{matrix} \right\rangle$$

$$\mathbf{CM_{NL}} = \left\langle \begin{matrix} & & (\mathbf{cm}^T_{L11}, \mathbf{cm}^I_{L11}, \mathbf{cm}^F_{L11}) & & \\ & (\mathbf{cm}^T_{L21}, \mathbf{cm}^I_{L21}, \mathbf{cm}^F_{L21}) & (\mathbf{cm}^T_{L11}, \mathbf{cm}_{L11}, \mathbf{cm}^F_{L11}) & (\mathbf{cm}^T_{L12}, \mathbf{cm}^I_{L12}, \mathbf{cm}^F_{L12}) & \\ (\mathbf{cm}^T_{L31}, \mathbf{cm}^I_{L31}, \mathbf{cm}^F_{L31}) & (\mathbf{cm}^T_{L21}, \mathbf{cm}^I_{L21}, \mathbf{cm}^F_{L21}) & (\mathbf{cm}^T_{L22}, \mathbf{cm}^I_{L22}, \mathbf{cm}^F_{L22}) & (\mathbf{cm}^T_{L12}, \mathbf{cm}^I_{L12}, \mathbf{cm}^F_{L12}) & (\mathbf{cm}^T_{L13}, \mathbf{cm}^I_{L13}, \mathbf{cm}^F_{L13}) \\ & (\mathbf{cm}^T_{L32}, \mathbf{cm}^I_{L32}, \mathbf{cm}^F_{L32}) & (\mathbf{cm}^T_{L22}, \mathbf{cm}^I_{L22}, \mathbf{cm}^F_{L22}) & (\mathbf{cm}^T_{L23}, \mathbf{cm}^I_{L23}, \mathbf{cm}^F_{L23}) & \\ & & (\mathbf{cm}^T_{L33}, \mathbf{cm}^I_{L33}, \mathbf{cm}^F_{L33}) & & \end{matrix} \right\rangle$$

Step 4 : The heart based multiplication is performed between the neutrosophic linguistic rhotrix to obtain the resultant matrix

$$\mathbf{R_{NL}} = \left\langle \begin{matrix} & & (\mathbf{d}^T_{L11}, \mathbf{d}^I_{L11}, \mathbf{d}^F_{L11}) & & \\ & (\mathbf{d}^T_{L21}, \mathbf{d}^I_{L21}, \mathbf{d}^F_{L21}) & (\mathbf{d}^T_{L11}, \mathbf{d}^I_{L11}, \mathbf{d}^F_{L11}) & (\mathbf{d}^T_{L12}, \mathbf{d}^I_{L12}, \mathbf{d}^F_{L12}) & \\ (\mathbf{d}^T_{L31}, \mathbf{d}^I_{L31}, \mathbf{d}^F_{L31}) & (\mathbf{d}^T_{L21}, \mathbf{d}^I_{L21}, \mathbf{d}^F_{L21}) & (\mathbf{d}^T_{L22}, \mathbf{d}^I_{L22}, \mathbf{d}^F_{L22}) & (\mathbf{d}^T_{L12}, \mathbf{d}^I_{L12}, \mathbf{d}^F_{L12}) & (\mathbf{d}^T_{L13}, \mathbf{d}^I_{L13}, \mathbf{d}^F_{L13}) \\ & (\mathbf{d}^T_{L32}, \mathbf{d}^I_{L32}, \mathbf{d}^F_{L32}) & (\mathbf{d}^T_{L22}, \mathbf{d}^I_{L22}, \mathbf{d}^F_{L22}) & (\mathbf{d}^T_{L23}, \mathbf{d}^I_{L23}, \mathbf{d}^F_{L23}) & \\ & & (\mathbf{d}^T_{L33}, \mathbf{d}^I_{L33}, \mathbf{d}^F_{L33}) & & \end{matrix} \right\rangle$$

Step 5: The complement of the neutrosophic linguistic matrices $\mathbf{CC_{NL}}$ and $\mathbf{CM_{NL}}$ is determined and the respective $\mathbf{CC^S_{NL}}$ and $\mathbf{CM^S_{NL}}$ are obtained by finding the difference between the falsity values and sum of truth and indeterminate values.

Step 6: The score rhotrix is obtained using the difference of $\mathbf{CC^S_{NL}}$ and $\mathbf{CM^S_{NL}}$ i.e $\mathbf{CC^S_{NL}}$ - $\mathbf{CM^S_{NL}}$. Based on the maximum values in each of the row, the companies and the methods are matched accordingly.

4. Illustration using Neutrosophic Quantifications of Linguistic Rhotrix

Let us consider a managerial decision-making situation where the companies have to choose the sustainable production methods based on criteria. Assume that there are seven companies and seven criteria. Out of these, four companies $(\mathbf{A_1}, \mathbf{A_3}, \mathbf{A_5}, \mathbf{A_7})$ give high preference to four criteria in specific $(\mathbf{T_2}, \mathbf{T_3}, \mathbf{T_5}, \mathbf{T_6})$ while the remaining three companies $(\mathbf{A_2}, \mathbf{A_4}, \mathbf{A_6})$ prefer three other criteria $(\mathbf{T_1}, \mathbf{T_4}, \mathbf{T_7})$. The linguistic rhotrix CC_L representing the relation between the companies and criteria is as follows.

$$CC = \left\langle \begin{array}{ccccccc} & & & \mathbf{A_1T_2} & & & \\ & & \mathbf{A_3T_2} & \mathbf{A_2T_1} & \mathbf{A_1T_3} & & \\ & \mathbf{A_5T_2} & \mathbf{A_4T_1} & \mathbf{A_3T_3} & \mathbf{A_2T_4} & \mathbf{A_1T_5} & \\ \mathbf{A_7T_2} & \mathbf{A_6T_1} & \mathbf{A_5T_3} & \mathbf{A_4T_4} & \mathbf{A_3T_5} & \mathbf{A_2T_7} & \mathbf{A_1T_6} \\ & \mathbf{A_7T_3} & \mathbf{A_6T_4} & \mathbf{A_5T_5} & \mathbf{A_4T_7} & \mathbf{A_3T_6} & \\ & & \mathbf{A_7T_5} & \mathbf{A_6T_7} & \mathbf{A_5T_6} & & \\ & & & \mathbf{A_7T_6} & & & \end{array} \right\rangle$$

The respective linguistic rhotrix with linguistic variables representing the degrees of preference of the criteria by the companies is

$$CC_L = \left\langle \begin{array}{ccccccc} & & & \boldsymbol{VL} & & & \\ & & \mathbf{M} & \mathbf{VL} & \mathbf{VH} & & \\ & \mathbf{M} & \mathbf{VL} & \boldsymbol{L} & \mathbf{M} & \mathbf{VL} & \\ \mathbf{VL} & \mathbf{VH} & \mathbf{M} & \mathbf{VH} & \mathbf{H} & \boldsymbol{VH} & \mathbf{H} \\ & \mathbf{M} & \mathbf{VL} & \mathbf{VH} & \mathbf{M} & \boldsymbol{L} & \\ & & \boldsymbol{L} & \mathbf{VL} & \mathbf{VH} & & \\ & & & \mathbf{VH} & & & \end{array} \right\rangle$$

The equivalent linguistic rhotrix with fuzzy, intuitionistic fuzzy and neutrosophic quantifications are presented below using the values in Table 1

$CC_{NL}=$

$$\left\langle \begin{array}{ccccccc} & & & (\mathbf{0.5,0.95,0.95}) & & & \\ & & (\mathbf{0.5,0.5,0.5}) & (\mathbf{0.5,0.95,0.95}) & (\mathbf{0.9,0.5,0.5}) & & \\ & (\mathbf{0.5,0.5,0.5}) & (\mathbf{0.5,0.95,0.95}) & (\mathbf{0.25,0.75,0.75}) & (\mathbf{0.5,0.5,0.5}) & (\mathbf{0.5,0.95,0.95}) & \\ (\mathbf{0.5,0.95,0.95}) & (\mathbf{0.9,0.5,0.5}) & (\mathbf{0.5,0.5,0.5}) & (\mathbf{0.9,0.5,0.5}) & (\mathbf{0.75,0.25,0.25}) & (\mathbf{0.9,0.5,0.5}) & (\mathbf{0.75,0.25,0.25}) \\ & (\mathbf{0.5,0.5,0.5} & (\mathbf{0.5,0.95,0.95}) & (\mathbf{0.9,0.5,0.5}) & (\mathbf{0.5,0.5,0.5}) & (\mathbf{0.25,0.75,0.75}) & \\ & & (\mathbf{0.25,0.75,0.75}) & (\mathbf{0.5,0.95,0.95}) & (\mathbf{0.9,0.5,0.5}) & & \\ & & & (\mathbf{0.9,0.5,0.5}) & & & \end{array} \right\rangle$$

Table 1: Quantification of the Linguistic Variable

Linguistic Variable	**Notation**	**Fuzzy Quantification**	**Intuitionistic Fuzzy Quantification**	**Neutrosophic Quantification**
Very Low	VL	0.2	(0.1,0.9)	(.25,.75,.75)
Very Low	L	0.4	(0.3,0.7)	(.5,.95,.95)
Moderate	M	0.6	(0.5,0.5)	(.5,.5,.5)
High	H	0.8	(0.8,0.2)	(.75,.25,.25)
Very High	VH	1	(0.9,0.1)	(.9,.5,.5)

Let us construct another linguistic rhotrix relating the methods and criteria. Let us assume that four methods namely $(\mathbf{M_1}, \mathbf{M_3}, \mathbf{M_5}, \mathbf{M_7})$ are known for the criteria $(\mathbf{A_2}, \mathbf{A_4}, \mathbf{A_5}, \mathbf{A_6})$,while the other three methods $(\mathbf{M_2}, \mathbf{M_4}, \mathbf{M_6})$ are known for the three criteria $(\mathbf{A_1}, \mathbf{A_3}, \mathbf{A_7})$.

$$CM= \left\langle \begin{array}{ccccccc} & & & \mathbf{M_1A_2} & & & \\ & & \mathbf{M_3A_2} & \mathbf{M_2A_1} & \mathbf{M_1A_4} & & \\ & \mathbf{M_5A_2} & \mathbf{M_4A_1} & \mathbf{M_3A_4} & \mathbf{M_2A_3} & \mathbf{M_1A_5} & \\ \mathbf{M_7A_2} & \mathbf{M_6A_1} & \mathbf{M_5A_4} & \mathbf{M_4A_3} & \mathbf{M_3A_5} & \mathbf{M_2A_7} & \mathbf{M_1A_6} \\ & \mathbf{M_7A_4} & \mathbf{M_6A_3} & \mathbf{M_5A_5} & \mathbf{M_4A_7} & \mathbf{M_3A_6} & \\ & & \mathbf{M_7A_5} & \mathbf{M_6A_7} & \mathbf{M_5A_6} & & \\ & & & \mathbf{M_7A_6} & & & \end{array} \right\rangle$$

The respective linguistic rhotrix with linguistic variables representing the degrees of fulfilment of the criteria by the methods is

$$CM_L = \left\langle \begin{array}{ccccccc} & & & \boldsymbol{VH} & & & \\ & & \mathbf{VH} & \mathbf{VL} & \mathbf{VH} & & \\ & \mathbf{VL} & \mathbf{M} & \boldsymbol{L} & \mathbf{M} & \mathbf{VH} & \\ \mathbf{VL} & \mathbf{VH} & \mathbf{M} & \mathbf{VL} & \mathbf{M} & \boldsymbol{H} & \mathbf{H} \\ & \mathbf{VL} & \mathbf{M} & \mathbf{VH} & \mathbf{M} & \boldsymbol{L} & \\ & & \boldsymbol{H} & \mathbf{VL} & \mathbf{H} & & \\ & & & \boldsymbol{H} & & & \end{array} \right\rangle$$

The equivalent linguistic rhotrix with neutrosophic quantifications is presented below using the values in Table 1.

CM_{NL} =

$$\left\langle \begin{array}{ccccccc} & & & (\mathbf{0.9,0.5,0.5}) & & & \\ & & (\mathbf{0.9,0.5,0.5}) & (\mathbf{0.5,0.95,0.95}) & (\mathbf{0.9,0.5,0.5}) & & \\ & (\mathbf{0.5,0.95,0.95}) & (\mathbf{0.5,0.5,0.5}) & (\mathbf{0.25,0.75,0.75}) & (\mathbf{0.5,0.5,0.5}) & (\mathbf{0.9,0.5,0.5}) & \\ (\mathbf{0.5,0.95,0.95}) & (\mathbf{0.9,0.5,0.5}) & (\mathbf{0.5,0.5,0.5}) & (\mathbf{0.5,0.95,0.95}) & (\mathbf{0.5,0.5,0.5}) & (\mathbf{0.75,0.25,0.25}) & (\mathbf{0.75,0.25,0.25}) \\ & (\mathbf{0.5,0.95,0.95}) & (\mathbf{0.5,0.5,0.5}) & (\mathbf{0.9,0.5,0.5}) & (\mathbf{0.5,0.5,0.5}) & (\mathbf{0.25,0.75,0.75}) & \\ & & ((\mathbf{0.75,0.25,0.25} & (\mathbf{0.5,0.95,0.95}) & (\mathbf{0.75,0.25,0.25}) & & \\ & & & (\mathbf{0.75,0.25,0.25}) & & & \end{array} \right\rangle$$

By applying the heart-based multiplication of two neutrosophic quantified linguistic rhotrices CC_{NL} and CM_{NL} we get

C_N=

$$\left\langle \begin{array}{ccccccc} & & & (\mathbf{0.9,0.95,.5}) & & & \\ & & (\mathbf{0.9,0.5,0.5}) & (\mathbf{0.5,0.95,0.95}) & (\mathbf{0.9,0.5,0.5}) & & \\ & (\mathbf{0.5,0.5,0.95}) & (\mathbf{0.5,0.0,0.0}) & (\mathbf{.25,.75,.75}) & (\mathbf{0.5,0.5,0.5}) & (\mathbf{.9,.95,.5}) & \\ (\mathbf{0.5,0.95,0.95}) & (\mathbf{0.9,0.5,0.5}) & (\mathbf{0.5,0.5,0.5}) & (\mathbf{0.5,0.5,0.95}) & (\mathbf{0.75,0.5,0.25}) & (\mathbf{.75,.5,.5}) & (\mathbf{.75,.25,.5}) \\ & (\mathbf{0.5,0.5,0.95}) & (\mathbf{.5,.95,.5}) & (\mathbf{.9,.5,.5}) & (\mathbf{.5,.5,.5}) & (\mathbf{.25,.75,.75}) & \\ & & (\mathbf{0.75,0.75,.5}) & (\mathbf{0.5,0.95,0.95}) & (\mathbf{.75,.5,.5}) & & \\ & & & (\mathbf{.9,.5,.25}) & & & \end{array} \right\rangle$$

The complement of CC_{NL} and CM_{NL} is also determined

$\boldsymbol{CC_{NL}^{C}}$=

$$\left\langle \begin{matrix} & & & & (0.5,0.05,0.05) & & \\ & & & (0.5,0.5,0.5) & (0.95,0.05,0.05) & (0.1,0.5,0.5) & \\ & & (0.5,0.5,0.5) & (0.5,0.05,0.05) & (0.75,0.25,0.25) & (0.5,0.5,0.5) & (0.5,0.05,0.05) \\ (0.5,0.05,0.05) & (0.1,0.5,0.5) & (0.5,0.5,0.5) & (0.1,0.5,0.5) & (0.25,0.75,0.75) & (0.1,0.5,0.5) & (0.25,0.75,0.75) \\ & & (0.5,0.5,0.5) & (0.5,0.05,0.05) & (0.1,0.5,0.5) & (0.5,0.5,0.5) & (0.75,0.25,0.25) \\ & & & (0.75,0.25,0.25) & (0.5,0.05,0.05) & (0.1,0.5,0.5) & \\ & & & & (0.1,0.5,0.5) & & \end{matrix} \right\rangle$$

$\boldsymbol{CM_{NL}^{C}}$ =

$$\left\langle \begin{matrix} & & & & (0.1,0.5,0.5) & & \\ & & & (0.1,0.5,0.5) & (0.5,0.05,0.05) & (0.1,0.5,0.5) & \\ & & (0.5,0.95,0.95) & (0.5,0.5,0.5) & (0.25,0.75,0.75) & (0.5,0.5,0.5) & (0.9,0.5,0.5) \\ (0.5,0.05,0.05) & (0.1,0.5,0.5) & (0.5,0.5,0.5) & (0.5,0.05,0.05) & (0.5,0.5,0.5) & (0.25,0.75,0.75) & (0.25,0.75,0.75) \\ & & (0.5,0.05,0.05) & (0.5,0.5,0.5) & (0.1,0.5,0.5) & (0.5,0.5,0.5) & (0.75,0.25,0.25) \\ & & & ((0.25,0.75,0.75) & (0.5,0.05,0.05) & (0.25,0.75,0.75) & \\ & & & & (0.25,0.75,0.75) & & \end{matrix} \right\rangle$$

By applying the heart-based multiplication of two neutrosophic quantified linguistic $\boldsymbol{CC_{NL}^{C}}$and $\boldsymbol{CM_{NL}^{C}}$ we get

D_{NL}=

$$\left\langle \begin{matrix} & & & & (.5,.5,.05) & & \\ & & & (.5,.5,.5) & (.5,.05,.05) & (.1,.5,.5) & \\ & & (.5,.5,.5) & (.5,0,0) & (.5,.5,.25) & (.5,.5,.25) & (.5,.5,.05) \\ (.5,.05,.05) & (.1,.5,.5) & (.5,.5,.5) & (0.1,0.05,0.5) & (.5,.75,.5) & (.1,.5,.05) & (.25,.5,.75) \\ & & (.5,.05,.5) & (.5,.5,.05) & (.1,.5,.5) & (.5,.5,.5) & (.5,.25,.25) \\ & & & (.5,.5,.25) & (.5,.05,.05) & (.1,.5,.5) & \\ & & & & (.1,.5,0) & & \end{matrix} \right\rangle$$

Then, by using step 5 of the aforementioned algorithm, the modified rhotrices are obtained

$$\boldsymbol{C_{NL}^{S}} = \left\langle \begin{matrix} & & & 1.35 & & & \\ & & 0.90 & 0.50 & 0.90 & & \\ & 0.05 & 0.50 & 0.25 & 0.50 & 0.90 & \\ 0.50 & 0.90 & 0.50 & 0.05 & 1.00 & 0.75 & 0.50 \\ & 0.05 & 0.95 & 0.90 & 0.50 & 0.25 & \\ & & 1.00 & 0.50 & 0.75 & & \\ & & & 1.15 & & & \end{matrix} \right\rangle$$

$$\boldsymbol{D_{NL}^{S}} = \left\langle \begin{matrix} & & & 0.95 & & & \\ & & 0.5 & 0.5 & 0.1 & & \\ & 0.5 & 0.5 & 0.75 & 0.75 & 0.95 & \\ 0.5 & 0.1 & 0.5 & -0.35 & 0.1 & 0.55 & 0 \\ & 0.05 & 0.95 & 0.1 & 0.75 & 0.5 & \\ & & 0.75 & 0.5 & 0.1 & & \\ & & & 0.6 & & & \end{matrix} \right\rangle$$

Using step 6, the score rhotrix is obtained.

The score rhotrix is $\left\langle \begin{matrix} & & & 0.400 & & & \\ & & 0.400 & 0.000 & 0.800 & & \\ & -0.45 & 0.000 & -0.50 & -0.25 & -0.05 & \\ 0.000 & 0.800 & 0.000 & 0.400 & 0.250 & 0.200 & 0.500 \\ & 0.000 & 0.000 & 0.800 & 0.000 & -0.25 & \\ & & 0.250 & 0.000 & 0.650 & & \\ & & & 0.550 & & & \end{matrix} \right\rangle$

The neutrosophic score values of the matchings between the methods and the companies are presented graphically in Fig. 1.

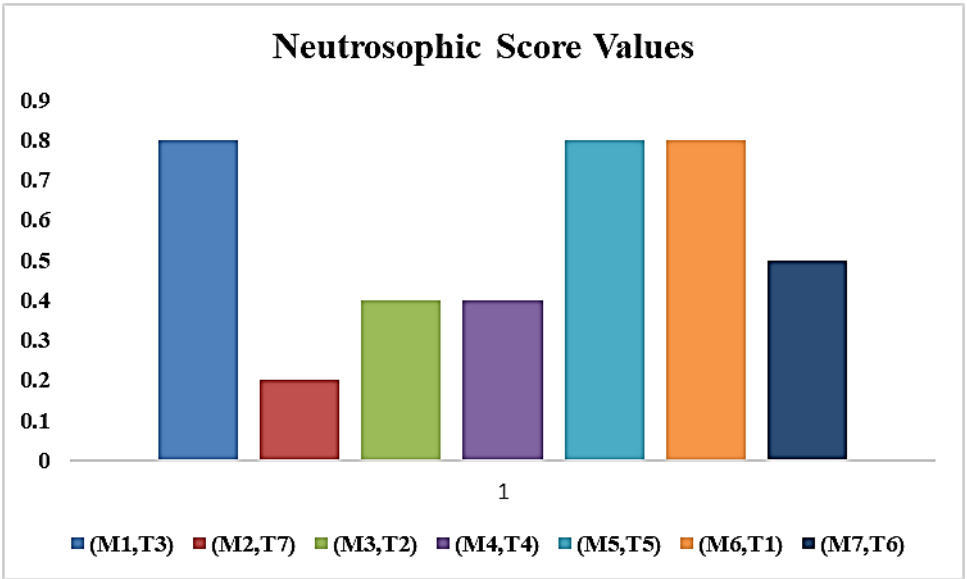

Fig. 1 Neutrosophic Score Values

The optimal matching results of the methods and the companies based on neutrosophic quantified linguistic rhotrix is presented in the below Table 2 and the graphical representation of the neutrosophic values is presented in Fig.1.

Table 2: Neutrosophic Results

Methods	M1	M2	M3	M4	M5	M6	M7
Companies	T3	T7	T2	T4	T5	T1	T6

From the results presented in Table 2, one can make a better inference on the matching of the methods to the demands and nature of the companies considered for decisioning. Since each company is different in its functioning, the optimal match of the methods with the companies appears to be more optimal and pragmatic.

5. Comparative Analysis with Fuzzy and Intuitionistic Fuzzy Quantifications

5.1 Decisioning with Fuzzy Quantification based Linguistic Rhotrix

The illustration discussed in section 3 with neutrosophic representations is discussed with fuzzy representations to make a comparative analysis of the results.

Using the fuzzy quantification presented in Table 1 the respective rhotrices CC_F and CM_F are obtained.

$$CC_F = \left\langle \begin{matrix} & & & 0.2 & \\ & 0.6 & 0.2 & 1.0 & \\ 0.6 & 0.2 & 0.4 & 0.6 & 0.2 \\ 0.2\ 1.0 & 0.6 & 1.0 & 1.0 & 1.0\ 0.8 \\ 0.6 & 0.2 & 0.1 & 0.6 & 0.4 \\ & 0.6 & 0.2 & 1.0 & \\ & & 1.0 & & \end{matrix} \right\rangle$$

$$CM_F = \left\langle \begin{matrix} & & & 0.8 & \\ & 1.0 & 0.2 & 0.6 & \\ 0.2 & 1.0 & 0.4 & 0.6 & 1.0 \\ 0.2\ 1.0 & 0.6 & 0.2 & 1.0 & 0.8\ 0.8 \\ 0.2 & 0.6 & 1.0 & 0.6 & 0.4 \\ & 0.8 & 0.2 & 0.8 & \\ & & 0.4 & & \end{matrix} \right\rangle$$

The heart based multiplication of CC_F and CM_F is

$$C_F = \left\langle \begin{matrix} & & & 0.8 & \\ & 1.0 & 1.0 & 0.6 & \\ 0.2 & 1.0 & 0.6 & 0.6 & 1.0 \\ 1.0\ 1.0 & 1.0 & 1.0 & 1.0 & 0.8\ 0.8 \\ 0.2 & 0.6 & 1.0 & 0.6 & 0.6 \\ & 0.8 & 1.0 & 1.0 & \\ & & 0.4 & & \end{matrix} \right\rangle$$

Build the complement matrix CC_F^C and CM_F^C of CC_F and CM_F

$$\boldsymbol{CC_F^C} = \left\langle \begin{matrix} & & & 0.8 & \\ & 0.4 & 0.8 & 0.0 & \\ 0.4 & 0.8 & 0.6 & 0.4 & 0.8 \\ 0.8\ 0.0 & 0.4 & 0.0 & 0.0 & 0.0\ 0.2 \\ 0.4 & 0.8 & 0.0 & 0.4 & 0.6 \\ & 0.4 & 0.8 & 0.0 & \\ & & 0.0 & & \end{matrix} \right\rangle$$

$$\boldsymbol{CM_F^C} = \left\langle \begin{matrix} & & & 0.2 & \\ & 0.0 & 0.8 & 0.4 & \\ 0.8 & 0.0 & 0.6 & 0.4 & 0.0 \\ 0.8\ 0.0 & 0.4 & 0.8 & 0.0 & 0.2\ 0.2 \\ 0.8 & 0.4 & 0.0 & 0.4 & 0.6 \\ & 0.2 & 0.8 & 0.2 & \\ & & 0.6 & & \end{matrix} \right\rangle$$

The heart - based multiplication of $\boldsymbol{CC_F^C}$ and $\boldsymbol{CM_F^C}$ is

$$D_F = \left\langle \begin{matrix} & & & 0.8 & & \\ & & 0.4 & 0.8 & 0.4 & \\ & 0.8 & 0.8 & 0.6 & 0.4 & 0.8 \\ 0.8 & 0.0 & 0.4 & 0.8 & 0.0 & 0.2 \\ & 0.8 & 0.8 & 0.0 & 0.4 & 0.6 \\ & & 0.4 & 0.8 & 0.2 & \\ & & & 0.6 & & \end{matrix} \right\rangle 0.2$$

The score rhotrix is

$$\left\langle \begin{matrix} & & & 0.00 & & \\ & & 0.60 & 0.20 & 0.20 & \\ & -0.6 & 0.20 & 0.00 & 0.20 & 0.20 \\ 0.20 & 1.00 & 0.60 & 0.20 & 1.00 & 0.60 \\ & -0.60 & -0.20 & 1.00 & 0.20 & 0.00 \\ & & 0.40 & 0.20 & 0.80 & \\ & & & -0.20 & & \end{matrix} \right\rangle 0.00$$

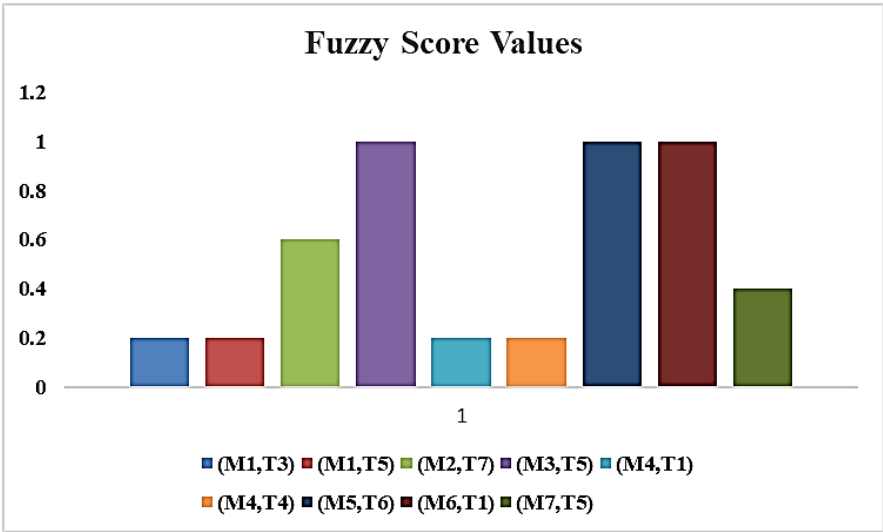

Fig. 2 Fuzzy Score Values

The optimal matching results of the methods and the companies based on fuzzy quantified linguistic rhotrix is presented in the below Table 3 and the fuzzy score values are presented graphically in Fig. 2.

Table 3: Fuzzy Results

Methods	M1	M2	M3	M4	M5	M6	M7
Companies	T3, T5	T7	T5	T1, T4	T5	T1	T5

From the results presented in Table 3, it is inferred that, no method is matched with company T2, also overlapping matchings arises in case of fuzzy quantifications. The decisioning results are not optimal for instance, the company T5 has several options in adopting the methods, in this case, the fuzzy

based quantifications are not facilitating the companies to adopt the most matching method subjected to their requirements.

5.2 Decisioning with Intuitionistic Fuzzy Quantification based Linguistic Rhotrix

The illustration discussed in section 3 with neutrosophic representations is discussed with intuitionistic fuzzy representations to make a comparative analysis of the results.

Using the intuitionistic fuzzy quantification presented in Table 1 the respective rhotrices CC_{IF} and CM_{IF} are obtained.

$$CC_{IF} = \left\langle \begin{matrix} & & & (.1,.9) & & \\ & & (.5,.5) & (.1,.9) & (.9,.1) & \\ & (.5,.5) & (.1,.9) & (.3,.7) & (.5,.5) & (.1,.9) \\ (.1,.9) & (.9,.1) & (.5,.5) & (.9,.1) & (.9,.1) & (.9,.1) \\ & (.5,.5) & (.1,.9) & (.9,.1) & (.5,.5) & (.3,.7) \\ & & (.3,.7) & (.1,.9) & (.9,.1) & \\ & & & (.9,.1) & & \end{matrix} \right\rangle (.8,.2)$$

$$CM_{IF} = \left\langle \begin{matrix} & & & (.8,.2) & & \\ & & (.9,.1) & (.1,.9) & (.5,.5) & \\ & (.1,.9) & (.9,.1) & (.3,.7) & (.5,.5) & (.9,.1) \\ (.1,.9) & (.9,.1) & (.5,.5) & (.1,.9) & (.9,.1) & (.8,.2) \\ & (.1,.9) & (.5,.5) & (.9,.1) & (.5,.5) & (.3,.7) \\ & & (.8,.2) & (.1,.9) & (.8,.2) & \\ & & & (.3,.7) & & \end{matrix} \right\rangle (.8,.2)$$

The heart-based multiplication of CC_F and CM_F is

$$C_F = \left\langle \begin{matrix} & & & 0.8 & & \\ & & 0.9 & 0.1 & 0.9 & \\ & 0.5 & 0.9 & 0.3 & 0.5 & 0.9 \\ 0.1 & 0.9 & 0.5 & 0.9 & 0.9 & 0.9 \\ & 0.5 & 0.5 & 0.9 & 0.5 & 0.3 \\ & & 0.8 & 0.1 & 0.9 & \\ & & & 0.9 & & \end{matrix} \right\rangle 0.8$$

Build the complement matrix CC_F^C and CM_F^C of CC_F and CM_F

$$\boldsymbol{CC_{IF}^{C}} = \left\langle \begin{matrix} & & & (.9,.1) & & \\ & & (.5,.5) & (.9,.1) & (.1,.9) & \\ & (.5,.5) & (.9,.1) & (.7,.3) & (.5,.5) & (.9,.1) \\ (.9,.1) & (.1,.9) & (.5,.5) & (.1,.9) & (.1,.9) & (.1,.9) \\ & (.5,.5) & (.9,.1) & (.1,.9) & (.5,.5) & (.7,.3) \\ & & (.7,.3) & (.9,.1) & (.1,.9) & \\ & & & (.1,.9) & & \end{matrix} \right\rangle (.2,.8)$$

$$\boldsymbol{CM_{IF}^{C}} = \left\langle \begin{matrix} & & & (.2,.8) & & \\ & & (.1,.9) & (.9,.1) & (.5,.5) & \\ & (.9,.1) & (.1,.9) & (.7,.3) & (.5,.5) & (.1,.9) \\ (.9,.1) & (.1,.9) & (.5,.5) & (.9,.1) & (.1,.9) & (.2,.8) \\ & (.9,.1) & (.5,.5) & (.1,.9) & (.5,.5) & (.7,.3) \\ & & (.2,.8) & (.9,.1) & (.2,.8) & \\ & & & (.7,.3) & & \end{matrix} \right\rangle (.2,.8)$$

The heart-based multiplication of $\boldsymbol{CC_{IF}^{C}}$ and $\boldsymbol{CM_{IF}^{C}}$ is

$$D_{IF} = \left\langle \begin{matrix} & & & 0.9 & & \\ & & 0.5 & 0.9 & 0.5 & \\ & 0.9 & 0.9 & 0.7 & 0.5 & 0.9 \\ 0.9 & 0.1 & 0.5 & 0.9 & 0.1 & 0.2 \\ & 0.9 & 0.9 & 0.1 & 0.5 & 0.7 \\ & & 0.7 & 0.9 & 0.2 & \\ & & & 0.7 & & \end{matrix} \right\rangle 0.2$$

The score rhotrix is

$$\left\langle \begin{matrix} & & & -0.1 & & \\ & & 0.4 & -0.8 & 0.4 & \\ & -0.4 & 0.0 & -0.4 & 0.0 & 0.0 \\ -0.8 & 0.8 & 0.0 & 0.0 & 0.8 & 0.7 \\ & -0.4 & -0.4 & 0.0 & 0.0 & -0.4 \\ & & 0.1 & -0.8 & 0.7 & \\ & & & 0.2 & & \end{matrix} \right\rangle 0.6$$

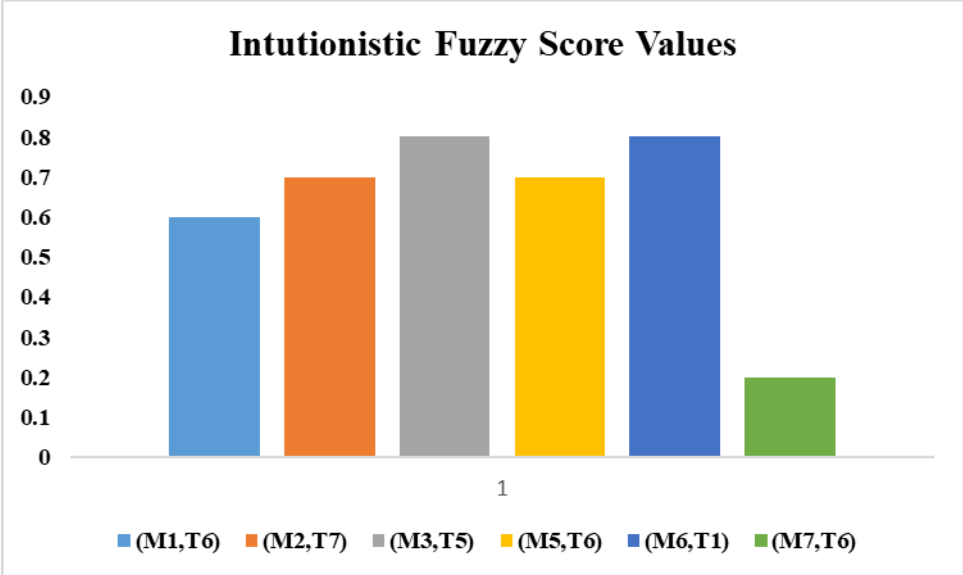

Fig. 3 Intuitionistic Fuzzy Score Values

The optimal matching results of the methods and the companies based on intuitionistic fuzzy quantified linguistic rhotrix is presented in the below Table 4 and the fuzzy score values are presented graphically in Fig. 3.

Table 3: Intuitionistic Fuzzy Results

Methods	M1	M2	M3	M4	M5	M6	M7
Companies	T6	T7	T5	-	T6	T1	T6

The table 3 clearly demonstrates the shortcomings of intuitionistic fuzzy quantifications in determining the optimal matching of the methods and the companies. For instance, the method M4 is not matched to any of the companies whereas the company T6 has three options of the methods to be chosen. Also, the companies T2, T3 and T4 are not assigned with any of the methods. This result does not facilitate in resolving the conflicts in decisioning.

5.3 Discussions

The managerial decisioning problem on optimal matching of the companies with the methods is discussed under three cases using linguistic rhotrix based on the quantifications of fuzzy, intuitionistic and neutrosophic respectively. Both fuzzy and intuitionistic results are not optimal as overlapping of the matchings take place and, in some instances, it is not possible to assign a method for the companies such as T2 under fuzzy and T2, T3 and T4 under intuitionistic fuzzy. However, on other hand the matchings based on neutrosophic is more compatible and comprehensive with each method is assigned to different companies. The result fulfills the objective of the decisioning problem of determining an optimal matching solution. The drawbacks of fuzzy and intuitionistic based quantifications are well handled by the neutrosophic representations. On comparing the results under all the three cases, it is inferred that the neutrosophic based quantifications are more competent in handling intricacies in data representations.

6. Conclusion

This research work proposes the novel concept of linguistic rhotrix and demonstrates the applications of neutrosophic quantified linguistic rhotrix in managerial decisioning. The comparisons of the results obtained using both fuzzy, intuitionistic fuzzy and neutrosophic based quantifications yield same results demonstrating the consistency of the proposed algorithmic framework. This research work shall be extended by considering quantifications using different kinds of sets, also, hypersoft set representations of criterion values shall be used in evolving new kind of linguistic rhotrices. The proposed kind of linguistic rhotrix decisioning approach shall be extended and discussed with neutrosophic hypersoft sets for facilitating the decision makers in designing furthermore optimum solutions to intricate managerial decisioning involving several interdependent entities characterized by uncertainty and indeterminacy.

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Group Decision-Making Model Based on Triangular Neutrosophic Sets for Service Quality Evaluation in Tourism Mobile E-Commerce

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Abstract: Triangular fuzzy numbers are frequently used by experts to assess their opinions in the group decision-making (GDM) paradigm. In this research, we used neutrosophic because preference connections with triangular fuzzy numbers are consistent. In GDM, it is crucial to consider the degree of consensus and the consistency of expert opinion. The idea of additive approximation consistency is put out for triangular neutrosophic Sets (TNSs) additive reciprocal matrices to differentiate the typical consistency. The GDM methodology is used to evaluate tourism mobile e-commerce services to select the best one. Two methods are used in this study such as SWARA is used to compute the criteria weights and the WASPAS method is used to rank the alternatives. 12 criteria and 7 alternatives are collected in this study to be evaluated. The results of the sensitivity analysis which is applied in this study show the stability of the rank of alternatives under different cases.

Keywords: Triangular Neutrosophic Sets (TNSs); Uncertainty; WASPAS; SWARA; Tourism Mobile E-Commerce.

1. Introduction

Group decision-making (GDM) is the process by which many businesses use several specialists to achieve a judgment because of the growing complexity of today's decision-making situations. An expert finds it challenging to consider every facet of a decision-making issue. Using preference representation forms, such as fuzzy preference relations, multiplicative preference relations, interval preference relations, and language frameworks for modeling GDM problems, all professionals can assess their conclusions[1], [2]. Furthermore, other formats of incomplete preference relations have also been used to address problems involving incomplete information. The imprecise understanding of the expert's preference level cannot be accurately reflected by a numerical value[3], [4]. Fuzzy set theory was created to explain the uncertainty that comes with vagueness. It uses set memberships to handle ambiguous data. L-R fuzzy numbers, trapezoidal fuzzy numbers, triangular fuzzy numbers, interval numbers, and others are examples of these set

memberships. Some of the fuzzy formats mentioned above have been used by experts in the GDM process[5], [6]. The fuzzy set has been expanded to create the intuitionistic fuzzy set to address ambiguity and uncertainty even more. Atanassov introduced the idea of intuitionistic fuzzy preference relations. The consistency of preference relations is one of the most crucial aspects of GDM problems to prevent decision-makers from contradicting themselves[7], [8]. In earlier research, the multiplicative consistency of intuitionistic fuzzy preference relations was the focus. By putting out a collective decision-making model in a neutrosophic setting, we were able to overcome all the shortcomings. The decision-making process appears to be characterized by confirmation, rejection, and indeterminacy since the problem domain should be precisely known; otherwise, people are unsure when determining the values for the preference evaluation. To distinguish between truth-membership function, indeterminacy-membership function, and falsity-membership function, Smarandache proposed the idea of a neutrosophic set[9], [10]. Therefore, the neutrosophic set theory ought to be applied to justify ambiguity-related uncertainty in a way that is comparable to human cognition. In terms of set memberships, it handles ambiguous data as possibility distributions. To ascertain their preference relations, the specialists ought to employ the neutrosophic set. The triangular neutrosophic additive reciprocal preference relation's approximate consistency is applied to the GDM problem in this study, and neutrosophic preference relations that lacked consistency have been fixed. The ideal method for group decision-making is consensus since it considers worries and opposing viewpoints without hostility or dread, but it is also far more difficult to resolve because of the divergent expert opinions and the varying weights given to them during the decision-making process[11], [12].

1.1 Research Gaps

It is clear from examining the earlier research that:

1. Because of the fuzzy set limitations, fuzzy preference relations have certain disadvantages.

2. In many real-world scenarios, a fuzzy set's single-valued function is employed to simultaneously represent evidence of acceptance and rejection.

3. The preference interactions with triangular fuzzy numbers are inherently inconsistent.

4. A few methods for enhancing consistency in intuitionistic fuzzy preference relations were also proposed.

1.2 Contributions of this study are organized as follows:

- ✓ We applied the GDM methodology to evaluate the types of Tourism Mobile E-Commerce Services.
- ✓ Two methods are used in this study, such as the SWARA methodology to compute the criteria weights and the WASPAS methodology is used to rank the alternatives.
- ✓ Twelve criteria and seven alternatives are collected to be evaluated by three experts and decision-makers.

- ✓ The sensitivity analysis is conducted to show the stability of rank in this study.

2. Preliminaries

Some definitions of the triangular neutrosophic sets (TNSs) are outlined in this part to show the operations of this uncertainty framework[11], [13].

2.1.Definition 1

TNSs can be defined as:

$$a = \left((a_1, a_2, a_3); T_a, I_a, F_a\right) \tag{1}$$

Three membership functions of TNSs as Truth, Indeterminacy, and Falsity can be defined:

$$T_a(x) = \begin{cases} T_a\left(\frac{x-a_1}{a_2-a_1}\right) & if\ a_1 \le x \le a_2 \\ T_a & if\ x = a_2 \\ T_a\left(\frac{a_3-x}{a_3-a_2}\right) & if\ a_2 \le x \le a_3 \\ 0 & otherwise \end{cases} \tag{2}$$

$$I_a(x) = \begin{cases} \frac{(a_2-x+I_a(x-a_1))}{(a_2-a_1)} & if\ a_1 \le x \le a_2 \\ I_a & if\ x = a_2 \\ \frac{(x-a_2+I_a(a_3-x))}{(a_3-a_2)} & if\ a_2 \le x \le a_3 \\ 1 & otherwise \end{cases} \tag{3}$$

$$F_a(x) = \begin{cases} \frac{(a_2-x+F_a(x-a_1))}{(a_2-a_1)} & if\ a_1 \le x \le a_2 \\ F_a & if\ x = a_2 \\ \frac{(x-a_2+F_a(a_3-x))}{(a_3-a_2)} & if\ a_2 \le x \le a_3 \\ 1 & otherwise \end{cases} \tag{4}$$

2.2. Definition 2

Let $a = \left((a_1, a_2, a_3); T_a, I_a, F_a\right)$ and $b = \left((b_1, b_2, b_3); T_b, I_b, F_b\right)$ Two triangular neutrosophic numbers (TNNs).

Addition

$$a + b = \left((a_1 + b_1, a_2 + b_2, a_3 + b_3);\ T_a \wedge T_b, I_a \vee I_b, F_a \vee F_b\right) \tag{5}$$

Subtraction

$$a - b = \left((a_1 - b_3, a_2 - b_2, a_3 - b_1);\ T_a \wedge T_b, I_a \wedge I_b, F_a \wedge F_b\right) \tag{6}$$

Inverse

$$a^{-1} = \left(\left(\frac{1}{a_3}, \frac{1}{a_2}, \frac{1}{a_{31}}\right);\ T_a, I_a, F_a\right) \tag{7}$$

Multiplication by constant

$$\sigma a = \begin{cases} ((\sigma a_1, \sigma a_2, \sigma a_3);\ T_a, I_a, F_a) & if\ \sigma > 0 \\ ((\sigma a_3, \sigma a_2, \sigma a_1);\ T_a, I_a, F_a) & if\ \sigma < 0 \end{cases} \tag{8}$$

Multiplication

$$ab = \begin{cases} ((a_1 b_1, a_2 b_2, a_3 b_3); T_a \wedge T_b, I_a \vee I_b, F_a \vee F_b) & if (a_3 > 0, b_3 > 0) \\ ((a_1 b_3, a_2 b_2, a_3 b_1); T_a \wedge T_b, I_a \vee I_b, F_a \vee F_b) & if (a_3 < 0, b_3 > 0) \\ ((a_3 b_3, a_2 b_2, a_1 b_1); T_a \wedge T_b, I_a \vee I_b, F_a \vee F_b) & if (a_3 < 0, b_3 < 0) \end{cases} \tag{9}$$

Multiplication

$$\frac{a}{b} = \begin{cases} \left(\left(\frac{a_1}{b_3}, \frac{a_2}{b_2}, \frac{a_3}{b_1}\right); T_a \wedge T_b, I_a \vee I_b, F_a \vee F_b\right) & if (a_3 > 0, b_3 > 0) \\ \left(\left(\left(\frac{a_3}{b_3}, \frac{a_2}{b_2}, \frac{a_1}{b_1}\right)\right); T_a \wedge T_b, I_a \vee I_b, F_a \vee F_b\right) & if (a_3 < 0, b_3 > 0) \\ \left(\left(\left(\frac{a_3}{b_1}, \frac{a_2}{b_2}, \frac{a_3}{b_3}\right)\right); T_a \wedge T_b, I_a \vee I_b, F_a \vee F_b\right) & if (a_3 < 0, b_3 < 0) \end{cases} \tag{10}$$

3. Methodology

In our study, we integrated the WASPAS and SWARA methods with TNSs to compute the criteria weights and rank the alternatives. Fig 1 shows the steps of the proposed approach.

3.1 SWARA Method

The SWARA method is used to compute the criteria weights.

Step 1. Initial prioritizing of the criteria

The criteria are ranked based on their importance by the decision-makers and experts.

Step 2. Compute the coefficient value

$$K_j = \begin{cases} 1 & if\ j = 1 \\ S_j + 1 & if\ j > 1 \end{cases} \tag{11}$$

Step 3. Compute the initial weight

$$U_j = \begin{cases} 1 & if\ j = 1 \\ \frac{U_j}{K_j} & if\ j > 1 \end{cases} \tag{12}$$

Step 4. Compute the criteria weights.

$$w_j = \frac{U_j}{\sum_{j=1}^{n} U_j} \tag{13}$$

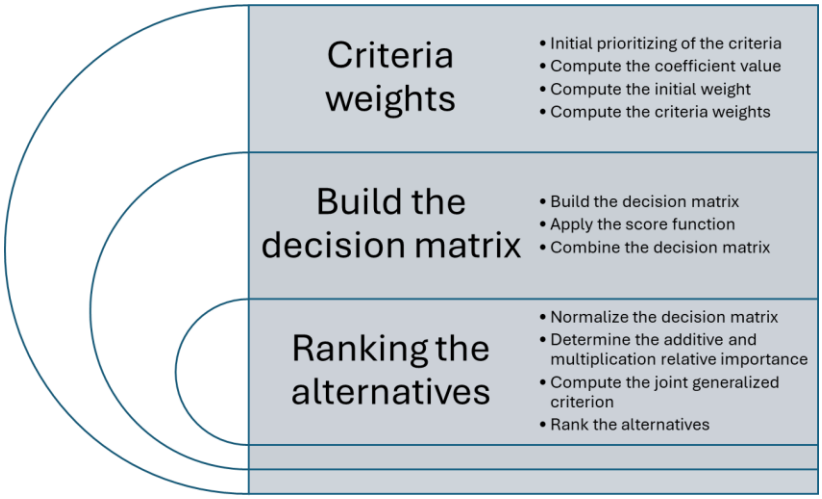
Criteria weights
• Initial prioritizing of the criteria
• Compute the coefficient value
• Compute the initial weight
• Compute the criteria weights
Build the decision matrix
• Build the decision matrix
• Apply the score function
• Combine the decision matrix
Ranking the alternatives
• Normalize the decision matrix
• Determine the additive and multiplication relative importance
• Compute the joint generalized criterion
• Rank the alternatives

Fig. 1. The steps of the SWARA and WASPAs methodology

3.2 WASPAS Method

This method is used to rank the alternatives. The steps of the WASPAS method are outlined such as:

Step 1. Build the decision matrix

The decision matrix is built using the opinions of the experts and decision-makers.

Step 2. Apply the score function

The score function is applied to convert the TNNs into single numbers.

Step 3. Combine the decision matrix.

The decision matrix is combined using the average method.

Step 4. Normalize the decision matrix

The decision matrix is normalized for positive and negative criteria.

$$n_{ij} = \frac{x_{ij}}{\max r_{ij}}; i = 1, \dots, m; j = 1, \dots, n \tag{14}$$

$$n_{ij} = \frac{\min x_{ij}}{x_{ij}}; i = 1, \dots, m; j = 1, \dots, n \tag{15}$$

Step 5. Determine the additive and multiplication relative importance

$$V_i^{(1)} = \sum_{j=1}^{n} n_{ij} w_j \tag{16}$$

$$V_i^{(2)} = \prod_{j=1}^{n} (n_{ij})^{w_j} \tag{17}$$

Step 6. Compute the joint generalized criterion

$$V_i = \pi V_i^{(1)} + (1-\pi) V_i^{(2)} \tag{18}$$

Value of π between 0 and 1.

Step 7. Rank the alternatives.

4. An Empirical Application

Mobile e-commerce service quality evaluation focuses on assessing the user experience and satisfaction with services provided through mobile platforms. Key factors influencing service quality include usability, reliability, responsiveness, and personalization. Usability refers to how easily users can navigate and interact with the mobile app or website, including interface design and functionality. Reliability ensures that the platform performs consistently without errors, providing accurate information and secure transactions. Responsiveness addresses the speed at which services are delivered, including loading times and customer support responses. Personalization involves tailoring the user experience based on preferences, behavior, or location, which can enhance customer satisfaction and loyalty. Additionally, trust and security play significant roles, as users expect safe payment systems and data protection. Effective evaluation often employs customer feedback, surveys, or performance metrics to identify areas for improvement and maintain competitive advantages. By prioritizing service quality, mobile e-commerce platforms can build stronger relationships with users and drive long-term success. This section shows the results of the case study. This study used the GDM approach for ranking the types of Tourism Mobile E-Commerce Services to select the best one. Three experts are using TNSs to evaluate the criteria and alternatives. These experts and decision-makers have experience in the field of mobile e-commerce services. We collected 12 criteria and seven alternatives [14], [15], [16], [17]in this study as shown in Table 1.

Table 1. The list of criteria and alternatives.

Criteria	Alternatives
Sustainability	Hotel Apps
Usability and Interface Design	Sustainable Travel Platforms
Loyalty Programs	Multi-Service Super Apps
Performance and Speed	Online Travel Agencies
Virtual Tours	Tour Booking Platforms
Personalization	Local Guide Apps
Responsiveness	Transportation Booking Apps
Reliability	
Integration with Other Services	
Payment Security	

User Feedback Integration Accuracy	

Step 1. We ranked the criteria based on the opinions of the experts and decision-makers.

Step 2. Eq. (11) is used to compute the coefficient value

Step 3. Eq. (12) is used to compute the initial weight

Step 4. Then Eq. (13) is used to compute the criteria weights as shown in Table 2.

Table 2. The criteria weights.

Criteria	K_j	U_j	w_j	Rank
C_1	1	1	0.032363	1
C_2	4.4625	4.4625	0.14442	12
C_3	5.5	1.232493	0.039887	3
C_4	5.4	4.381364	0.141794	11
C_5	4.9	1.118373	0.036194	2
C_6	4.075	3.643685	0.117921	9
C_7	6.225	1.708435	0.05529	6
C_8	5.2125	3.051037	0.098741	7
C_9	4.4625	1.462617	0.047335	4
C_{10}	5.5	3.760382	0.121697	10
C_{11}	5.8875	1.565665	0.05067	5
C_{12}	5.5	3.512884	0.113688	8

Then we apply the WASPAS method to rank the alternatives.

Step 1. We build the decision matrix using the TNNs as shown in Tables 3-5.

Step 2. Then we apply the score function to a single number.

Step 3. Then we combine the decision matrix into a single matrix.

Step 4. All criteria are positive. Then we normalize the decision matrix using Eq. (14) as shown in Table 6.

Step 5. Then we determine the additive and multiplication relative importance using Eqs. (16 and 17).

Step 6. Then we compute the joint generalized criterion using Eq. (18).

Step 7. Then rank the alternatives as shown in Table 7.

Table 3. First decision matrix.

	A_1	A_2	A_3	A_4	A_5	A_6	A_7

C_1	((1,2,3);0.4,0.60,0.65)	((4,5,6);0.8,0.15,0.20)	((5,6,7);0.70,0.25,0.30)	((7,8,9);0.85,0.10,0.15)	((1,2,3);0.4,0.60,0.65)	((9,9,9);1.00,0.00,0.00)	((6,7,8);0.9,0.10,0.10)
C_2	((2,3,4);0.3,0.75,0.70)	((5,6,7);0.70,0.25,0.30)	((7,8,9);0.85,0.10,0.15)	((6,7,8);0.9,0.10,0.10)	((1,1,1);0.5,0.5,0.5)	((2,3,4);0.3,0.75,0.70)	((4,5,6);0.8,0.15,0.20)
C_3	((4,5,6);0.8,0.15,0.20)	((1,2,3);0.4,0.60,0.65)	((5,6,7);0.70,0.25,0.30)	((7,8,9);0.85,0.10,0.15)	((6,7,8);0.9,0.10,0.10)	((1,1,1);0.5,0.5,0.5)	((2,3,4);0.3,0.75,0.70)
C_4	((6,7,8);0.9,0.10,0.10)	((5,6,7);0.70,0.25,0.30)	((7,8,9);0.85,0.10,0.15)	((4,5,6);0.8,0.15,0.20)	((6,7,8);0.9,0.10,0.10)	((1,1,1);0.5,0.5,0.5)	((1,1,1);0.5,0.5,0.5)
C_5	((5,6,7);0.70,0.25,0.30)	((7,8,9);0.85,0.10,0.15)	((4,5,6);0.8,0.15,0.20)	((6,7,8);0.9,0.10,0.10)	((1,1,1);0.5,0.5,0.5)	((9,9,9);1.00,0.00,0.00)	((1,2,3);0.4,0.60,0.65)
C_6	((1,2,3);0.4,0.60,0.65)	((4,5,6);0.8,0.15,0.20)	((6,7,8);0.9,0.10,0.10)	((1,1,1);0.5,0.5,0.5)	((5,6,7);0.70,0.25,0.30)	((1,2,3);0.4,0.60,0.65)	((9,9,9);1.00,0.00,0.00)
C_7	((3,4,5);0.35,0.60,0.40)	((5,6,7);0.70,0.25,0.30)	((7,8,9);0.85,0.10,0.15)	((4,5,6);0.8,0.15,0.20)	((6,7,8);0.9,0.10,0.10)	((1,1,1);0.5,0.5,0.5)	((6,7,8);0.9,0.10,0.10)
C_8	((5,6,7);0.70,0.25,0.30)	((7,8,9);0.85,0.10,0.15)	((6,7,8);0.9,0.10,0.10)	((1,1,1);0.5,0.5,0.5)	((2,3,4);0.3,0.75,0.70)	((5,6,7);0.70,0.25,0.30)	((4,5,6);0.8,0.15,0.20)
C_9	((1,1,1);0.5,0.5,0.5)	((5,6,7);0.70,0.25,0.30)	((7,8,9);0.85,0.10,0.15)	((6,7,8);0.9,0.10,0.10)	((1,1,1);0.5,0.5,0.5)	((1,1,1);0.5,0.5,0.5)	((2,3,4);0.3,0.75,0.70)
C_{10}	((2,3,4);0.3,0.75,0.70)	((3,4,5);0.35,0.60,0.40)	((1,2,3);0.4,0.60,0.65)	((4,5,6);0.8,0.15,0.20)	((6,7,8);0.9,0.10,0.10)	((1,1,1);0.5,0.5,0.5)	((1,1,1);0.5,0.5,0.5)
C_{11}	((4,5,6);0.8,0.15,0.20)	((5,6,7);0.70,0.25,0.30)	((7,8,9);0.85,0.10,0.15)	((6,7,8);0.9,0.10,0.10)	((1,1,1);0.5,0.5,0.5)	((4,5,6);0.8,0.15,0.20)	((5,6,7);0.70,0.25,0.30)
C_{12}	((5,6,7);0.70,0.25,0.30)	((7,8,9);0.85,0.10,0.15)	((6,7,8);0.9,0.10,0.10)	((1,1,1);0.5,0.5,0.5)	((5,6,7);0.70,0.25,0.30)	((6,7,8);0.9,0.10,0.10)	((3,4,5);0.35,0.60,0.40)

Table 4. Second decision matrix.

	A_1	A_2	A_3	A_4	A_5	A_6	A_7
C_1	((5,6,7);0.70,0.25,0.30)	((7,8,9);0.85,0.10,0.15)	((3,4,5);0.35,0.60,0.40)	((5,6,7);0.70,0.25,0.30)	((1,1,1);0.5,0.5,0.5)	((9,9,9);1.00,0.00,0.00)	((6,7,8);0.9,0.10,0.10)
C_2	((2,3,4);0.3,0.75,0.70)	((5,6,7);0.70,0.25,0.30)	((7,8,9);0.85,0.10,0.15)	((1,2,3);0.4,0.60,0.65)	((3,4,5);0.35,0.60,0.40)	((5,6,7);0.70,0.25,0.30)	((1,1,1);0.5,0.5,0.5)
C_3	((4,5,6);0.8,0.15,0.20)	((1,2,3);0.4,0.60,0.65)	((5,6,7);0.70,0.25,0.30)	((7,8,9);0.85,0.10,0.15)	((5,6,7);0.70,0.25,0.30)	((1,1,1);0.5,0.5,0.5)	((2,3,4);0.3,0.75,0.70)
C_4	((6,7,8);0.9,0.10,0.10)	((1,2,3);0.4,0.60,0.65)	((3,4,5);0.35,0.60,0.40)	((5,6,7);0.70,0.25,0.30)	((7,8,9);0.85,0.10,0.15)	((6,7,8);0.9,0.10,0.10)	((1,1,1);0.5,0.5,0.5)
C_5	((9,9,9);1.00,0.00,0.00)	((1,2,3);0.4,0.60,0.65)	((3,4,5);0.35,0.60,0.40)	((5,6,7);0.70,0.25,0.30)	((1,1,1);0.5,0.5,0.5)	((9,9,9);1.00,0.00,0.00)	((1,2,3);0.4,0.60,0.65)
C_6	((1,2,3);0.4,0.60,0.65)	((3,4,5);0.35,0.60,0.40)	((5,6,7);0.70,0.25,0.30)	((7,8,9);0.85,0.10,0.15)	((5,6,7);0.70,0.25,0.30)	((1,2,3);0.4,0.60,0.65)	((9,9,9);1.00,0.00,0.00)
C_7	((3,4,5);0.35,0.60,0.40)	((1,2,3);0.4,0.60,0.65)	((3,4,5);0.35,0.60,0.40)	((5,6,7);0.70,0.25,0.30)	((1,1,1);0.5,0.5,0.5)	((3,4,5);0.35,0.60,0.40)	((6,7,8);0.9,0.10,0.10)
C_8	((5,6,7);0.70,0.25,0.30)	((1,1,1);0.5,0.5,0.5)	((1,2,3);0.4,0.60,0.65)	((3,4,5);0.35,0.60,0.40)	((5,6,7);0.70,0.25,0.30)	((1,1,1);0.5,0.5,0.5)	((4,5,6);0.8,0.15,0.20)
C_9	((1,1,1);0.5,0.5,0.5)	((5,6,7);0.70,0.25,0.30)	((5,6,7);0.70,0.25,0.30)	((7,8,9);0.85,0.10,0.15)	((3,4,5);0.35,0.60,0.40)	((5,6,7);0.70,0.25,0.30)	((1,1,1);0.5,0.5,0.5)
C_{10}	((2,3,4);0.3,0.75,0.70)	((5,6,7);0.70,0.25,0.30)	((7,8,9);0.85,0.10,0.15)	((3,4,5);0.35,0.60,0.40)	((5,6,7);0.70,0.25,0.30)	((1,1,1);0.5,0.5,0.5)	((1,1,1);0.5,0.5,0.5)
C_{11}	((4,5,6);0.8,0.15,0.20)	((1,2,3);0.4,0.60,0.65)	((5,6,7);0.70,0.25,0.30)	((7,8,9);0.85,0.10,0.15)	((3,4,5);0.35,0.60,0.40)	((5,6,7);0.70,0.25,0.30)	((1,1,1);0.5,0.5,0.5)
C_{12}	((6,7,8);0.9,0.10,0.10)	((9,9,9);1.00,0.00,0.00)	((5,6,7);0.70,0.25,0.30)	((7,8,9);0.85,0.10,0.15)	((5,6,7);0.70,0.25,0.30)	((1,1,1);0.5,0.5,0.5)	((3,4,5);0.35,0.60,0.40)

Table 5. Third decision matrix.

	A_1	A_2	A_3	A_4	A_5	A_6	A_7
C_1	((5,6,7);0.70,0.25,0.30)	((7,8,9);0.85,0.10,0.15)	((5,6,7);0.70,0.25,0.30)	((3,4,5);0.35,0.60,0.40)	((1,2,3);0.4,0.60,0.65)	((9,9,9);1.00,0.00,0.00)	((6,7,8);0.9,0.10,0.10)
C_2	((2,3,4);0.3,0.75,0.70)	((5,6,7);0.70,0.25,0.30)	((7,8,9);0.85,0.10,0.15)	((5,6,7);0.70,0.25,0.30)	((1,1,1);0.5,0.5,0.5)	((2,3,4);0.3,0.75,0.70)	((4,5,6);0.8,0.15,0.20)
C_3	((4,5,6);0.8,0.15,0.20)	((1,2,3);0.4,0.60,0.65)	((1,1,1);0.5,0.5,0.5)	((5,6,7);0.70,0.25,0.30)	((7,8,9);0.85,0.10,0.15)	((4,5,6);0.8,0.15,0.20)	((2,3,4);0.3,0.75,0.70)
C_4	((6,7,8);0.9,0.10,0.10)	((9,9,9);1.00,0.00,0.00)	((2,3,4);0.3,0.75,0.70)	((5,6,7);0.70,0.25,0.30)	((7,8,9);0.85,0.10,0.15)	((6,7,8);0.9,0.10,0.10)	((1,1,1);0.5,0.5,0.5)
C_5	((9,9,9);1.00,0.00,0.00)	((6,7,8);0.9,0.10,0.10)	((5,6,7);0.70,0.25,0.30)	((7,8,9);0.85,0.10,0.15)	((3,4,5);0.35,0.60,0.40)	((9,9,9);1.00,0.00,0.00)	((1,2,3);0.4,0.60,0.65)
C_6	((1,2,3);0.4,0.60,0.65)	((4,5,6);0.8,0.15,0.20)	((5,6,7);0.70,0.25,0.30)	((7,8,9);0.85,0.10,0.15)	((5,6,7);0.70,0.25,0.30)	((1,2,3);0.4,0.60,0.65)	((9,9,9);1.00,0.00,0.00)
C_7	((3,4,5);0.35,0.60,0.40)	((2,3,4);0.3,0.75,0.70)	((9,9,9);1.00,0.00,0.00)	((5,6,7);0.70,0.25,0.30)	((5,6,7);0.70,0.25,0.30)	((7,8,9);0.85,0.10,0.15)	((6,7,8);0.9,0.10,0.10)
C_8	((5,6,7);0.70,0.25,0.30)	((1,1,1);0.5,0.5,0.5)	((1,2,3);0.4,0.60,0.65)	((5,6,7);0.70,0.25,0.30)	((7,8,9);0.85,0.10,0.15)	((5,6,7);0.70,0.25,0.30)	((4,5,6);0.8,0.15,0.20)
C_9	((5,6,7);0.70,0.25,0.30)	((7,8,9);0.85,0.10,0.15)	((3,4,5);0.35,0.60,0.40)	((1,2,3);0.4,0.60,0.65)	((4,5,6);0.8,0.15,0.20)	((1,1,1);0.5,0.5,0.5)	((2,3,4);0.3,0.75,0.70)
C_{10}	((2,3,4);0.3,0.75,0.70)	((3,4,5);0.35,0.60,0.40)	((5,6,7);0.70,0.25,0.30)	((7,8,9);0.85,0.10,0.15)	((6,7,8);0.9,0.10,0.10)	((2,3,4);0.3,0.75,0.70)	((1,1,1);0.5,0.5,0.5)
C_{11}	((4,5,6);0.8,0.15,0.20)	((1,2,3);0.4,0.60,0.65)	((1,1,1);0.5,0.5,0.5)	((5,6,7);0.70,0.25,0.30)	((7,8,9);0.85,0.10,0.15)	((4,5,6);0.8,0.15,0.20)	((5,6,7);0.70,0.25,0.30)
C_{12}	((5,6,7);0.70,0.25,0.30)	((7,8,9);0.85,0.10,0.15)	((2,3,4);0.3,0.75,0.70)	((4,5,6);0.8,0.15,0.20)	((5,6,7);0.70,0.25,0.30)	((6,7,8);0.9,0.10,0.10)	((3,4,5);0.35,0.60,0.40)

Table 6. Normalized decision matrix.

	A_1	A_2	A_3	A_4	A_5	A_6	A_7
C_1	0.346914	0.664815	0.385185	0.482716	0.075309	1	0.7
C_2	0.122596	0.620192	1	0.546474	0.134615	0.288462	0.416667
C_3	0.674312	0.126606	0.500917	1	0.965138	0.279817	0.140367
C_4	0.93719	0.697521	0.475207	0.628926	1	0.649587	0.07438
C_5	0.825926	0.518519	0.37716	0.649383	0.103704	1	0.085185
C_6	0.085185	0.369136	0.551852	0.532099	0.477778	0.085185	1
C_7	0.285714	0.313051	0.938272	0.671076	0.587302	0.488536	1
C_8	1	0.614987	0.607235	0.511628	0.936693	0.705426	0.949612
C_9	0.341202	1	0.839056	0.901288	0.410944	0.341202	0.141631
C_{10}	0.150888	0.467456	0.710059	0.758383	1	0.109467	0.088757
C_{11}	0.698669	0.3327	0.669202	1	0.526616	0.711027	0.519011
C_{12}	0.651603	1	0.500729	0.503644	0.56414	0.572886	0.236152

Table 7. The rank of alternatives.

	$V_i^{(1)}$	$V_i^{(2)}$	V_i	Ranks
A_1	0.487188	0.340673	0.41393	6
A_2	0.590135	0.539274	0.269637	1
A_3	0.651773	0.624326	0.312163	2
A_4	0.640364	0.624128	0.513726	7
A_5	0.619193	0.487046	0.374887	4
A_6	0.450826	0.347591	0.325877	3
A_7	0.439686	0.282643	0.391602	5

5. Analysis

This section shows the sensitivity analysis to show the different ranks of alternatives. In the ranking of the alternatives by the WASPAS method we the π value with 0.5 and we ranked the alternatives in the joint generalized criterion. But in this section, we change this value between 0 and 0.9 to show different ranks of alternatives. Table 8 shows the joint generalized criterion values of each alternative.

Table 8. The joint generalized criterion values of each alternative.

	$\pi = 0$	$\pi = 0.1$	$\pi = 0.2$	$\pi = 0.3$	$\pi = 0.4$	$\pi = 0.5$	$\pi = 0.6$	$\pi = 0.7$	$\pi = 0.8$	$\pi = 0.9$
A_1	0.340673	0.355325	0.369976	0.384627	0.399279	0.41393	0.428582	0.443233	0.457885	0.472536
A_2	0.539274	0.485347	0.431419	0.377492	0.323564	0.269637	0.21571	0.161782	0.107855	0.053927
A_3	0.624326	0.561893	0.499461	0.437028	0.374595	0.312163	0.24973	0.187298	0.124865	0.062433
A_4	0.7901	0.734825	0.67955	0.624275	0.569001	0.513726	0.458451	0.403176	0.347901	0.292627
A_5	0.749774	0.674797	0.599819	0.524842	0.449864	0.374887	0.29991	0.224932	0.149955	0.074977
A_6	0.651755	0.586579	0.521404	0.456228	0.391053	0.325877	0.260702	0.195526	0.130351	0.065175
A_7	0.66757	0.612376	0.557183	0.501989	0.446796	0.391602	0.336409	0.281215	0.226022	0.170828

Then we ranked the alternatives to show the best and the worst alternatives. We show that alternative 4 is the best of the seven cases as shown in Fig. 2. The alternative one is the best of the three cases as shown in Fig. 3.

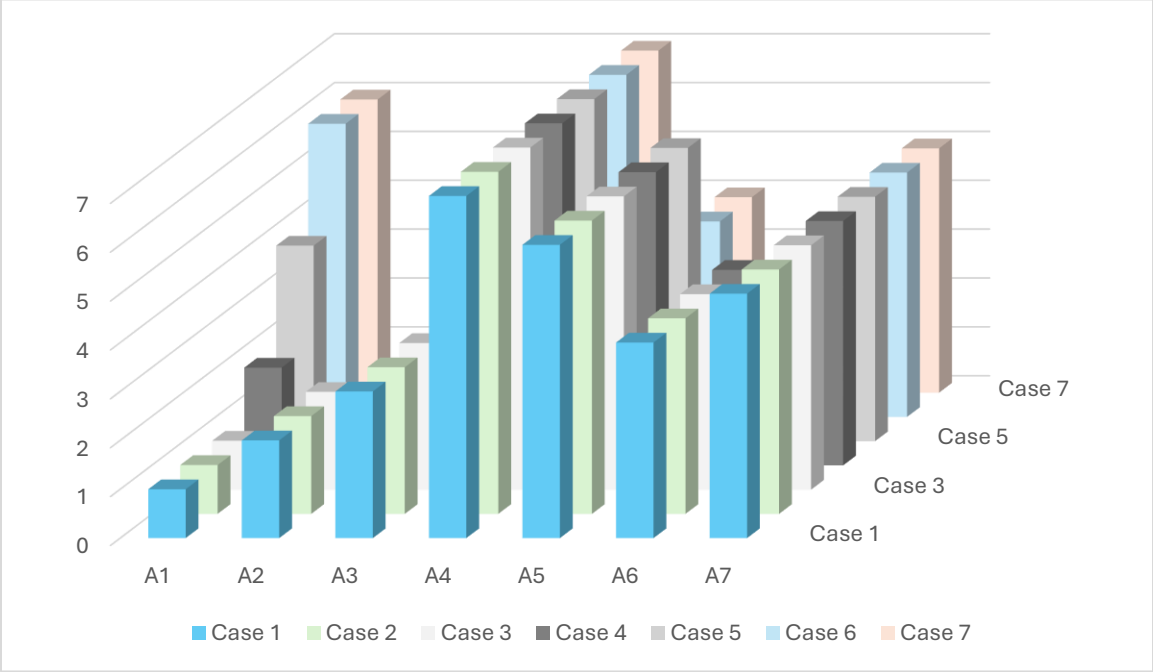

Fig. 2. The rank of seven cases.

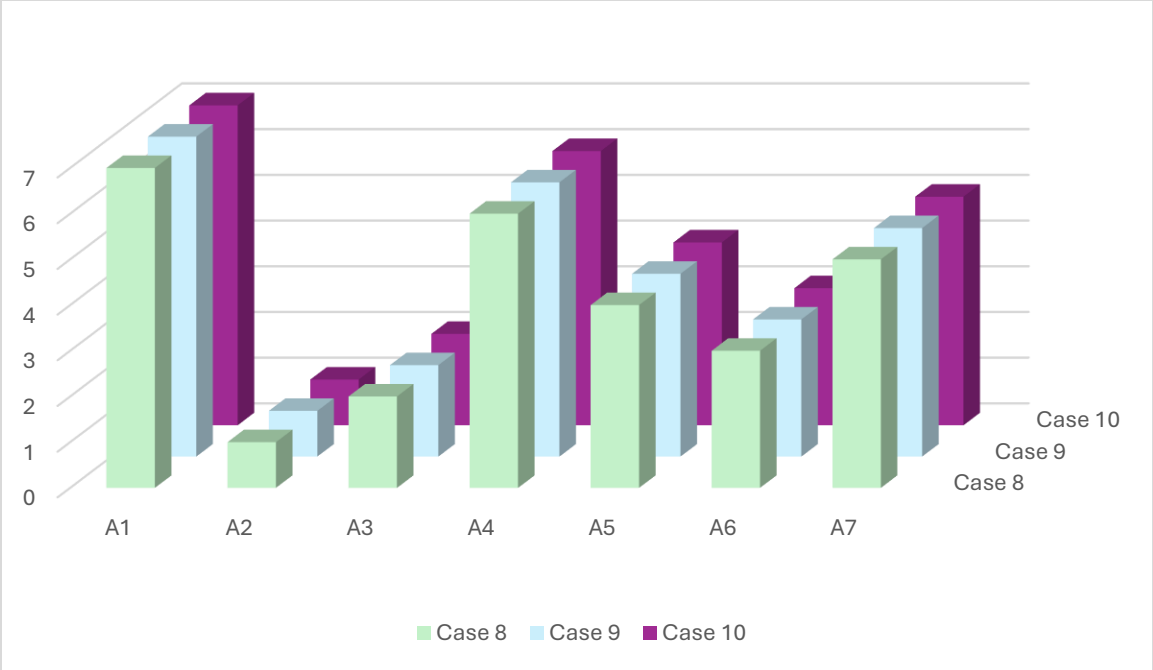

Fig. 3. The rank of three cases.

In the last three cases, we show alternative 1 is the best, followed by alternative 4, and alternative 2 is the worst.

6. Conclusions and Future Works.

This study proposed a GDM approach to evaluate tourism mobile e-commerce services with different criteria and rankings. Two GDM approaches are used in this study such as the SWARA method to compute the criteria weights and rank the alternatives. The WASPAS method is used to rank the alternatives. Three experts have evaluated the criteria and alternatives by using the triangular neutrosophic sets (TNSs) to deal with vague and uncertain data. 12 criteria and seven alternatives are collected to be evaluated in this study. The sensitivity analysis is conducted to show the different ranks of the alternatives. The results show the rank of alternatives is stable in different cases.

In future work, the proposed approach can be applied to different GDM issues to compute the criteria weights and rank the alternatives. The differences in neutrosophic extension can be used in this study to deal with vague data.

Acknowledgment

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Bipolar Neutrosophic Driven Approach for Assessing the Teaching Quality of English Language and Literature Learning Outcomes

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Abstract: The quality of teaching in English Language and Literature is a crucial factor in enhancing student learning outcomes, fostering critical thinking, and promoting linguistic and literary proficiency. Nevertheless, the procedure of assessing the quality of English teaching remains a thought-provoking task since it typically depends on individual judgments, variety in criteria, and inconsistency in human observation, resulting in intrinsic uncertainty. Bipolar Neutrosophic Sets (BNS) offer an expressive way to model ambiguity in a bipolar manner using three positive and three negative memberships. Motivated by that, we propose to explore the novel approach based on BNS to improve the judgmental process of evaluating the quality of teaching of Quality of English Language and related literature. To assess the relative importance of aggregated teaching criteria, we apply the Analytic Hierarchy Process (AHP) to drive representative weights that guide the later decision-making. We drive a new weighting scheme to apply the AHP weights into a bipolar decision matrix. Following, we introduce multi-criteria decision-making (MCDM) to determine the bipolar ideal solutions in a bipolar decision matrix, and then identify the relative closeness teaching techniques. Based on a numerical study, we conduct a holistic analysis of the results of the proposed framework, which demonstrates remarkable power in making appropriate decisions about the best scenario English teaching approach that achieves the optimal benefit for students.

Keywords: Bipolar neutrosophic sets (BNSs), Uncertainty, Neutrosophic Theory, Teaching Quality, English Literature.

1. Introduction

Continuous evaluation of the quality of teaching of English language and literature has been gaining much attention due to the important role it plays in shaping learners' linguistic capability, critical thinking, and cultural awareness of students [1]. The practice of Teaching English usually entails improving the students' proficiency in different language skills—listening, speaking, reading, and writing—which is not only a medium of communication but also a allows interpreting the literary traditions, and historical backgrounds, as well as human experiences [2]. Accordingly, judging the English teaching quality within different domains not only necessitates surface-level evaluation practices; instead, it requires a nuanced method that deliberates

pedagogical efficiency, student engagement, background, level of education, and the level of interaction between creativity and academic rigor. This, in turn, makes the traditional evaluation techniques are no longer effective and might fail to consider the variation in individual learner experiences and wider pedagogical effects [3].

Florentin Smarandache [4] introduced the Neutrosophic set (NS) as a theoretical extension to the standard and fuzzy sets by integrating three basic elements namely truth (Tr), falsity (F), and indeterminacy (II). A part from these traditional sets, NSs were designed to offer a more elastic representation of data, by simultaneously considering the ambiguity, indistinctness, and illogicalities. Each of these consisting of three elements is self-determining and usually represented using a value between 0 and 1, which makes the NS predominantly operative when it comes to dealing with the complex process of evaluating the teaching quality of English and literature.

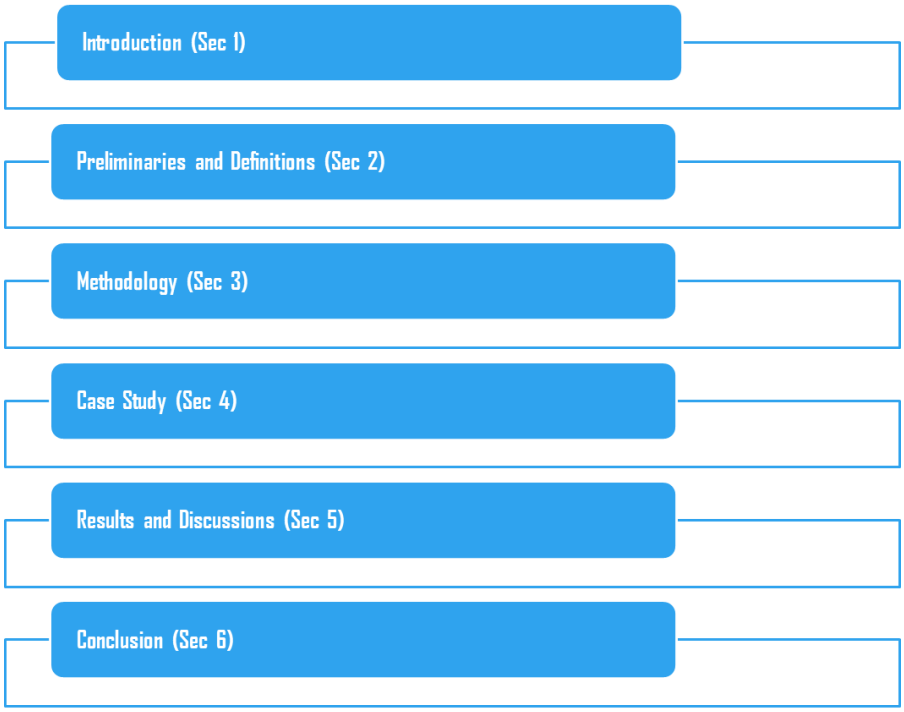

Figure 1. Paper Outline

As an extension of Neutrosophic theory, Bipolar NSs (BNSs) were proposed to include dual degrees for each component of NSs, one in the positive direction, and another in the negative direction [5]. This bipolar nature enables the modeling of information originating from circumstances with bipolar interaction, which provides an expressive representation of multifaceted phenomena [6], [7], [8]. When it comes to assessing teaching superiority, student feedback capacity concurrently replicates optimistic facets (e.g., appealing teaching method) and bad facets (e.g., unclear descriptions). BNS can effectually handle such dualistic problems and

help develop a reliable decision-making approach for analyzing the indeterminacy in assessing contradictory feedback about English teaching [5], [9].

To the best of the author's knowledge, the study is the first to explore the potential of BNSs driving a new and fair approach to assessing teaching quality for the English language and literature. The main contributions can be summarized as follows:

- This study introduces a novel logarithmic weighting mechanism based on the AHP to determine the relative importance of criteria within a Bipolar Neutrosophic framework.
- A novel methodology is proposed for defining ideal solutions and computing distances to them within a Bipolar Neutrosophic matrix, which allows for handling uncertainty, contradiction, and hesitation, providing a more precise and comprehensive evaluation framework compared to existing methods.
- We propose a numerical application to prove and analyze the validity of the proposed approach for providing useful insights about the English language and literature being taught to students, and the results demonstrated demonstrate significant advantages over conventional evaluation methods.

The outcomes of the current research project will have significant implications for the administration and management of the teaching process in classes on the English language and related literature. In Figure 1, we present the outline of the remainder of this study to improve the readability of the paper.

2. Preliminaries & Definition

Definition 1. BNSs [5] define dual directed degree for each of the three components of the original neutrosophic theory.

$$\overleftrightarrow{B} = \{e, \langle Tr^{+}(e), II^{+}(e), F^{+}(c), Tr^{-}(e), II^{-}(c), F^{-}(e)\rangle | e \in E\}, \quad (1)$$

Whereas

$$Tr^{+}(e), II^{+}(e), F^{+}(e): E \to [0,1]$$

$$Tr^{-}(e), II^{-}(e), F^{-}(e): E \to [-1,0]$$

Definition 2. Assume two BNSs $\mathcal{S}_1 = \{e, \langle {Tr_1}^{+}(e), {II_1}^{+}(e), {F_1}^{+}(e), {Tr_1}^{-}(e), II_1^{-}(e), F_1^{-}(e)\rangle | e \in E\}$, and
$\mathcal{S}_2 = \{e, \langle Tr^{+}(e), II^{+}(e), F^{+}(e), Tr^{-}(e), II^{-}(e), F^{-}(e)\rangle | \ e \in E\}$, The following are the main operations for BNSs:

1) The complement of $\mathcal{S}_1$ is computed as follows:

$$\mathcal{A}_1^{c} = \begin{Bmatrix} (\{1^{+}\} - Tr^{+}(e)), (\{1^{+}\} - II^{+}(e)), (\{1^{+}\} - F^{+}(e)), \\ (\{1^{-}\} - Tr^{-}(e)), (\{1^{-}\} - II^{-}(e)), (\{1^{-}\} - F^{-}(e)) \end{Bmatrix} \quad (2)$$

2) $\mathcal{S}_1 \oplus \mathcal{S}_2$ if, and only if,

$$\mathcal{S}_1 \oplus \mathcal{S}_2 = \begin{Bmatrix} \left(\{1^+\} - Tr^+(e)\right), \left(\{1^+\} - II^+(e)\right), \left(\{1^+\} - F^+(e)\right), \\ \left(\{1^-\} - Tr^-(e)\right), \left(\{1^-\} - II^-(e)\right), \left(\{1^-\} - F^-(e)\right) \end{Bmatrix} \tag{3}$$

3) $\mathcal{S}_1 \otimes \mathcal{S}_2$ if, and only if,

$$\mathcal{S}_1 \otimes \mathcal{S}_2 = \begin{Bmatrix} \left(\{1^+\} - Tr^+(e)\right), \left(\{1^+\} - II^+(e)\right), \left(\{1^+\} - F^+(e)\right), \\ \left(\{1^-\} - Tr^-(e)\right), \left(\{1^-\} - II^-(e)\right), \left(\{1^-\} - F^-(e)\right) \end{Bmatrix} \tag{4}$$

4) $\mathcal{S}_1 \subseteq \mathcal{S}_2$ if, and only if,

$$Tr^+(e) < Tr^+(e), F^+(e) < F^+(e), II^+(e) \geq II^+(e) \tag{5}$$

and

$$Tr^+(e) < Tr^+(e), F^+(e) < F^+(e), II^+(e) \geq II^+(e) \tag{6}$$

5) $\mathcal{S}_1 = \mathcal{S}_2$ if, and only if,

$$Tr^+(e) = Tr^+(e), F^+(e) = F^+(e), II^+(e) = II^+(e) > \tag{7}$$

and

$$Tr^+(e) = Tr^+(e), F^+(e) = F^+(e), II^+(e) = II^+(c) > \tag{8}$$

6) The intersection of both $\mathcal{S}_1$ and $\mathcal{S}_2$ is defined as:

$$\mathcal{S}_1 \cap \mathcal{S}_2 = \left\{ e, \min\left(Tr^+(e), Tr^+(e)\right), \frac{II^+(e) + II^+(e)}{2}, \max\left(F^+(e), F^+(e)\right), \max\left(Tr^-(e), Tr^-(e)\right), \frac{II^-(e) + II^-(e)}{2}, \min\left(F^-(e), F^-(e)\right) | e \in E \right\} \tag{9}$$

7) The union of both $\mathcal{S}_1$ and $\mathcal{S}_2$ is defined as:

$$\mathcal{S}_1 \cup \mathcal{S}_2 = \left\{ c, \max\left(Tr^+(e), Tr^+(e)\right), \frac{II^+(e) + II^+(e)}{2}, \min\left(F^+(e), F^+(e)\right), \min\left(Tr^-(c), Tr^-(e)\right), \frac{II_1^-(e) + II^-(e)}{2}, \max\left(F^-(e), F^-(e)\right) | e \in E \right\} \tag{10}$$

Definition 3. Given a BNS $\mathcal{S}_1 = \left\{e, \langle {Tr_1}^+(e), {II_1}^+(e), {F_1}^+(e), {Tr_1}^-(e), II_1^-(e), F_1^-(e)\rangle \middle| e \in E\right\}$,the de-neutrosophication of BNS refers to the act of transforming the BNS into a crisp number. This was originally defined using the following equation [10], [11]:

(11)

$$S(\mathcal{S}_1) = \frac{\left(Tr_1^+(e) + 1 - II_1^+(e) + 1 - F_1^+(e) + 1 + Tr_1^-(e) - II_1^-(e) - F_1^-(e)\right)}{6}.$$

However, this definition can be flowed in many ways. First, it ignores possible synergies or conflicts, such as, how $Tr_1{}^+$ interacts with $II_1{}^+$, or how $Tr_1^-(e)$ affects $F_1^+(e)$. Second, by linear aggregation, the summed components deliver a comprehensive representation, which might not reflect the relative changes.

3. Material and Methods

The evaluation of teaching quality in English language and literature focuses on assessing the effectiveness of instructional methods, curriculum design, and student outcomes. Key indicators include the clarity of teaching objectives, the engagement and participation of students, and the development of core skills such as reading, writing, speaking, and critical thinking. A well-structured curriculum should balance language proficiency with literary analysis, encouraging students to appreciate cultural and historical contexts while improving their communication skills. Effective teaching methods utilize interactive activities, discussions, and creative assignments to foster deeper understanding and active learning. Assessment strategies, such as tests, essays, presentations, and class participation, help measure student progress and comprehension. Additionally, teaching quality is influenced by teacher expertise, the use of technology, and the availability of resources like books and multimedia tools. Feedback from students and peer evaluations also play a crucial role in identifying areas for improvement. By maintaining high teaching standards, educators can inspire students to excel in both language mastery and literary appreciation, fostering lifelong skills and intellectual growth. The evaluation of the quality of teaching the English Language and related literature is no longer an easy task, but it is a complex process requiring consideration of multiple pedagogical, technological, and student-centered aspects. According to a recent research study on English teaching [12], we found six main criteria governing teaching quality of English and related literature, namely student feedback (C1), fulfilling workspace expectation (C2), inefficient resource usage (C3), personal traits (C4), classroom engagements(C5), instructional rigidity (C6), English and content knowledge (C7), and complaint rate (C8). These criteria replicate the main dimensions of teaching quality and form the basis for evaluating a set of alternative teaching techniques, such as direct method (A1), task-based learning (A2), communicative language teaching (A3), grammar-translation method (A4), audio-Lingual Method (A5), silent way (A6), content and language integrated learning (A7), and total physical response (A8) [13]. These methods are selected for their exceptional approaches to language education, ranging from traditional to modern, technology-enhanced methods. Each technique offers distinct advantages and challenges, making them perfect applicants for comparative analysis within our BNSs-based decision-making approach. Three English language experts are involved into the process of determining the representative values linguistic variable as defined in Table 1. These experts evaluated the linguistic variables of different criteria for different teaching alternatives across different study semesters, guaranteeing an inclusive and representative analysis of teaching quality. The process

of aggregating the expert opinions and semester-wise evaluations enabled achieving balanced and nuanced interpretation, taking into account both the positive and negative facet for each criterion.

Table 1. BNS-based definition of linguistic variables for criteria in assessing teaching quality.

Criteria	Linguistic Variable	BNS Representation	Criteria	Linguistic Variable	BNS Representation
C1	Excellent	⟨0.9, 0.07, 0.06, -0.06, -0.06, -0.91⟩	**C5**	Highly Engaging	⟨0.92, 0.07, 0.07, -0.09, -0.08, -0.91⟩
	Good	⟨0.80, 0.14, 0.13, -0.10, -0.11, -0.81⟩		Engaging	⟨0.84, 0.12, 0.12, -0.13, -0.11, -0.82⟩
	Fair	⟨0.60, 0.22, 0.24, -0.21, -0.23, -0.65⟩		Neutral	⟨0.51, 0.33, 0.21, -0.24, -0.31, -0.52⟩
	Poor	⟨0.32, 0.23, 0.54, -0.53, -0.25, -0.30⟩		Disengaging	⟨0.35, 0.23, 0.53, -0.50, -0.25, -0.31⟩
	Very Poor	⟨0.11, 0.11, 0.85, -0.81, -0.11, -0.11⟩		Highly Disengaging	⟨0.13, 0.15, 0.83, -0.81, -0.12, -0.15⟩
C2	Highly Relevant	⟨0.85, 0.13, 0.08, -0.05, -0.14, -0.90⟩	**C6**	Flexible	⟨0.89, 0.11, 0.08, -0.06, -0.11, -0.89⟩
	Relevant	⟨0.78, 0.15, 0.14, -0.12, -0.19, -0.77⟩		Moderately Flexible	⟨0.74, 0.19, 0.15, -0.16, -0.18, -0.71⟩
	Moderately Relevant	⟨0.61, 0.22, 0.25, -0.22, -0.24, -0.64⟩		Rigid	⟨0.32, 0.24, 0.53, -0.51, -0.21, -0.34⟩
	Irrelevant	⟨0.33, 0.25, 0.52, -0.51, -0.25, -0.34⟩		Highly Rigid	⟨0.11, 0.11, 0.82, -0.82, -0.10, -0.10⟩
	Highly Irrelevant	⟨0.10, 0.11, 0.83, -0.81, -0.11, -0.12⟩			
C3	Highly Efficient	⟨0.94, 0.07, 0.05, -0.06, -0.08, -0.94⟩	**C7**	Advanced	{0.90, 0.05, 0.05, -0.05, -0.05, -0.90}
	Efficient	⟨0.84, 0.11, 0.13, -0.13, -0.11, -0.84⟩		Proficient	⟨0.93, 0.05, 0.09, -0.06, -0.09, -0.92⟩
	Moderately Efficient	⟨0.64, 0.23, 0.22, -0.22, -0.20, -0.65⟩		Competent	⟨0.84, 0.11, 0.15, -0.14, -0.13, -0.82⟩
	Inefficient	⟨0.31, 0.23, 0.53, -0.53, -0.21, -0.31⟩		Inadequate	⟨0.63, 0.22, 0.23, -0.25, -0.22, -0.62⟩
	Highly Inefficient	⟨0.14, 0.15, 0.82, -0.82, -0.15, -0.14⟩		Very Poor	⟨0.34, 0.20, 0.55, -0.52, -0.22, -0.35⟩
C4	Exemplary	⟨0.98, 0.07, 0.03, -0.06, -0.07, -0.96⟩	**C8**	Very Low	⟨0.92, 0.08, 0.08, -0.09, -0.06, -0.94⟩
	Very Positive	⟨0.82, 0.15, 0.11, -0.11, -0.15, -0.82⟩		Low	⟨0.82, 0.14, 0.11, -0.14, -0.11, -0.83⟩
	Neutral	⟨0.50, 0.31, 0.21, -0.25, -0.34, -0.52⟩		Neutral	⟨0.54, 0.31, 0.22, -0.24, -0.32, -0.54⟩
	Negative	⟨0.31, 0.22, 0.54, -0.50, -0.24, -0.34⟩		High	⟨0.35, 0.24, 0.55, -0.51, -0.20, -0.32⟩
	Highly Negative	⟨0.11, 0.10, 0.80, -0.81, -0.15, -0.14⟩		Very Highly	⟨0.15, 0.11, 0.81, -0.83, -0.11, -0.10⟩

In the following, we provide our proposed approach to BNSs for making appropriate decisions about the quality of teaching of English language and related literature.

Step 1: we define BNSs linguistic variables. In other words, Firstly, we formulate the linguistic variables and determine the corresponding BNSs that better characterize their degree of interaction among different aspects, as shown in Figure 2. Then, the definition of linguistic variables follows the definitions in Section 2, and the corresponding memberships should satisfy the requirements in Definition 1.

Step 2**:** Choose the teaching scenarios and aggregate the bipolar valuations assessed by English experts. In particular, we specify the alternative and ask experts to evaluate different teaching criteria based on the abovementioned linguistic variables. Assume that we have K experts, $\mathrm{E_e} = \{\mathrm{E_1}, \mathrm{E_2}, \ldots, \mathrm{E_k}\}(\mathrm{k} = 1,2,\ldots,\mathrm{K})$, assessing a number, $\mathrm{N_c}$, of teaching criteria, such that $\Omega_\mathrm{i} = \{\Omega_1, \Omega_2, \ldots, \Omega_{\mathrm{N_c}}$. The expert evaluations can be expressed as follows:

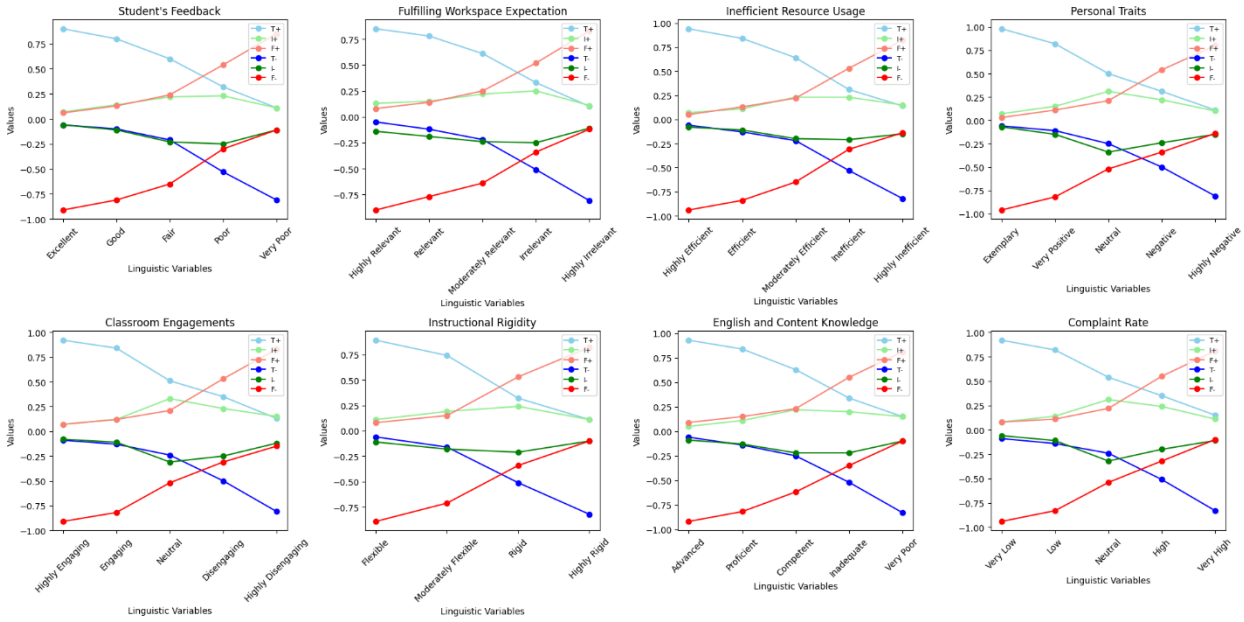

Figure 2. visualization of BNS representation of linguistic variables.

$$D^k_{N_A \times N_c} = d^k_{ij} = \begin{bmatrix} d^k_{11} & \cdots & d^k_{1N_c} \\ \vdots & \ddots & \vdots \\ d^k_{N_A 1} & \cdots & d^k_{N_A N_c} \end{bmatrix}, k \in K, \tag{12}$$

where $\mathrm{D^k_{N_A \times N_c}}$ denotes the BNS decision matrix of the $\mathrm{K-th}$ experts. $d^k_{ij} = \langle (Tr^+)^k_{ij}, (II^+)^k_{ij}, (F^+)^k_{ij}, (Tr^-)^k_{ij}, (II^-)^k_{ij}, (F^-)^k_{ij} \rangle, k = 1, \cdots, K; i = 1, \cdots, N_A; j = 1, \cdots, N_c.$

Step 3, we aggregate evaluations from different experts, then, the aggregated decision matrix can be expressed as follows:

$$D_{N_A \times N_c} = d_{ij} = \begin{bmatrix} d_{11} & \cdots & d_{1N_c} \\ \vdots & \ddots & \vdots \\ d_{N_A 1} & \cdots & d_{N_A N_c} \end{bmatrix}, \tag{13}$$

where $d_{ij} = \langle (Tr^+)_{ij}, (II^+)_{ij}, (F^+)_{ij}, (Tr^-)_{ij}, (II^-)_{ij}, (F^-)_{ij} \rangle, i = 1, \cdots, N_A; j = 1, \cdots, N_c$. The language expert evaluations are collected based on majority voting between linguistic variables.

In step 4, we transform the aggregated BNSs decision matrix into the crisp valued matrix by applying the de-neutrosophication operation as previously described in Definition 4. However, this definition can be flowed in many ways. First, it ignores possible synergies or conflicts, such as, how Tr_1^{+} interacts with II_1^{+}, or how $Tr_1^{-}(e)$ affects $F_1^{+}(e)$. Second, by linear aggregation, the summed components deliver a comprehensive representation, which might not reflect the relative changes. To avoid these limitations, we propose the de-neutrosophication method based on logarithms to emphasize diminishing returns and highlight the relative changes instead of absolute magnitudes. This can be expressed with the following de-neutrosophication equation:

$$S(d_{ij}) = \left(\ln\left(1+Tr^{+}(d_{ij})\right) + \ln\left(1+\left(1-II^{+}(d_{ij})\right)\right) + \ln\left(1+\left(1-F^{+}(d_{ij})\right)\right) - \ln\left(1+Tr^{-}(d_{ij})\right) - \ln\left(1+II^{-}(d_{ij})\right) - \ln\left(1+F^{-}(d_{ij})\right)\right)/6 \quad (14)$$

By applying this formula, we obtain a representative decision matrix consisting of crisp values.

Figure 2. visualization of Saaty's scale

Let B_{ij} be a crisp number matrix, then the matrix is denoted as:

$$B_{N_A \times N_c} = b_{ij} = \begin{bmatrix} b_{11} & \cdots & b_{1N_c} \\ \vdots & \ddots & \vdots \\ b_{N_A 1} & \cdots & b_{N_A N_c} \end{bmatrix}, \quad (15)$$

Step 5: we apply the AHP to calculate the relative importance of each criterion.

Step 5.1, we create a pairwise comparison matrix of size. $N_c \times N_c$, where N_c is the number of criteria, and $a_{i,j}$ Symbolize the relative importance of $i-th$ criterion and $j-th$ criterion. If $i = j$ then $a_{i,j} = 1$. In building our pairwise comparison, we used Saaty's scale [14], [15] displayed in Figure 2.

Reciprocal values ($a_{i,j} = 1$) for inverse comparisons.

Step 5.2: we normalize the pairwise comparison matrix, and calculate the sum for each column in $B_{N_A \times N_c}$:

$$P_{N_c \times N_c} = p_{ij} = \begin{bmatrix} b_{11} & \cdots & b_{1N_c} \\ \vdots & \ddots & \vdots \\ b_{N_c 1} & \cdots & b_{N_c N_c} \end{bmatrix}, \tag{16}$$

$$\text{Sum}_j = \sum_{i=1}^{N_c} p_{ij},\ \forall j \in \{1, \ldots, N_c\} \tag{17}$$

Then, we divide each element, p_{ij}, by the corresponding column sum to get the normalized matrix P_{norm} :

$$p_{\text{norm},ij} = \frac{p_{ij}}{\text{Sum}_j},\ \forall i, j \tag{18}$$

In step 5.3, we calculate the relative Weights based on the priority vector $\mathcal{w} = [\mathcal{w}_1, \mathcal{w}_2, \ldots, \mathcal{w}_{N_c}]^T$ representing the average of the rows of P_{norm} :

$$\mathcal{w}_i = \frac{\sum_{j=1}^{N_c} p_{\text{norm},ij}}{N_c},\ \forall i \in \{1, \cdots, N_c\} \tag{19}$$

Step 5.4, we can check consistency by computing the weighted sum vector $\lambda = P \cdot \mathcal{w}$, then find the largest eigenvalue. $\lambda_{\max}$:

$$\lambda_{\max} = \frac{\sum_{i=1}^{N_c} \left(\frac{\lambda_i}{w_i}\right)}{N_c} \tag{20}$$

Next, we calculate the Consistency Index (Cl), and Consistency Ratio (CR) as follows:

$$\text{CI} = \frac{\lambda_{\max} - N_c}{N_c - 1} \tag{21}$$

$$\text{CR} = \frac{\text{CI}}{\text{RI}} \tag{22}$$

Where RI (Random Index) depends on N_c .

Based on the computed weights, we can drive a weighted decision matrix as follows:

$$\begin{aligned} b_{ij}^{\mathcal{w}_j} &= \left\{Tr_{ij}^{\mathcal{w}_j +}, II_{ij}^{\mathcal{w}_j +}, F_{ij}^{\mathcal{w}_j +}, Tr_{ij}^{\mathcal{w}_j -}, II_{ij}^{\mathcal{w}_j -}, F_{ij}^{\mathcal{w}_j -}\right\} \\ &= \left\{1 - \left(1 - Tr_{ij}^{+}\right)^{\mathcal{w}_j}, \left(II_{ij}^{+}\right)^{\mathcal{w}_j}, \left(F_{ij}^{+}\right)^{\mathcal{w}_j}, -\left(-Tr_{ij}^{-}\right)^{\mathcal{w}_j}, -\left(-II_{ij}^{-}\right)^{\mathcal{w}_j}, -\left(\left(1 - \left(-F_{ij}^{-}\right)\right)^{\mathcal{w}_j}\right)\right\}, \end{aligned} \tag{23}$$

Step 6, we compute BNS relative positive ideal solution ($\mathfrak{P}$) as formulated below:

$$\mathfrak{P} = \left\{ \begin{matrix} \langle {}^{+}Tr_1^{w_1+}, {}^{+}II_1^{w_1+}, {}^{+}F_1^{w_1+}, {}^{+}Tr_1^{w_1-}, {}^{+}II_1^{w_1-}, {}^{+}F_1^{w_1-} \rangle, \langle {}^{+}Tr_2^{w_2+}, {}^{+}II_2^{w_2+}, {}^{+}F_2^{w_2+}, {}^{+}Tr_2^{w_2-}, II_2^{w_2-}, {}^{+}F_2^{w_2-} \rangle, \\ \ldots, \langle {}^{+}Tr_{N_c}^{w_n+}, {}^{+}II_{N_c}^{w_n+}, {}^{+}F_{N_c}^{w_n+}, {}^{+}Tr_{N_c}^{w_n-}, {}^{+}II_{N_c}^{w_n-}, {}^{+}F_{N_c}^{w_n-} \rangle \end{matrix} \right\} \quad (24)$$

and compute BNS relative negative ideal solution ($\mathfrak{N}$) as formulated below:

$$\mathfrak{N} = \left\{ \begin{matrix} \langle {}^{-}Tr_1^{w_1+}, {}^{-}II_1^{w_1+}, {}^{-}F_1^{w_1+}, {}^{-}Tr_1^{w_1-}, {}^{-}II_1^{w_1-}, {}^{-}F_1^{w_1-} \rangle, \langle {}^{-}Tr_2^{w_2+}, {}^{-}II_2^{w_2+}, {}^{-}F_2^{w_2+}, {}^{-}Tr_2^{w_2-}, {}^{-}II_2^{w_2-}, {}^{-}F_2^{w_2-} \rangle, \\ \ldots, \langle {}^{-}Tr_n^{w_n+}, {}^{-}II_n^{w_n+}, {}^{-}F_n^{w_n+}, {}^{-}Tr_n^{w_n-}, {}^{-}II_n^{w_n-}, {}^{-}F_n^{w_n-} \rangle \end{matrix} \right\} \quad (25)$$

For benefit criteria, $j = 1,2, \ldots, N_c$,

$$\begin{aligned} \langle {}^{+}Tr_j^{w_j+}, {}^{+}II_j^{w_j+}, {}^{+}F_j^{w_j+}, {}^{+}Tr_j^{w_j-}, {}^{+}II_j^{w_j-}, {}^{+}F_j^{w_j-} \rangle &= \langle \max\left(Tr_{ij}^{w_j+}\right), \min\left(II_{ij}^{w_j+}\right), \min\left(F_{ij}^{w_j+}\right), \\ &\quad \min\left(Tr_{ij}^{w_j-}\right), \max\left(II_{ij}^{w_j-}\right), \max\left(F_{ij}^{w_j-}\right) \rangle, \\ \langle {}^{-}Tr_j^{w_j+}, {}^{-}II_j^{w_j+}, {}^{-}F_j^{w_j+}, {}^{-}Tr_j^{w_j-}, {}^{-}II_j^{w_j-}, {}^{-}F_j^{w_j-} \rangle &= \langle \min\left(Tr_{ij}^{w_j+}\right), \max\left(II_{ij}^{w_j+}\right), \max\left(F_{ij}^{w_j+}\right), \\ &\quad \max\left(Tr_{ij}^{w_j-}\right), \min\left(II_{ij}^{w_j-}\right), \min\left(F_{ij}^{w_j-}\right) \rangle. \end{aligned} \quad (26)$$

While cost criteria, $j = 1,2, \ldots, N_c$

$$\begin{aligned} \langle {}^{+}Tr_j^{w_j+}, {}^{+}II_j^{w_j+}, {}^{+}F_j^{w_j+}, {}^{+}Tr_j^{w_j-}, {}^{+}II_j^{w_j-}, {}^{+}F_j^{w_j-} \rangle &= \langle \min\left(Tr_{ij}^{w_j+}\right), \max\left(II_{ij}^{w_j+}\right), \max\left(F_{ij}^{w_j+}\right), \\ &\quad \max\left(Tr_{ij}^{w_j-}\right), \min\left(II_{ij}^{w_j-}\right), \min\left(F_{ij}^{w_j-}\right) \rangle, \\ \langle {}^{-}Tr_j^{w_j+}, {}^{-}II_j^{w_j+}, {}^{-}F_j^{w_j+}, {}^{-}Tr_j^{w_j-}, {}^{-}II_j^{w_j-}, {}^{-}F_j^{w_j-} \rangle &= \langle \max\left(Tr_{ij}^{w_j+}\right), \min\left(II_{ij}^{w_j+}\right), \min\left(F_{ij}^{w_j+}\right), \\ &\quad \min\left(Tr_{ij}^{w_j-}\right), \max\left(II_{ij}^{w_j-}\right), \max\left(F_{ij}^{w_j-}\right) \rangle. \end{aligned} \quad (27)$$

Step 7, we compute the distance between each alternative w.r.t $\mathfrak{P}$,

$$D_{mins}(\mathcal{S}_i, \mathfrak{P}) = \left(\frac{1}{6N_c} \sum_{j=1}^{N_c} \left\{ \begin{matrix} \left| Tr_{ij}^{w_j+} - {}^{+}Tr_j^{w_j+} \right|^p + \left| II_{ij}^{w_j+} - {}^{+}II_j^{w_j+} \right|^p + \left| F_{ij}^{w_j+} - {}^{+}F_j^{w_j+} \right|^p + \\ \left| Tr_{ij}^{w_j-} - {}^{+}Tr_j^{w_j-} \right|^p + \left| II_{ij}^{w_j-} - {}^{+}II_j^{w_j-} \right|^p + \left| F_{ij}^{w_j-} - {}^{+}F_j^{w_j-} \right|^p \end{matrix} \right\} \right)^{\frac{1}{p}}, \quad (28)$$

and we compute the distance between each alternative w.r.t $\mathfrak{N}$,

$$D_{mins}(\mathcal{S}_i, \mathfrak{N}) = \left(\frac{1}{6N_c} \sum_{j=1}^{N_c} \left\{ \begin{matrix} \left| Tr_{ij}^{w_j+} - {}^{-}Tr_j^{w_j+} \right|^p + \left| II_{ij}^{w_j+} - {}^{-}II_j^{w_j+} \right|^p + \left| F_{ij}^{w_j+} - {}^{-}F_j^{w_j+} \right|^p + \\ \left| Tr_{ij}^{w_j-} - {}^{-}Tr_j^{w_j-} \right|^p + \left| II_{ij}^{w_j-} - {}^{-}II_j^{w_j-} \right|^p + \left| F_{ij}^{w_j-} - {}^{-}F_j^{w_j-} \right|^p \end{matrix} \right\} \right). \quad (29)$$

Step 8, for each alternative, we calculate the level of nearness, μ_i, w.r.t $\mathfrak{P}$ as formulated below:

(30)

$$\mu_i(S_i) = \frac{D_{mins}(S_i, \mathfrak{N})}{max\{D_{mins}(S_i, \mathfrak{N})\} + D_{mins}(S_i, \mathfrak{P})}, i = 1,2, \dots, N_A.$$

Step 9, for each alternative, we use the above degrees to calculate the inferior ratio, $\vartheta(i)$ as below:

$$\vartheta(i) = \frac{\mu_i(S_i)}{\min\limits_{1 \le i \le m} \left(\mu_i(S_i)\right)}. \quad (31)$$

where the value of $\vartheta(i)$ belong to interval [0,1]. In Last, the alternatives are ranked according to the order of values of $\vartheta(i)$, in which the finest alternative is nominated according to the lowest value of $\vartheta(i)$.

4. Results and Analysis

Result-set 1: according to the definition of linguistic variables in Table 1, the BNS-based decision matrix is driven as shown in Table 2.

Table 2. BNS-based decision matrix.

Alternative \ Criteria	C1	C2	C3	C4
S_1	⟨ 0.9 , 0.07 , 0.06 , -0.06 , -0.06 , -0.91 ⟩	⟨ 0.61 , 0.22 , 0.25 , -0.22 , -0.24 , -0.64 ⟩	⟨ 0.14 , 0.15 , 0.82 , -0.82 , -0.15 , -0.14 ⟩	⟨ 0.98 , 0.07 , 0.03 , -0.06 , -0.07 , -0.96 ⟩
S_2	⟨ 0.6 , 0.22 , 0.24 , -0.21 , -0.23 , -0.65 ⟩	⟨ 0.85 , 0.13 , 0.08 , -0.05 , -0.14 , -0.9 ⟩	⟨ 0.31 , 0.23 , 0.53 , -0.53 , -0.21 , -0.31 ⟩	⟨ 0.82 , 0.15 , 0.11 , -0.11 , -0.15 , -0.82 ⟩
S_3	⟨ 0.8 , 0.14 , 0.13 , -0.1 , -0.11 , -0.81 ⟩	⟨ 0.33 , 0.25 , 0.52 , -0.51 , -0.25 , -0.34 ⟩	⟨ 0.84 , 0.11 , 0.13 , -0.13 , -0.11 , -0.84 ⟩	⟨ 0.5 , 0.31 , 0.21 , -0.25 , -0.34 , -0.52 ⟩
S_4	⟨ 0.9 , 0.07 , 0.06 , -0.06 , -0.06 , -0.91 ⟩	⟨ 0.78 , 0.15 , 0.14 , -0.12 , -0.19 , -0.77 ⟩	⟨ 0.14 , 0.15 , 0.82 , -0.82 , -0.15 , -0.14 ⟩	⟨ 0.31 , 0.22 , 0.54 , -0.5 , -0.24 , -0.34 ⟩
S_5	⟨ 0.11 , 0.11 , 0.85 , -0.81 , -0.11 , -0.11 ⟩	⟨ 0.1 , 0.11 , 0.83 , -0.81 , -0.11 , -0.12 ⟩	⟨ 0.94 , 0.07 , 0.05 , -0.06 , -0.08 , -0.94 ⟩	⟨ 0.11 , 0.1 , 0.8 , -0.81 , -0.15 , -0.14 ⟩
S_6	⟨ 0.32 , 0.23 , 0.54 , -0.53 , -0.25 , -0.3 ⟩	⟨ 0.78 , 0.15 , 0.14 , -0.12 , -0.19 , -0.77 ⟩	⟨ 0.31 , 0.23 , 0.53 , -0.53 , -0.21 , -0.31 ⟩	⟨ 0.82 , 0.15 , 0.11 , -0.11 , -0.15 , -0.82 ⟩
S_7	⟨ 0.6 , 0.22 , 0.24 , -0.21 , -0.23 , -0.65 ⟩	⟨ 0.1 , 0.11 , 0.83 , -0.81 , -0.11 , -0.12 ⟩	⟨ 0.84 , 0.11 , 0.13 , -0.13 , -0.11 , -0.84 ⟩	⟨ 0.31 , 0.22 , 0.54 , -0.5 , -0.24 , -0.34 ⟩
S_8	⟨ 0.8 , 0.14 , 0.13 , -0.1 , -0.11 , -0.81 ⟩	⟨ 0.33, 0.25 , 0.52 , -0.51 , 0.25 , -0.34 ⟩	⟨ 0.64 , 0.23 , 0.22 , -0.22 , -0.2 , -0.65 ⟩	⟨ 0.98 , 0.07 , 0.03 , -0.06 , -0.07 , -0.96 ⟩

Table 2. BNS based decision matrix (cont..).

Alternative \ Criteria	C5	C6	C7	C8
S_1	⟨0.92, 0.07 , 0.07 , -0.09 , -0.08 , -0.91 ⟩	⟨0.11 , 0.11 , 0.82 , -0.82 , -0.1 , -0.1 ⟩	⟨0.93 , 0.05 , 0.09 , -0.06 , -0.09 , -0.92 ⟩	⟨0.92, 0.08 , 0.08 , -0.09 , -0.06 , -0.94⟩
S_2	⟨0.84, 0.12 , 0.12 , -0.13 , -0.11 , -0.82 ⟩	⟨0.74 , 0.19 , 0.15 , -0.16 , -0.18 , -0.71 ⟩	⟨0.84 , 0.11 , 0.15 , -0.14 , -0.13 , -0.82 ⟩	⟨0.82, 0.14 , 0.11 , -0.14 , -0.11 , -0.83⟩
S_3	⟨0.51, 0.33 , 0.21 , -0.24 , -0.31 , -0.52 ⟩	⟨0.32 , 0.24 , 0.53 , -0.51 , -0.21 , -0.34 ⟩	⟨0.15 , 0.15 , 0.81 , -0.83 , -0.1 , -0.1 ⟩	⟨0.54, 0.31 , 0.22 , -0.24 , -0.32 , -0.54⟩
S_4	⟨0.35, 0.23 , 0.53 , -0.5 , -0.25 , -0.31 ⟩	⟨0.11 , 0.11 , 0.82 , -0.82 , -0.1 , -0.1 ⟩	⟨0.34 , 0.2 , 0.55 , -0.52 , -0.22 , -0.35 ⟩	⟨0.35, 0.24 , 0.55 , -0.51 , -0.2 , -0.32⟩
S_5	⟨0.13, 0.15 , 0.83 , -0.81 , -0.12 , -0.15 ⟩	⟨0.74 , 0.19 , 0.15 , -0.16 , -0.18 , -0.71 ⟩	⟨0.15 , 0.15 , 0.81 , -0.83 , -0.1 , -0.1 ⟩	⟨0.15, 0.11 , 0.81 , -0.83 , -0.11 , -0.1⟩
S_6	⟨0.35, 0.23 , 0.53 , -0.5 , -0.25 , -0.31 ⟩	⟨0.89, 0.11 , 0.08 , -0.06 , -0.11 , -0.89 ⟩	⟨0.93 , 0.05 , 0.09 , -0.06 , -0.09 , -0.92 ⟩	⟨0.54, 0.31 , 0.22 , -0.24 , -0.32 , -0.54⟩

S_7	⟨0.51, 0.33 , 0.21 , -0.24 , -0.31 , -0.52 ⟩	⟨0.32 , 0.24 , 0.53 , -0.51 , -0.21 , -0.34 ⟩	⟨0.84 , 0.11 , 0.15 , -0.14 , -0.13 , -0.82 ⟩	⟨0.82, 0.14 , 0.11 , -0.14 , -0.11 , -0.83⟩
S_8	⟨0.84, 0.12 , 0.12 , -0.13 , -0.11 , -0.82 ⟩	⟨0.74 , 0.19 , 0.15 , -0.16 , -0.18 , -0.71 ⟩	⟨0.63 , 0.22 , 0.23 , -0.25 , -0.22 , -0.62 ⟩	⟨0.35, 0.24 , 0.55 , -0.51 , -0.2 , -0.32⟩

Result-set 2, the pairwise comparison matrix is given in Table 3.

Table 3. Aggregated pairwise comparison matrix for different criteria

Criteria	**C1**	**C2**	**C3**	**C4**	**C5**	**C6**	**C7**	**C8**
C1	1	3	5	4	2	6	3	7
C2	$\frac{1}{3}$	1	3	2	$\frac{1}{2}$	4	2	5
C3	$\frac{1}{5}$	$\frac{1}{3}$	1	$\frac{1}{2}$	$\frac{1}{3}$	3	$\frac{1}{2}$	4
C4	$\frac{1}{4}$	$\frac{1}{2}$	2	1	$\frac{1}{3}$	3	$\frac{1}{2}$	4
C5	$\frac{1}{2}$	2	3	3	1	5	2	6
C6	$\frac{1}{6}$	$\frac{1}{4}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{5}$	1	$\frac{1}{2}$	3
C7	$\frac{1}{3}$	$\frac{1}{2}$	2	2	$\frac{1}{2}$	2	1	4
C8	$\frac{1}{7}$	$\frac{1}{5}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{4}$	1

Result-set 3, the numerical results representing weights from the AHP method are given in Table 3.

Table 3. AHP weights for various English teaching criteria.

Criteria	C1	C2	C3	C4	C5	C6	C7	C8
Weights	0.315	0.15	0.067	0.087	0.208	0.042	0.107	0.25

Result-set 4, the decision matrix of weighted BNS is shown in Table 4.

Table 4. AHP-weighted decision matrix.

Criteria / Alternative	**C1**	**C2**	**C3**	**C4**
S_1	(0.5157, 0.4329, 0.4124, -0.4124, -0.4124, -0.5315)	(0.1317, 0.7969, 0.8123, -0.7969, -0.8074, -0.142)	(0.0101, 0.88, 0.9867, -0.9867, -0.88, -0.0101)	(0.2876, 0.7941, 0.7378, -0.7835, -0.7941, -0.2435)
S_2	(0.2506, 0.6208, 0.6381, -0.6118, -0.6296, -0.2815)	(0.2476, 0.7365, 0.6848, -0.6382, -0.7447, -0.292)	(0.0247, 0.9057, 0.9581, -0.9581, -0.9002, -0.0247)	(0.1381, 0.8483, 0.8258, -0.8258, -0.8483, -0.1381)
S_3	(0.3975, 0.5385, 0.5261, -0.4843, -0.4991, -0.4072)	(0.0583, 0.8123, 0.9066, -0.904, -0.8123, -0.0604)	(0.1162, 0.8618, 0.8715, -0.8715, -0.8618, -0.1162)	(0.0583, 0.9034, 0.8734, -0.8868, -0.9107, -0.0617)
S_4	(0.5157, 0.4329, 0.4124, -0.4124, -0.4124, -0.5315)	(0.2031, 0.7524, 0.7447, -0.7277, -0.7796, -0.1978)	(0.0101, 0.88, 0.9867, -0.9867, -0.88, -0.0101)	(0.0317, 0.877, 0.948, -0.9417, -0.8836, -0.0354)
S_5	(0.036, 0.4991, 0.9501, -0.9358, -0.4991, -0.036)	(0.0157, 0.7182, 0.9724, -0.9689, -0.7182, -0.019)	(0.1727, 0.8359, 0.8172, -0.8273, -0.8435, -0.1727)	(0.0101, 0.819, 0.9808, -0.9819, -0.8483, -0.013)
S_6	(0.1143, 0.6296, 0.8237, -0.8188, -0.6463, -0.1062)	(0.2031, 0.7524, 0.7447, -0.7277, -0.7796, -0.1978)	(0.0247, 0.9057, 0.9581, -0.9581, -0.9002, -0.0247)	(0.1381, 0.8483, 0.8258, -0.8258, -0.8483, -0.1381)
S_7	(0.2506, 0.6208, 0.6381, -0.6118, -0.6296, -0.2815)	(0.0157, 0.7182, 0.9724, -0.9689, -0.7182, -0.019)	(0.1162, 0.8618, 0.8715, -0.8715, -0.8618, -0.1162)	(0.0317, 0.877, 0.948, -0.9417, -0.8836, -0.0354)

S_8	(0.3975, 0.5385, 0.5261, -0.4843, -0.4991, -0.4072)	(0.0583, 0.8123, 0.9066, -0.904, -0.8123, -0.0604)	(0.0665, 0.9057, 0.903, -0.903, -0.8972, -0.0683)	(0.2876, 0.7941, 0.7378, -0.7835, -0.7941, -0.2435)

Table 4. AHP-weighted decision matrix (cont..).

Alternative \ Criteria	C5	C6	C7	C8
S_1	(0.408, 0.5758, 0.5758, -0.6066, -0.592, -0.3934)	(0.0048, 0.9124, 0.9918, -0.9918, -0.9088, -0.0044)	(0.2474, 0.726, 0.7731, -0.7403, -0.7731, -0.2366)	(0.0615, 0.9385, 0.9385, -0.9413, -0.9318, -0.0682)
S_2	(0.3164, 0.6439, 0.6439, -0.6547, -0.6324, -0.2995)	(0.0544, 0.9333, 0.9242, -0.9267, -0.9312, -0.0501)	(0.1779, 0.7898, 0.8165, -0.8105, -0.8041, -0.1675)	(0.0422, 0.9518, 0.9461, -0.9518, -0.9461, -0.0435)
S_3	(0.1376, 0.7944, 0.7233, -0.7436, -0.7842, -0.1413)	(0.0159, 0.9424, 0.974, -0.9724, -0.9372, -0.0171)	(0.0172, 0.8165, 0.9777, -0.9803, -0.7818, -0.0112)	(0.0193, 0.971, 0.9627, -0.9648, -0.9718, -0.0193)
S_4	(0.0855, 0.7371, 0.8765, -0.866, -0.7499, -0.0741)	(0.0048, 0.9124, 0.9918, -0.9918, -0.9088, -0.0044)	(0.0434, 0.842, 0.9381, -0.9325, -0.8506, -0.045)	(0.0108, 0.9648, 0.9851, -0.9832, -0.9604, -0.0096)
S_5	(0.0285, 0.6745, 0.9621, -0.9572, -0.6439, -0.0332)	(0.0544, 0.9333, 0.9242, -0.9267, -0.9312, -0.0501)	(0.0172, 0.8165, 0.9777, -0.9803, -0.7818, -0.0112)	(0.0041, 0.9461, 0.9947, -0.9953, -0.9461, -0.0026)
S_6	(0.0855, 0.7371, 0.8765, -0.866, -0.7499, -0.0741)	(0.0876, 0.9124, 0.9004, -0.8897, -0.9124, -0.0876)	(0.2474, 0.726, 0.7731, -0.7403, -0.7731, -0.2366)	(0.0193, 0.971, 0.9627, -0.9648, -0.9718, -0.0193)
S_7	(0.1376, 0.7944, 0.7233, -0.7436, -0.7842, -0.1413)	(0.0159, 0.9424, 0.974, -0.9724, -0.9372, -0.0171)	(0.1779, 0.7898, 0.8165, -0.8105, -0.8041, -0.1675)	(0.0422, 0.9518, 0.9461, -0.9518, -0.9461, -0.0435)
S_8	(0.3164, 0.6439, 0.6439, -0.6547, -0.6324, -0.2995)	(0.0544, 0.9333, 0.9242, -0.9267, -0.9312, -0.0501)	(0.1008, 0.8506, 0.8546, -0.8623, -0.8506, -0.0982)	(0.0108, 0.9648, 0.9851, -0.9832, -0.9604, -0.0096)

Result-set 4, the bipolar ideal solutions are shown in Table 5.

Table 5. The ideal solutions are based on a weighted BNS decision matrix.

$\mathfrak{P}$	$\mathfrak{N}$
(0.5158, 0.0464, 0.0429, -0.9358, -0.0334, -0.1709)	(0.0360, 0.6294, 0.9501, -0.0305, -0.6462, -0.9640)
(0.2477, 0.0360, 0.2064, -0.9689, -0.0334, -0.0363)	(0.0157, 0.8123, 0.9724, -0.1386, -0.8123, -0.9810)
(0.0101, 0.9062, 0.9868, -0.0392, -0.9007, -0.9899)	(0.4386, 0.0360, 0.0429, -0.9868, -0.0334, -0.1709)
(0.7084, 0.0226, 0.0095, -0.9818, -0.0215, -0.0966)	(0.0101, 0.9031, 0.9808, -0.0185, -0.9104, -0.9870)
(0.4386, 0.0395, 0.0395, -0.9571, -0.0334, -0.1410)	(0.0286, 0.7941, 0.9620, -0.0392, -0.7838, -0.9668)
(0.0239, 0.7432, 0.9596, -0.0479, -0.7228, -0.9783)	(0.3682, 0.0642, 0.0499, -0.9596, -0.0535, -0.0966)
(0.4386, 0.0360, 0.0499, -0.9620, -0.0392, -0.1641)	(0.0332, 0.7298, 0.9571, -0.0421, -0.7298, -0.9783)
(0.0332, 0.7838, 0.9571, -0.0421, -0.7890, -0.9783)	(0.4173, 0.0464, 0.0360, -0.9620, -0.0334, -0.0914)

Result-set 4, in Table 6, the quantitative distances w.r.t $\mathfrak{P}$ and $\mathfrak{N}$ are computed for each for each alternative.

Table 6. The distances of alternative w.r.t ideal solutions.

Alternatives	$D_{mins}(S_i, \mathfrak{P})$	$D_{mins}(S_i, \mathfrak{N})$
S_1	0.387762	0.378713
S_2	0.406064	0.360411
S_3	0.418849	0.347626
S_4	0.403471	0.363004
S_5	0.445369	0.321106
S_6	0.427612	0.338863
S_7	0.354894	0.41158
S_8	0.327378	0.439097

Result-set 5, the calculated inferior ratio, and the related ranking are displayed in Table 7.

Table 7. Ranking of different teaching criteria based on Inferior ratio.

	$\vartheta(i)$	Rank
s_1	0.494097	8
s_2	0.470219	7
s_3	0.453538	1
s_4	0.473602	4
s_5	0.418939	2
s_6	0.442106	3
s_7	0.536978	6
s_8	0.572879	5

Result-set 6, we conducted a comparative analysis against different MCDM approaches namely CoCoSo [16], CODAS [17], and TOPSIS [18], [19], then, we compared their ranking outcomes in Table 8.

Table 8. Results of comparative analysis.

	CoCoSo	Proposed	CODAS	TOPSIS
s_1	1	8	1	1
s_2	2	7	3	4
s_3	6	1	6	5
s_4	5	4	5	3
s_5	8	2	8	8
s_6	4	3	4	6
s_7	7	6	7	7
s_8	3	5	2	2

Result-set 7, we conduct sensitivity analysis by randomly varying the weight of criteria with 10% and 20% of increment and decrement, then, we display the results as shown in Figure 4.

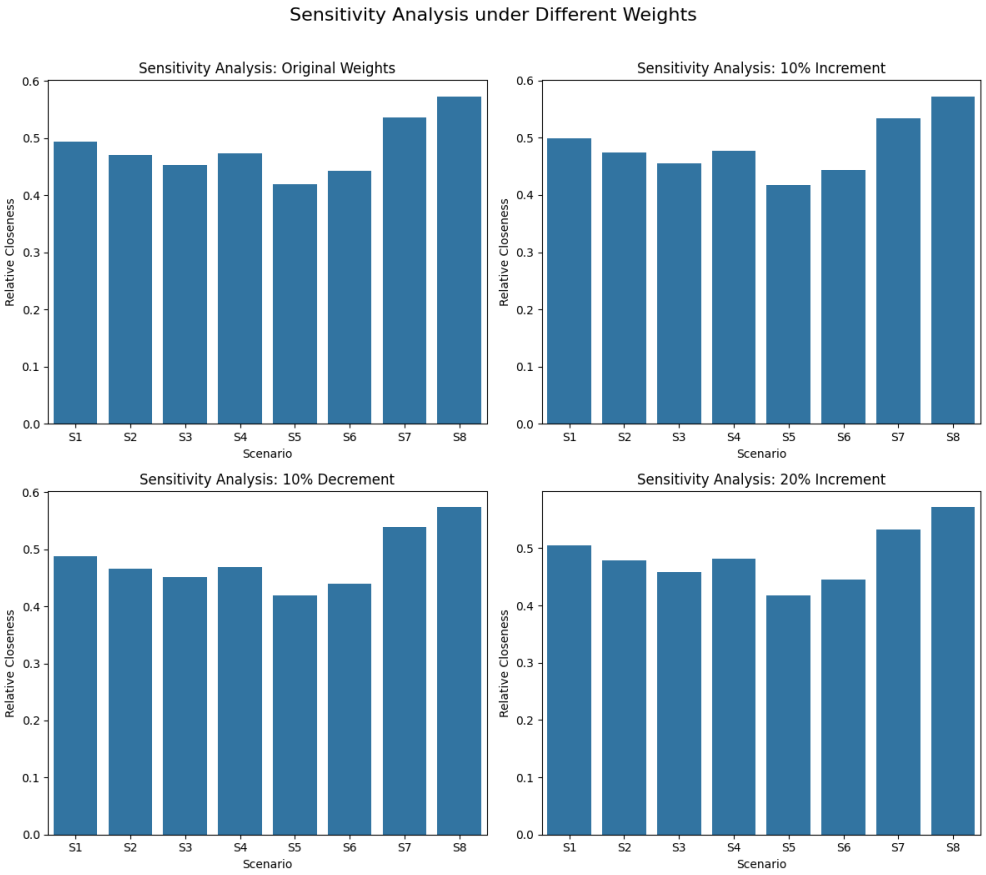

Figure 4. Quantitative results of sensitivity analysis.

5. Conclusions

In conclusion, this study introduces a Bipolar Neutrosophic-driven approach for assessing the teaching quality of English Language and Literature, incorporating a customized AHP-based weighting technique, a novel method for identifying ideal solutions and computing distances within a Bipolar Neutrosophic matrix, and a comprehensive comparative and sensitivity analysis. The proposed model effectively captures uncertainty, hesitation, and contradiction in decision-making, offering a more robust and reliable assessment framework. The results validate the superiority of our approach over traditional methods, highlighting its practical applicability and adaptability in educational evaluation systems.

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Andean Epistemology, Ch'ixi, and Neutrosophic Logic

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Abstract: This article explores the potential of Andean epistemology—rooted in principles of complementarity (yanantin), interdependence, and indeterminacy—to enrich contemporary science and artificial intelligence (AI). By transcending binary logic through concepts like ch'ixi (simultaneous coexistence of opposites) and yanantin, Andean thought aligns with Florentin Smarandache's neutrosophic logic, a system that formalizes truth (T), falsehood (F), and indeterminacy (I) to navigate ambiguity. Smarandache's MultiAlism, which emphasizes the coexistence of multiple perspectives, further resonates with Andean cosmovision, offering a framework to reinterpret mythology and folklore beyond rigid dualities. Integrating these principles into AI could foster systems that better handle uncertainty, cultural context, and ethical complexity. However, current models like GPT-4 exhibit critical limitations: biases toward overestimating truth values (T) and artificially clustering indeterminacy (I ≈ 0.7), flattening nuanced concepts such as yanantin and ch'ixi. Additionally, the low semantic similarity (cosine = 0.17) between AI-generated justifications and original Andean texts reveals gaps in replicating cultural terminology and relational reasoning, underscoring the need for epistemologically grounded AI design. Future work should prioritize expanding datasets with contextualized examples—including regional variations, ritual practices, and neutrosophic scales (T, I, F)—employing advanced language models aligned with Andean epistemological annotations and integrating hybrid post-processing strategies such as bilingual (Quechua/Spanish) fine-tuning and expert collaboration for culturally and technically rigorous validation. By bridging Andean philosophy with Smarandache's neutrosophic frameworks, this research advocates for culturally inclusive AI capable of honoring pluralistic worldviews, fostering equitable knowledge systems, and advancing technologies that serve humanity and the planet holistically.

Keywords: Andean epistemology, neutrosophy, artificial intelligence, ch'ixi, yanantin

1. Introduction

The Aristotelian framework of two-valued logic, which categorizes phenomena into binary oppositions (true/false, existence/non-existence), has profoundly influenced Western scientific and philosophical thought [1]. While this paradigm has enabled systematic advancements, its rigidity often struggles to accommodate the complexity, fluidity, and interdependence inherent in many natural and cultural systems [2]. Contemporary interdisciplinary research increasingly highlights the limitations of binary models, particularly in fields such as quantum physics, ecology, and decolonial studies, where indeterminacy, reciprocity, and hybridity are central [3–5]. This growing

recognition underscores the urgency of exploring alternative logical frameworks that transcend traditional dichotomies.

Non-Western ontologies, such as Andean and Aymara epistemologies, offer rich conceptual tools for rethinking reality through principles of complementarity and dynamic balance. For instance, the Andean concept of ch'ixi—an Aymara term describing the coexistence of opposites without synthesis, akin to a speckled weave of contrasting colors (Figure 1)—challenges static binaries by affirming simultaneous existence and non-existence [6,7]. Similarly, Ayni, a principle of reciprocal interdependence, reframes relationships not as hierarchical dualities but as mutually sustaining interactions [8]. These ideas resonate with modern neutrosophic logic, proposed by Smarandache [9], which introduces a third, indeterminate value (neither true nor false) to formal systems, enabling nuanced modeling of paradox and ambiguity.

Despite these parallels, the integration of Indigenous knowledge systems with formal logic remains underexplored, often hindered by epistemological divides between Western and non-Western traditions [10]. While some scholars argue for the universality of classical logic [11], others advocate for pluralistic frameworks that honor diverse ontological perspectives [12,13]. This study engages this debate by examining how Andean and Aymara epistemologies align with neutrosophic logic, positing that their convergence offers transformative tools for addressing complexity in fields ranging from artificial intelligence to cultural studies [14].

Figure 1. Ch'ixi tapestry illustrating the juxtaposition of contrasting colors, embodying coexistence without synthesis

Authors: Edward Cooper and Bernardo Jarrín

Source: https://www.grafitat.com/2012/08/08/chxi-la-yuxtaposicion-de-los-contrastes/

The aim of this work is twofold: (1) to demonstrate the structural and philosophical synergies between Indigenous South American ontologies and neutrosophic logic, and (2) to illustrate their collective potential for fostering more inclusive, interdisciplinary approaches to knowledge. Through analysis of Andean myths, rituals, and textual sources, we argue that principles like ch'ixi and Ayni prefigure modern logical innovations, emphasizing balance and indeterminacy over exclusionary binaries. Our findings suggest that bridging these traditions not only enriches logical theory but also supports decolonial efforts to diversify epistemological paradigms in academia.

2. Preliminaries Section

2.1 Andean Epistemology and Neutrosophic Logic

This article explores the implications of Andean epistemology—embedded in the cultural and philosophical traditions of Indigenous Andean peoples—for contemporary logical systems. Unlike Western frameworks grounded in Aristotelian binary logic [1], Andean thought transcends the "being/non-being" dichotomy, emphasizing principles of complementarity (yanantin) and coexistence (ch'ixi) [2]. This perspective interprets reality as a dynamic interplay of opposites rather than mutually exclusive categories. The Andean cultural area (Figure 2), which aligns with the territorial expanse of the Inca Empire, spans modern-day Colombia, Ecuador, Peru, Bolivia, Chile, and Argentina [15], underscoring the broad influence of its epistemological principles.

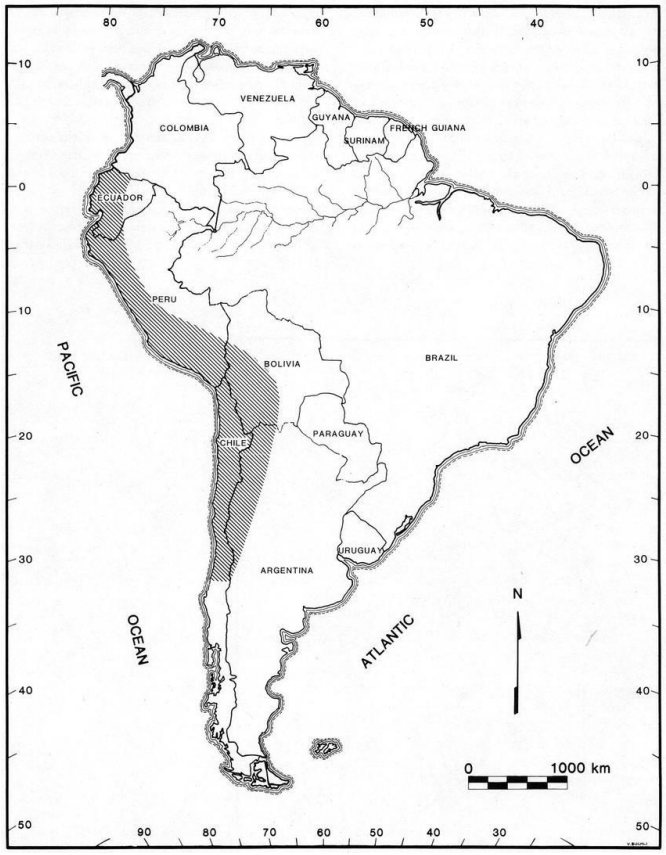

Figure 2. Representation of the Andean cultural area [15].

Central to Andean epistemology is the Aymara concept of ch'ixi—a state where opposites coexist without synthesis [6]. This principle aligns with neutrosophic logic [16, 17], which introduces a third value (indeterminacy) to model complex realities. Mathematically, it is expressed as:

$$\boldsymbol{N(x) = (T_x, I_x, F_x)} \quad \textbf{(1)}$$

where $\boldsymbol{T_x}, \boldsymbol{I_x}$, and $\boldsymbol{F_x}$ represent degrees of truth, indeterminacy, and falsehood, respectively.

Yanantin, another pillar of Andean cosmology, asserts that opposites (e.g., hanan/hurin, masculine/feminine) are not contradictory but complementary [18, 19]. This principle rejects hierarchies and manifests in rituals and social practices that balance dualities [20-22]. MultiAlism—an extension of neutrosophic logic—formalizes this interdependence through:

$$\langle(\boldsymbol{multi})\boldsymbol{A}\rangle + \langle(\boldsymbol{multi})\boldsymbol{neutA}\rangle + \langle(\boldsymbol{multi})\boldsymbol{antiA}\rangle = \infty \quad (2)$$

capturing the dynamic coexistence of affirmative, neutral, and opposing elements.

Andean myths do not perceive the world in absolute terms of good and evil; instead, they emphasize balance and reciprocity, known as Ayni. This perspective aligns with neutrosophy, which acknowledges the coexistence of truth, falsehood, and indeterminacy, offering a more inclusive and holistic understanding of the universe. In Andean culture, rituals often integrate elements that blend the sacred and the profane, the human and the divine. Neutrosophy suggests that such practices reflect humanity's intrinsic need to navigate uncertainty and ambiguity through hybrid narratives and cultural expressions [23].

2.2 Relationship Between Andean Epistemology and Artificial Intelligence

Artificial intelligence (AI) struggles with ambiguity and cultural contexts due to its reliance on binary logic [1]. Andean epistemology in conjunction with neutrosophic logic offers tools to address these limitations:

- Ch'ixi and AI: By acknowledging simultaneous truths, ch'ixi could enhance natural language processing in intercultural settings.
- Ayni (Reciprocity): This principle emphasizes contextual relationships, urging AI models to account for interdependence in predictive analytics.

Table 1. Biases Due to the Exclusion of Andean Epistemology

Bias	Impact
Binary Logic	Oversimplifies decisions in culturally complex contexts.
Lack of Contextualization	Ignores social dynamics in Indigenous communities.
Exclusion of Non-Western Knowledge	Limits epistemological diversity in AI.
Inability to Manage Uncertainty	Errors in fields like medicine or justice.

Integrating Andean frameworks into AI fosters more inclusive and ethical systems. For example, Ayni-informed models could prioritize communal balance over individualistic optimization.

2. Materials and Methods

Dataset

This study utilized a dataset of 20 Andean philosophical statements, each evaluated using a neutrosophic framework (Truth T, Indeterminacy I, Falsity F) with academic/cultural justifications. The dataset was derived from a combination of expert interviews and a comprehensive literature review. A structured overview of the dataset is presented in Table 2.

Table 2. Dataset Excerpt.

ID	Andean Philosophical Statement (Spanish)	T	I	F	Unified Justification (Key Points)
1	"El ch'ixi representa la coexistencia de lo indígena y lo colonial sin síntesis."	0.8	0.3	0.1	T: Rivera Cusicanqui (2012). I: Ambiguity in urban contexts. F: Rare critiques.
2	El yanantin solo se aplica a la dualidad hombre-mujer en comunidades aymaras	0.4	0.1	0.6	T: Gender studies (Arnold & Yapita, 2006). F: Yanantin extends to ecology (Estermann, 2006).
3	"El ayni es una práctica obsoleta en las ciudades modernas andinas."	0.2	0.5	0.8	T: Symbolic ayni persists (Canessa, 2012). F: Thrives in informal networks (Stobart, 2020).
4	"Los rituales de la Pachamama garantizan cosechas abundantes científicamente."	0.3	0.6	0.5	T: Ethnobotanical validity (Bastien, 1987). F: No direct scientific correlation.
5	"El concepto de suma qamaña (vivir bien) rechaza el desarrollo económico."	0.5	0.4	0.4	T: Critiques developmentalism (Huanacuni, 2010). F: Hybrid models exist.
…	*.[Truncated for brevity]*	…	…	…	…

Each statement was evaluated using the following prompts with the GPT-4 model [24]:

- Truth Value: "Evaluate the following statement: [statement]. Based on academic sources or documented cultural evidence, assign a truth value between 0 and 1, where 0 indicates the statement is completely false and 1 indicates it is completely true. Provide only the numerical value without any further explanation."
- Indeterminacy Value: "Assign a numerical indeterminacy value between 0 and 1 to the following statement: [statement], where 0 represents absolute certainty and 1 indicates maximum uncertainty, based on the existence of academic debates, cultural variations, and availability of documented evidence. Return only the numerical value."
- Justification: "Analyze the following statement: [statement]. Assign numerical values between 0 and 1 for the following criteria:

  T (Truth): Indicate to what extent the statement is true according to academic sources or documented cultural evidence.

  I (Indeterminacy): Evaluate the degree of ambiguity or lack of clarity in the statement.

  F (Falsehood): Determine to what extent the statement is false according to academic sources or documented cultural evidence.

  Provide a detailed justification for each assigned value, citing relevant academic or cultural references."

The resulting dataset was analyzed using a Python script. The script performed the following analyses:

- Correlation Analysis: Spearman's rank correlation coefficient [25] was calculated to assess the relationships between the truth (T), indeterminacy (I), and falsehood (F) values assigned by humans and GPT-4[24].
- Regression Analysis: Mean absolute error (MAE), mean squared error (MSE), and R-squared (R2) were calculated to evaluate the agreement between human and GPT-4 assigned values [26].
- Distribution Analysis: Kernel density estimation plots were used to visualize the distributions of human-assigned and GPT-4 assigned values.
- Principal Component Analysis (PCA) and Clustering: PCA was used to reduce the dimensionality of the data, and K-means clustering was performed to identify potential groupings of statements [27].
- Semantic Similarity Analysis: Sentence embeddings were generated using the 'paraphrase-MiniLM-L6-v2' model [28] to assess the semantic similarity between human-generated and GPT-4 generated justifications. Cosine similarity was used to quantify the similarity [29].

3. Results

The resulting data is obtained using GPT-4 model, an excerpt is shown below (Table 3)

Table 3. Results of the application of GPT-4 model

ID	GPT4_T	GPT4_I	GPT4_F	GPT4 Justification (Key Points)
1	0,9	0,2	0,1	T=0.9: Theory validated. I=0.2: Clear concept. F=0.1: Academic consensus.
2	0,3	0,2	0,7	T=0.3: Overly narrow scope. F=0.7: Broader cultural application (Classen, 1993).
3	0,3	0,2	0,7	F=0.7: Adapts to urban contexts.
4	0,1	0,2	0,9	F=0.9: No empirical evidence.
5	0,7	0,3	0,3	T=0.7: Critique of Western models (Gudynas, 2011).
…	*[Truncated for brevity]*	…	…	…

Analysis of Data Distribution

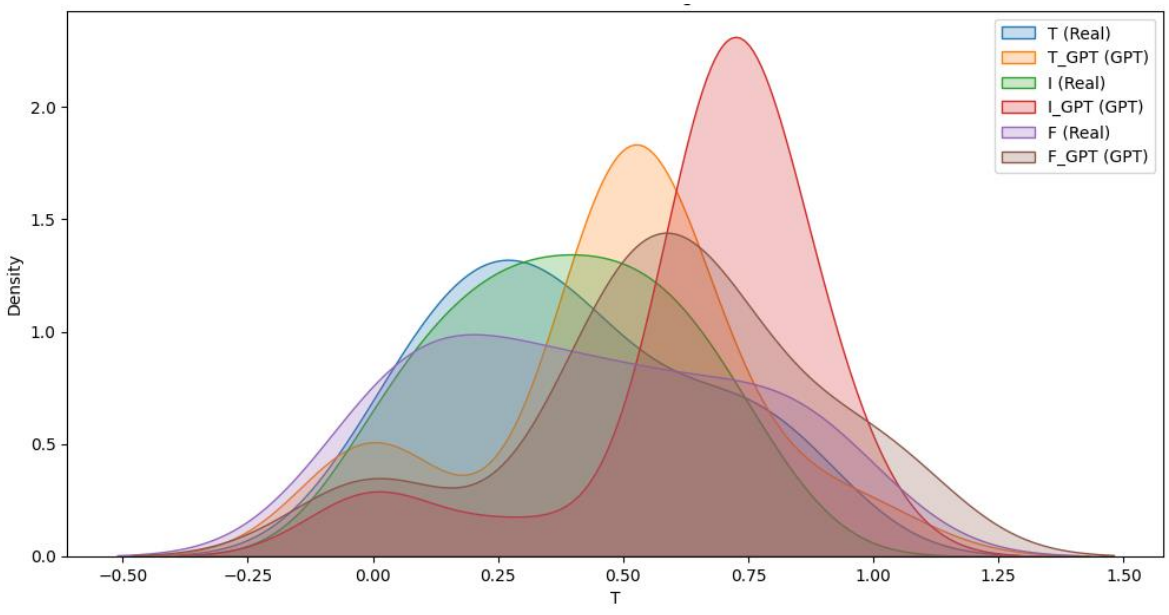

Figure 3. Distribution of Real vs. GPT Variables

The comparative histogram analysis of real and GPT-generated neutrosophic scores (Truth T, Indeterminacy I, Falsity F) reveals both alignment and divergence in value distributions. While certain ranges exhibit similarity, systematic discrepancies are evident. For Truth (T), GPT-generated scores skew toward higher values (e.g., clustering near 0.8–1.0) compared to the original dataset, which displays a broader spread across moderate ranges. In Indeterminacy (I), GPT outputs concentrate heavily around 0.7, contrasting with the real data's more dispersed distribution (0.3–0.9). Similarly, Falsity (F) scores generated by GPT show a propensity for extreme values (near 1.0), whereas human-assigned scores are more evenly distributed. These patterns suggest GPT's tendency to amplify certainty (high T) and ambiguity (clustered I) while overemphasizing definitive falsehoods (extreme F), deviating from the nuanced variability observed in culturally contextualized human evaluations.

Interpretation of PCA and Clustering Results

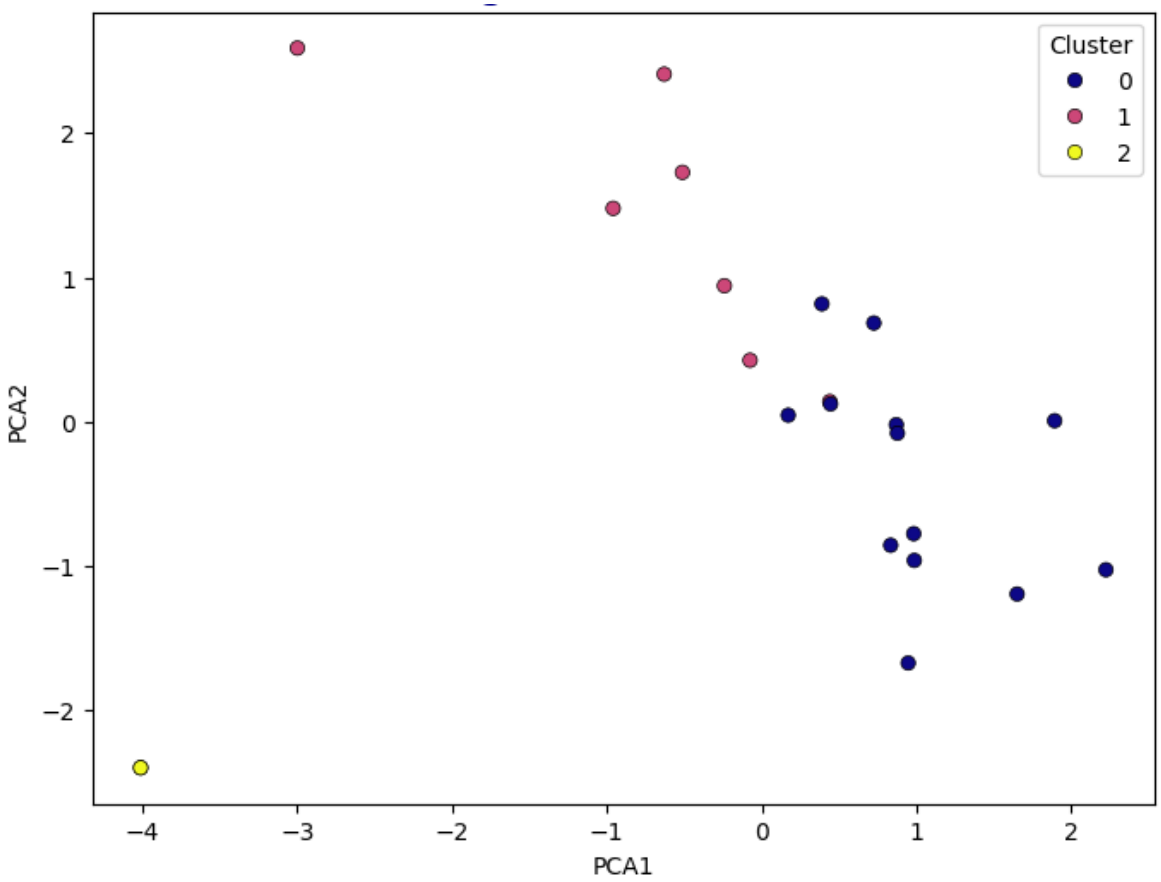

Figure 4. K-Means Clustering on PCA Data After Final Cleaning

The application of clustering techniques and PCA reduction to the neutrosophic scores (T, I, F) revealed three distinct clusters, visually differentiated by color in the projected space. These clusters suggest shared patterns between GPT-generated and original human-assigned values, implying partial alignment in how both approaches categorize Andean philosophical statements. Notably, the clustered points reflect overlapping similarities in specific subsets of statements, where GPT's outputs approximate the original dataset's structure. However, scattered points outside the primary clusters highlight statements with unique characteristics that resist straightforward classification, underscoring inherent complexities in modeling culturally nuanced concepts. The clear separation of clusters in the reduced PCA space further demonstrates that both original and GPT-generated

variables contribute meaningful information for segmenting the statements, albeit with divergent emphases.

While GPT partially replicates the latent structure of the original data, the persistence of multiple distinct clusters signals significant divergences in value assignments. The compact grouping of certain statements in the PCA projection suggests that GPT generates more consistent scores for specific thematic or conceptual categories (e.g., widely recognized principles like Ayni), indicating areas where the model aligns closely with human judgment. Conversely, the scattered outliers reveal limitations in GPT's ability to capture idiosyncratic or context-dependent aspects of Andean philosophy, such as fluid dualities or regionally specific practices. These findings underscore the need for targeted model refinement, including training on culturally annotated datasets and integrating domain-specific constraints—to improve coherence in handling unique or ambiguous statements. Future work could leverage hybrid frameworks, combining AI-generated outputs with expert validation, to better preserve the epistemological diversity of Indigenous knowledge systems.

Correlation Analysis

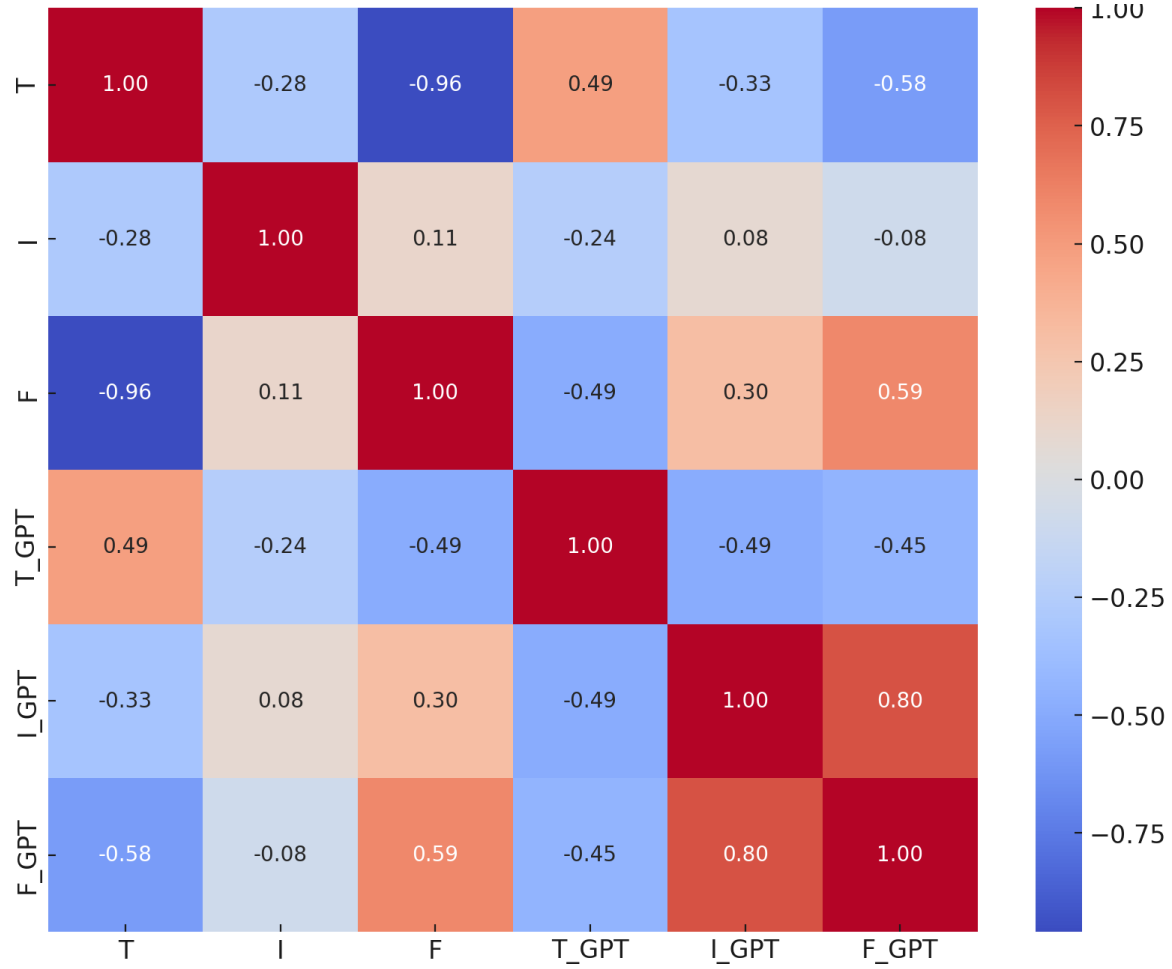

Figure 5. Correlation Between Original and GPT-Generated Variables

The Spearman correlation matrix reveals critical insights into the alignment between GPT-generated and human-assigned neutrosophic scores (Truth T, Indeterminacy I, Falsity F). Strong correlations dominate expected relationships, such as the near-perfect inverse link between real truth (T) and real falsehood (F) (ϱ = -0.96), confirming logical consistency in the original dataset.

However, GPT exhibits notable biases: a strong positive correlation between its indeterminacy (I_GPT) and falsehood (F_GPT) scores (ϱ = +0.80) suggests an artificial conflation of uncertainty and falsity, diverging from human interpretations. Moderate correlations, such as between real and GPT-generated falsehood (FFF vs. F_GPT, ϱ = +0.59) and partial alignment in truth (T vs. T_GPT, ϱ = +0.48), indicate limited fidelity in replicating nuanced judgments. Weak correlations, particularly between real and GPT indeterminacy (I vs. IGPT, ϱ = +0.08), highlight GPT's inability to interpret uncertainty in culturally contextualized frameworks.

The analysis underscores systemic gaps in GPT's capacity to emulate human-like reasoning within Andean philosophical contexts. The model's near-negligible correlation with indeterminacy values and its skewed association of falsehood with uncertainty reflect oversimplified probabilistic assumptions, potentially erasing culturally specific interpretations of ambiguity (e.g., Yanantin dualities). Additionally, the inverse relationship between real falsehood (F) and GPT truth (T_GPT, ϱ = -0.49) reveals partial adherence to logical truth-falsehood dichotomies but lacks full coherence. Coupled with low cosine similarity in textual justifications—indicating stylistic mismatches—these findings emphasize the need for domain-specific fine-tuning. Enhancing GPT's cultural fidelity may require training on annotated datasets that explicitly encode indeterminacy and contextual falsehood, alongside hybrid frameworks integrating expert validation to bridge epistemic gaps in Indigenous knowledge representation.

Semantic Similarity Analysis

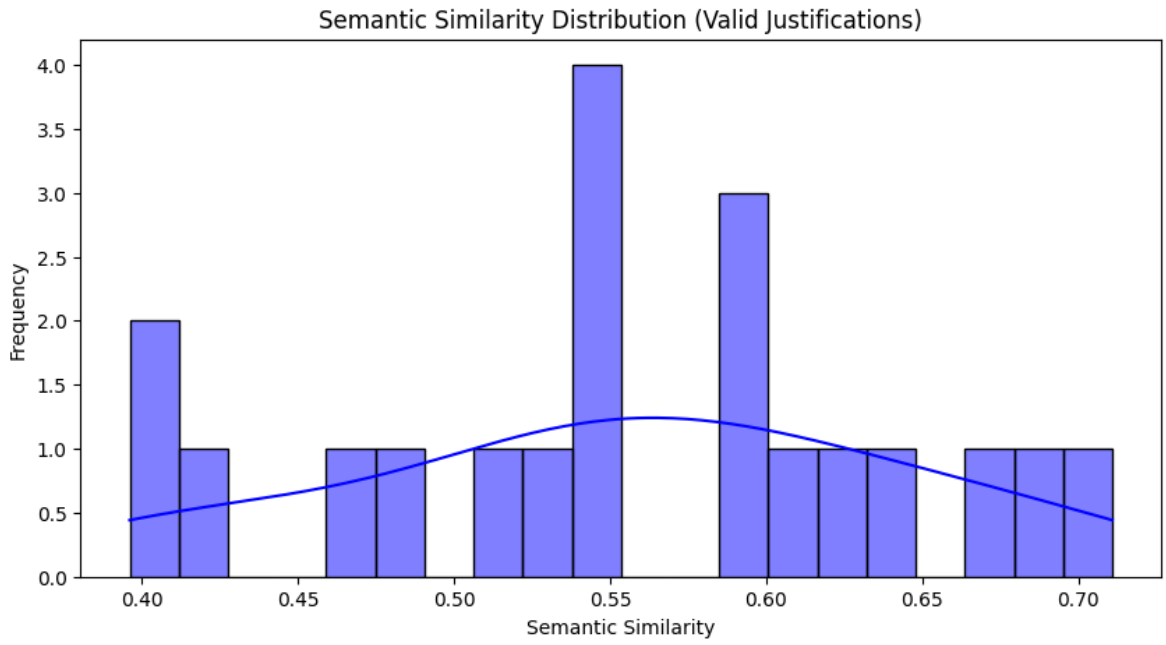

Figure 6. Distribution of Semantic Similarity

The semantic comparison via cosine similarity between GPT-generated and original justifications reveals moderate alignment, with an average similarity score of 0.5558 after filtering invalid responses (2 out of 22 removed). This intermediate value suggests that while GPT captures some aspects of the original justifications, there remain notable differences in expression and content depth when addressing Andean philosophical concepts. The distribution analysis demonstrates that GPT's outputs maintain a baseline level of semantic coherence, though with varying degrees of success in replicating the nuanced cultural content.

The semantic similarity scores span a range of 0.40 to 0.71, with a bimodal distribution showing peaks around 0.55 and 0.60, as evidenced in the histogram. The highest similarity score (0.71) was achieved in responses discussing the suma qamaña concept, indicating GPT's stronger performance when engaging with well-defined philosophical principles. The lower range (0.40-0.45) typically corresponds to cases where GPT defaulted to formulaic analytical frameworks (e.g., "Para poder asignar valores a la afirmación..."), rather than engaging directly with the cultural content. The clustering of scores in the 0.55-0.65 range suggests that GPT maintains consistent but moderate fidelity to the original justifications' semantic content.

The analysis reveals two key patterns: (1) GPT achieves higher similarity scores (0.65-0.71) when discussing specific cultural concepts or cited works, as seen in the top-performing justifications about suma qamaña and Silvia Rivera Cusicanqui's work, and (2) lower scores (0.40-0.48) correlate with responses that adopt a generic analytical framework rather than engaging with the specific cultural context. The bimodal distribution visible in the histogram suggests two distinct response patterns: one maintaining closer semantic alignment with cultural content (higher peak) and another falling back on more generic analytical structures (lower peak). These findings indicate that while GPT can engage meaningfully with Andean philosophical concepts, its performance varies significantly based on the specificity of the cultural content and its ability to avoid defaulting to generic analytical frameworks.

To enhance GPT-4's alignment with Andean philosophical concepts, targeted strategies are recommended: additional training with culturally specific examples to deepen contextual understanding, refined generation prompts emphasizing replication of the original dataset's style and content, and post-processing techniques like paraphrasing to improve semantic similarity. While GPT-4 demonstrates the potential for analyzing Andean epistemology, critical gaps persist, including inconsistent handling of indeterminacy, limited semantic fidelity in justifications, and biases in truth-value assignments. These limitations underscore the need for domain-specific fine-tuning, integration of cultural expertise, and hybrid validation frameworks to bridge algorithmic outputs with Indigenous knowledge systems. Future research should prioritize collaborative approaches with local scholars to ensure both technical and epistemic accuracy in cross-cultural computational analyses [30].

The study presents key limitations, such as the use of a small dataset (20 Andean philosophical statements), which restricts the generalization of findings, and the reliance on GPT-4. To address these aspects, the following improvements are recommended: (1) expand the dataset with more contextualized examples, including regional variations and ritual practices; (2) utilize more recent language models and refine prompts with Andean epistemological annotations; (3) integrate post-processing techniques (e.g., fine-tuning with Quechua/Spanish bilingual corpora) and implement [31] technical and cultural accuracy in representing uncertainty and complementary dualities.

5. Conclusions

This study has explored the profound interconnection between Andean epistemology and neutrosophic logic, demonstrating its potential to enrich science and artificial intelligence (AI). The ability of neutrosophy to encompass truth, falsehood, and indeterminacy makes it an exceptional tool for capturing the complexities of Andean thought, which transcends traditional binary logic by embracing the complementarity of opposites, interdependence, and coexistence. However, key limitations emerged in the use of GPT-4, including its biases toward overestimating truth values (T) and artificially clustering indeterminacy ($I \approx 0.7$), which oversimplify nuanced concepts such as yanantin (complementarity) and ch'ixi (simultaneous coexistence). Furthermore, the low average semantic similarity (cosine = 0.17) between AI-generated and original justifications highlights the

model's current limitations in replicating cultural nuances and domain-specific terminology, underscoring the need for culturally grounded AI frameworks.

Andean epistemology, with its emphasis on reciprocity and relationality, offers an invaluable perspective for designing ethical and responsible AI. Incorporating concepts such as Ayni (reciprocity) and Yanantin (complementarity) allows us to develop systems that consider not only individual data points but also the relationships and contexts in which they are generated, promoting social justice and sustainability. To advance this integration, future work should prioritize:

- Expanding the dataset with contextualized examples, including regional variations and ritual practices, to enhance cultural representativeness.
- Leveraging newer language models and refining prompts with Andean epistemological annotations to better align AI outputs with Indigenous ontologies.
- Integrating post-processing techniques, such as fine-tuning with Quechua/Spanish bilingual corpora, and implementing hybrid validation through collaboration with Andean worldview experts, ensuring both technical rigor and cultural fidelity in representing uncertainty and complementary dualities.

By addressing these challenges, AI systems can evolve to better reflect the richness of Andean cosmovision, fostering a more inclusive technological future where technology serves humanity and the planet holistically

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Inverse fractional function setting sine trigonometric neutrosophic set approach to interaction aggregating operators

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Abstract. In this paper we present novel techniques for the interacting aggregating operator of the inverse fractional function sine trigonometric neutrosophic set. Swapping the input and output variables and solving for the original input variable in terms of the original output variable are the steps involved in determining the inverse of a function. The innovative averaging and geometric operations of inverse fractional function sine trigonometric neutrosophic numbers are studied using the universal aggregation function. The inverse fractional function sine trigonometric neutrosophic set is idempotent, boundedness compatible, associative and commutative. Four new aggregating operators are introduced: inverse fractional function sine trigonometric neutrosophic weighted averaging, inverse fractional function sine trigonometric neutrosophic weighted geometric, generalized inverse fractional function sine trigonometric neutrosophic weighted averaging, and generalized inverse fractional function sine trigonometric neutrosophic weighted geometric. The aggregation functions are frequently thought to be represented by the Euclidean distance, Hamming distance and score values.

Keywords: weighted averaging, weighted geometric, generalized weighted averaging, generalized weighted geometric.

1. Introduction

The uncertainties have led to the development of the fuzzy set (FS) [1], intuitionistic FS (IFS) [2], Pythagorean FS (PFS) [3,4], neutrosophic set (NSS) [5], and Fermatean FS (FFS) [6]. For decision-makers, Zadeh's FS [1] suggests a membership value (MV). An IFS notion was introduced by Atanassov [2] because each object has MV τ and non-membership value (NMV)

ψ and meets $0 \leq \tau + \psi \leq 1$, for $\tau, \psi \in [0, 1]$. Yager introduced the condition that $\tau^2 + \psi^2 \leq 1$, which states that PFSs are defined by their MVs and NMVs [3]. Both PFSs and IFSs have been widely utilized and investigated in a wide range of fields. Cuong and associates [7] by developing the concept of picture FSs (PFSs). PFSs has been found to accommodate several more ambiguity because it is an enhanced version of IFSs. PFSs remark that the MV τ, neutral ψ, and NMV α have $0 \leq \tau+\psi+\alpha \leq 1$; for $\tau, \psi, \alpha \in [0, 1]$. "Yes," "abstain," "no," and "refusal" are expert opinion messages that will be ensured to be transmitted via the PFS. It will also ensure consistency between the assessment data and the actual decision environment and avoid evaluation information from being left out. The idea has not been fully explored, despite the fact that PiFSs have several uses and studies. The spherical FS (SFS) was defined by Shahzaib et al. [8] for certain AOs with MADM. An alternative to $0 \leq \tau + \psi + \alpha \leq 1$ is required by the SFS: $0 \leq \tau^2 + \psi^2 + \alpha^2 \leq 1$. Various algebraic structures and aggregation techniques with applications were studied by Palanikumar et al. [10–12]. In MADM problems, the linguistic SFS AOs were discussed by Jin et al. [9]. SFSs and their applications were introduced by Rafiq et al. in DM [13] and $\tau^2 + \psi^2 \geq 1$ and decision-making (DM) are troublesome. The concept of an FFS was first proposed by Senapati and associates [6] in 2019. It is a feature of both the MV and NMV that $0 \leq \tau^3 + \psi^3 \leq 1$. The following contributions are made as a result of this endeavor: Idempotency, commutativity, and associativity are only a few of the many characteristics of algebra that have been shown. IFFSTNs have two features: HD and ED. The purpose of this method is to calculate the ED distance between two IFFSTNs. Ibraheem et al. [14] discussed the concept of complex NSSs using various AOs. Al-Husban et al. [15] deals that Type-I extension Diophantine IVNS. Recently, Udhayakumar et al. discussed the many fuzzy applications and its generalization [16–20]. Below are the major objectives of this work:

(1) Some basic algebraic structure satisfied the IFFSTNN aggregation operators.
(2) We appeal to the weighted operator using IFFSTNNWA, IFFSTNNWG, GIFFSTNNWA, and GIFFSTNNWG.

2. **Basic concepts**

Definition 2.1. [3] Let L be the universe set. The PFS $\Xi = \left\{x, \langle \tau\{x\}, \chi(x)\rangle | x \in L\right\}$, where $\tau, \chi : L \to (0, 1)$ refers the MV and NMV of $x \in L$ to Ξ, respectively and $0 \leq (\tau(x))^2 + (\chi(x))^2 \leq 1$. For, $\Xi = \langle \tau, \chi \rangle$ is called the Pythagorean fuzzy number (PFN).

Definition 2.2. [6] A FFS $\Xi = \left\{x, \langle \tau\{x\}, \chi\{x\}\rangle | x \in L\right\}$, where $\tau\{x\}$ and $\chi\{x\}$ denote MV and NMV of u respectively, where $\tau, \chi : L \to \{0, 1\}$ and $0 \leq \{\tau\{x\}\}^3 + \{\chi\{x\}\}^3 \leq 1$. Here, $\Xi = \langle \tau, \chi \rangle$ is represent a Fermatean fuzzy number {IFFN}.

Definition 2.3. For any PFNs, $\Xi = \langle \tau, \chi \rangle$, $\Xi_1 = \langle \tau_1, \chi_1 \rangle$ and $\Xi_2 = \langle \tau_2, \chi_2 \rangle$, τ, χ denote MV and NMV of u respectively. Then

(1) $\Xi_1 \bigsqcup \Xi_2 = \left[\sqrt{\{\tau_1\}^2 + \{\tau_2\}^2 - \{\tau_1\}^2 \cdot \{\tau_2\}^2}, \{\chi_1 \cdot \chi_2\} \right]$
(2) $\Xi_1 \divideontimes \Xi_2 = \left[\{\tau_1 \cdot \tau_2\}, \sqrt{\{\chi_1\}^2 + \{\chi_2\}^2 - \{\chi_1\}^2 \cdot \{\chi_2\}^2} \right]$
(3) $\aleph \cdot \Xi = \left[\sqrt{\imath - \left\{\imath - \{\tau\}^2\right\}^{\aleph}}, \{\chi\}^{\aleph} \right]$
(4) $\Xi^{\aleph} = \left[\{\tau\}^{\aleph}, \sqrt{\imath - \left\{\imath - \{\chi\}^2\right\}^{\aleph}} \right]$

Definition 2.4. If $\Xi_1 = \langle \tau_1, \chi_1 \rangle$ and $\Xi_2 = \langle \tau_2, \chi_2 \rangle$ are any two PFNs. Then the interaction AO is defined as

(1) $\Xi_1 \bigsqcup \Xi_2 = \begin{bmatrix} \sqrt{\{\tau_1\}^2 + \{\tau_2\}^2 - \{\tau_1\}^2 \cdot \{\tau_2\}^2}, \\ \sqrt{\{\chi_1\}^2 + \{\chi_2\}^2 - \{\chi_1\}^2 \cdot \{\chi_2\}^2 - \{\chi_1\}^2 \cdot \{\tau_2\}^2 - \{\tau_1\}^2 \cdot \{\chi_2\}^2} \end{bmatrix}$
(2) $\Xi_1 \divideontimes \Xi_2 = \begin{bmatrix} \sqrt{\{\tau_1\}^2 + \{\tau_2\}^2 - \{\tau_1\}^2 \cdot \{\tau_2\}^2 - \{\tau_1\}^2 \cdot \{\chi_2\}^2 - \{\chi_1\}^2 \cdot \{\tau_2\}^2}, \\ \sqrt{\{\chi_1\}^2 + \{\chi_2\}^2 - \{\chi_1\}^2 \cdot \{\chi_2\}^2} \end{bmatrix}$
(3) $\aleph \cdot \Xi_1 = \left[\sqrt{\imath - \left\{\imath - \{\tau_1\}^2\right\}^{\aleph}}, \sqrt{\left\{\imath - \{\tau_1\}^2\right\}^{\aleph} - \left\{\imath - \{\tau_1 + \chi_1\}^2\right\}^{\aleph}} \right]$
(4) $\Xi_1^{\aleph} = \left[\sqrt{\left\{\imath - \{\chi_1\}^2\right\}^{\aleph} - \left\{\imath - \{\tau_1 + \chi_1\}^2\right\}^{\aleph}}, \sqrt{\imath - \left\{\imath - \{\chi_1\}^2\right\}^{\aleph}} \right]$

where $\aleph$ be a positive integers.

3. Different AOs for IFFSTNN

Another way to define a fractional part function is as the difference between a real number and its greatest integer value, which is found using the greatest integer function. For example, if ϵ is a fractional part function, then the fractional part of x is expressed as $\epsilon\{x\} = \{x\} = x - \lfloor x \rfloor$, where x is a real number. In the event that x is an integer, its fractional component equals 0. Therefore, x cannot be an integer in order for $\epsilon\{x\} = \frac{1}{x}$ to be defined. Therefore, all real numbers, with the exception of integers, are included in the domain of $\epsilon\{x\} = \frac{1}{x}$. Here, $\theta = \sin \pi/2$.

Definition 3.1. Suppose that $\Xi_1 = \langle \theta\tau_1, \theta\omega_1, \theta\chi_1 \rangle$ and $\Xi_2 = \langle \theta\tau_2, \theta\omega_2, \theta\chi_2 \rangle$ be the any two IFFSTNNs. Then

(1) $\Xi_1 \bigsqcup \Xi_2 = \begin{bmatrix} \sqrt[\frac{1}{\epsilon}]{\{\theta\tau_1\}^{\frac{1}{\epsilon}} + \{\theta\tau_2\}^{\frac{1}{\epsilon}} - \{\theta\tau_1\}^{\frac{1}{\epsilon}} \cdot \{\theta\tau_2\}^{\frac{1}{\epsilon}}}, \\ \sqrt[\frac{1}{\epsilon}]{\{\theta\omega_1\}^{\frac{1}{\epsilon}} + \{\theta\omega_2\}^{\frac{1}{\epsilon}} - \{\theta\omega_1\}^{\frac{1}{\epsilon}} \cdot \{\theta\omega_2\}^{\frac{1}{\epsilon}}}, \\ \sqrt[\frac{1}{\epsilon}]{\begin{matrix} \{\theta\chi_1\}^{\frac{1}{\epsilon}} + \{\theta\chi_2\}^{\frac{1}{\epsilon}} - \{\theta\chi_1\}^{\frac{1}{\epsilon}} \cdot \{\theta\chi_2\}^{\frac{1}{\epsilon}} \\ -\{\theta\chi_1\}^{\frac{1}{\epsilon}} \cdot \{\theta\tau_2\}^{\frac{1}{\epsilon}} - \{\theta\tau_1\}^{\frac{1}{\epsilon}} \cdot \{\theta\chi_2\}^{\frac{1}{\epsilon}} \end{matrix}} \end{bmatrix}$

$$(2)\ \Xi_1 \divideontimes \Xi_2 = \begin{bmatrix} \sqrt[\frac{1}{\epsilon}]{\{\theta\tau_1\}^{\frac{1}{\epsilon}} + \{\theta\tau_2\}^{\frac{1}{\epsilon}} - \{\theta\tau_1\}^{\frac{1}{\epsilon}} \cdot \{\theta\tau_2\}^{\frac{1}{\epsilon}}} \\ -\{\theta\tau_1\}^{\frac{1}{\epsilon}} \cdot \{\theta\chi_2\}^{\frac{1}{\epsilon}} - \{\theta\chi_1\}^{\frac{1}{\epsilon}} \cdot \{\theta\tau_2\}^{\frac{1}{\epsilon}}, \\ \sqrt[\frac{1}{\epsilon}]{\{\theta\omega_1\}^{\frac{1}{\epsilon}} + \{\theta\omega_2\}^{\frac{1}{\epsilon}} - \{\theta\omega_1\}^{\frac{1}{\epsilon}} \cdot \{\theta\omega_2\}^{\frac{1}{\epsilon}}} \\ \sqrt[\frac{1}{\epsilon}]{\{\theta\chi_1\}^{\frac{1}{\epsilon}} + \{\theta\chi_2\}^{\frac{1}{\epsilon}} - \{\theta\chi_1\}^{\frac{1}{\epsilon}} \cdot \{\theta\chi_2\}^{\frac{1}{\epsilon}}} \end{bmatrix}$$

$$(3)\ \aleph \cdot \Xi_1 = \begin{bmatrix} \sqrt[\frac{1}{\epsilon}]{\imath - \left\{\imath - \{\theta\tau_1\}^{\frac{1}{\epsilon}}\right\}^{\aleph}},\ \sqrt[\frac{1}{\epsilon}]{\imath - \left\{\imath - \{\theta\omega_1\}^{\frac{1}{\epsilon}}\right\}^{\aleph}}, \\ \sqrt[\frac{1}{\epsilon}]{\left\{\imath - \{\theta\tau_1\}^{\frac{1}{\epsilon}}\right\}^{\aleph} - \left\{\imath - \{\theta\tau_1 + \theta\chi_1\}^{\frac{1}{\epsilon}}\right\}^{\aleph}} \end{bmatrix}$$

$$(4)\ \Xi_1^{\aleph} = \begin{bmatrix} \sqrt[\frac{1}{\epsilon}]{\left\{\imath - \{\theta\chi_1\}^{\frac{1}{\epsilon}}\right\}^{\aleph} - \left\{\imath - \{\theta\tau_1 + \theta\chi_1\}^{\frac{1}{\epsilon}}\right\}^{\aleph}} \\ \sqrt[\frac{1}{\epsilon}]{\imath - \left\{\imath - \{\theta\omega_1\}^{\frac{1}{\epsilon}}\right\}^{\aleph}},\ \sqrt[\frac{1}{\epsilon}]{\imath - \left\{\imath - \{\theta\chi_1\}^{\frac{1}{\epsilon}}\right\}^{\aleph}} \end{bmatrix}$$

3.1. *IFFSTNWA operator*

Definition 3.2. Let $\Xi_\jmath = \langle \theta\tau_\jmath, \theta\omega_\jmath, \theta\chi_\jmath \rangle$ be the IFFSTNNs, $\jmath = 1, 2, ..., \ell$, $\psi_\jmath$ be the weight of $\Xi_\jmath$ and $\psi_\jmath \geq 0$, $\varnothing_{\jmath=1}^{k} \psi_\jmath = 1$. Then the IFFSTNWA operator $\{\Xi_1, \Xi_2, ..., \Xi_\ell\} = \varnothing_{\jmath=1}^{k} \psi_\jmath \Xi_\jmath$.

Theorem 3.3. *Let $\Xi_\jmath = \langle \theta\tau_\jmath, \theta\omega_\jmath, \theta\chi_\jmath \rangle$ be the IFFSTNNs, $\jmath = 1, 2, ..., \ell$. Then,*

$$IFFSTNWA\{\Xi_1, \Xi_2, ..., \Xi_\ell\} = \begin{bmatrix} \sqrt[\frac{1}{\epsilon}]{\imath - \Diamond_{\jmath=1}^{k} \left\{\imath - \{\theta\tau_\jmath\}^{\frac{1}{\epsilon}}\right\}^{\psi_\jmath}},\ \sqrt[\frac{1}{\epsilon}]{\imath - \Diamond_{\jmath=1}^{k} \left\{\imath - \{\theta\omega_\jmath\}^{\frac{1}{\epsilon}}\right\}^{\psi_\jmath}}, \\ \sqrt[\frac{1}{\epsilon}]{\Diamond_{\jmath=1}^{k} \left\{\imath - \{\theta\tau_\jmath\}^{\frac{1}{\epsilon}}\right\}^{\psi_\jmath} - \Diamond_{\jmath=1}^{k} \left\{\imath - \{\theta\tau_\jmath + \theta\chi_\jmath\}^{\frac{1}{\epsilon}}\right\}^{\psi_\jmath}} \end{bmatrix}.$$

Proof. If $\jmath = 2$, IFFSTNWA$\{\Xi_1, \Xi_2\} = \psi_1\Xi_1 \bigsqcup \psi_2\Xi_2$,

where,

$$\psi_1\Xi_1 = \begin{bmatrix} \sqrt[\frac{1}{\epsilon}]{\imath - \left\{\imath - \{\theta\tau_1\}^{\frac{1}{\epsilon}}\right\}^{\psi_1}},\ \sqrt[\frac{1}{\epsilon}]{\imath - \left\{\imath - \{\theta\omega_1\}^{\frac{1}{\epsilon}}\right\}^{\psi_1}}, \\ \sqrt[\frac{1}{\epsilon}]{\left\{\imath - \{\theta\tau_1\}^{\frac{1}{\epsilon}}\right\}^{\psi_1} - \left\{\imath - \{\theta\tau_1 + \theta\chi_1\}^{\frac{1}{\epsilon}}\right\}^{\psi_1}} \end{bmatrix}$$

and

$$\psi_2\Xi_2 = \begin{bmatrix} \sqrt[\frac{1}{\epsilon}]{\imath - \left\{\imath - \{\theta\tau_2\}^{\frac{1}{\epsilon}}\right\}^{\psi_2}},\ \sqrt[\frac{1}{\epsilon}]{\imath - \left\{\imath - \{\theta\omega_2\}^{\frac{1}{\epsilon}}\right\}^{\psi_2}}, \\ \sqrt[\frac{1}{\epsilon}]{\left\{\imath - \{\theta\tau_2\}^{\frac{1}{\epsilon}}\right\}^{\psi_2} - \left\{\imath - \{\theta\tau_2 + \theta\chi_2\}^{\frac{1}{\epsilon}}\right\}^{\psi_2}} \end{bmatrix}$$

We get

$$\psi_1\Xi_1 \bigsqcup \psi_2\Xi_2 = \begin{bmatrix} \sqrt[\frac{1}{\epsilon}]{\left\{\imath-\left\{\imath-\{\theta\tau_1\}^{\frac{1}{\epsilon}}\right\}^{\psi_1}\right\}+\left\{\imath-\left\{\imath-\{\theta\tau_2\}^{\frac{1}{\epsilon}}\right\}^{\psi_2}\right\}} \\ -\left\{\imath-\left\{\imath-\{\theta\tau_1\}^{\frac{1}{\epsilon}}\right\}^{\psi_1}\right\}\cdot\left\{\imath-\left\{\imath-\{\theta\tau_2\}^{\frac{1}{\epsilon}}\right\}^{\psi_2}\right\}, \\ \sqrt[\frac{1}{\epsilon}]{\left\{\imath-\left\{\imath-\{\theta\omega_1\}^{\frac{1}{\epsilon}}\right\}^{\psi_1}\right\}+\left\{\imath-\left\{\imath-\{\theta\omega_2\}^{\frac{1}{\epsilon}}\right\}^{\psi_2}\right\}} \\ -\left\{\imath-\left\{\imath-\{\theta\omega_1\}^{\frac{1}{\epsilon}}\right\}^{\psi_1}\right\}\cdot\left\{\imath-\left\{\imath-\{\theta\omega_2\}^{\frac{1}{\epsilon}}\right\}^{\psi_2}\right\}, \\ \sqrt[\frac{1}{\epsilon}]{\left\{\imath-\left\{\imath-\theta\chi_1\}^{\frac{1}{\epsilon}}\right\}^{\psi_1}\right\}+\left\{\imath-\left\{\imath-\theta\chi_2\}^{\frac{1}{\epsilon}}\right\}^{\psi_2}\right\}} \\ -\left\{\imath-\left\{\imath-\theta\chi_1\}^{\frac{1}{\epsilon}}\right\}^{\psi_1}\right\}\cdot\left\{\imath-\left\{\imath-\theta\chi_2\}^{\frac{1}{\epsilon}}\right\}^{\psi_2}\right\} \\ \sqrt[\frac{1}{\epsilon}]{-\{\imath-\{\theta\tau_1+\theta\chi_1\}^{\frac{1}{\epsilon}}\}^{\psi_1}\cdot\{\imath-\{\theta\tau_2+\theta\chi_2\}^{\frac{1}{\epsilon}}\}^{\psi_2}} \end{bmatrix}$$

$$= \begin{bmatrix} \sqrt[\frac{1}{\epsilon}]{\imath-\Diamond_{\jmath=1}^{\frac{1}{\epsilon}}\left\{\imath-\{\theta\tau_\jmath\}^{\frac{1}{\epsilon}}\right\}^{\psi_\jmath}}, \sqrt[\frac{1}{\epsilon}]{\imath-\Diamond_{\jmath=1}^{\frac{1}{\epsilon}}\left\{\imath-\{\theta\omega_\jmath\}^{\frac{1}{\epsilon}}\right\}^{\psi_\jmath}}, \\ \sqrt[\frac{1}{\epsilon}]{\Diamond_{\jmath=1}^{\frac{1}{\epsilon}}\left\{\imath-\{\theta\tau_\jmath\}^{\frac{1}{\epsilon}}\right\}^{\psi_\jmath}-\Diamond_{\jmath=1}^{\frac{1}{\epsilon}}\left\{\imath-\{\theta\tau_\jmath+\theta\chi_\jmath\}^{\frac{1}{\epsilon}}\right\}^{\psi_\jmath}} \end{bmatrix}.$$

Using induction $\jmath \geq 3$, IFFSTNWA$\{\Xi_1, \Xi_2, ..., \Xi_\ell\}$

$$= \begin{bmatrix} \sqrt[\frac{1}{\epsilon}]{\imath-\Diamond_{\jmath=1}^{\ell}\left\{\imath-\{\theta\tau_\jmath\}^{\frac{1}{\epsilon}}\right\}^{\psi_\jmath}}, \sqrt[\frac{1}{\epsilon}]{\imath-\Diamond_{\jmath=1}^{\ell}\left\{\imath-\{\theta\omega_\jmath\}^{\frac{1}{\epsilon}}\right\}^{\psi_\jmath}}, \\ \sqrt[\frac{1}{\epsilon}]{\Diamond_{\jmath=1}^{\ell}\left\{\imath-\{\theta\tau_\jmath\}^{\frac{1}{\epsilon}}\right\}^{\psi_\jmath}-\Diamond_{\jmath=1}^{\ell}\left\{\imath-\{\theta\tau_\jmath+\theta\chi_\jmath\}^{\frac{1}{\epsilon}}\right\}^{\psi_\jmath}}, \end{bmatrix}.$$

If $\jmath = \ell + 1$, then $IFFSTNWA\{\Xi_1, \Xi_2, ..., \Xi_\ell, \Xi_{\ell+1}\}$

$$= \begin{bmatrix} \sqrt[\frac{1}{\epsilon}]{\varnothing_{\jmath=1}^{\ell}\left\{\imath-\left\{\imath-\{\theta\tau_\jmath\}^{\frac{1}{\epsilon}}\right\}^{\psi_\jmath}\right\}+\left\{\imath-\left\{\imath-\{\theta\tau_{\ell+1}\}^{\frac{1}{\epsilon}}\right\}^{\psi_{\ell+1}}\right\}} \\ -\Diamond_{\jmath=1}^{\ell}\left\{\imath-\left\{\imath-\{\theta\tau_\jmath\}^{\frac{1}{\epsilon}}\right\}^{\psi_\jmath}\right\}\cdot\left\{\imath-\left\{\imath-\{\theta\tau_{\ell+1}\}^{\frac{1}{\epsilon}}\right\}^{\psi_{\ell+1}}\right\}, \\ \sqrt[\frac{1}{\epsilon}]{\varnothing_{\jmath=1}^{\ell}\left\{\imath-\left\{\imath-\{\theta\omega_\jmath\}^{\frac{1}{\epsilon}}\right\}^{\psi_\jmath}\right\}+\left\{\imath-\left\{\imath-\{\theta\omega_{\ell+1}\}^{\frac{1}{\epsilon}}\right\}^{\psi_{\ell+1}}\right\}} \\ -\Diamond_{\jmath=1}^{\ell}\left\{\imath-\left\{\imath-\{\theta\omega_\jmath\}^{\frac{1}{\epsilon}}\right\}^{\psi_\jmath}\right\}\cdot\left\{\imath-\left\{\imath-\{\theta\omega_{\ell+1}\}^{\frac{1}{\epsilon}}\right\}^{\psi_{\ell+1}}\right\}, \\ \sqrt[\frac{1}{\epsilon}]{\varnothing_{\jmath=1}^{\ell}\left\{\imath-\left\{\imath-\theta\chi_\jmath\}^{\frac{1}{\epsilon}}\right\}^{\psi_\jmath}\right\}+\left\{\imath-\left\{\imath-\theta\chi_{\ell+1}\}^{\frac{1}{\epsilon}}\right\}^{\psi_{\ell+1}}\right\}} \\ -\Diamond_{\jmath=1}^{\ell}\left\{\imath-\left\{\imath-\theta\chi_\jmath\}^{\frac{1}{\epsilon}}\right\}^{\psi_\jmath}\right\}\cdot\left\{\imath-\left\{\imath-\theta\chi_{\ell+1}\}^{\frac{1}{\epsilon}}\right\}^{\psi_{\ell+1}}\right\} \\ \sqrt[\frac{1}{\epsilon}]{-\Diamond_{\jmath=1}^{\ell}\left\{\imath-\{\theta\tau_\jmath+\theta\chi_\jmath\}^{\frac{1}{\epsilon}}\right\}^{\psi_\jmath}\cdot\left\{\imath-\{\theta\tau_{\ell+1}+\theta\chi_{\ell+1}\}^{\frac{1}{\epsilon}}\right\}^{\psi_{\ell+1}}} \end{bmatrix}$$

$$= \begin{bmatrix} \sqrt[\frac{1}{\epsilon}]{\imath - \Diamond_{\jmath=1}^{\ell}\left\{\imath - \{\theta\tau_\jmath\}^{\frac{1}{\epsilon}}\right\}^{\psi_\jmath} \cdot \left\{\imath - \{\theta\tau_{\ell+1}\}^{\frac{1}{\epsilon}}\right\}^{\psi_{\ell+1}}}, \\ \sqrt[\frac{1}{\epsilon}]{\imath - \Diamond_{\jmath=1}^{\ell}\left\{\imath - \{\theta\omega_\jmath\}^{\frac{1}{\epsilon}}\right\}^{\psi_\jmath} \cdot \left\{\imath - \{\theta\omega_{\ell+1}\}^{\frac{1}{\epsilon}}\right\}^{\psi_{\ell+1}}}, \\ \sqrt[\frac{1}{\epsilon}]{\left\{\Diamond_{\jmath=1}^{\ell}\left\{\imath - \{\theta\tau_\jmath\}^{\frac{1}{\epsilon}}\right\}^{\psi_\jmath} - \Diamond_{\jmath=1}^{\ell}\left\{\imath - \{\theta\tau_\jmath + \theta\chi_\jmath\}^{\frac{1}{\epsilon}}\right\}^{\psi_\jmath}\right\} \cdot \left\{\left\{\{\theta\tau_{\ell+1}\}^{\frac{1}{\epsilon}}\right\}^{\psi_{\ell+1}} - \left\{\{\theta\tau_{\ell+1} + \theta\chi_{\ell+1}\}^{\frac{1}{\epsilon}}\right\}^{\psi_{\ell+1}}\right\}} \end{bmatrix}$$

$$= \begin{bmatrix} \sqrt[\frac{1}{\epsilon}]{\imath - \Diamond_{\jmath=1}^{\ell+1}\left\{\imath - \{\theta\tau_\jmath\}^{\frac{1}{\epsilon}}\right\}^{\psi_\jmath}}, \sqrt[\frac{1}{\epsilon}]{\imath - \Diamond_{\jmath=1}^{\ell+1}\left\{\imath - \{\theta\omega_\jmath\}^{\frac{1}{\epsilon}}\right\}^{\psi_\jmath}}, \\ \sqrt[\frac{1}{\epsilon}]{\Diamond_{\jmath=1}^{\ell+1}\left\{\imath - \{\theta\tau_\jmath\}^{\frac{1}{\epsilon}}\right\}^{\psi_\jmath} - \Diamond_{\jmath=1}^{\ell+1}\left\{\imath - \{\theta\tau_\jmath + \theta\chi_\jmath\}^{\frac{1}{\epsilon}}\right\}^{\psi_\jmath}} \end{bmatrix}$$

Theorem 3.4. *If $\Xi_\jmath = \langle\theta\tau_\jmath, \theta\omega_\jmath, \theta\chi_\jmath\rangle$ be the IFFSTNNs and $\Xi_\jmath = \Xi$, then the IFFSTNWA$\{\Xi_1, \Xi_2, ..., \Xi_\ell\} = \Xi$, $\jmath = 1, 2, ..., \ell$.*

Proof. Note that, $\{\theta\tau_\jmath, \theta\omega_\jmath, \theta\chi_\jmath\} = \{\theta\tau, \theta\omega, \theta\chi\}$, $\jmath = 1, 2, ..., \ell$ and $\varnothing_{\jmath=1}^{k}\psi_\jmath = 1$. We get, $IFFSTNWA\{\Xi_1, \Xi_2, ..., \Xi_\ell\}$

$$= \begin{bmatrix} \sqrt[\frac{1}{\epsilon}]{\imath - \Diamond_{\jmath=1}^{k}\left\{\imath - \{\theta\tau\}^{\frac{1}{\epsilon}}\right\}^{\psi_\jmath}}, \sqrt[\frac{1}{\epsilon}]{\imath - \Diamond_{\jmath=1}^{k}\left\{\imath - \{\theta\omega\}^{\frac{1}{\epsilon}}\right\}^{\psi_\jmath}}, \\ \sqrt[\frac{1}{\epsilon}]{\Diamond_{\jmath=1}^{k}\left\{\imath - \{\theta\tau\}^{\frac{1}{\epsilon}}\right\}^{\psi_\jmath} - \Diamond_{\jmath=1}^{k}\left\{\imath - \{\theta\tau + \theta\chi\}^{\frac{1}{\epsilon}}\right\}^{\psi_\jmath}} \end{bmatrix}$$

$$= \begin{bmatrix} \sqrt[\frac{1}{\epsilon}]{\imath - \left\{\imath - \{\theta\tau\}^{\frac{1}{\epsilon}}\right\}^{\varnothing_{\jmath=1}^{k}\psi_\jmath}}, \sqrt[\frac{1}{\epsilon}]{\imath - \left\{\imath - \{\theta\omega\}^{\frac{1}{\epsilon}}\right\}^{\varnothing_{\jmath=1}^{k}\psi_\jmath}}, \\ \sqrt[\frac{1}{\epsilon}]{\left\{\imath - \{\theta\tau\}^{\frac{1}{\epsilon}}\right\}^{\varnothing_{\jmath=1}^{k}\psi_\jmath} - \left\{\imath - \{\theta\tau + \theta\chi\}^{\frac{1}{\epsilon}}\right\}^{\varnothing_{\jmath=1}^{k}\psi_\jmath}} \end{bmatrix}$$

$$= \begin{bmatrix} \sqrt[\frac{1}{\epsilon}]{\imath - \left\{\imath - \{\theta\tau\}^{\frac{1}{\epsilon}}\right\}}, \sqrt[\frac{1}{\epsilon}]{\imath - \left\{\imath - \{\theta\omega\}^{\frac{1}{\epsilon}}\right\}}, \\ \sqrt[\frac{1}{\epsilon}]{\left\{\imath - \{\theta\tau\}^{\frac{1}{\epsilon}}\right\} - \left\{\imath - \{\theta\tau + \theta\chi\}^{\frac{1}{\epsilon}}\right\}} \end{bmatrix}$$

$$= \left\{\theta\tau, \theta\omega, \theta\chi\right\} = \Xi$$

3.2. *Interaction weighted geometric(IFFSTNWG) operator*

Definition 3.5. Let $\Xi_\jmath = \langle\theta\tau_\jmath, \theta\omega_\jmath, \theta\chi_\jmath\rangle$ be the IFFSTNNs, $\psi_\jmath$ be the weight of $\Xi_\jmath$. Then the IFFSTNWG operator $\{\Xi_1, \Xi_2, ..., \Xi_\ell\} = \Diamond_{\jmath=1}^{k}\Xi_\jmath^{\psi_\jmath}$.

Theorem 3.6. *If $\Xi_\jmath = \langle\theta\tau_\jmath, \theta\omega_\jmath, \theta\chi_\jmath\rangle$ be the IFFSTNNs. Then,*

$$IFFSTNWG\{\Xi_1, \Xi_2, ..., \Xi_\ell\} = \begin{bmatrix} \sqrt[\frac{1}{\epsilon}]{\Diamond_{\jmath=1}^{k}\left\{\imath - \{\theta\chi_\jmath\}^{\frac{1}{\epsilon}}\right\}^{\psi_\jmath} - \Diamond_{\jmath=1}^{k}\left\{\imath - \{\theta\tau_\jmath + \theta\chi_\jmath\}^{\frac{1}{\epsilon}}\right\}^{\psi_\jmath}}, \\ \sqrt[\frac{1}{\epsilon}]{\imath - \Diamond_{\jmath=1}^{k}\left\{\imath - \{\theta\omega_\jmath\}^{\frac{1}{\epsilon}}\right\}^{\psi_\jmath}}, \sqrt[\frac{1}{\epsilon}]{\imath - \Diamond_{\jmath=1}^{k}\left\{\imath - \{\theta\chi_\jmath\}^{\frac{1}{\epsilon}}\right\}^{\psi_\jmath}} \end{bmatrix}$$

Proof. If $\jmath = 2$, then

$$\Xi_1^{\psi_1} \circledast \Xi_2^{\psi_2} = \left[\begin{array}{c} \sqrt[\frac{1}{\epsilon}]{\begin{array}{l} \left\{\imath - \left\{\imath - \{\theta\chi_1\}^{\frac{1}{\epsilon}}\right\}^{\psi_1}\right\} + \left\{\imath - \left\{\imath - \{\theta\chi_2\}^{\frac{1}{\epsilon}}\right\}^{\psi_2}\right\} \\ -\left\{\imath - \left\{\imath - \{\theta\chi_1\}^{\frac{1}{\epsilon}}\right\}^{\psi_1}\right\} \cdot \left\{\imath - \left\{\imath - \{\theta\chi_2\}^{\frac{1}{\epsilon}}\right\}^{\psi_2}\right\} \\ -\left\{\left\{\imath - \{\theta\tau_1 + \theta\chi_1\}^{\frac{1}{\epsilon}}\right\}^{\psi_1} \cdot \left\{\imath - \{\theta\tau_2 + \theta\chi_2\}^{\frac{1}{\epsilon}}\right\}^{\psi_2}\right\} \end{array}}, \\ \sqrt[\frac{1}{\epsilon}]{\begin{array}{l} \left\{\imath - \left\{\imath - \{\theta\omega_1\}^{\frac{1}{\epsilon}}\right\}^{\psi_1}\right\} + \left\{\imath - \left\{\imath - \{\theta\omega_2\}^{\frac{1}{\epsilon}}\right\}^{\psi_2}\right\} \\ -\left\{\imath - \left\{\imath - \{\theta\omega_1\}^{\frac{1}{\epsilon}}\right\}^{\psi_1}\right\} \cdot \left\{\imath - \left\{\imath - \{\theta\omega_2\}^{\frac{1}{\epsilon}}\right\}^{\psi_2}\right\} \end{array}}, \\ \sqrt[\frac{1}{\epsilon}]{\begin{array}{l} \left\{\imath - \left\{\imath - \{\theta\chi_1\}^{\frac{1}{\epsilon}}\right\}^{\psi_1}\right\} + \left\{\imath - \left\{\imath - \{\theta\chi_2\}^{\frac{1}{\epsilon}}\right\}^{\psi_2}\right\} \\ -\left\{\imath - \left\{\imath - \{\theta\chi_1\}^{\frac{1}{\epsilon}}\right\}^{\psi_1}\right\} \cdot \left\{\imath - \left\{\imath - \{\theta\chi_2\}^{\frac{1}{\epsilon}}\right\}^{\psi_2}\right\} \end{array}} \end{array}\right]$$

Hence, $\text{IFFSTNWG}\{\Xi_1, \Xi_2\} = \left[\begin{array}{c} \sqrt[\frac{1}{\epsilon}]{\Diamond_{\jmath=1}^{\frac{1}{\epsilon}}\left\{\imath - \{\theta\chi_\jmath\}^{\frac{1}{\epsilon}}\right\}^{\psi_\jmath} - \Diamond_{\jmath=1}^{\frac{1}{\epsilon}}\left\{\imath - \{\theta\tau_\jmath + \theta\chi_\jmath\}^{\frac{1}{\epsilon}}\right\}^{\psi_\jmath}}, \\ \sqrt[\frac{1}{\epsilon}]{\imath - \Diamond_{\jmath=1}^{\frac{1}{\epsilon}}\left\{\imath - \{\theta\omega_\jmath\}^{\frac{1}{\epsilon}}\right\}^{\psi_\jmath}}, \sqrt[\frac{1}{\epsilon}]{\imath - \Diamond_{\jmath=1}^{\frac{1}{\epsilon}}\left\{\imath - \{\theta\chi_\jmath\}^{\frac{1}{\epsilon}}\right\}^{\psi_\jmath}} \end{array}\right]$

$\text{IFFSTNWG}\{\Xi_1, \Xi_2, ..., \Xi_\ell\} = \left[\begin{array}{c} \sqrt[\frac{1}{\epsilon}]{\Diamond_{\jmath=1}^{\ell}\left\{\imath - \{\theta\chi_\jmath\}^{\frac{1}{\epsilon}}\right\}^{\psi_\jmath} - \Diamond_{\jmath=1}^{\ell}\left\{\imath - \{\theta\tau_\jmath + \theta\chi_\jmath\}^{\frac{1}{\epsilon}}\right\}^{\psi_\jmath}}, \\ \sqrt[\frac{1}{\epsilon}]{\imath - \Diamond_{\jmath=1}^{\ell}\left\{\imath - \{\theta\omega_\jmath\}^{\frac{1}{\epsilon}}\right\}^{\psi_\jmath}}, \sqrt[\frac{1}{\epsilon}]{\imath - \Diamond_{\jmath=1}^{\ell}\left\{\imath - \{\theta\chi_\jmath\}^{\frac{1}{\epsilon}}\right\}^{\psi_\jmath}} \end{array}\right]$

If $\jmath = \ell + 1$, then $\text{IFFSTNWG}\{\Xi_1, ..., \Xi_\ell, \Xi_{\ell+1}\}$

$$= \left[\begin{array}{c} \sqrt[\frac{1}{\epsilon}]{\begin{array}{l} \varnothing_{\jmath=1}^{\ell}\left\{\imath - \left\{\imath - \{\theta\chi_\jmath\}^{\frac{1}{\epsilon}}\right\}^{\psi_\jmath}\right\} + \left\{\imath - \left\{\imath - \{\theta\chi_{\ell+1}\}^{\frac{1}{\epsilon}}\right\}^{\psi_{\ell+1}}\right\} \\ -\Diamond_{\jmath=1}^{\ell}\left\{\imath - \left\{\imath - \{\theta\chi_\jmath\}^{\frac{1}{\epsilon}}\right\}^{\psi_\jmath}\right\} \cdot \left\{\imath - \left\{\imath - \{\theta\chi_{\ell+1}\}^{\frac{1}{\epsilon}}\right\}^{\psi_{\ell+1}}\right\} \\ -\Diamond_{\jmath=1}^{\ell}\left\{\imath - \{\theta\tau_\jmath + \theta\chi_\jmath\}^{\frac{1}{\epsilon}}\right\}^{\psi_\jmath} \cdot \left\{\imath - \{\theta\tau_{\ell+1} + \theta\chi_{\ell+1}\}^{\frac{1}{\epsilon}}\right\}^{\psi_{\ell+1}} \end{array}}, \\ \sqrt[\frac{1}{\epsilon}]{\begin{array}{l} \varnothing_{\jmath=1}^{\ell}\left\{\imath - \left\{\imath - \{\theta\omega_\jmath\}^{\frac{1}{\epsilon}}\right\}^{\psi_\jmath}\right\} + \left\{\imath - \left\{\imath - \{\theta\omega_{\ell+1}\}^{\frac{1}{\epsilon}}\right\}^{\psi_{\ell+1}}\right\} \\ -\Diamond_{\jmath=1}^{\ell}\left\{\imath - \left\{\imath - \{\theta\omega_\jmath\}^{\frac{1}{\epsilon}}\right\}^{\psi_\jmath}\right\} \cdot \left\{\imath - \left\{\imath - \{\theta\omega_{\ell+1}\}^{\frac{1}{\epsilon}}\right\}^{\psi_{\ell+1}}\right\} \end{array}}, \\ \sqrt[\frac{1}{\epsilon}]{\begin{array}{l} \varnothing_{\jmath=1}^{\ell}\left\{\imath - \left\{\imath - \{\theta\chi_\jmath\}^{\frac{1}{\epsilon}}\right\}^{\psi_\jmath}\right\} + \left\{\imath - \left\{\imath - \{\theta\chi_{\ell+1}\}^{\frac{1}{\epsilon}}\right\}^{\psi_{\ell+1}}\right\} \\ -\Diamond_{\jmath=1}^{\ell}\left\{\imath - \left\{\imath - \{\theta\chi_\jmath\}^{\frac{1}{\epsilon}}\right\}^{\psi_\jmath}\right\} \cdot \left\{\imath - \left\{\imath - \{\theta\chi_{\ell+1}\}^{\frac{1}{\epsilon}}\right\}^{\psi_{\ell+1}}\right\} \end{array}} \end{array}\right]$$

$$= \left[\begin{array}{c} \sqrt[\frac{1}{\epsilon}]{\left\{ \Diamond_{\jmath=1}^{\ell}\left\{\imath - \{\theta\chi_\jmath\}^{\frac{1}{\epsilon}}\right\}^{\psi_\jmath} - \Diamond_{\jmath=1}^{\ell}\left\{\imath - \{\theta\tau_\jmath + \theta\chi_\jmath\}^{\frac{1}{\epsilon}}\right\}^{\psi_\jmath} \right\} \cdot \left\{ \left\{ \{\theta\chi_{\ell+1}\}^{\frac{1}{\epsilon}} \right\}^{\psi_{\ell+1}} - \{\theta\tau_{\ell+1} + \theta\chi_{\ell+1}\}^{\frac{1}{\epsilon}} \right\}^{\psi_{\ell+1}} \right\}}, \\ \sqrt[\frac{1}{\epsilon}]{\imath - \Diamond_{\jmath=1}^{\ell}\left\{\imath - \{\theta\omega_\jmath\}^{\frac{1}{\epsilon}}\right\}^{\psi_\jmath} \cdot \left\{\imath - \{\theta\omega_{\ell+1}\}^{\frac{1}{\epsilon}}\right\}^{\psi_{\ell+1}}}, \\ \sqrt[\frac{1}{\epsilon}]{\imath - \Diamond_{\jmath=1}^{\ell}\left\{\imath - \{\theta\chi_\jmath\}^{\frac{1}{\epsilon}}\right\}^{\psi_\jmath} \cdot \left\{\imath - \{\theta\chi_{\ell+1}\}^{\frac{1}{\epsilon}}\right\}^{\psi_{\ell+1}}} \end{array} \right]$$

$$= \left[\begin{array}{c} \sqrt[\frac{1}{\epsilon}]{\Diamond_{\jmath=1}^{\ell+1}\left\{\imath - \{\theta\chi_\jmath\}^{\frac{1}{\epsilon}}\right\}^{\psi_\jmath} - \Diamond_{\jmath=1}^{\ell+1}\left\{\imath - \{\theta\tau_\jmath + \theta\chi_\jmath\}^{\frac{1}{\epsilon}}\right\}^{\psi_\jmath}}, \\ \sqrt[\frac{1}{\epsilon}]{\imath - \Diamond_{\jmath=1}^{\ell+1}\left\{\imath - \{\theta\omega_\jmath\}^{\frac{1}{\epsilon}}\right\}^{\psi_\jmath}} \sqrt[\frac{1}{\epsilon}]{\imath - \Diamond_{\jmath=1}^{\ell+1}\left\{\imath - \{\theta\chi_\jmath\}^{\frac{1}{\epsilon}}\right\}^{\psi_\jmath}} \end{array} \right]$$

Corollary 3.7. *Let* $\Xi_\jmath = \langle \theta\tau_\jmath, \theta\omega_\jmath, \theta\chi_\jmath \rangle$ *be the IFFSTNNs and all are equal and and* $\theta\tau \cdot \theta\chi = 0$. *Then IFFSTNWG*$\{\Xi_1, \Xi_2, ..., \Xi_\ell\} = \Xi$.

3.3. *generalized IFFSTNWA (GIFFSTNWA) operator*

Definition 3.8. Let $\Xi_\jmath = \langle \theta\tau_\jmath, \theta\tau_\jmath, \theta\chi_\jmath \rangle$ be the IFFSTNNs, $\psi_\jmath$ be a weight of $\Xi_\jmath$. Then, the GIFFSTNWA operator $\{\Xi_1, \Xi_2, ..., \Xi_\ell\} = \left\{ \varnothing_{\jmath=1}^{k} \psi_\jmath \Xi_\jmath^{\frac{1}{\epsilon}} \right\}^{1/2}$.

Theorem 3.9. *Let* $\Xi_\jmath = \langle \theta\tau_\jmath, \theta\tau_\jmath, \theta\chi_\jmath \rangle$ *be the IFFSTNNs. Then GIFFSTNWA* $\{\Xi_1, \Xi_2, ..., \Xi_\ell\} =$

$$\left[\begin{array}{c} \left\{ \sqrt[\frac{1}{\epsilon}]{\imath - \Diamond_{\jmath=1}^{k}\left\{\imath - \left\{\{\theta\tau_\jmath\}^{\frac{1}{\epsilon}}\right\}^{\frac{1}{\epsilon}}\right\}^{\psi_\jmath}} \right\}, \left\{ \sqrt[\frac{1}{\epsilon}]{\imath - \Diamond_{\jmath=1}^{k}\left\{\imath - \left\{\{\theta\omega_\jmath\}^{\frac{1}{\epsilon}}\right\}^{\frac{1}{\epsilon}}\right\}^{\psi_\jmath}} \right\}, \\ \left\{ \sqrt[\frac{1}{\epsilon}]{\Diamond_{\jmath=1}^{k}\left\{\imath - \left\{\{\theta\tau_\jmath\}^{\frac{1}{\epsilon}}\right\}^{\frac{1}{\epsilon}}\right\}^{\psi_\jmath} - \Diamond_{\jmath=1}^{k}\left\{\imath - \left\{\{\theta\tau_\jmath + \theta\chi_\jmath\}^{\frac{1}{\epsilon}}\right\}^{\frac{1}{\epsilon}}\right\}^{\psi_\jmath}} \right\} \end{array} \right].$$

Proof. First, we have find that

$$\varnothing_{\jmath=1}^{k} \psi_\jmath \Xi_\jmath^{\frac{1}{\epsilon}} = \left[\begin{array}{c} \sqrt[\frac{1}{\epsilon}]{\imath - \Diamond_{\jmath=1}^{k}\left\{\imath - \left\{\{\theta\tau_\jmath\}^{\frac{1}{\epsilon}}\right\}^{\frac{1}{\epsilon}}\right\}^{\psi_\jmath}}, \sqrt[\frac{1}{\epsilon}]{\imath - \Diamond_{\jmath=1}^{k}\left\{\imath - \left\{\{\theta\omega_\jmath\}^{\frac{1}{\epsilon}}\right\}^{\frac{1}{\epsilon}}\right\}^{\psi_\jmath}}, \\ \sqrt[\frac{1}{\epsilon}]{\Diamond_{\jmath=1}^{k}\left\{\imath - \left\{\{\theta\tau_\jmath\}^{\frac{1}{\epsilon}}\right\}^{\frac{1}{\epsilon}}\right\}^{\psi_\jmath} - \Diamond_{\jmath=1}^{k}\left\{\imath - \left\{\{\theta\tau_\jmath + \theta\chi_\jmath\}^{\frac{1}{\epsilon}}\right\}^{\frac{1}{\epsilon}}\right\}^{\psi_\jmath}} \end{array} \right].$$

If $\jmath = 2$, then $\psi_1 \Xi_1^{\frac{1}{\epsilon}} = \left[\begin{array}{c} \sqrt[\frac{1}{\epsilon}]{\imath - \left\{\imath - \left\{\{\theta\tau_1\}^{\frac{1}{\epsilon}}\right\}^{\frac{1}{\epsilon}}\right\}^{\psi_1}}, \sqrt[\frac{1}{\epsilon}]{\imath - \left\{\imath - \left\{\{\theta\omega_1\}^{\frac{1}{\epsilon}}\right\}^{\frac{1}{\epsilon}}\right\}^{\psi_1}}, \\ \sqrt[\frac{1}{\epsilon}]{\left\{\imath - \left\{\{\theta\tau_1\}^{\frac{1}{\epsilon}}\right\}^{\frac{1}{\epsilon}}\right\}^{\psi_1} - \left\{\imath - \left\{\{\theta\tau_1 + \theta\chi_1\}^{\frac{1}{\epsilon}}\right\}^{\frac{1}{\epsilon}}\right\}^{\psi_1}} \end{array} \right]$$

and

$$\psi_2 \Xi_2^{\frac{1}{\epsilon}} = \left[\begin{array}{c} \sqrt[\frac{1}{\epsilon}]{\imath - \left\{\imath - \left\{\{\theta\tau_2\}^{\frac{1}{\epsilon}}\right\}^{\frac{1}{\epsilon}}\right\}^{\psi_1}}, \sqrt[\frac{1}{\epsilon}]{\imath - \left\{\imath - \left\{\{\theta\omega_2\}^{\frac{1}{\epsilon}}\right\}^{\frac{1}{\epsilon}}\right\}^{\psi_1}}, \\ \sqrt[\frac{1}{\epsilon}]{\left\{\imath - \left\{\{\theta\tau_2\}^{\frac{1}{\epsilon}}\right\}^{\frac{1}{\epsilon}}\right\}^{\psi_1} - \left\{\imath - \left\{\{\theta\tau_2 + \theta\chi_2\}^{\frac{1}{\epsilon}}\right\}^{\frac{1}{\epsilon}}\right\}^{\psi_1}} \end{array} \right].$$

We get, $\psi_1\Xi_1 \bigsqcup \psi_2\Xi_2 =$

$$\left[\begin{array}{l}
\sqrt[\frac{1}{\epsilon}]{\left\{\sqrt[\frac{1}{\epsilon}]{\imath-\left\{\imath-\left\{\{\theta\tau_1\}^{\frac{1}{\epsilon}}\right\}^{\frac{1}{\epsilon}}\right\}^{\psi_1}}\right\}^{\frac{1}{\epsilon}}+\left\{\sqrt[\frac{1}{\epsilon}]{\imath-\left\{\imath-\left\{\{\theta\tau_2\}^{\frac{1}{\epsilon}}\right\}^{\frac{1}{\epsilon}}\right\}^{\psi_1}}\right\}^{\frac{1}{\epsilon}},} \\
\quad -\left\{\sqrt[\frac{1}{\epsilon}]{\imath-\left\{\imath-\left\{\{\theta\tau_1\}^{\frac{1}{\epsilon}}\right\}^{\frac{1}{\epsilon}}\right\}^{\psi_1}}\right\}^{\frac{1}{\epsilon}}\cdot\left\{\sqrt[\frac{1}{\epsilon}]{\imath-\left\{\imath-\left\{\{\theta\tau_2\}^{\frac{1}{\epsilon}}\right\}^{\frac{1}{\epsilon}}\right\}^{\psi_1}}\right\}^{\frac{1}{\epsilon}} \\
\sqrt[\frac{1}{\epsilon}]{\left\{\sqrt[\frac{1}{\epsilon}]{\imath-\left\{\imath-\left\{\{\theta\omega_1\}^{\frac{1}{\epsilon}}\right\}^{\frac{1}{\epsilon}}\right\}^{\psi_1}}\right\}^{\frac{1}{\epsilon}}+\left\{\sqrt[\frac{1}{\epsilon}]{\imath-\left\{\imath-\left\{\{\theta\omega_2\}^{\frac{1}{\epsilon}}\right\}^{\frac{1}{\epsilon}}\right\}^{\psi_1}}\right\}^{\frac{1}{\epsilon}},} \\
\quad -\left\{\sqrt[\frac{1}{\epsilon}]{\imath-\left\{\imath-\left\{\{\theta\omega_1\}^{\frac{1}{\epsilon}}\right\}^{\frac{1}{\epsilon}}\right\}^{\psi_1}}\right\}^{\frac{1}{\epsilon}}\cdot\left\{\sqrt[\frac{1}{\epsilon}]{\imath-\left\{\imath-\left\{\{\theta\omega_2\}^{\frac{1}{\epsilon}}\right\}^{\frac{1}{\epsilon}}\right\}^{\psi_1}}\right\}^{\frac{1}{\epsilon}} \\
\sqrt[\frac{1}{\epsilon}]{\left\{\sqrt[\frac{1}{\epsilon}]{\imath-\left\{\imath-\left\{\{\theta\chi_1\}^{\frac{1}{\epsilon}}\right\}^{\frac{1}{\epsilon}}\right\}^{\psi_1}}\right\}^{\frac{1}{\epsilon}}+\left\{\sqrt[\frac{1}{\epsilon}]{\imath-\left\{\imath-\left\{\{\theta\chi_2\}^{\frac{1}{\epsilon}}\right\}^{\frac{1}{\epsilon}}\right\}^{\psi_1}}\right\}^{\frac{1}{\epsilon}},} \\
\quad -\left\{\sqrt[\frac{1}{\epsilon}]{\imath-\left\{\imath-\left\{\{\theta\chi_1\}^{\frac{1}{\epsilon}}\right\}^{\frac{1}{\epsilon}}\right\}^{\psi_1}}\right\}^{\frac{1}{\epsilon}}\cdot\left\{\sqrt[\frac{1}{\epsilon}]{\imath-\left\{\imath-\left\{\{\theta\chi_2\}^{\frac{1}{\epsilon}}\right\}^{\frac{1}{\epsilon}}\right\}^{\psi_1}}\right\}^{\frac{1}{\epsilon}} \\
\quad -\left[\begin{array}{l}\left\{\sqrt[\frac{1}{\epsilon}]{\left\{\imath-\left\{\{\theta\tau_1\}^{\frac{1}{\epsilon}}\right\}^{\frac{1}{\epsilon}}\right\}^{\psi_1}-\left\{\imath-\left\{\{\theta\tau_1+\theta\chi_1\}^{\frac{1}{\epsilon}}\right\}^{\frac{1}{\epsilon}}\right\}^{\psi_1}}\right\}^{\frac{1}{\epsilon}}\cdot \\ \left\{\sqrt[\frac{1}{\epsilon}]{\left\{\imath-\left\{\{\theta\tau_2\}^{\frac{1}{\epsilon}}\right\}^{\frac{1}{\epsilon}}\right\}^{\psi_1}-\left\{\imath-\left\{\{\theta\tau_2+\theta\chi_2\}^{\frac{1}{\epsilon}}\right\}^{\frac{1}{\epsilon}}\right\}^{\psi_1}}\right\}^{\frac{1}{\epsilon}}\end{array}\right]
\end{array}\right]$$

$$= \left[\begin{array}{l}\sqrt[\frac{1}{\epsilon}]{\imath-\Diamond_{\jmath=1}^{\frac{1}{\epsilon}}\left\{\imath-\left\{\{\theta\tau_1\}^{\frac{1}{\epsilon}}\right\}^{\frac{1}{\epsilon}}\right\}^{\psi_\jmath}},\ \sqrt[\frac{1}{\epsilon}]{\imath-\Diamond_{\jmath=1}^{\frac{1}{\epsilon}}\left\{\imath-\left\{\{\theta\omega_1\}^{\frac{1}{\epsilon}}\right\}^{\frac{1}{\epsilon}}\right\}^{\psi_\jmath}}, \\ \sqrt[\frac{1}{\epsilon}]{\Diamond_{\jmath=1}^{\frac{1}{\epsilon}}\left\{\imath-\left\{\{\theta\tau_1\}^{\frac{1}{\epsilon}}\right\}^{\frac{1}{\epsilon}}\right\}^{\psi_\jmath}-\Diamond_{\jmath=1}^{\frac{1}{\epsilon}}\left\{\imath-\left\{\{\theta\tau_1+\theta\chi_1\}^{\frac{1}{\epsilon}}\right\}^{\frac{1}{\epsilon}}\right\}^{\psi_\jmath}}\end{array}\right]$$

In general,

$$= \left[\begin{array}{l}\sqrt[\frac{1}{\epsilon}]{\imath-\Diamond_{\jmath=1}^{\ell}\left\{\imath-\left\{\{\theta\tau_1\}^{\frac{1}{\epsilon}}\right\}^{\frac{1}{\epsilon}}\right\}^{\psi_\jmath}},\ \sqrt[\frac{1}{\epsilon}]{\imath-\Diamond_{\jmath=1}^{\ell}\left\{\imath-\left\{\{\theta\omega_1\}^{\frac{1}{\epsilon}}\right\}^{\frac{1}{\epsilon}}\right\}^{\psi_\jmath}} \\ \sqrt[\frac{1}{\epsilon}]{\Diamond_{\jmath=1}^{\ell}\left\{\imath-\left\{\{\theta\tau_1\}^{\frac{1}{\epsilon}}\right\}^{\frac{1}{\epsilon}}\right\}^{\psi_\jmath}-\Diamond_{\jmath=1}^{\ell}\left\{\imath-\left\{\{\theta\tau_1+\theta\chi_1\}^{\frac{1}{\epsilon}}\right\}^{\frac{1}{\epsilon}}\right\}^{\psi_\jmath}}\end{array}\right].$$

If $\jmath=\ell+1$, then $\varnothing_{\jmath=1}^{\ell}\psi_\jmath\Xi_\jmath^{\frac{1}{\epsilon}}+\psi_{\ell+1}\Xi_{\ell+1}^{\frac{1}{\epsilon}}=\varnothing_{\jmath=1}^{\ell+1}\psi_\jmath\Xi_\jmath^{\frac{1}{\epsilon}}$.

Now, $\varnothing_{\jmath=1}^{\ell}\psi_\jmath\Xi_\jmath^{\frac{1}{\epsilon}}+\psi_{\ell+1}\Xi_{\ell+1}^{\frac{1}{\epsilon}}=\psi_1\Xi_1^{\frac{1}{\epsilon}}\bigsqcup\psi_2\Xi_2^{\frac{1}{\epsilon}}\bigsqcup\ldots\bigsqcup\psi_\ell\Xi_\ell^{\frac{1}{\epsilon}}\bigsqcup\psi_{\ell+1}\Xi_{\ell+1}^{\frac{1}{\epsilon}}$

$$= \left[\begin{array}{l} \sqrt[\frac{1}{\epsilon}]{\left\{ \sqrt[\frac{1}{\epsilon}]{\imath - \Diamond_{\jmath=1}^{\ell}\left\{\imath - \left\{\{\theta\tau_\jmath\}^{\frac{1}{\epsilon}}\right\}^{\frac{1}{\epsilon}}\right\}^{\psi_\jmath}} \right\}^{\frac{1}{\epsilon}} + \left\{ \sqrt[\frac{1}{\epsilon}]{\imath - \left\{\imath - \left\{\{\theta\tau_{\ell+1}\}^{\frac{1}{\epsilon}}\right\}^{\frac{1}{\epsilon}}\right\}^{\psi_1}} \right\}^{\frac{1}{\epsilon}}, \\ -\left\{ \sqrt[\frac{1}{\epsilon}]{\imath - \Diamond_{\jmath=1}^{\ell}\left\{\imath - \left\{\{\theta\tau_\jmath\}^{\frac{1}{\epsilon}}\right\}^{\frac{1}{\epsilon}}\right\}^{\psi_\jmath}} \right\}^{\frac{1}{\epsilon}} \cdot \left\{ \sqrt[\frac{1}{\epsilon}]{\imath - \left\{\imath - \left\{\{\theta\tau_{\ell+1}\}^{\frac{1}{\epsilon}}\right\}^{\frac{1}{\epsilon}}\right\}^{\psi_1}} \right\}^{\frac{1}{\epsilon}} \end{array} \right]$$

$$\left[\begin{array}{l} \sqrt[\frac{1}{\epsilon}]{\left\{ \sqrt[\frac{1}{\epsilon}]{\imath - \Diamond_{\jmath=1}^{\ell}\left\{\imath - \left\{\{\theta\omega_\jmath\}^{\frac{1}{\epsilon}}\right\}^{\frac{1}{\epsilon}}\right\}^{\psi_\jmath}} \right\}^{\frac{1}{\epsilon}} + \left\{ \sqrt[\frac{1}{\epsilon}]{\imath - \left\{\imath - \left\{\{\theta\omega_{\ell+1}\}^{\frac{1}{\epsilon}}\right\}^{\frac{1}{\epsilon}}\right\}^{\psi_1}} \right\}^{\frac{1}{\epsilon}}, \\ -\left\{ \sqrt[\frac{1}{\epsilon}]{\imath - \Diamond_{\jmath=1}^{\ell}\left\{\imath - \left\{\{\theta\omega_\jmath\}^{\frac{1}{\epsilon}}\right\}^{\frac{1}{\epsilon}}\right\}^{\psi_\jmath}} \right\}^{\frac{1}{\epsilon}} \cdot \left\{ \sqrt[\frac{1}{\epsilon}]{\imath - \left\{\imath - \left\{\{\theta\omega_{\ell+1}\}^{\frac{1}{\epsilon}}\right\}^{\frac{1}{\epsilon}}\right\}^{\psi_1}} \right\}^{\frac{1}{\epsilon}} \end{array} \right]$$

$$\left[\begin{array}{l} \sqrt[\frac{1}{\epsilon}]{\left\{ \sqrt[\frac{1}{\epsilon}]{\imath - \Diamond_{\jmath=1}^{\ell}\left\{\imath - \left\{\{\theta\tau_\jmath\}^{\frac{1}{\epsilon}}\right\}^{\frac{1}{\epsilon}}\right\}^{\psi_\jmath}} \right\}^{\frac{1}{\epsilon}} + \left\{ \sqrt[\frac{1}{\epsilon}]{\imath - \left\{\imath - \left\{\{\theta\tau_{\ell+1}\}^{\frac{1}{\epsilon}}\right\}^{\frac{1}{\epsilon}}\right\}^{\psi_1}} \right\}^{\frac{1}{\epsilon}}, \\ -\left\{ \sqrt[\frac{1}{\epsilon}]{\imath - \Diamond_{\jmath=1}^{\ell}\left\{\imath - \left\{\{\theta\tau_\jmath\}^{\frac{1}{\epsilon}}\right\}^{\frac{1}{\epsilon}}\right\}^{\psi_\jmath}} \right\}^{\frac{1}{\epsilon}} \cdot \left\{ \sqrt[\frac{1}{\epsilon}]{\imath - \left\{\imath - \left\{\{\theta\tau_{\ell+1}\}^{\frac{1}{\epsilon}}\right\}^{\frac{1}{\epsilon}}\right\}^{\psi_1}} \right\}^{\frac{1}{\epsilon}} \\ \quad - \left[\begin{array}{l} \left\{ \sqrt[\frac{1}{\epsilon}]{\Diamond_{\jmath=1}^{\ell}\left\{\imath - \left\{\{\theta\tau_\jmath\}^{\frac{1}{\epsilon}}\right\}^{\frac{1}{\epsilon}}\right\}^{\psi_\jmath}} - \Diamond_{\jmath=1}^{\ell}\left\{\imath - \left\{\{\theta\tau_\jmath + \theta\chi_\jmath\}^{\frac{1}{\epsilon}}\right\}^{\frac{1}{\epsilon}}\right\}^{\psi_\jmath}} \right\}^{\frac{1}{\epsilon}} \cdot \\ \left\{ \sqrt[\frac{1}{\epsilon}]{\left\{\imath - \left\{\{\theta\tau_{\ell+1}\}^{\frac{1}{\epsilon}}\right\}^{\frac{1}{\epsilon}}\right\}^{\psi_1}} - \left\{\imath - \left\{\{\theta\tau_{\ell+1} + \theta\chi_{\ell+1}\}^{\frac{1}{\epsilon}}\right\}^{\frac{1}{\epsilon}}\right\}^{\psi_1}} \right\}^{\frac{1}{\epsilon}} \end{array} \right] \end{array} \right]$$

$$= \left[\begin{array}{l} \sqrt[\frac{1}{\epsilon}]{\imath - \Diamond_{\jmath=1}^{\ell+1}\left\{\imath - \left\{\{\theta\tau_1\}^{\frac{1}{\epsilon}}\right\}^{\frac{1}{\epsilon}}\right\}^{\psi_\jmath}}, \sqrt[\frac{1}{\epsilon}]{\imath - \Diamond_{\jmath=1}^{\ell+1}\left\{\imath - \left\{\{\theta\omega_1\}^{\frac{1}{\epsilon}}\right\}^{\frac{1}{\epsilon}}\right\}^{\psi_\jmath}}, \\ \sqrt[\frac{1}{\epsilon}]{\Diamond_{\jmath=1}^{\ell+1}\left\{\imath - \left\{\{\theta\tau_1\}^{\frac{1}{\epsilon}}\right\}^{\frac{1}{\epsilon}}\right\}^{\psi_\jmath}} - \Diamond_{\jmath=1}^{\ell+1}\left\{\imath - \left\{\{\theta\tau_1 + \theta\chi_1\}^{\frac{1}{\epsilon}}\right\}^{\frac{1}{\epsilon}}\right\}^{\psi_\jmath}} \end{array} \right]$$

and $\oslash_{\jmath=1}^{\ell+1}\left\{\psi_\jmath \Xi_\jmath^{\frac{1}{\epsilon}}\right\} = \left[\begin{array}{c} \left\{ \sqrt[\frac{1}{\epsilon}]{\imath - \Diamond_{\jmath=1}^{\ell+1}\left\{\imath - \left\{\{\theta\tau_\jmath\}^{\frac{1}{\epsilon}}\right\}^{\frac{1}{\epsilon}}\right\}^{\psi_\jmath}} \right\}, \\ \left\{ \sqrt[\frac{1}{\epsilon}]{\imath - \Diamond_{\jmath=1}^{\ell+1}\left\{\imath - \left\{\{\theta\omega_\jmath\}^{\frac{1}{\epsilon}}\right\}^{\frac{1}{\epsilon}}\right\}^{\psi_\jmath}} \right\}, \\ \left\{ \sqrt[\frac{1}{\epsilon}]{\Diamond_{\jmath=1}^{\ell+1}\left\{\imath - \left\{\{\theta\tau_\jmath\}^{\frac{1}{\epsilon}}\right\}^{\frac{1}{\epsilon}}\right\}^{\psi_\jmath}} - \Diamond_{\jmath=1}^{\ell+1}\left\{\imath - \left\{\{\theta\tau_\jmath + \theta\chi_\jmath\}^{\frac{1}{\epsilon}}\right\}^{\frac{1}{\epsilon}}\right\}^{\psi_\jmath}} \right\} \end{array} \right].$

Corollary 3.10. *Let $\Xi_\jmath = \langle \theta\tau_\jmath, \theta\omega_\jmath, \theta\chi_\jmath \rangle$ be the IFFSTNNs and all are equal. Then GIFFSTNWA $\{\Xi_1, \Xi_2, ..., \Xi_\ell\} = \Xi$.*

3.4. *Generalized IFFSTNWG (GIFFSTNWG) operator*

Definition 3.11. Let $\Xi_\jmath = \langle \theta\tau_\jmath, \theta\tau_\jmath, \theta\chi_\jmath \rangle$ be the IFFSTNNs, $\psi_\jmath\}$ be the weight of $\Xi_\jmath$, where $\jmath = 1, 2, ..., \ell$. Then, the GIFFSTNWG$\{\Xi_1, \Xi_2, ..., \Xi_\ell\} = \frac{1}{\aleph}\left\{\Diamond_{\jmath=1}^{k}\{\aleph\Xi_\jmath\}^{\psi_\jmath}\right\}$.

Theorem 3.12. *Let* $\Xi_\jmath = \left\langle \theta\tau_\jmath, \theta\tau_\jmath, \theta\chi_\jmath \right\rangle$ *be the collection of IFFSTNNs. Then the GIFFSTNWG operator* $\{\Xi_1, \Xi_2, ..., \Xi_\ell\} =$

$$\left[\begin{array}{c} \left\{ \sqrt[\frac{1}{\epsilon}]{\Diamond_{\jmath=1}^{k} \left\{ \imath - \left\{ \{\theta\chi_\jmath\}^{\frac{1}{\epsilon}} \right\}^{\frac{1}{\epsilon}} \right\}^{\psi_\jmath}} - \Diamond_{\jmath=1}^{k} \left\{ \imath - \left\{ \{\theta\tau_\jmath + \theta\chi_\jmath\}^{\frac{1}{\epsilon}} \right\}^{\frac{1}{\epsilon}} \right\}^{\psi_\jmath} \right\}, \\ \left\{ \sqrt[\frac{1}{\epsilon}]{\imath - \Diamond_{\jmath=1}^{k} \left\{ \imath - \left\{ \{\theta\omega_\jmath\}^{\frac{1}{\epsilon}} \right\}^{\frac{1}{\epsilon}} \right\}^{\psi_\jmath}} \right\} \left\{ \sqrt[\frac{1}{\epsilon}]{\imath - \Diamond_{\jmath=1}^{k} \left\{ \imath - \left\{ \{\theta\chi_\jmath\}^{\frac{1}{\epsilon}} \right\}^{\frac{1}{\epsilon}} \right\}^{\psi_\jmath}} \right\} \end{array} \right]$$

Proof. Using the induction method,

$$\Diamond_{\jmath=1}^{k} \{\aleph\Xi_\jmath\}^{\psi_\jmath} = \left[\begin{array}{c} \sqrt[\frac{1}{\epsilon}]{\Diamond_{\jmath=1}^{k} \left\{ \imath - \left\{ \{\theta\chi_\jmath\}^{\frac{1}{\epsilon}} \right\}^{\frac{1}{\epsilon}} \right\}^{\psi_\jmath}} - \Diamond_{\jmath=1}^{k} \left\{ \imath - \left\{ \{\theta\tau_\jmath + \theta\chi_\jmath\}^{\frac{1}{\epsilon}} \right\}^{\frac{1}{\epsilon}} \right\}^{\psi_\jmath}}, \\ \sqrt[\frac{1}{\epsilon}]{\imath - \Diamond_{\jmath=1}^{k} \left\{ \imath - \left\{ \{\theta\omega_\jmath\}^{\frac{1}{\epsilon}} \right\}^{\frac{1}{\epsilon}} \right\}^{\psi_\jmath}} \sqrt[\frac{1}{\epsilon}]{\imath - \Diamond_{\jmath=1}^{k} \left\{ \imath - \left\{ \{\theta\chi_\jmath\}^{\frac{1}{\epsilon}} \right\}^{\frac{1}{\epsilon}} \right\}^{\psi_\jmath}} \end{array} \right]$$

If $\jmath = 2$, then

$$\{\aleph\Xi_1\}^{\psi_1} = \left[\begin{array}{c} \sqrt[\frac{1}{\epsilon}]{\left\{ \imath - \left\{ \{\theta\chi_1\}^{\frac{1}{\epsilon}} \right\}^{\frac{1}{\epsilon}} \right\}^{\psi_1}} - \left\{ \imath - \left\{ \{\theta\tau_1 + \theta\chi_1\}^{\frac{1}{\epsilon}} \right\}^{\frac{1}{\epsilon}} \right\}^{\psi_1}} \\ \sqrt[\frac{1}{\epsilon}]{\imath - \left\{ \imath - \left\{ \{\theta\omega_1\}^{\frac{1}{\epsilon}} \right\}^{\frac{1}{\epsilon}} \right\}^{\psi_1}} \sqrt[\frac{1}{\epsilon}]{\imath - \left\{ \imath - \left\{ \{\theta\chi_1\}^{\frac{1}{\epsilon}} \right\}^{\frac{1}{\epsilon}} \right\}^{\psi_1}} \end{array} \right]$$

and

$$\{\aleph\Xi_2\}^{\psi_2} = \left[\begin{array}{c} \sqrt[\frac{1}{\epsilon}]{\left\{ \imath - \left\{ \{\theta\chi_2\}^{\frac{1}{\epsilon}} \right\}^{\frac{1}{\epsilon}} \right\}^{\psi_1}} - \left\{ \imath - \left\{ \{\theta\tau_2 + \theta\chi_2\}^{\frac{1}{\epsilon}} \right\}^{\frac{1}{\epsilon}} \right\}^{\psi_1}} \\ \sqrt[\frac{1}{\epsilon}]{\imath - \left\{ \imath - \left\{ \{\theta\omega_2\}^{\frac{1}{\epsilon}} \right\}^{\frac{1}{\epsilon}} \right\}^{\psi_1}} \sqrt[\frac{1}{\epsilon}]{\imath - \left\{ \imath - \left\{ \{\theta\chi_2\}^{\frac{1}{\epsilon}} \right\}^{\frac{1}{\epsilon}} \right\}^{\psi_1}} \end{array} \right]$$

We get, $\{\aleph\Xi_1\}^{\psi_1} \divideontimes \{\aleph\Xi_2\}^{\psi_2}$

$$= \left[\begin{array}{l} \sqrt[\frac{1}{\epsilon}]{\left\{\sqrt[\frac{1}{\epsilon}]{\imath - \left\{\imath - \left\{\{\theta\chi_1\}^{\frac{1}{\epsilon}}\right\}^{\frac{1}{\epsilon}}\right\}^{\psi_1}}\right\}^{\frac{1}{\epsilon}} + \left\{\sqrt[\frac{1}{\epsilon}]{\imath - \left\{\imath - \left\{\{\theta\chi_2\}^{\frac{1}{\epsilon}}\right\}^{\frac{1}{\epsilon}}\right\}^{\psi_1}}\right\}^{\frac{1}{\epsilon}} - \left\{\sqrt[\frac{1}{\epsilon}]{\imath - \left\{\imath - \left\{\{\theta\chi_1\}^{\frac{1}{\epsilon}}\right\}^{\frac{1}{\epsilon}}\right\}^{\psi_1}}\right\}^{\frac{1}{\epsilon}} \cdot \left\{\sqrt[\frac{1}{\epsilon}]{\imath - \left\{\imath - \left\{\{\theta\chi_2\}^{\frac{1}{\epsilon}}\right\}^{\frac{1}{\epsilon}}\right\}^{\psi_1}}\right\}^{\frac{1}{\epsilon}} - \left[\begin{array}{l} \left\{\sqrt[\frac{1}{\epsilon}]{\left\{\imath - \left\{\{\theta\chi_1\}^{\frac{1}{\epsilon}}\right\}^{\frac{1}{\epsilon}}\right\}^{\psi_1} - \left\{\imath - \left\{\{\theta\tau_1 + \theta\tau_1\}^{\frac{1}{\epsilon}}\right\}^{\frac{1}{\epsilon}}\right\}^{\psi_1}}\right\}^{\frac{1}{\epsilon}} \cdot \\ \left\{\sqrt[\frac{1}{\epsilon}]{\left\{\imath - \left\{\{\theta\chi_2\}^{\frac{1}{\epsilon}}\right\}^{\frac{1}{\epsilon}}\right\}^{\psi_1} - \left\{\imath - \left\{\{\theta\tau_2 + \theta\chi_2\}^{\frac{1}{\epsilon}}\right\}^{\frac{1}{\epsilon}}\right\}^{\psi_1}}\right\}^{\frac{1}{\epsilon}} \end{array} \right], \\ \sqrt[\frac{1}{\epsilon}]{\left\{\sqrt[\frac{1}{\epsilon}]{\imath - \left\{\imath - \left\{\{\theta\omega_1\}^{\frac{1}{\epsilon}}\right\}^{\frac{1}{\epsilon}}\right\}^{\psi_1}}\right\}^{\frac{1}{\epsilon}} + \left\{\sqrt[\frac{1}{\epsilon}]{\imath - \left\{\imath - \left\{\{\theta\omega_2\}^{\frac{1}{\epsilon}}\right\}^{\frac{1}{\epsilon}}\right\}^{\psi_1}}\right\}^{\frac{1}{\epsilon}} - \left\{\sqrt[\frac{1}{\epsilon}]{\imath - \left\{\imath - \left\{\{\theta\omega_1\}^{\frac{1}{\epsilon}}\right\}^{\frac{1}{\epsilon}}\right\}^{\psi_1}}\right\}^{\frac{1}{\epsilon}} \cdot \left\{\sqrt[\frac{1}{\epsilon}]{\imath - \left\{\imath - \left\{\{\theta\omega_2\}^{\frac{1}{\epsilon}}\right\}^{\frac{1}{\epsilon}}\right\}^{\psi_1}}\right\}^{\frac{1}{\epsilon}}} \\ \sqrt[\frac{1}{\epsilon}]{\left\{\sqrt[\frac{1}{\epsilon}]{\imath - \left\{\imath - \left\{\{\theta\chi_1\}^{\frac{1}{\epsilon}}\right\}^{\frac{1}{\epsilon}}\right\}^{\psi_1}}\right\}^{\frac{1}{\epsilon}} + \left\{\sqrt[\frac{1}{\epsilon}]{\imath - \left\{\imath - \left\{\{\theta\chi_2\}^{\frac{1}{\epsilon}}\right\}^{\frac{1}{\epsilon}}\right\}^{\psi_1}}\right\}^{\frac{1}{\epsilon}} - \left\{\sqrt[\frac{1}{\epsilon}]{\imath - \left\{\imath - \left\{\{\theta\chi_1\}^{\frac{1}{\epsilon}}\right\}^{\frac{1}{\epsilon}}\right\}^{\psi_1}}\right\}^{\frac{1}{\epsilon}} \cdot \left\{\sqrt[\frac{1}{\epsilon}]{\imath - \left\{\imath - \left\{\{\theta\chi_2\}^{\frac{1}{\epsilon}}\right\}^{\frac{1}{\epsilon}}\right\}^{\psi_1}}\right\}^{\frac{1}{\epsilon}}} \end{array} \right]$$

$$= \left[\begin{array}{l} \sqrt[\frac{1}{\epsilon}]{\Diamond_{\jmath=1}^{\frac{1}{\epsilon}}\left\{\imath - \left\{\{\theta\chi_\jmath\}^{\frac{1}{\epsilon}}\right\}^{\frac{1}{\epsilon}}\right\}^{\psi_\jmath} - \Diamond_{\jmath=1}^{\frac{1}{\epsilon}}\left\{\imath - \left\{\{\theta\tau_\jmath + \theta\chi_\jmath\}^{\frac{1}{\epsilon}}\right\}^{\frac{1}{\epsilon}}\right\}^{\psi_\jmath}}, \\ \sqrt[\frac{1}{\epsilon}]{\imath - \Diamond_{\jmath=1}^{\frac{1}{\epsilon}}\left\{\imath - \left\{\{\theta\omega_\jmath\}^{\frac{1}{\epsilon}}\right\}^{\frac{1}{\epsilon}}\right\}^{\psi_\jmath}}, \sqrt[\frac{1}{\epsilon}]{\imath - \Diamond_{\jmath=1}^{\frac{1}{\epsilon}}\left\{\imath - \left\{\{\theta\chi_\jmath\}^{\frac{1}{\epsilon}}\right\}^{\frac{1}{\epsilon}}\right\}^{\psi_\jmath}} \end{array} \right]$$

If $\jmath = \ell$, then

$$= \left[\begin{array}{l} \sqrt[\frac{1}{\epsilon}]{\Diamond_{\jmath=1}^{\ell}\left\{\imath - \left\{\{\theta\chi_\jmath\}^{\frac{1}{\epsilon}}\right\}^{\frac{1}{\epsilon}}\right\}^{\psi_\jmath} - \Diamond_{\jmath=1}^{\ell}\left\{\imath - \left\{\{\theta\tau_\jmath + \theta\chi_\jmath\}^{\frac{1}{\epsilon}}\right\}^{\frac{1}{\epsilon}}\right\}^{\psi_\jmath}}, \\ \sqrt[\frac{1}{\epsilon}]{\imath - \Diamond_{\jmath=1}^{\ell}\left\{\imath - \left\{\{\theta\omega_\jmath\}^{\frac{1}{\epsilon}}\right\}^{\frac{1}{\epsilon}}\right\}^{\psi_\jmath}}, \sqrt[\frac{1}{\epsilon}]{\imath - \Diamond_{\jmath=1}^{\ell}\left\{\imath - \left\{\{\theta\chi_\jmath\}^{\frac{1}{\epsilon}}\right\}^{\frac{1}{\epsilon}}\right\}^{\psi_\jmath}} \end{array} \right]$$

If $\jmath = \ell + 1$, then $\Diamond_{\jmath=1}^{\ell}\{\aleph\Xi_\jmath\}^{\psi_\jmath} \cdot \{\aleph\Xi_{\ell+1}\}^{\psi_{\ell+1}} = \Diamond_{\jmath=1}^{\ell+1}\{\aleph\Xi_\jmath\}^{\psi_\jmath}$.

Now, $\Diamond_{\jmath=1}^{\ell}\{\aleph\Xi_\jmath\}^{\psi_\jmath} \cdot \{\aleph\Xi_{\ell+1}\}^{\psi_{\ell+1}} = \{\aleph\Xi_1\}^{\psi_1} \divideontimes \{\aleph\Xi_2\}^{\psi_2} \divideontimes ... \divideontimes \{\aleph\Xi_\ell\}^{\psi_\ell} \divideontimes \{\aleph\Xi_{\ell+1}\}^{\psi_{\ell+1}}$

$$= \left[\begin{array}{l} \sqrt[\frac{1}{\epsilon}]{\left\{\sqrt[\frac{1}{\epsilon}]{\imath - \Diamond_{\jmath=1}^{\ell}\left\{\imath - \left\{\{\theta\chi_\jmath\}^{\frac{1}{\epsilon}}\right\}^{\frac{1}{\epsilon}}\right\}^{\psi_\jmath}}\right\}^{\frac{1}{\epsilon}} + \left\{\sqrt[\frac{1}{\epsilon}]{\imath - \left\{\imath - \left\{\{\theta\chi_{\ell+1}\}^{\frac{1}{\epsilon}}\right\}^{\frac{1}{\epsilon}}\right\}^{\psi_1}}\right\}^{\frac{1}{\epsilon}} - \left\{\sqrt[\frac{1}{\epsilon}]{\imath - \Diamond_{\jmath=1}^{\ell}\left\{\imath - \left\{\{\theta\chi_\jmath\}^{\frac{1}{\epsilon}}\right\}^{\frac{1}{\epsilon}}\right\}^{\psi_\jmath}}\right\}^{\frac{1}{\epsilon}} \cdot \left\{\sqrt[\frac{1}{\epsilon}]{\imath - \left\{\imath - \left\{\{\theta\chi_{\ell+1}\}^{\frac{1}{\epsilon}}\right\}^{\frac{1}{\epsilon}}\right\}^{\psi_1}}\right\}^{\frac{1}{\epsilon}} - \left[\begin{array}{l} \left\{\sqrt[\frac{1}{\epsilon}]{\Diamond_{\jmath=1}^{\ell}\left\{\imath - \left\{\{\theta\chi_\jmath\}^{\frac{1}{\epsilon}}\right\}^{\frac{1}{\epsilon}}\right\}^{\psi_\jmath}} - \Diamond_{\jmath=1}^{\ell}\left\{\imath - \left\{\{\theta\tau_\jmath + \theta\chi_\jmath\}^{\frac{1}{\epsilon}}\right\}^{\frac{1}{\epsilon}}\right\}^{\psi_\jmath}}\right\}^{\frac{1}{\epsilon}} \cdot \\ \left\{\sqrt[\frac{1}{\epsilon}]{\left\{\imath - \left\{\{\theta\chi_{\ell+1}\}^{\frac{1}{\epsilon}}\right\}^{\frac{1}{\epsilon}}\right\}^{\psi_1}} - \left\{\imath - \left\{\{\theta\tau_{\ell+1} + \theta\chi_{\ell+1}\}^{\frac{1}{\epsilon}}\right\}^{\frac{1}{\epsilon}}\right\}^{\psi_1}}\right\}^{\frac{1}{\epsilon}} \end{array}\right] \end{array}, \right.$$

$$\left. \begin{array}{l} \sqrt[\frac{1}{\epsilon}]{\left\{\sqrt[\frac{1}{\epsilon}]{\imath - \Diamond_{\jmath=1}^{\ell}\left\{\imath - \left\{\{\theta\omega_\jmath\}^{\frac{1}{\epsilon}}\right\}^{\frac{1}{\epsilon}}\right\}^{\psi_\jmath}}\right\}^{\frac{1}{\epsilon}} + \left\{\sqrt[\frac{1}{\epsilon}]{\imath - \left\{\imath - \left\{\{\theta\omega_{\ell+1}\}^{\frac{1}{\epsilon}}\right\}^{\frac{1}{\epsilon}}\right\}^{\psi_1}}\right\}^{\frac{1}{\epsilon}} - \left\{\sqrt[\frac{1}{\epsilon}]{\imath - \Diamond_{\jmath=1}^{\ell}\left\{\imath - \left\{\{\theta\omega_\jmath\}^{\frac{1}{\epsilon}}\right\}^{\frac{1}{\epsilon}}\right\}^{\psi_\jmath}}\right\}^{\frac{1}{\epsilon}} \cdot \left\{\sqrt[\frac{1}{\epsilon}]{\imath - \left\{\imath - \left\{\{\theta\omega_{\ell+1}\}^{\frac{1}{\epsilon}}\right\}^{\frac{1}{\epsilon}}\right\}^{\psi_1}}\right\}^{\frac{1}{\epsilon}} \\ \sqrt[\frac{1}{\epsilon}]{\left\{\sqrt[\frac{1}{\epsilon}]{\imath - \Diamond_{\jmath=1}^{\ell}\left\{\imath - \left\{\{\theta\chi_\jmath\}^{\frac{1}{\epsilon}}\right\}^{\frac{1}{\epsilon}}\right\}^{\psi_\jmath}}\right\}^{\frac{1}{\epsilon}} + \left\{\sqrt[\frac{1}{\epsilon}]{\imath - \left\{\imath - \left\{\{\theta\chi_{\ell+1}\}^{\frac{1}{\epsilon}}\right\}^{\frac{1}{\epsilon}}\right\}^{\psi_1}}\right\}^{\frac{1}{\epsilon}} - \left\{\sqrt[\frac{1}{\epsilon}]{\imath - \Diamond_{\jmath=1}^{\ell}\left\{\imath - \left\{\{\theta\chi_\jmath\}^{\frac{1}{\epsilon}}\right\}^{\frac{1}{\epsilon}}\right\}^{\psi_\jmath}}\right\}^{\frac{1}{\epsilon}} \cdot \left\{\sqrt[\frac{1}{\epsilon}]{\imath - \left\{\imath - \left\{\{\theta\chi_{\ell+1}\}^{\frac{1}{\epsilon}}\right\}^{\frac{1}{\epsilon}}\right\}^{\psi_1}}\right\}^{\frac{1}{\epsilon}} \end{array}} \right]$$

$$= \left[\begin{array}{l} \sqrt[\frac{1}{\epsilon}]{\Diamond_{\jmath=1}^{\ell+1}\left\{\imath - \left\{\{\theta\chi_1\}^{\frac{1}{\epsilon}}\right\}^{\frac{1}{\epsilon}}\right\}^{\psi_\jmath}} - \Diamond_{\jmath=1}^{\ell+1}\left\{\imath - \left\{\{\theta\tau_1 + \theta\chi_1\}^{\frac{1}{\epsilon}}\right\}^{\frac{1}{\epsilon}}\right\}^{\psi_\jmath}}, \\ \sqrt[\frac{1}{\epsilon}]{\imath - \Diamond_{\jmath=1}^{\ell+1}\left\{\imath - \left\{\{\theta\omega_1\}^{\frac{1}{\epsilon}}\right\}^{\frac{1}{\epsilon}}\right\}^{\psi_\jmath}}, \sqrt[\frac{1}{\epsilon}]{\imath - \Diamond_{\jmath=1}^{\ell+1}\left\{\imath - \left\{\{\theta\chi_1\}^{\frac{1}{\epsilon}}\right\}^{\frac{1}{\epsilon}}\right\}^{\psi_\jmath}} \end{array} \right]$$

Hence

$$\frac{1}{\aleph}\left\{\Diamond_{\jmath=1}^{\ell+1}\{\aleph\Xi_\jmath\}^{\psi_\jmath}\right\} = \left[\begin{array}{l} \left\{\sqrt[\frac{1}{\epsilon}]{\Diamond_{\jmath=1}^{\ell+1}\left\{\imath - \left\{\{\theta\chi_\jmath\}^{\frac{1}{\epsilon}}\right\}^{\frac{1}{\epsilon}}\right\}^{\psi_\jmath}} - \Diamond_{\jmath=1}^{\ell+1}\left\{\imath - \left\{\{\theta\tau_\jmath + \theta\chi_\jmath\}^{\frac{1}{\epsilon}}\right\}^{\frac{1}{\epsilon}}\right\}^{\psi_\jmath}}\right\}, \\ \left\{\sqrt[\frac{1}{\epsilon}]{\imath - \Diamond_{\jmath=1}^{\ell+1}\left\{\imath - \left\{\{\theta\omega_\jmath\}^{\frac{1}{\epsilon}}\right\}^{\frac{1}{\epsilon}}\right\}^{\psi_\jmath}}\right\}\left\{\sqrt[\frac{1}{\epsilon}]{\imath - \Diamond_{\jmath=1}^{\ell+1}\left\{\imath - \left\{\{\theta\chi_\jmath\}^{\frac{1}{\epsilon}}\right\}^{\frac{1}{\epsilon}}\right\}^{\psi_\jmath}}\right\} \end{array} \right]$$

Corollary 3.13. *Let $\Xi_\jmath = \langle \theta\tau_\jmath, \theta\omega_\jmath, \theta\chi_\jmath \rangle$ be the collection of IFFSTNNs and all are equal. Then GIFFSTNWG $\{\Xi_1, \Xi_2, ..., \Xi_\ell\} = \Xi$.*

4. Conclusion:

The algebraic accessibility of ED and HD for IFFSTNNs is a significant advantage. HD of IFFSTNNs may offer significant improvements in data analysis. The utilization of compelling statistics illustrates the advantages of HD. We provided IFFSTNWA, IFFSTNWG, GIFFSTNWA, and GIFFSTNWG examples and suggested models. The following subjects will be covered in more detail later on: A more detailed treatment of the following topics will be covered: (1) Interaction AOs establish a link between the cubic NS and IVPFS. (2) The issue may be resolved by using complex IFFSTNWA, complex IFFSTNWG, complex GIFFSTNWA, and complex GIFFSTNWG. (3) The following topics will be discussed in more detail as Soft and expert sets were explored in terms of the Diophantine vague NSS, complex vague NSS, q-Rung interval valued NSS and complex cubic q-Rung NSS.

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Advanced Machine Learning Approaches for Breast Cancer Detection with Neutrosophic Sets

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Abstract — Breast Cancer (BC) remains a significant health challenge for women and is one of the leading causes of mortality worldwide. Accurate diagnosis is critical for successful therapy and increased survival rates. Recent advances in medical imaging and computational technologies have enabled more precise methods of detecting and evaluating breast cancer. Accurate analysis and diagnosis utilizing medical imaging have developed as essential research topics, providing important help in clinical decision-making for various illnesses, including breast cancer. Machine learning (ML) can accurately predict breast cancer. But the breast cancer data has vague and uncertainty information. So, the neutrosophic sets (NSs) are used in this study to deal with uncertainty data. We convert the original dataset into neutrosophic data with three components such as truth, indeterminacy, and falsity values. Then we applied four ML models with N-data such as logistic regression, gradient boosting (GB), k-nearest neighbor (KNN), and support vector machines (SVM), to improve diagnostic accuracy. Then we compared the ML models with and without using N-data. The results show the logistic regression has higher accuracy with 98.6% with the N-data and 95.80% without N-data. So, the NSs can improve the accuracy of ML models.

Keywords: Machine Learning; Neutrosophic Sets; Uncertainty, Breast Cancer; Prediction Task.

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1. Introduction

Breast cancer is still the second most prevalent type of cancer in the world to be diagnosed in women. Usually, this cancer develops in breast cells. An infected patient may exhibit early symptoms such as edema, skin irritation, breast lump, nipple retraction, redness, lymph node alterations, and more. Reliable approaches that are highly efficient in diagnosing and treating this type of cancer have been consistently developed over the past few decades [1].

The primary classification challenge in breast cancer diagnosis still involves distinguishing between a malignant tumor, which is cancerous in nature, and a benign tumor, which is not cancerous. The primary classification challenge in breast cancer diagnosis still involves distinguishing between a malignant tumor, which is cancerous in nature, and a benign

tumor, which is not cancerous. Researchers in the field of computer science are now able to apply different data mining [2] and machine learning [3] techniques to perform accurate classification on real-life data, thanks to an exponential increase in the amount of data available and the introduction of upgraded computational hardware.

To diagnose breast cancer, a variety of machine-learning techniques have been employed [4], including deep learning [5], fuzzy logic [6], data mining [7], artificial neural networks [8], and support vector machines (SVMs) [9]. Because of its benefits in terms of computing efficiency, versatility, and capacity to address large-dimensional datasets in a shorter amount of time, the SVM offers state-of-the-art performance among all these approaches [10].

In medical applications, NSs particularly single-valued neutrosophic sets (SVNSs)improve the management of uncertainty and imprecision. More efficient solutions for information fusion DM, and medical data processing are obtained by combining NSs with ML approaches. These techniques have demonstrated effectiveness in medical data. NSs are perfect for handling inconsistent or incomplete data since they provide a framework for representing ambiguity and uncertainty. Numerous medical applications highlight the value of NS and offer a framework for utilizing it to improve medical image processing and diagnosis.

The combination of neuromorphic sets and deep learning models creates a powerful framework for breast cancer classification that addresses uncertainty while improving model accuracy. Neuromorphic sets, which manage truth, indeterminacy, and falsity, are used to preprocess medical data like mammograms and histopathology pictures, minimizing noise and dealing with partial information. Deep learning algorithms, such as convolutional neural networks (CNNs), use these refined inputs to extract and classify features accurately, discriminating between benign and malignant tumors or cancer subtypes. This hybrid technique not only enhances diagnostic accuracy but also provides a more nuanced study of complicated medical data, with performance evaluated using metrics such as precision, recall, and ROC-AUC [11].

Neutrosophic-based machine learning algorithms offer a more sophisticated approach to breast cancer analysis and diagnosis by tackling the inherent uncertainty and vagueness of medical data. These strategies improve the preparation of complicated datasets, such as imaging or clinical records, by including neutrosophic logic, which analyzes truth, indeterminacy, and falsity components. Machine learning models combined with neutrosophic principles can detect subtle patterns and correlations in data, assisting in the accurate classification of benign and malignant tumors, among other diagnostic tasks. This hybrid method not only enhances diagnostic accuracy, but also provides a strong and dependable foundation for breast cancer detection and prognosis [12].

The major goal of employing ML models to identify breast cancer is to create a professionally constructed ML model with a low classification error for the unclear circumstances of newly diagnosed patients. The capacity to generalize is known to be significantly influenced by two challenges: feature selection and parameter optimization [13]. Using feature selection, one may choose which subset of the given features is most important for precise categorization. Computational efficiency, learning convergence, and generalization performance are the three key factors that have emphasized the significance of feature selection. Proper feature selection can provide models with improved generalization ability in addition to saving a substantial amount of computing time [14, 15]. Greater accuracy in breast cancer detection methods is still required.

This study's main goal is to provide a method that can efficiently identify and categorize breast cancer while requiring little processing time. The detection and classification of breast cancer is suggested using four ML models with N-data to deal with uncertainty information. An intelligent cancer classification method is proposed based on the selection of a feature subset and the concurrent optimization of relevant four ML classifier parameters through an intelligent algorithm to classify breast cancer. This increases the accuracy of the four ML classifier's classification for breast cancer diagnosis.

The remainder of the paper is organized as follows: related work in Section 2, followed by proposed methodology in Section 3, and the results are explained in Section 4. Discussion and conclusions are reported in Section 5.

2. Related Work

With the advancement of medical science, many novel technologies for diagnosing breast cancer have been created. The following is a quick summary of the research in this field.

Smith et al. [16] presented an enhanced neutrosophic set theory integrated with machine learning for breast cancer diagnosis. Using the WDBC dataset, the study converted data into neutrosophic representations and employed algorithms like Decision Tree, Random Forest, and AdaBoost. The neutrosophic-AdaBoost approach achieved 99.12% accuracy and 100% precision, showcasing the efficacy of neutrosophic preprocessing in reducing noise and uncertainty. This framework highlighted the potential of neutrosophic logic for improving diagnostic reliability in healthcare.

Johnson and Lee [17] introduced a hybrid method combining neutrosophic sets with an optimized Fast Fuzzy C-Means algorithm to segment thermogram images for breast cancer prediction. Their approach modeled noise and uncertainty effectively during preprocessing. The method enhanced the accuracy of thermogram analysis compared to conventional techniques, as demonstrated through experimental results. The study proved the capability of neutrosophic clustering methods in medical diagnostics.

Miller et al. [18] investigated the application of neutrosophic sets in deep learning for breast cancer detection. By integrating neutrosophic logic with convolutional neural networks (CNNs), they addressed challenges such as noise and incomplete mammogram data. Their approach led to improved accuracy and robustness compared to traditional deep learning models, highlighting neutrosophic logic as a powerful tool in medical imaging.

Garcia and Ahmed [19] proposed a neutrosophic-based machine-learning framework for breast cancer detection. The study applied neutrosophic preprocessing with algorithms like SVM and kNN to handle noisy and missing data. Results demonstrated significant improvements in classification accuracy, emphasizing the role of neutrosophic logic in managing uncertainty and enhancing decision-making in diagnostics.
Cheng et al. [20] developed a novel breast cancer classification framework by incorporating neutrosophic techniques with deep neural networks. Neutrosophic preprocessing was used to handle low-contrast and noisy mammographic images, which were then fed into a deep neural network for classification. The method achieved superior diagnostic performance, demonstrating the complementary benefits of neutrosophic logic and deep learning.

Patel et al. [21] presented a neutrosophic segmentation approach for breast lesion detection in computed tomography (CT) images. Their method converted CT data into a neutrosophic domain to differentiate noise from significant patterns. Experimental

results showed enhanced segmentation quality, improving diagnostic outcomes and proving the potential of neutrosophic processing in CT imaging.

Kumar and Tan [22] designed a deep-learning model for breast cancer detection in ultrasound images, focusing on real-time applications. Their custom neural network architecture achieved 99.29% accuracy on a large ultrasound dataset. The study also introduced preprocessing techniques like noise reduction and feature enhancement, contributing to the model's high performance. This work highlighted the role of AI in improving ultrasound-based diagnostics.

Huang et al. [23] explored the use of deep learning for breast cancer detection via microwave imaging. Their reconstruction-based algorithm processed microwave scan data to identify early-stage cancer with high accuracy and sensitivity. The study emphasized the potential of combining microwave imaging with AI for non-invasive and cost-effective diagnostics.

3. Proposed Methodology

In this study, we present a comprehensive approach for breast cancer detection utilizing ML models and neutrosophic sets. We convert the original breast cancer dataset into neutrosophic numbers to deal with vague and uncertain data in the original dataset to enhance the accuracy of the ML models under uncertainty information.

3.1 Data Preprocessing

Several preprocessing steps are applied to the dataset before training the ML models. Firstly, the dataset is loaded from a CSV file. The dataset employed in the current work is a subset of the Breast Cancer Dataset on Kaggle (https://www.kaggle.com/datasets/yasserh/breast-cancer-dataset/data).
It has 2 positive and negative classes. The dataset is divided into 70% ratio of training and 30% for testing set preserving the directory structure. The dataset contains 570 rows and 32 features.
This process involves reading the data into a suitable structure, such as a Pandas Data Frame. The dataset is checked for missing values. However, the "Unnamed: 32" column seems to contain all missing values.
The target variable "diagnosis" is encoded into numerical values using label encoder methodology.
Standardization is performed on the feature variables to put the dataset into the same range. The data processing steps are shown in detail in Figure 1.

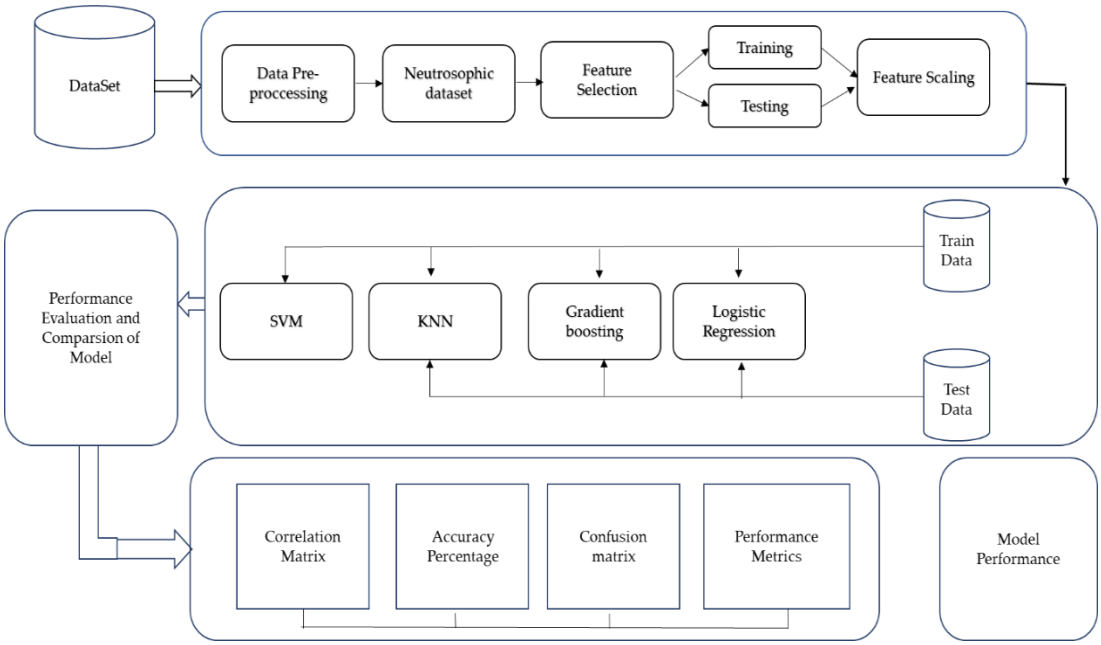

Figure 1. The framework of the study

3.2 Create Neutrosophic Data (N-data)

By putting forth a neutrosophic set, Smarandache expanded fuzzy sets even further. Truth-membership, falsity-membership, and indeterminacy-membership functions are the three independent membership functions that make up the neutrosophic set.

The augmented neutrosophic set and machine learning technique provide a comprehensive strategy for breast cancer prediction that successfully manages the uncertainty and imprecision found in medical information. This approach enhances the standard neutrosophic set by incorporating modern techniques for better modeling truth, indeterminacy, and falsity, hence enhancing data preparation quality. When combined with machine learning algorithms, it allows for the extraction of crucial patterns and features that can predict breast cancer outcomes accurately. This integration not only improves diagnostic accuracy but also allows for early detection and individualized treatment, making it an essential tool in modern oncology.

The integration of machine learning and neutrosophic multi-criteria decision-making (MCDM) methodology offers a robust framework for colorectal cancer prediction. By leveraging the neutrosophic set's ability to handle uncertainty, indeterminacy, and imprecision, this approach enhances data preprocessing and analysis, ensuring that the input features are of high quality. The MCDM methodology complements this by prioritizing and ranking critical factors involved in cancer prediction, such as genetic markers, clinical history, and lifestyle variables. Machine learning models then analyze these refined inputs to identify patterns and predict outcomes with high accuracy. This case study demonstrates the potential of combining these methodologies to improve early detection, personalized treatment planning, and overall decision-making in colorectal cancer diagnosis.

The breast cancer evaluation can contain vague and uncertainty data. So, the results of the ML models cannot be accurate. So, in this section, we convert the breast cancer dataset into neutrosophic data with three membership functions such as truth A, indeterminacy B, and falsity C.

The N-data can be defined as $< A_Y, B_Y, C_Y >$ where every component of the set $U = \{u_1, \dots, u_n\}$ As:

$$\forall u(A, B, C) \in < A_Y, B_Y, C_Y > \quad (1)$$

We change the original dataset of breast cancer into the neutrosophic data. We compute the average vectors for the dataset. M^{all}, the positive class M^{+}, and negative class as M^{-}. We can compute the average vectors as:

$$M^{all} = \sum n^{all}\, u \quad (2)$$

$$M^{+} = \sum n^{+}\, u \quad (3)$$

$$M^{-} = \sum n^{-}\, u \quad (4)$$

Then we compute the components of neutrosophic sets as:

$$A = 1 - \frac{|| \ u - M^{+} \ ||}{\max(|| \ u - M^{+} \ ||)} \quad (5)$$

$$B = 1 - \frac{|| \ u - M^{all} \ ||}{\max(|| \ u - M^{all} \ ||)} \quad (6)$$

$$C = 1 - \frac{|| \ u - M^{-} \ ||}{\max(|| \ u - M^{-} \ ||)} \quad (7)$$

We obtain the three components of neutrosophic data into the training and testing dataset. These values can improve the ML models.

3.3 Machine Learning Models

Four different techniques are showcased: logistic regression (LG), Gradient boosting (GB), k-nearest neighbor (KNN), and support vector machine (SVM). Logistic regression, a fundamental classification algorithm, is utilized to model the probability of a binary outcome.

Logistic regression is a statistical method used for binary classification tasks, where the target variable has only two possible outcomes. Logistic Regression is a type of linear model. Despite its name, it's a classification algorithm rather than a regression algorithm. Logistic Regression is widely used for binary classification tasks because of its simplicity and efficiency [24].

Gradient Boosting is a machine learning technique that creates predictive models by successively combining numerous weak learners, usually decision trees, to form a stronger model [25].

K-Nearest Neighbors (KNN) is a simple and intuitive machine learning algorithm used for classification and regression tasks. [26].

Support Vector Machine (SVM) is a supervised machine learning algorithm used for classification and regression tasks. The basic idea behind SVM is to find the hyperplane that separates the different classes in the feature space. [27].

4. Results

The correlation matrix visualizes the relationships between various features used in breast cancer analysis. Each cell represents the correlation coefficient between a pair of

features, ranging from -1 to 1. Positive values (closer to 1) indicate a strong positive correlation, where an increase in one feature corresponds to an increase in the other. Negative values (closer to -1) show an inverse relationship, where an increase in one feature correlates with a decrease in the other. Values near 0 imply little to no correlation.

From the matrix, we observe that certain features, such as radius mean, perimeter mean, and area mean, exhibit a strong positive correlation, which is expected as these metrics are geometrically related. Conversely, features like fractal_dimension_mean have weak correlations with most other features, indicating lower interdependence. Such patterns can inform feature selection during the model development process, ensuring redundant features are minimized for efficient computation and enhanced accuracy shown in Figure 2.

The diagonal values of the matrix are all 1, as each feature is perfectly correlated with itself. This matrix plays a vital role in understanding data relationships, aiding in preprocessing steps like dimensionality reduction or feature engineering.

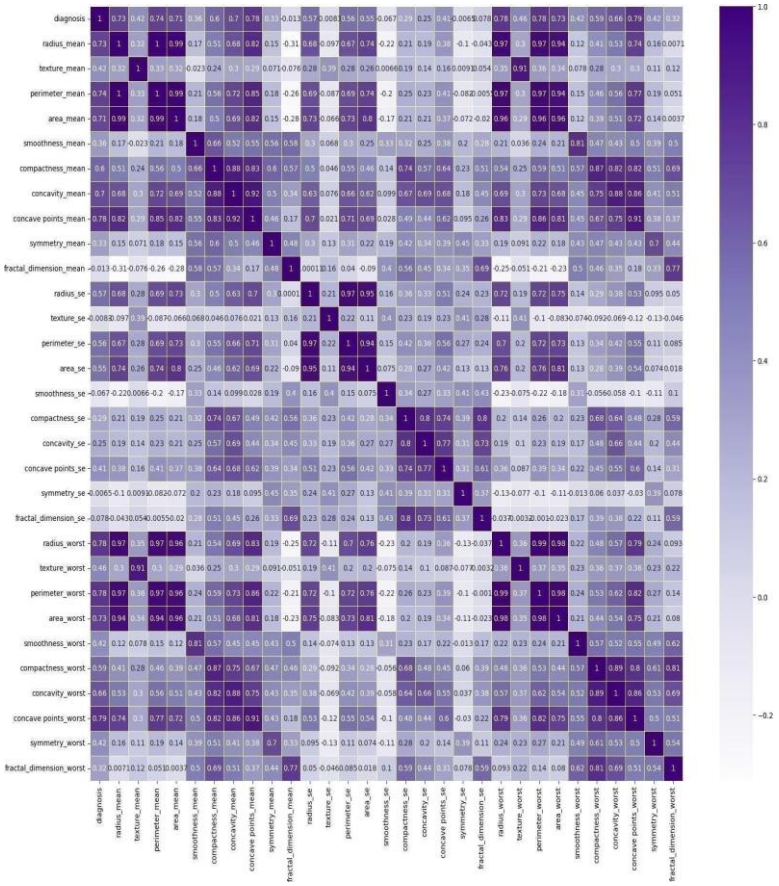

Figure 2. Correlation Matrix Between Features

4.1 Accuracy Percentage of the proposed Machine Learning Models

In the assessment stage, the model exhibited outstanding predictive capabilities, achieving higher impressive test accuracy. This remarkable performance underscores the model's proficiency in generalizing to new, unseen data, highlighting its reliability and effectiveness in real-world scenarios.

The high level of accuracy attained by the model instills confidence in its ability to make precise predictions consistently, making it an asset in various applications where accuracy and reliability are paramount. This high accuracy demonstrates that the model has successfully learned to distinguish between positive and negative images with minimal errors. Such a high level of performance is particularly crucial in medical diagnostics, where accurate predictions can significantly impact patient outcomes as shown in Table 1. We showed the logistic regression has higher accuracy with the N-dataset.

Table 1. Accuracy Percentage of the proposed Machine Learning Models

Method	Accuracy (without N-Dataset)	Accuracy (with N-Dataset)
Logistic Regression	95.80%	98.60%
SVM	51.75%	96.50%
GB	95.10%	97.20%
KNN	74.83%	95.80%

4.2 Confusion Matrix

A confusion matrix is a table that summarizes the performance of a classification algorithm. It shows the number of true positives (TP), true negatives (TN), false positives (FP), and false negatives (FN).
It's a useful tool for understanding the types of errors made by the classifier. We obtained the confusion matrix for the four ML models with and without N-Data. We show the N-Data obtained higher accuracy.

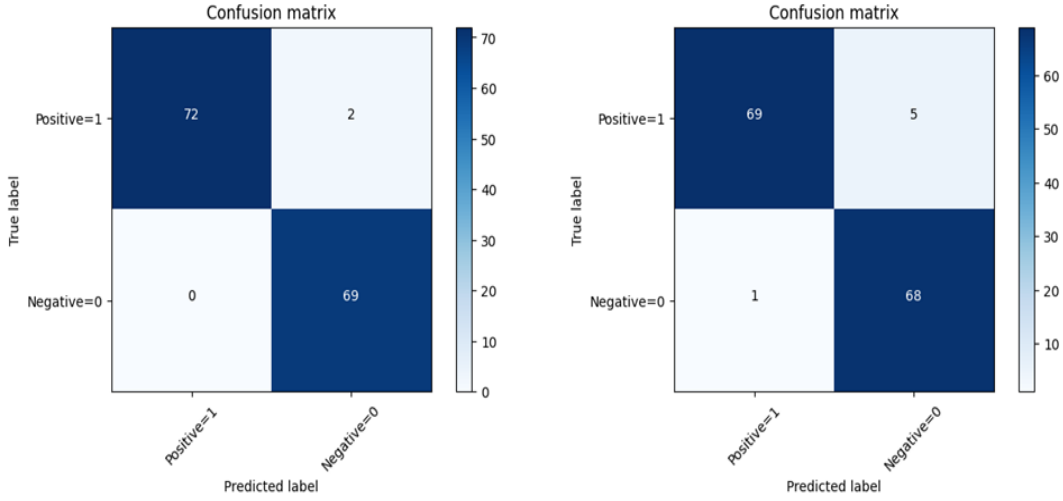

Figure 3: Confusion Matrix for logistic regression with and without N-data'

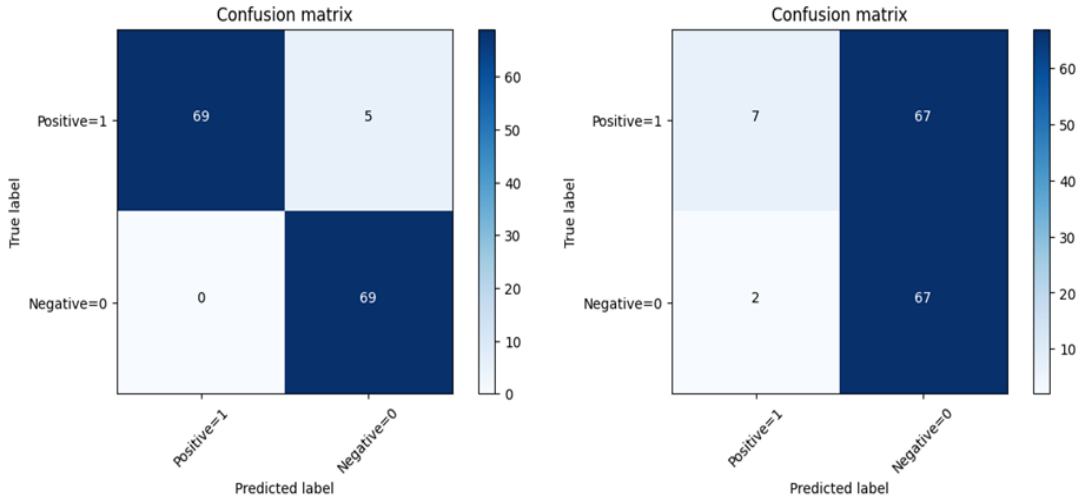

Figure 4: Confusion Matrix for SVM with and without N-data

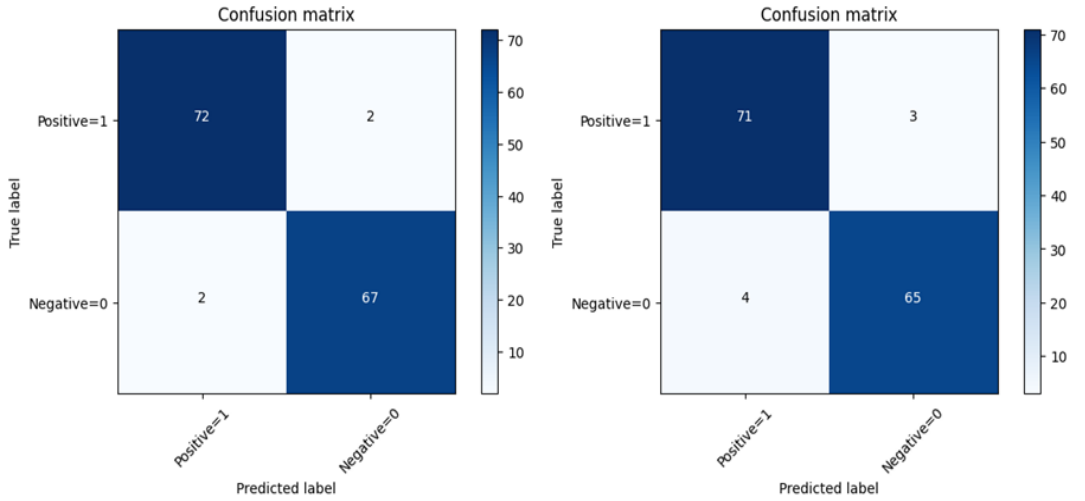

Figure 5: Confusion Matrix for GB with and without N-data

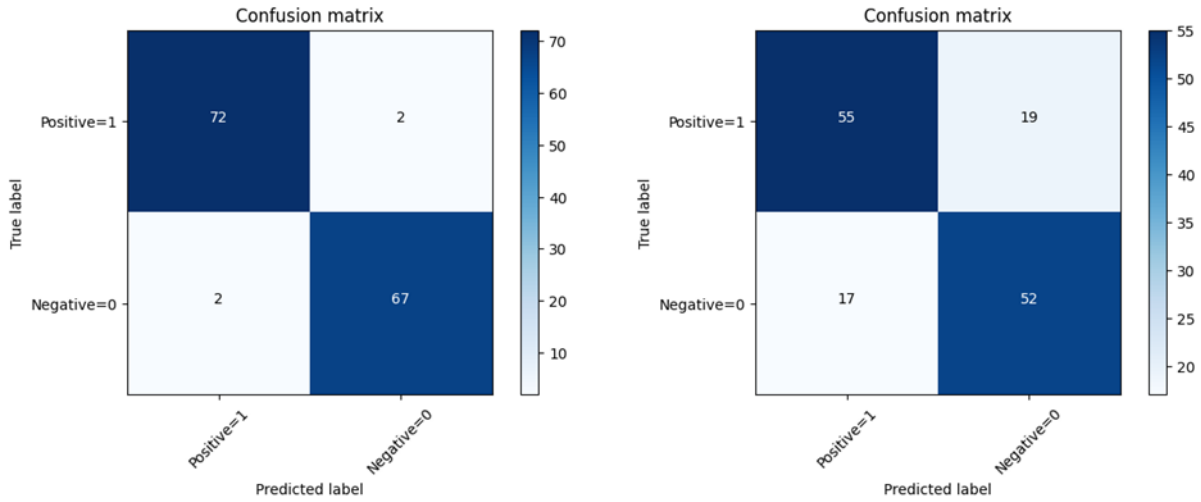

Figure 6: Confusion Matrix for KNN with and without N-data

4.3 Performance Metrics

Table 2 shows the performance characteristics of a classification model for breast cancer detection with and without N-data. It provides important measures including precision, recall, F1-score, and support for two classes: 0 (presumably non-cancerous instances) and 1 (certainly malignant ones). Precision denotes the fraction of successfully detected cases among all expected cases for a particular class, whereas recall denotes the ability to identify all actual cases inside a class.

Table 2: Classification report with and without the N-Data.

Logistic regression without N-dataset	Precision	Recall	F1 score
0	0.93	0.99	0.96
1	0.99	0.93	0.96
macro avg	0.96	0.96	0.96
weighted avg	0.96	0.96	0.96
SVM without N-dataset	Precision	Recall	F1 score
0	0.5	0.97	0.66
1	0.78	0.09	0.17
macro avg	0.64	0.53	0.41
weighted avg	0.64	0.52	0.41
GB without N-dataset	Precision	Recall	F1 score
0	0.97	0.96	0.96
1	0.96	0.97	0.97
macro avg	0.97	0.96	0.96
weighted avg	0.97	0.97	0.97
KNN without N-dataset	Precision	Recall	F1 score
0	0.73	0.75	0.74
1	0.76	0.74	0.75
macro avg	0.75	0.75	0.75
weighted avg	0.75	0.75	0.75
Logistic regression with N-dataset	Precision	Recall	F1 score
0	0.97	1	0.99
1	1	0.97	0.99
macro avg	0.99	0.99	0.99
weighted avg	0.99	0.99	0.99
SVM with N-dataset	Precision	Recall	F1 score
0	0.93	1	0.97
1	1	0.93	0.97
macro avg	0.97	0.97	0.97
weighted avg	0.97	0.97	0.97
GB with N-dataset	Precision	Recall	F1 score
0	0.97	0.97	0.97
1	0.97	0.97	0.97
macro avg	0.97	0.97	0.97
weighted avg	0.97	0.97	0.97
KNN with N-dataset	Precision	Recall	F1 score
0	0.93	0.99	0.96
1	0.99	0.93	0.96
macro avg	0.96	0.96	0.96
weighted avg	0.96	0.96	0.96

5. Conclusion and Future Work

Breast cancer remains a global health issue, ranking as one of the leading causes of death among women. Early and accurate detection is crucial to improving patient outcomes and lowering the disease's impact. Recent advances in medical imaging and machine learning have created new prospects for improving diagnostic accuracy and efficiency. In this paper we developed machine learning techniques for breast cancer detection, focusing on logistic regression, gradient boosting, KNN, and support vector machines (SVM) to enhance diagnostic accuracy. These models are applied with Neutrosophic-data and without Neutrosophic-data to deal with uncertainty information. A comprehensive strategy was developed for breast cancer diagnosis, leveraging SVM algorithms. The proposed approach obtained an improved accuracy of 98.60% with the Neutrosophic-data, highlighting its potential to considerably improve diagnostic precision and clinical decision-making procedures. So, the Neutrosophic-data is the better to deal with the uncertainty under ML models. Further work might investigate the use of deep learning approaches, such as convolutional neural networks (CNNs), to

improve feature extraction from complex imaging datasets. Furthermore, including larger and more diverse.

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Characterization of interaction aggregating operators setting interval-valued Pythogorean neutrosophic set

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Abstract. In this work, we present novel techniques for the interval-valued Pythogorean neutrosophic interaction aggregating operator. A hybrid of the neutrosophic set and the interval-valued Pythogorean fuzzy set. The innovative averaging and geometric operations of interval-valued Pythogorean neutrosophic interaction numbers are studied using aggregation operator. The interval-valued Pythogorean neutrosophic interaction is boundedness compatible, idempotent, associative, and commutative. Four new aggregating operators are introduced: IPNI weighted averaging operator, interval-valued Pythogorean neutrosophic interaction weighted geometric operator, generalized interval-valued Pythogorean neutrosophic interaction weighted averaging and generalized interval-valued Pythogorean neutrosophic interaction weighted geometric.

Keywords: weighted averaging, weighted geometric, generalized weighted averaging, generalized weighted geometric.

1. Introduction

The uncertainties led to the development of the fuzzy set (FS) [1], intuitionistic FS (IFS) [2], Pythagorean FS (PFS) [3, 4], neutrosophic set (NS) [5], Fuzzy extension and its application [6–14] and Fermatean FS (FFS) [15]. FS suggests that decision-makers use Zadeh's [1] with the membership degree (MD). Each object has MD $\measuredangle$ and non-membership degree (NMD)

ɜ, and it fulfills $0 \leq \measuredangle + \text{ɜ} \leq 1$, for $\measuredangle, \text{ɜ} \in (0,1)$. Atanassov developed an IFS idea [2]. PFSs are defined by their MDs and NMDs under the condition that $\measuredangle^2 + \text{ɜ}^2 \leq 1$, which was devised by Yager [3]. IFSs and PFSs are widely utilized and have been explored in a wide range of fields. Cuong and associates [16] by developing the concept of picture FSs (PiFSs). PiFSs has been found to accommodate certain more ambiguity because it is an expanded version of IFSs. For $\measuredangle, \text{ɜ}, \varrho \in (0,1)$, it is observed in PiFSs that the MD $\measuredangle$, the neutral ɜ, and the NMD ϱ have $0 \leq \measuredangle + \text{ɜ} + \varrho \leq 1$. "Yes," "abstain," "no," and "refusal" are expert opinion messages that will be ensured to be transmitted via the PiFS. Furthermore, it will ensure consistency between the assessment data and the actual decision environment and avoid evaluation information from being left out. Despite the fact that PiFSs have been the subject of several applications and studies, the concept has not been fully explored. Raed Hatamleh et al. discussed the various applications such as Uryson's Operator, Sine-Gordon System and Traces Class Pertubations [17–22]. Shahzaib and colleagues [23] defined the spherical FS (SFS) for certain AOs with MADM. An alternative to $0 \leq \measuredangle + \text{ɜ} + \varrho \leq 1$ is required by the SFS: $0 \leq \measuredangle^2 + \text{ɜ}^2 + \varrho^2 \leq 1$. Linguistic SFS AOs were introduced by Jin et al. [24] and discussed in MADM problems. Rafiq and colleagues introduced SFSs and their applications in DM [25] with the condition that $\measuredangle^2 + \text{ɜ}^2 \geq 1$ and decision-making (DM) are troublesome. The concept of an FFS was first proposed by Senapati and associates [15] in 2019. When $0 \leq \measuredangle^3 + \text{ɜ}^3 \leq 1$, both the MD and NMD have this characteristic. Al-Husban et al. discussed the algebraic structures via complex Fuzzy Hyperring, Fuzzy Soft Groups based on Fuzzy Space and Structures of fibers of groups actions on graphs [26–33].

2. **Different AOs for IPNN**

Definition 2.1. Suppose that $\dashv_1 = \left\langle (\measuredangle_1^{ta}, \measuredangle_1^{tb}), (\sigma_1^{ia}, \sigma_1^{ib}), (\varrho_1^{fa}, \varrho_1^{fb}) \right\rangle$ and $\dashv_2 = \left\langle (\measuredangle_2^{ta}, \measuredangle_2^{tb}), (\sigma_2^{ia}, \sigma_2^{ib}), (\varrho_2^{fa}, \varrho_2^{fb}) \right\rangle$ be the any two IPNNs. Then

$$(1)\ \dashv_1 \veebar \dashv_2 = \begin{bmatrix} \begin{bmatrix} \sqrt{(\measuredangle_1^{ta})^2 + (\measuredangle_2^{ta})^2 - (\measuredangle_1^{ta})^2 \cdot (\measuredangle_2^{ta})^2}, \\ \sqrt{(\measuredangle_1^{tb})^2 + (\measuredangle_2^{tb})^2 - (\measuredangle_1^{tb})^2 \cdot (\measuredangle_2^{tb})^2} \end{bmatrix} \\ \begin{bmatrix} \sqrt{(\sigma_1^{ia})^2 + (\sigma_2^{ia})^2 - (\sigma_1^{ia})^2 \cdot (\sigma_2^{ia})^2}, \\ \sqrt{(\sigma_1^{ib})^2 + (\sigma_2^{ib})^2 - (\sigma_1^{ib})^2 \cdot (\sigma_2^{ib})^2} \end{bmatrix} \\ \begin{bmatrix} \sqrt{\begin{matrix} (\varrho_1^{fa})^2 + (\varrho_2^{fa})^2 - (\varrho_1^{fa})^2 \cdot (\varrho_2^{fa})^2 \\ -(\varrho_1^{fa})^2 \cdot (\measuredangle_2^{ta})^2 - (\measuredangle_1^{ta})^2 \cdot (\varrho_2^{fa})^2 \end{matrix}}, \\ \sqrt{\begin{matrix} (\varrho_1^{fb})^2 + (\varrho_2^{fb})^2 - (\varrho_1^{fb})^2 \cdot (\varrho_2^{fb})^2 \\ -(\varrho_1^{fb})^2 \cdot (\measuredangle_2^{tb})^2 - (\measuredangle_1^{tb})^2 \cdot (\varrho_2^{fb})^2 \end{matrix}} \end{bmatrix} \end{bmatrix}$$

$$(2)\ \exists_1 \overline{\wedge}\ \exists_2 = \begin{bmatrix} \begin{bmatrix} \sqrt{\begin{matrix}(\measuredangle_1^{ta})^2+(\measuredangle_2^{ta})^2-(\measuredangle_1^{ta})^2\cdot(\measuredangle_2^{ta})^2 \\ -(\measuredangle_1^{ta})^2\cdot(\varrho_2^{fa})^2-(\varrho_1^{fa})^2\cdot(\measuredangle_2^{ta})^2\text{'}\end{matrix}} \\ \sqrt{\begin{matrix}(\measuredangle_1^{tb})^2+(\measuredangle_2^{tb})^2-(\measuredangle_1^{tb})^2\cdot(\measuredangle_2^{tb})^2 \\ -(\measuredangle_1^{tb})^2\cdot(\varrho_2^{fb})^2-(\varrho_1^{fb})^2\cdot(\measuredangle_2^{tb})^2\end{matrix}} \end{bmatrix} \\ \begin{bmatrix} \sqrt{(\sigma_1^{ia})^2+(\sigma_2^{ia})^2-(\sigma_1^{ia})^2\cdot(\sigma_2^{ia})^2}, \\ \sqrt{(\sigma_1^{ib})^2+(\sigma_2^{ib})^2-(\sigma_1^{ib})^2\cdot(\sigma_2^{ib})^2} \end{bmatrix} \\ \begin{bmatrix} \sqrt{(\varrho_1^{fa})^2+(\varrho_2^{fa})^2-(\varrho_1^{fa})^2\cdot(\varrho_2^{fa})^2}, \\ \sqrt{(\varrho_1^{fb})^2+(\varrho_2^{fb})^2-(\varrho_1^{fb})^2\cdot(\varrho_2^{fb})^2} \end{bmatrix} \end{bmatrix}$$

$$(3)\ \zeta\cdot\exists_1 = \begin{bmatrix} \left[\sqrt{1-\left(1-(\measuredangle_1^{ta})^2\right)^{\zeta}}, \sqrt{1-\left(1-(\measuredangle_1^{tb})^2\right)^{\zeta}}\right] \\ \left[\sqrt{1-\left(1-(\sigma_1^{ia})^2\right)^{\zeta}}, \sqrt{1-\left(1-(\sigma_1^{ib})^2\right)^{\zeta}}\right] \\ \left[\sqrt{\left(1-(\measuredangle_1^{ta})^2\right)^{\zeta}-\left(1-(\measuredangle_1^{ta}+\varrho_1^{fa})^2\right)^{\zeta}}, \right. \\ \left. \sqrt{\left(1-(\measuredangle_1^{tb})^2\right)^{\zeta}-\left(1-(\measuredangle_1^{tb}+\varrho_1^{fb})^2\right)^{\zeta}}\right] \end{bmatrix}$$

$$(4)\ \exists_1^{\zeta} = \begin{bmatrix} \left[\sqrt{\left(1-(\varrho_1^{fa})^2\right)^{\zeta}-\left(1-(\measuredangle_1^{ta}+\varrho_1^{fa})^2\right)^{\zeta}}, \right. \\ \left. \sqrt{\left(1-(\varrho_1^{fb})^2\right)^{\zeta}-\left(1-(\measuredangle_1^{tb}+\varrho_1^{fb})^2\right)^{\zeta}}\right] \\ \left[\sqrt{1-\left(1-(\sigma_1^{ia})^2\right)^{\zeta}}, \sqrt{1-\left(1-(\sigma_1^{ib})^2\right)^{\zeta}}, \right] \\ \left[\sqrt{1-\left(1-(\varrho_1^{fa})^2\right)^{\zeta}}, \sqrt{1-\left(1-(\varrho_1^{fb})^2\right)^{\zeta}}\right] \end{bmatrix}$$

2.1. *IPNIWAO*

Definition 2.2. Let $\exists_\partial = \left\langle (\measuredangle_\partial^{ta}, \measuredangle_\partial^{tb}), (\sigma_\partial^{ia}, \sigma_\partial^{ib}), (\varrho_\partial^{fa}, \varrho_\partial^{fb}) \right\rangle$ be the IPNNs, $\partial = 1, 2, ..., \propto, \ni_\partial$ be the weight of $\exists_\partial$ and $\ni_\partial \geq 0, \Diamond_{\partial=1}^{\propto} \ni_\partial = 1$. Then the IPNIWAO $(\exists_1, \exists_2, ..., \exists_\propto) = \Diamond_{\partial=1}^{\propto} \ni_\partial\ \exists_\partial$.

Theorem 2.3. *Let* $\exists_\partial = \left\langle (\measuredangle_\partial^{ta}, \measuredangle_\partial^{tb}), (\sigma_\partial^{ia}, \sigma_\partial^{ib}), (\varrho_\partial^{fa}, \varrho_\partial^{fb}) \right\rangle$ *be the IPNNs,* $\partial = 1, 2, ..., \propto$.

$$\textit{Then,IPNIWAO}(\exists_1, \exists_2, ..., \exists_\propto) = \begin{bmatrix} \left[\sqrt{1-\square_{\partial=1}^{\propto}\left(1-(\measuredangle_\partial^{ta})^2\right)^{\ni_\partial}}, \sqrt{1-\square_{\partial=1}^{\propto}\left(1-(\measuredangle_\partial^{tb})^2\right)^{\ni_\partial}}\right] \\ \left[\sqrt{1-\square_{\partial=1}^{\propto}\left(1-(\sigma_\partial^{ia})^2\right)^{\ni_\partial}}, \sqrt{1-\square_{\partial=1}^{\propto}\left(1-(\sigma_\partial^{ib})^2\right)^{\ni_\partial}}\right] \\ \left[\sqrt{\square_{\partial=1}^{\propto}\left(1-(\measuredangle_\partial^{ta})^2\right)^{\ni_\partial}-\square_{\partial=1}^{\propto}\left(1-(\measuredangle_\partial^{ta}+\varrho_\partial^{fa})^2\right)^{\ni_\partial}}, \right. \\ \left. \sqrt{\square_{\partial=1}^{\propto}\left(1-(\measuredangle_\partial^{tb})^2\right)^{\ni_\partial}-\square_{\partial=1}^{\propto}\left(1-(\measuredangle_\partial^{tb}+\varrho_\partial^{fb})^2\right)^{\ni_\partial}}\right] \end{bmatrix}.$$

Proof. If $\partial = 2$, IPNIWAO$(\dashv_1, \dashv_2) = \jmath_1 \dashv_1 \veebar \jmath_2 \dashv_2$,

where,

$$\jmath_1 \dashv_1 = \begin{bmatrix} \left[\sqrt{1-\left(1-(\measuredangle_1^{ta})^2\right)^{\jmath_1}}, \sqrt{1-\left(1-(\measuredangle_1^{tb})^2\right)^{\jmath_1}}\right] \\ \left[\sqrt{1-\left(1-(\sigma_1^{ia})^2\right)^{\jmath_1}}, \sqrt{1-\left(1-(\sigma_1^{ib})^2\right)^{\jmath_1}}\right] \\ \left[\sqrt{\left(1-(\measuredangle_1^{ta})^2\right)^{\jmath_1}-\left(1-(\measuredangle_1^{ta}+\varrho_1^{fa})^2\right)^{\jmath_1}}, \right. \\ \left. \sqrt{\left(1-(\measuredangle_1^{tb})^2\right)^{\jmath_1}-\left(1-(\measuredangle_1^{tb}+\varrho_1^{fb})^2\right)^{\jmath_1}}\right] \end{bmatrix}$$

and

$$\jmath_2 \dashv_2 = \begin{bmatrix} \left[\sqrt{1-\left(1-(\measuredangle_2^{ta})^2\right)^{\jmath_2}}, \sqrt{1-\left(1-(\measuredangle_2^{tb})^2\right)^{\jmath_2}}\right] \\ \left[\sqrt{1-\left(1-(\sigma_2^{ia})^2\right)^{\jmath_2}}, \sqrt{1-\left(1-(\sigma_2^{ib})^2\right)^{\jmath_2}}\right] \\ \left[\sqrt{\left(1-(\measuredangle_2^{ta})^2\right)^{\jmath_2}-\left(1-(\measuredangle_2^{ta}+\varrho_2^{fa})^2\right)^{\jmath_2}}, \right. \\ \left. \sqrt{\left(1-(\measuredangle_2^{tb})^2\right)^{\jmath_2}-\left(1-(\measuredangle_2^{tb}+\varrho_2^{fb})^2\right)^{\jmath_2}}\right] \end{bmatrix}$$

We get

$$\jmath_1 \dashv_1 \veebar \jmath_2 \dashv_2 = \begin{bmatrix} \left[\sqrt{\left(1-\left(1-(\measuredangle_1^{ta})^2\right)^{\jmath_1}\right)+\left(1-\left(1-(\measuredangle_2^{ta})^2\right)^{\jmath_2}\right)} \right. \\ \left. \sqrt{-\left(1-\left(1-(\measuredangle_1^{ta})^2\right)^{\jmath_1}\right)\cdot\left(1-\left(1-(\measuredangle_2^{ta})^2\right)^{\jmath_2}\right)}, \right. \\ \left. \sqrt{\left(1-\left(1-(\measuredangle_1^{tb})^2\right)^{\jmath_1}\right)+\left(1-\left(1-(\measuredangle_2^{tb})^2\right)^{\jmath_2}\right)} \right. \\ \left. \sqrt{-\left(1-\left(1-(\measuredangle_1^{tb})^2\right)^{\jmath_1}\right)\cdot\left(1-\left(1-(\measuredangle_2^{tb})^2\right)^{\jmath_2}\right)}\right] \\ \left[\sqrt{\left(1-\left(1-(\sigma_1^{ia})^2\right)^{\jmath_1}\right)+\left(1-\left(1-(\sigma_2^{ia})^2\right)^{\jmath_2}\right)} \right. \\ \left. \sqrt{-\left(1-\left(1-(\sigma_1^{ia})^2\right)^{\jmath_1}\right)\cdot\left(1-\left(1-(\sigma_2^{ia})^2\right)^{\jmath_2}\right)}, \right. \\ \left. \sqrt{\left(1-\left(1-(\sigma_1^{ib})^2\right)^{\jmath_1}\right)+\left(1-\left(1-(\sigma_2^{ib})^2\right)^{\jmath_2}\right)} \right. \\ \left. \sqrt{-\left(1-\left(1-(\sigma_1^{ib})^2\right)^{\jmath_1}\right)\cdot\left(1-\left(1-(\sigma_2^{ib})^2\right)^{\jmath_2}\right)}\right] \\ \left[\sqrt{\left(1-\left(1-\varrho_1^{fa})^2\right)^{\jmath_1}\right)+\left(1-\left(1-\varrho_2^{fa})^2\right)^{\jmath_2}\right)} \right. \\ \left. -\left(1-\left(1-\varrho_1^{fa})^2\right)^{\jmath_1}\right)\cdot\left(1-\left(1-\varrho_2^{fa})^2\right)^{\jmath_2}\right) \right. \\ \left. \sqrt{-\left(1-(\measuredangle_1^{ta}+\varrho_1^{fa})^2\right)^{\jmath_1}\cdot\left(1-(\measuredangle_2^{ta}+\varrho_2^{fa})^2\right)^{\jmath_2}} \right. \\ \left. \sqrt{\left(1-\left(1-\varrho_1^{fb})^2\right)^{\jmath_1}\right)+\left(1-\left(1-\varrho_2^{fb})^2\right)^{\jmath_2}\right)} \right. \\ \left. -\left(1-\left(1-\varrho_1^{fb})^2\right)^{\jmath_1}\right)\cdot\left(1-\left(1-\varrho_2^{fb})^2\right)^{\jmath_2}\right) \right. \\ \left. \sqrt{-\left(1-(\measuredangle_1^{tb}+\varrho_1^{fb})^2\right)^{\jmath_1}\cdot\left(1-(\measuredangle_2^{tb}+\varrho_2^{fb})^2\right)^{\jmath_2}}\right] \end{bmatrix}$$

$$= \begin{bmatrix} \left[\sqrt{1-\square_{\partial=1}^{2}\left(1-(\measuredangle_{\partial}^{ta})^{2}\right)^{\jmath_{\partial}}}, \sqrt{1-\square_{\partial=1}^{2}\left(1-(\measuredangle_{\partial}^{tb})^{2}\right)^{\jmath_{\partial}}}\right] \\ \left[\sqrt{1-\square_{\partial=1}^{2}\left(1-(\sigma_{\partial}^{ia})^{2}\right)^{\jmath_{\partial}}}, \sqrt{1-\square_{\partial=1}^{2}\left(1-(\sigma_{\partial}^{ib})^{2}\right)^{\jmath_{\partial}}}\right] \\ \left[\sqrt{\square_{\partial=1}^{2}\left(1-(\measuredangle_{\partial}^{ta})^{2}\right)^{\jmath_{\partial}}-\square_{\partial=1}^{2}\left(1-(\measuredangle_{\partial}^{ta}+\varrho_{\partial}^{fa})^{2}\right)^{\jmath_{\partial}}}, \right. \\ \left. \sqrt{\square_{\partial=1}^{2}\left(1-(\measuredangle_{\partial}^{tb})^{2}\right)^{\jmath_{\partial}}-\square_{\partial=1}^{2}\left(1-(\measuredangle_{\partial}^{tb}+\varrho_{\partial}^{fb})^{2}\right)^{\jmath_{\partial}}}\right] \end{bmatrix}.$$

Using induction $\partial \geq 3$,IPNIWAO$(\beth_1, \beth_2, ..., \beth_{\propto})$

$$= \begin{bmatrix} \left[\sqrt{1-\square_{\partial=1}^{\propto}\left(1-(\measuredangle_{\partial}^{ta})^{2}\right)^{\jmath_{\partial}}}, \sqrt{1-\square_{\partial=1}^{\propto}\left(1-(\measuredangle_{\partial}^{tb})^{2}\right)^{\jmath_{\partial}}}\right] \\ \left[\sqrt{1-\square_{\partial=1}^{\propto}\left(1-(\sigma_{\partial}^{ia})^{2}\right)^{\jmath_{\partial}}}, \sqrt{1-\square_{\partial=1}^{\propto}\left(1-(\measuredangle_{\partial}^{tb})^{2}\right)^{\jmath_{\partial}}}\right] \\ \left[\sqrt{\square_{\partial=1}^{\propto}\left(1-(\measuredangle_{\partial}^{ta})^{2}\right)^{\jmath_{\partial}}-\square_{\partial=1}^{\propto}\left(1-(\measuredangle_{\partial}^{ta}+\varrho_{\partial}^{fa})^{2}\right)^{\jmath_{\partial}}}, \right. \\ \left. \sqrt{\square_{\partial=1}^{\propto}\left(1-(\measuredangle_{\partial}^{tb})^{2}\right)^{\jmath_{\partial}}-\square_{\partial=1}^{\propto}\left(1-(\measuredangle_{\partial}^{tb}+\varrho_{\partial}^{fb})^{2}\right)^{\jmath_{\partial}}}\right] \end{bmatrix}.$$

If $\partial = \propto +1$, then $IPNIWAO(\beth_1, \beth_2, ..., \beth_{\propto}, \beth_{\propto+1})$

$$= \begin{bmatrix} \left[\sqrt{1-\square_{\partial=1}^{\propto}\left(1-(\measuredangle_{\partial}^{ta})^{2}\right)^{\jmath_{\partial}}\cdot\left(1-(\measuredangle_{\propto+1}^{ta})^{2}\right)^{\jmath_{\propto+1}}}, \right. \\ \left. \sqrt{1-\square_{\partial=1}^{\propto}\left(1-(\measuredangle_{\partial}^{tb})^{2}\right)^{\jmath_{\partial}}\cdot\left(1-(\measuredangle_{\propto+1}^{tb})^{2}\right)^{\jmath_{\propto+1}}}\right] \\ \left[\sqrt{1-\square_{\partial=1}^{\propto}\left(1-(\sigma_{\partial}^{ia})^{2}\right)^{\jmath_{\partial}}\cdot\left(1-(\sigma_{\propto+1}^{ia})^{2}\right)^{\jmath_{\propto+1}}}, \right. \\ \left. \sqrt{1-\square_{\partial=1}^{\propto}\left(1-(\sigma_{\partial}^{ib})^{2}\right)^{\jmath_{\partial}}\cdot\left(1-(\sigma_{\propto+1}^{ib})^{2}\right)^{\jmath_{\propto+1}}}\right] \\ \left[\sqrt{\begin{array}{r}\left(\square_{\partial=1}^{\propto}\left(1-(\measuredangle_{\partial}^{ta})^{2}\right)^{\jmath_{\partial}}-\square_{\partial=1}^{\propto}\left(1-(\measuredangle_{\partial}^{ta}+\varrho_{\partial}^{fa})^{2}\right)^{\jmath_{\partial}}\right)\cdot \\ \left(\left((\measuredangle_{\propto+1}^{ta})^{2}\right)^{\jmath_{\propto+1}}-(\measuredangle_{\propto+1}^{ta}+\varrho_{\propto+1}^{fa})^{2}\right)^{\jmath_{\propto+1}}\end{array}}, \right. \\ \left. \sqrt{\begin{array}{r}\left(\square_{\partial=1}^{\propto}\left(1-(\measuredangle_{\partial}^{tb})^{2}\right)^{\jmath_{\partial}}-\square_{\partial=1}^{\propto}\left(1-(\measuredangle_{\partial}^{tb}+\varrho_{\partial}^{fb})^{2}\right)^{\jmath_{\partial}}\right)\cdot \\ \left(\left((\measuredangle_{\propto+1}^{tb})^{2}\right)^{\jmath_{\propto+1}}-(\measuredangle_{\propto+1}^{tb}+\varrho_{\propto+1}^{fb})^{2}\right)^{\jmath_{\propto+1}}\end{array}}\right] \end{bmatrix}$$

$$= \begin{bmatrix} \left[\sqrt{1-\square_{\partial=1}^{\propto+1}\left(1-(\measuredangle_{\partial}^{ta})^{2}\right)^{\jmath_{\partial}}}, \sqrt{1-\square_{\partial=1}^{\propto+1}\left(1-(\measuredangle_{\partial}^{tb})^{2}\right)^{\jmath_{\partial}}}\right] \\ \left[\sqrt{1-\square_{\partial=1}^{\propto+1}\left(1-(\sigma_{\partial}^{ia})^{2}\right)^{\jmath_{\partial}}}, \sqrt{1-\square_{\partial=1}^{\propto+1}\left(1-(\sigma_{\partial}^{ib})^{2}\right)^{\jmath_{\partial}}}\right] \\ \left[\sqrt{\square_{\partial=1}^{\propto+1}\left(1-(\measuredangle_{\partial}^{ta})^{2}\right)^{\jmath_{\partial}}-\square_{\partial=1}^{\propto+1}\left(1-(\measuredangle_{\partial}^{ta}+\varrho_{\partial}^{fa})^{2}\right)^{\jmath_{\partial}}}, \right. \\ \left. \sqrt{\square_{\partial=1}^{\propto+1}\left(1-(\measuredangle_{\partial}^{tb})^{2}\right)^{\jmath_{\partial}}-\square_{\partial=1}^{\propto+1}\left(1-(\measuredangle_{\partial}^{tb}+\varrho_{\partial}^{fb})^{2}\right)^{\jmath_{\partial}}}\right] \end{bmatrix}$$

Theorem 2.4. *If* $\beth_{\partial} = \left\langle (\measuredangle_{\partial}^{ta}, \measuredangle_{\partial}^{tb}), (\sigma_{\partial}^{ia}, \sigma_{\partial}^{ib}), (\varrho_{\partial}^{fa}, \varrho_{\partial}^{fb}) \right\rangle$ *be the IPNNs and* $\beth_{\partial} = \beth$ *and* $\measuredangle^{tb} \cdot \varrho^{fb} = 0$,*then the* $IPNIWAO(\beth_1, \beth_2, ..., \beth_{\propto}) = \beth, \partial = 1, 2, ..., \propto$.

Proof. Note that,$\left\langle(\measuredangle_{\partial}^{ta},\measuredangle_{\partial}^{tb}),(\sigma_{\partial}^{ia},\sigma_{\partial}^{ib}),(\varrho_{\partial}^{fa},\varrho_{\partial}^{fb})\right\rangle = \left\langle(\measuredangle^{ta},\measuredangle^{tb}),(\sigma^{ia},\sigma^{ib}),(\varrho^{fa},\varrho^{fb})\right\rangle$, $\partial = 1,2,...,\propto$ and $\Diamond_{\partial=1}^{\propto}\jmath_{\partial} = 1$. We get, $IPNIWAO(\exists_1,\exists_2,...,\exists_{\propto})$

$$= \begin{bmatrix} \left[\sqrt{1-\square_{\partial=1}^{\propto}\left(1-(\measuredangle^{ta})^2\right)^{\jmath_{\partial}}}, \sqrt{1-\square_{\partial=1}^{\propto}\left(1-(\measuredangle^{tb})^2\right)^{\jmath_{\partial}}}\right] \\ \left[\sqrt{1-\square_{\partial=1}^{\propto}\left(1-(\sigma^{ia})^2\right)^{\jmath_{\partial}}}, \sqrt{1-\square_{\partial=1}^{\propto}\left(1-(\sigma^{ib})^2\right)^{\jmath_{\partial}}}\right] \\ \left[\sqrt{\square_{\partial=1}^{\propto}\left(1-(\measuredangle^{ta})^2\right)^{\jmath_{\partial}}-\square_{\partial=1}^{\propto}\left(1-(\measuredangle^{ta}+\varrho^{fa})^2\right)^{\jmath_{\partial}}}, \\ \sqrt{\square_{\partial=1}^{\propto}\left(1-(\measuredangle^{tb})^2\right)^{\jmath_{\partial}}-\square_{\partial=1}^{\propto}\left(1-(\measuredangle^{tb}+\varrho^{fb})^2\right)^{\jmath_{\partial}}}\end{bmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} \left[\sqrt{1-\left(1-(\measuredangle^{ta})^2\right)^{\Diamond_{\partial=1}^{\propto}\jmath_{\partial}}}, \sqrt{1-\left(1-(\measuredangle^{tb})^2\right)^{\Diamond_{\partial=1}^{\propto}\jmath_{\partial}}},\right] \\ \left[\sqrt{1-\left(1-(\sigma^{ia})^2\right)^{\Diamond_{\partial=1}^{\propto}\jmath_{\partial}}}, \sqrt{1-\left(1-(\sigma^{ib})^2\right)^{\Diamond_{\partial=1}^{\propto}\jmath_{\partial}}}\right] \\ \left[\sqrt{\left(1-(\measuredangle^{ta})^2\right)^{\Diamond_{\partial=1}^{\propto}\jmath_{\partial}}-\left(1-(\measuredangle^{ta}+\varrho^{fa})^2\right)^{\Diamond_{\partial=1}^{\propto}\jmath_{\partial}}}, \\ \sqrt{\left(1-(\measuredangle^{tb})^2\right)^{\Diamond_{\partial=1}^{\propto}\jmath_{\partial}}-\left(1-(\measuredangle^{tb}+\varrho^{fb})^2\right)^{\Diamond_{\partial=1}^{\propto}\jmath_{\partial}}}\end{bmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} \left[\sqrt{1-\left(1-(\measuredangle^{ta})^2\right)}, \sqrt{1-\left(1-(\measuredangle^{tb})^2\right)}\right] \\ \left[\sqrt{1-\left(1-(\sigma^{ia})^2\right)}, \sqrt{1-\left(1-(\sigma^{ib})^2\right)}\right] \\ \left[\sqrt{\left(1-(\measuredangle^{ta})^2\right)-\left(1-(\measuredangle^{ta}+\varrho^{fa})^2\right)}, \\ \sqrt{\left(1-(\measuredangle^{tb})^2\right)-\left(1-(\measuredangle^{tb}+\varrho^{fb})^2\right)}\end{bmatrix} \end{bmatrix}$$

$$= \left((\measuredangle^{ta},\measuredangle^{tb}),(\sigma^{ia},\sigma^{ib}),(\varrho^{fa},\varrho^{fb})\right) = \exists$$

2.2. *Interaction weighted geometric(IPNIWGO)*

Definition 2.5. Let $\exists_{\partial} = \left\langle(\measuredangle_{\partial}^{ta},\measuredangle_{\partial}^{tb}),(\sigma_{\partial}^{ia},\sigma_{\partial}^{ib}),(\varrho_{\partial}^{fa},\varrho_{\partial}^{fb})\right\rangle$ be the IPNNs,$\jmath_{\partial}$ be the weight of $\exists_{\partial}$. Then the IPNIWGO $(\exists_1,\exists_2,...,\exists_{\propto}) = \square_{\partial=1}^{\propto}\exists_{\partial}^{\jmath_{\partial}}$.

Theorem 2.6. *If* $\exists_{\partial} = \left\langle(\measuredangle_{\partial}^{ta},\measuredangle_{\partial}^{tb}),(\sigma_{\partial}^{ia},\sigma_{\partial}^{ib}),(\varrho_{\partial}^{fa},\varrho_{\partial}^{fb})\right\rangle$ *be the IPNNs. Then,*

$$IPNIWGO(\exists_1,\exists_2,...,\exists_{\propto}) = \begin{bmatrix} \left[\sqrt{\square_{\partial=1}^{\propto}\left(1-(\varrho_{\partial}^{fa})^2\right)^{\jmath_{\partial}}-\square_{\partial=1}^{\propto}\left(1-(\measuredangle_{\partial}^{ta}+\varrho_{\partial}^{fa})^2\right)^{\jmath_{\partial}}}, \\ \sqrt{\square_{\partial=1}^{\propto}\left(1-(\varrho_{\partial}^{fb})^2\right)^{\jmath_{\partial}}-\square_{\partial=1}^{\propto}\left(1-(\measuredangle_{\partial}^{tb}+\varrho_{\partial}^{fb})^2\right)^{\jmath_{\partial}}}\end{bmatrix} \\ \left[\sqrt{1-\square_{\partial=1}^{\propto}\left(1-(\sigma_{\partial}^{ia})^2\right)^{\jmath_{\partial}}}, \sqrt{1-\square_{\partial=1}^{\propto}\left(1-(\sigma_{\partial}^{ib})^2\right)^{\jmath_{\partial}}}\right] \\ \left[\sqrt{1-\square_{\partial=1}^{\propto}\left(1-(\varrho_{\partial}^{fa})^2\right)^{\jmath_{\partial}}}, \sqrt{1-\square_{\partial=1}^{\propto}\left(1-(\varrho_{\partial}^{fb})^2\right)^{\jmath_{\partial}}}\right] \end{bmatrix}$$

Proof. If $\partial = 2$,then $\mathrm{IPNIWGO}(\dashv_1, \dashv_2) = \dashv_1^{\jmath_1} \barwedge \dashv_2^{\jmath_2}$,

where,

$$\dashv_1^{\jmath_1} = \begin{bmatrix} \left[\sqrt{\left(1-(\varrho_1^{fa})^2\right)^{\jmath_1} - \left(1-(\measuredangle_1^{ta}+\varrho_1^{fa})^2\right)^{\jmath_1}}, \right. \\ \left. \sqrt{\left(1-(\varrho_1^{fb})^2\right)^{\jmath_1} - \left(1-(\measuredangle_1^{tb}+\varrho_1^{fb})^2\right)^{\jmath_1}} \right] \\ \left[\sqrt{1-\left(1-(\sigma_1^{ia})^2\right)^{\jmath_1}}, \sqrt{1-\left(1-(\sigma_1^{ib})^2\right)^{\jmath_1}} \right] \\ \left[\sqrt{1-\left(1-(\varrho_1^{fa})^2\right)^{\jmath_1}}, \sqrt{1-\left(1-(\varrho_1^{fb})^2\right)^{\jmath_1}} \right] \end{bmatrix}$$

$$\dashv_2^{\jmath_2} = \begin{bmatrix} \left[\sqrt{\left(1-(\varrho_2^{fa})^2\right)^{\jmath_2} - \left(1-(\measuredangle_2^{ta}+\varrho_2^{fa})^2\right)^{\jmath_2}}, \right. \\ \left. \sqrt{\left(1-(\varrho_2^{fb})^2\right)^{\jmath_2} - \left(1-(\measuredangle_2^{tb}+\varrho_2^{fb})^2\right)^{\jmath_2}} \right] \\ \left[\sqrt{1-\left(1-(\sigma_2^{ia})^2\right)^{\jmath_2}}, \sqrt{1-\left(1-(\sigma_2^{ib})^2\right)^{\jmath_2}} \right] \\ \left[\sqrt{1-\left(1-(\varrho_2^{fa})^2\right)^{\jmath_2}}, \sqrt{1-\left(1-(\varrho_2^{fb})^2\right)^{\jmath_2}} \right] \end{bmatrix}$$

We get,

$$\dashv_1^{\jmath_1} \barwedge \dashv_2^{\jmath_2} = \begin{bmatrix} \left[\sqrt{\begin{array}{r} \left(1-\left(1-(\varrho_1^{fa})^2\right)^{\jmath_1}\right) + \left(1-\left(1-(\varrho_2^{fa})^2\right)^{\jmath_2}\right) \\ -\left(1-\left(1-(\varrho_1^{fa})^2\right)^{\jmath_1}\right) \cdot \left(1-\left(1-(\varrho_2^{fa})^2\right)^{\jmath_2}\right) \\ -\left(\left(1-(\measuredangle_1^{ta}+\varrho_1^{fa})^2\right)^{\jmath_1} \cdot \left(1-(\measuredangle_2^{ta}+\varrho_2^{fa})^2\right)^{\jmath_2}\right) \end{array}}, \right. \\ \left. \sqrt{\begin{array}{r} \left(1-\left(1-(\varrho_1^{fb})^2\right)^{\jmath_1}\right) + \left(1-\left(1-(\varrho_2^{fb})^2\right)^{\jmath_2}\right) \\ -\left(1-\left(1-(\varrho_1^{fb})^2\right)^{\jmath_1}\right) \cdot \left(1-\left(1-(\varrho_2^{fb})^2\right)^{\jmath_2}\right) \\ -\left(\left(1-(\measuredangle_1^{tb}+\varrho_1^{fb})^2\right)^{\jmath_1} \cdot \left(1-(\measuredangle_2^{tb}+\varrho_2^{fb})^2\right)^{\jmath_2}\right) \end{array}} \right] \\ \left[\sqrt{\begin{array}{r} \left(1-\left(1-(\sigma_1^{ia})^2\right)^{\jmath_1}\right) + \left(1-\left(1-(\sigma_2^{ia})^2\right)^{\jmath_2}\right) \\ -\left(1-\left(1-(\sigma_1^{ia})^2\right)^{\jmath_1}\right) \cdot \left(1-\left(1-(\sigma_2^{ia})^2\right)^{\jmath_2}\right) \end{array}}, \right. \\ \left. \sqrt{\begin{array}{r} \left(1-\left(1-(\sigma_1^{ib})^2\right)^{\jmath_1}\right) + \left(1-\left(1-(\sigma_2^{ib})^2\right)^{\jmath_2}\right) \\ -\left(1-\left(1-(\sigma_1^{ib})^2\right)^{\jmath_1}\right) \cdot \left(1-\left(1-(\sigma_2^{ib})^2\right)^{\jmath_2}\right) \end{array}} \right] \\ \left[\sqrt{\begin{array}{r} \left(1-\left(1-(\varrho_1^{fa})^2\right)^{\jmath_1}\right) + \left(1-\left(1-(\varrho_2^{fa})^2\right)^{\jmath_2}\right) \\ -\left(1-\left(1-(\varrho_1^{fa})^2\right)^{\jmath_1}\right) \cdot \left(1-\left(1-(\varrho_2^{fa})^2\right)^{\jmath_2}\right) \end{array}}, \right. \\ \left. \sqrt{\begin{array}{r} \left(1-\left(1-(\varrho_1^{fb})^2\right)^{\jmath_1}\right) + \left(1-\left(1-(\varrho_2^{fb})^2\right)^{\jmath_2}\right) \\ -\left(1-\left(1-(\varrho_1^{fb})^2\right)^{\jmath_1}\right) \cdot \left(1-\left(1-(\varrho_2^{fb})^2\right)^{\jmath_2}\right) \end{array}} \right] \end{bmatrix},$$

$$\text{Hence, IPNIWGO}(\exists_1,\exists_2)=\left[\begin{array}{c}\left[\begin{array}{c}\sqrt{\square_{\partial=1}^{2}\left(1-(\varrho_{\partial}^{fa})^{2}\right)^{\jmath\partial}-\square_{\partial=1}^{2}\left(1-(\measuredangle_{\partial}^{ta}+\varrho_{\partial}^{fa})^{2}\right)^{\jmath\partial}},\\ \sqrt{\square_{\partial=1}^{2}\left(1-(\varrho_{\partial}^{fb})^{2}\right)^{\jmath\partial}-\square_{\partial=1}^{2}\left(1-(\measuredangle_{\partial}^{tb}+\varrho_{\partial}^{fb})^{2}\right)^{\jmath\partial}},\end{array}\right]\\ \left[\sqrt{1-\square_{\partial=1}^{2}\left(1-(\sigma_{\partial}^{ia})^{2}\right)^{\jmath\partial}},\sqrt{1-\square_{\partial=1}^{2}\left(1-(\sigma_{\partial}^{ib})^{2}\right)^{\jmath\partial}}\right]\\ \left[\sqrt{1-\square_{\partial=1}^{2}\left(1-(\varrho_{\partial}^{fa})^{2}\right)^{\jmath\partial}},\sqrt{1-\square_{\partial=1}^{2}\left(1-(\varrho_{\partial}^{fb})^{2}\right)^{\jmath\partial}}\right]\end{array}\right]$$

$$\text{IPNIWGO}(\exists_1,\exists_2,...,\exists_{\propto})=\left[\begin{array}{c}\left[\begin{array}{c}\sqrt{\square_{\partial=1}^{\propto}\left(1-(\varrho_{\partial}^{fa})^{2}\right)^{\jmath\partial}-\square_{\partial=1}^{\propto}\left(1-(\measuredangle_{\partial}^{ta}+\varrho_{\partial}^{fa})^{2}\right)^{\jmath\partial}},\\ \sqrt{\square_{\partial=1}^{\propto}\left(1-(\varrho_{\partial}^{fb})^{2}\right)^{\jmath\partial}-\square_{\partial=1}^{\propto}\left(1-(\measuredangle_{\partial}^{tb}+\varrho_{\partial}^{fb})^{2}\right)^{\jmath\partial}},\end{array}\right]\\ \left[\sqrt{1-\square_{\partial=1}^{\propto}\left(1-(\sigma_{\partial}^{ia})^{2}\right)^{\jmath\partial}},\sqrt{1-\square_{\partial=1}^{\propto}\left(1-(\sigma_{\partial}^{ib})^{2}\right)^{\jmath\partial}}\right]\\ \left[\sqrt{1-\square_{\partial=1}^{\propto}\left(1-(\varrho_{\partial}^{fa})^{2}\right)^{\jmath\partial}},\sqrt{1-\square_{\partial=1}^{\propto}\left(1-(\varrho_{\partial}^{fb})^{2}\right)^{\jmath\partial}}\right]\end{array}\right]$$

If $\partial=\propto+1$,then $\text{IPNIWGO}(\exists_1,...,\exists_{\propto},\exists_{\propto+1})$

$$=\left[\begin{array}{c}\left[\begin{array}{c}\sqrt{\begin{array}{c}\Diamond_{\partial=1}^{\propto}\left(1-\left(1-(\varrho_{\partial}^{fa})^{2}\right)^{\jmath\partial}\right)+\left(1-\left(1-(\varrho_{\propto+1}^{fa})^{2}\right)^{\jmath\propto+1}\right)\\ -\square_{\partial=1}^{\propto}\left(1-\left(1-(\varrho_{\partial}^{fa})^{2}\right)^{\jmath\partial}\right)\cdot\left(1-\left(1-(\varrho_{\propto+1}^{fa})^{2}\right)^{\jmath\propto+1}\right)\end{array}},\\ \sqrt{\begin{array}{c}-\square_{\partial=1}^{\propto}\left(1-(\measuredangle_{\partial}^{ta}+\varrho_{\partial}^{fa})^{2}\right)^{\jmath\partial}\cdot\left(1-(\measuredangle_{\propto+1}^{ta}+\varrho_{\propto+1}^{fa})^{2}\right)^{\jmath\propto+1}\end{array}}\end{array}\right]\\ \left[\begin{array}{c}\sqrt{\begin{array}{c}\Diamond_{\partial=1}^{\propto}\left(1-\left(1-(\varrho_{\partial}^{fb})^{2}\right)^{\jmath\partial}\right)+\left(1-\left(1-(\varrho_{\propto+1}^{fb})^{2}\right)^{\jmath\propto+1}\right)\\ -\square_{\partial=1}^{\propto}\left(1-\left(1-(\varrho_{\partial}^{fb})^{2}\right)^{\jmath\partial}\right)\cdot\left(1-\left(1-(\varrho_{\propto+1}^{fb})^{2}\right)^{\jmath\propto+1}\right)\end{array}},\\ \sqrt{\begin{array}{c}-\square_{\partial=1}^{\propto}\left(1-(\measuredangle_{\partial}^{tb}+\varrho_{\partial}^{fb})^{2}\right)^{\jmath\partial}\cdot\left(1-(\measuredangle_{\propto+1}^{tb}+\varrho_{\propto+1}^{fb})^{2}\right)^{\jmath\propto+1}\end{array}}\end{array}\right]\\ \left[\begin{array}{c}\sqrt{\begin{array}{c}\Diamond_{\partial=1}^{\propto}\left(1-\left(1-(\sigma_{\partial}^{ia})^{2}\right)^{\jmath\partial}\right)+\left(1-\left(1-(\sigma_{\propto+1}^{ia})^{2}\right)^{\jmath\propto+1}\right)\\ -\square_{\partial=1}^{\propto}\left(1-\left(1-(\sigma_{\partial}^{ia})^{2}\right)^{\jmath\partial}\right)\cdot\left(1-\left(1-(\sigma_{\propto+1}^{ia})^{2}\right)^{\jmath\propto+1}\right)\end{array}},\\ \sqrt{\begin{array}{c}\Diamond_{\partial=1}^{\propto}\left(1-\left(1-(\sigma_{\partial}^{ib})^{2}\right)^{\jmath\partial}\right)+\left(1-\left(1-(\sigma_{\propto+1}^{ib})^{2}\right)^{\jmath\propto+1}\right)\\ -\square_{\partial=1}^{\propto}\left(1-\left(1-(\sigma_{\partial}^{ib})^{2}\right)^{\jmath\partial}\right)\cdot\left(1-\left(1-(\sigma_{\propto+1}^{ib})^{2}\right)^{\jmath\propto+1}\right)\end{array}}\end{array}\right]\\ \left[\begin{array}{c}\sqrt{\begin{array}{c}\Diamond_{\partial=1}^{\propto}\left(1-\left(1-(\varrho_{\partial}^{fa})^{2}\right)^{\jmath\partial}\right)+\left(1-\left(1-(\varrho_{\propto+1}^{fa})^{2}\right)^{\jmath\propto+1}\right)\\ -\square_{\partial=1}^{\propto}\left(1-\left(1-(\varrho_{\partial}^{fa})^{2}\right)^{\jmath\partial}\right)\cdot\left(1-\left(1-(\varrho_{\propto+1}^{fa})^{2}\right)^{\jmath\propto+1}\right)\end{array}},\\ \sqrt{\begin{array}{c}\Diamond_{\partial=1}^{\propto}\left(1-\left(1-(\varrho_{\partial}^{fb})^{2}\right)^{\jmath\partial}\right)+\left(1-\left(1-(\varrho_{\propto+1}^{fb})^{2}\right)^{\jmath\propto+1}\right)\\ -\square_{\partial=1}^{\propto}\left(1-\left(1-(\varrho_{\partial}^{fb})^{2}\right)^{\jmath\partial}\right)\cdot\left(1-\left(1-(\varrho_{\propto+1}^{fb})^{2}\right)^{\jmath\propto+1}\right)\end{array}}\end{array}\right]\end{array}\right]$$

$$= \left[\begin{array}{c} \left[\begin{array}{c} \sqrt{\begin{array}{r}\left(\square_{\partial=1}^{\propto}\left(1-(\varrho_{\partial}^{fa})^2\right)^{\jmath\partial}-\square_{\partial=1}^{\propto}\left(1-(\measuredangle_{\partial}^{ta}+\varrho_{\partial}^{fa})^2\right)^{\jmath\partial}\right)\cdot \\ \left(\left((\varrho_{\propto+1}^{fa})^2\right)^{\jmath\propto+1}-(\measuredangle_{\propto+1}^{ta}+\varrho_{\propto+1}^{fa})^2\right)^{\jmath\propto+1}\end{array}\left(\right)}, \\ \sqrt{\begin{array}{r}\left(\square_{\partial=1}^{\propto}\left(1-(\varrho_{\partial}^{fb})^2\right)^{\jmath\partial}-\square_{\partial=1}^{\propto}\left(1-(\measuredangle_{\partial}^{tb}+\varrho_{\partial}^{fb})^2\right)^{\jmath\partial}\right)\cdot \\ \left(\left((\varrho_{\propto+1}^{fb})^2\right)^{\jmath\propto+1}-(\measuredangle_{\propto+1}^{tb}+\varrho_{\propto+1}^{fb})^2\right)^{\jmath\propto+1}\end{array}\left(\right)} \end{array}\right], \\ \left[\begin{array}{c}\sqrt{1-\square_{\partial=1}^{\propto}\left(1-(\sigma_{\partial}^{ia})^2\right)^{\jmath\partial}\cdot\left(1-(\sigma_{\propto+1}^{ia})^2\right)^{\jmath\propto+1}}, \\ \sqrt{1-\square_{\partial=1}^{\propto}\left(1-(\sigma_{\partial}^{ib})^2\right)^{\jmath\partial}\cdot\left(1-(\sigma_{\propto+1}^{ib})^2\right)^{\jmath\propto+1}}\end{array}\right] \\ \left[\begin{array}{c}\sqrt{1-\square_{\partial=1}^{\propto}\left(1-(\varrho_{\partial}^{fa})^2\right)^{\jmath\partial}\cdot\left(1-(\varrho_{\propto+1}^{fa})^2\right)^{\jmath\propto+1}}, \\ \sqrt{1-\square_{\partial=1}^{\propto}\left(1-(\varrho_{\partial}^{fb})^2\right)^{\jmath\partial}\cdot\left(1-(\varrho_{\propto+1}^{fb})^2\right)^{\jmath\propto+1}}\end{array}\right] \end{array}\right]$$

$$= \left[\begin{array}{c}\left[\begin{array}{c}\sqrt{\square_{\partial=1}^{\propto+1}\left(1-(\varrho_{\partial}^{fa})^2\right)^{\jmath\partial}-\square_{\partial=1}^{\propto+1}\left(1-(\measuredangle_{\partial}^{ta}+\varrho_{\partial}^{fa})^2\right)^{\jmath\partial}}, \\ \sqrt{\square_{\partial=1}^{\propto+1}\left(1-(\varrho_{\partial}^{fb})^2\right)^{\jmath\partial}-\square_{\partial=1}^{\propto+1}\left(1-(\measuredangle_{\partial}^{tb}+\varrho_{\partial}^{fb})^2\right)^{\jmath\partial}}\end{array}\right] \\ \left[\sqrt{1-\square_{\partial=1}^{\propto+1}\left(1-(\sigma_{\partial}^{ia})^2\right)^{\jmath\partial}},\sqrt{1-\square_{\partial=1}^{\propto+1}\left(1-(\sigma_{\partial}^{ib})^2\right)^{\jmath\partial}}\right] \\ \left[\sqrt{1-\square_{\partial=1}^{\propto+1}\left(1-(\varrho_{\partial}^{fa})^2\right)^{\jmath\partial}},\sqrt{1-\square_{\partial=1}^{\propto+1}\left(1-(\varrho_{\partial}^{fb})^2\right)^{\jmath\partial}}\right] \end{array}\right]$$

Corollary 2.7. *Let* $\dashv_{\partial}=\left\langle(\measuredangle_{\partial}^{ta},\measuredangle_{\partial}^{tb}),(\sigma_{\partial}^{ia},\sigma_{\partial}^{ib}),(\varrho_{\partial}^{fa},\varrho_{\partial}^{fb})\right\rangle$ *be the IPNNs and all are equal and* $\measuredangle^{tb}\cdot\varrho^{fb}=0$. *Then IPNIWGO*$(\dashv_1,\dashv_2,...,\dashv_{\propto})=\dashv$.

2.3. *generalized IPNIWAO (GIPNIWAO)*

Definition 2.8. Let $\dashv_{\partial}=\left\langle(\measuredangle_{\partial}^{ta},\measuredangle_{\partial}^{tb}),(\sigma_{\partial}^{ia},\sigma_{\partial}^{ib}),(\varrho_{\partial}^{fa},\varrho_{\partial}^{fb})\right\rangle$ be the IPNNs,$\jmath_{\partial}$ be a weight of $\dashv_{\partial}$. Then,the GIPNIWAO $(\dashv_1,\dashv_2,...,\dashv_{\propto})=\left(\Diamond_{\partial=1}^{\propto}\ \jmath_{\partial}\ \dashv_{\partial}^2\right)^{1/2}$.

Theorem 2.9. *Let* $\dashv_{\partial}=\left\langle\left\langle(\measuredangle_{\partial}^{ta},\measuredangle_{\partial}^{tb}),(\sigma_{\partial}^{ia},\sigma_{\partial}^{ib}),(\varrho_{\partial}^{fa},\varrho_{\partial}^{fb})\right\rangle\right.$ *be the IPNNs. Then GIPNIWAO* $(\dashv_1,\dashv_2,...,\dashv_{\propto})=$

$$\left[\begin{array}{c}\left[\sqrt{1-\square_{\partial=1}^{\propto}\left(1-\left((\measuredangle_{\partial}^{ta})^2\right)^2\right)^{\jmath\partial}},\sqrt{1-\square_{\partial=1}^{\propto}\left(1-\left((\measuredangle_{\partial}^{tb})^2\right)^2\right)^{\jmath\partial}}\right] \\ \left[\sqrt{1-\square_{\partial=1}^{\propto}\left(1-\left((\sigma_{\partial}^{ia})^2\right)^2\right)^{\jmath\partial}},\sqrt{1-\square_{\partial=1}^{\propto}\left(1-\left((\sigma_{\partial}^{ib})^2\right)^2\right)^{\jmath\partial}}\right] \\ \left[\begin{array}{c}\sqrt{\square_{\partial=1}^{\propto}\left(1-\left((\measuredangle_{\partial}^{ta})^2\right)^2\right)^{\jmath\partial}-\square_{\partial=1}^{\propto}\left(1-\left((\measuredangle_{\partial}^{ta}+\varrho_{\partial}^{fa})^2\right)^2\right)^{\jmath\partial}}, \\ \sqrt{\square_{\partial=1}^{\propto}\left(1-\left((\measuredangle_{\partial}^{tb})^2\right)^2\right)^{\jmath\partial}-\square_{\partial=1}^{\propto}\left(1-\left((\measuredangle_{\partial}^{tb}+\varrho_{\partial}^{fb})^2\right)^2\right)^{\jmath\partial}}\end{array}\right] \end{array}\right].$$

Proof. First, we have to prove that

$$\Diamond_{\partial=1}^{\propto} \ \ni_{\partial} \ \dashv_{\partial}^{2} = \left[\begin{array}{c} \left[\sqrt{1-\square_{\partial=1}^{\propto}\left(1-\left((\measuredangle_{\partial}^{ta})^2\right)^2\right)^{\ni_{\partial}}}, \sqrt{1-\square_{\partial=1}^{\propto}\left(1-\left((\measuredangle_{\partial}^{tb})^2\right)^2\right)^{\ni_{\partial}}}\right] \\ \left[\sqrt{1-\square_{\partial=1}^{\propto}\left(1-\left((\sigma_{\partial}^{ia})^2\right)^2\right)^{\ni_{\partial}}}, \sqrt{1-\square_{\partial=1}^{\propto}\left(1-\left((\sigma_{\partial}^{ib})^2\right)^2\right)^{\ni_{\partial}}}\right] \\ \left[\sqrt{\square_{\partial=1}^{\propto}\left(1-\left((\measuredangle_{\partial}^{ta})^2\right)^2\right)^{\ni_{\partial}}-\square_{\partial=1}^{\propto}\left(1-\left((\measuredangle_{\partial}^{ta}+\varrho_{\partial}^{fa})^2\right)^2\right)^{\ni_{\partial}}}, \right. \\ \left. \sqrt{\square_{\partial=1}^{\propto}\left(1-\left((\measuredangle_{\partial}^{tb})^2\right)^2\right)^{\ni_{\partial}}-\square_{\partial=1}^{\propto}\left(1-\left((\measuredangle_{\partial}^{tb}+\varrho_{\partial}^{fb})^2\right)^2\right)^{\ni_{\partial}}}\right] \end{array}\right].$$

If $\partial = 2$,then $\ni_1 \ \dashv_1^2 =$

$$\left[\begin{array}{c} \left[\sqrt{1-\left(1-\left((\measuredangle_{1}^{ta})^2\right)^2\right)^{\ni_1}}, \sqrt{1-\left(1-\left((\measuredangle_{1}^{tb})^2\right)^2\right)^{\ni_1}},\right] \\ \left[\sqrt{1-\left(1-\left((\sigma_{1}^{ia})^2\right)^2\right)^{\ni_1}}, \sqrt{1-\left(1-\left((\sigma_{1}^{ib})^2\right)^2\right)^{\ni_1}}\right] \\ \left[\sqrt{\left(1-\left((\measuredangle_{1}^{ta})^2\right)^2\right)^{\ni_1}-\left(1-\left((\measuredangle_{1}^{ta}+\varrho_{1}^{fa})^2\right)^2\right)^{\ni_1}}, \right. \\ \left. \sqrt{\left(1-\left((\measuredangle_{1}^{tb})^2\right)^2\right)^{\ni_1}-\left(1-\left((\measuredangle_{1}^{tb}+\varrho_{1}^{fb})^2\right)^2\right)^{\ni_1}}\right] \end{array}\right]$$

and

$$\ni_2 \ \dashv_2^2 = \left[\begin{array}{c} \left[\sqrt{1-\left(1-\left((\measuredangle_{2}^{ta})^2\right)^2\right)^{\ni_1}}, \sqrt{1-\left(1-\left((\measuredangle_{2}^{tb})^2\right)^2\right)^{\ni_1}},\right] \\ \left[\sqrt{1-\left(1-\left((\sigma_{2}^{ia})^2\right)^2\right)^{\ni_1}}, \sqrt{1-\left(1-\left((\sigma_{2}^{ib})^2\right)^2\right)^{\ni_1}}\right] \\ \left[\sqrt{\left(1-\left((\measuredangle_{2}^{ta})^2\right)^2\right)^{\ni_1}-\left(1-\left((\measuredangle_{2}^{ta}+\varrho_{2}^{fa})^2\right)^2\right)^{\ni_1}}, \right. \\ \left. \sqrt{\left(1-\left((\measuredangle_{2}^{tb})^2\right)^2\right)^{\ni_1}-\left(1-\left((\measuredangle_{2}^{tb}+\varrho_{2}^{fb})^2\right)^2\right)^{\ni_1}}\right] \end{array}\right].$$

We get, $\ni_1 \dashv_1 \veebar \ni_2 \dashv_2 =$

$$\left[\begin{array}{l}
\left[\begin{array}{l}
\sqrt{\begin{array}{l}\left(\sqrt{1-\left(1-\left((\measuredangle_1^{ta})^2\right)^2\right)^{\ni_1}}\right)^2+\left(\sqrt{1-\left(1-\left((\measuredangle_2^{ta})^2\right)^2\right)^{\ni_1}}\right)^2,\\ -\left(\sqrt{1-\left(1-\left((\measuredangle_1^{ta})^2\right)^2\right)^{\ni_1}}\right)^2\cdot\left(\sqrt{1-\left(1-\left((\measuredangle_2^{ta})^2\right)^2\right)^{\ni_1}}\right)^2\end{array}},\\
\sqrt{\begin{array}{l}\left(\sqrt{1-\left(1-\left((\measuredangle_1^{tb})^2\right)^2\right)^{\ni_1}}\right)^2+\left(\sqrt{1-\left(1-\left((\measuredangle_2^{tb})^2\right)^2\right)^{\ni_1}}\right)^2,\\ -\left(\sqrt{1-\left(1-\left((\measuredangle_1^{tb})^2\right)^2\right)^{\ni_1}}\right)^2\cdot\left(\sqrt{1-\left(1-\left((\measuredangle_2^{tb})^2\right)^2\right)^{\ni_1}}\right)^2\end{array}}
\end{array}\right],\\
\left[\begin{array}{l}
\sqrt{\begin{array}{l}\left(\sqrt{1-\left(1-\left((\sigma_1^{ia})^2\right)^2\right)^{\ni_1}}\right)^2+\left(\sqrt{1-\left(1-\left((\sigma_2^{ia})^2\right)^2\right)^{\ni_1}}\right)^2,\\ -\left(\sqrt{1-\left(1-\left((\sigma_1^{ia})^2\right)^2\right)^{\ni_1}}\right)^2\cdot\left(\sqrt{1-\left(1-\left((\sigma_2^{ia})^2\right)^2\right)^{\ni_1}}\right)^{2'}\end{array}},\\
\sqrt{\begin{array}{l}\left(\sqrt{1-\left(1-\left((\sigma_1^{ib})^2\right)^2\right)^{\ni_1}}\right)^2+\left(\sqrt{1-\left(1-\left((\sigma_2^{ib})^2\right)^2\right)^{\ni_1}}\right)^2,\\ -\left(\sqrt{1-\left(1-\left((\sigma_1^{ib})^2\right)^2\right)^{\ni_1}}\right)^2\cdot\left(\sqrt{1-\left(1-\left((\sigma_2^{ib})^2\right)^2\right)^{\ni_1}}\right)^2\end{array}}
\end{array}\right],\\
\left[\begin{array}{l}
\sqrt{\begin{array}{l}\left(\sqrt{1-\left(1-\left((\varrho_1^{fa})^2\right)^2\right)^{\ni_1}}\right)^2+\left(\sqrt{1-\left(1-\left((\varrho_2^{fa})^2\right)^2\right)^{\ni_1}}\right)^2,\\ -\left(\sqrt{1-\left(1-\left((\varrho_1^{fa})^2\right)^2\right)^{\ni_1}}\right)^2\cdot\left(\sqrt{1-\left(1-\left((\varrho_2^{fa})^2\right)^2\right)^{\ni_1}}\right)^2\\ \quad -\left[\begin{array}{l}\left(\sqrt{\left(1-\left((\measuredangle_1^{ta})^2\right)^2\right)^{\ni_1}-\left(1-\left((\measuredangle_1^{ta}+\varrho_1^{fa})^2\right)^2\right)^{\ni_1}}\right)^2\cdot\\ \left(\sqrt{\left(1-\left((\measuredangle_2^{ta})^2\right)^2\right)^{\ni_1}-\left(1-\left((\measuredangle_2^{ta}+\varrho_2^{fa})^2\right)^2\right)^{\ni_1}}\right)^2\end{array}\right],\end{array}}\\
\sqrt{\begin{array}{l}\left(\sqrt{1-\left(1-\left((\varrho_1^{fb})^2\right)^2\right)^{\ni_1}}\right)^2+\left(\sqrt{1-\left(1-\left((\varrho_2^{fb})^2\right)^2\right)^{\ni_1}}\right)^2,\\ -\left(\sqrt{1-\left(1-\left((\varrho_1^{fb})^2\right)^2\right)^{\ni_1}}\right)^2\cdot\left(\sqrt{1-\left(1-\left((\varrho_2^{fb})^2\right)^2\right)^{\ni_1}}\right)^2\\ \quad -\left[\begin{array}{l}\left(\sqrt{\left(1-\left((\measuredangle_1^{tb})^2\right)^2\right)^{\ni_1}-\left(1-\left((\measuredangle_1^{tb}+\varrho_1^{fb})^2\right)^2\right)^{\ni_1}}\right)^2\cdot\\ \left(\sqrt{\left(1-\left((\measuredangle_2^{tb})^2\right)^2\right)^{\ni_1}-\left(1-\left((\measuredangle_2^{tb}+\varrho_2^{fb})^2\right)^2\right)^{\ni_1}}\right)^2\end{array}\right],\end{array}}
\end{array}\right]
\end{array}\right]$$

$$= \begin{bmatrix} \left[\sqrt{1-\square_{\partial=1}^{2}\left(1-\left((\measuredangle_1^{ta})^2\right)^2\right)^{\jmath_\partial}}, \sqrt{1-\square_{\partial=1}^{2}\left(1-\left((\measuredangle_1^{tb})^2\right)^2\right)^{\jmath_\partial}} \right] \\ \left[\sqrt{1-\square_{\partial=1}^{2}\left(1-\left((\sigma_1^{ia})^2\right)^2\right)^{\jmath_\partial}}, \sqrt{1-\square_{\partial=1}^{2}\left(1-\left((\sigma_1^{ib})^2\right)^2\right)^{\jmath_\partial}}, \right] \\ \left[\sqrt{\square_{\partial=1}^{2}\left(1-\left((\measuredangle_1^{ta})^2\right)^2\right)^{\jmath_\partial} - \square_{\partial=1}^{2}\left(1-\left((\measuredangle_1^{ta}+\varrho_1^{fa})^2\right)^2\right)^{\jmath_\partial}}, \right] \\ \left[\sqrt{\square_{\partial=1}^{2}\left(1-\left((\measuredangle_1^{tb})^2\right)^2\right)^{\jmath_\partial} - \square_{\partial=1}^{2}\left(1-\left((\measuredangle_1^{tb}+\varrho_1^{fb})^2\right)^2\right)^{\jmath_\partial}} \right] \end{bmatrix}$$

In general,

$$= \begin{bmatrix} \left[\sqrt{1-\square_{\partial=1}^{\propto}\left(1-\left((\measuredangle_1^{ta})^2\right)^2\right)^{\jmath_\partial}}, \sqrt{1-\square_{\partial=1}^{\propto}\left(1-\left((\measuredangle_1^{tb})^2\right)^2\right)^{\jmath_\partial}}, \right] \\ \left[\sqrt{1-\square_{\partial=1}^{\propto}\left(1-\left((\sigma_1^{ia})^2\right)^2\right)^{\jmath_\partial}}, \sqrt{1-\square_{\partial=1}^{\propto}\left(1-\left((\sigma_1^{ib})^2\right)^2\right)^{\jmath_\partial}} \right] \\ \left[\sqrt{\square_{\partial=1}^{\propto}\left(1-\left((\measuredangle_1^{ta})^2\right)^2\right)^{\jmath_\partial} - \square_{\partial=1}^{\propto}\left(1-\left((\measuredangle_1^{ta}+\varrho_1^{fa})^2\right)^2\right)^{\jmath_\partial}}, \right] \\ \left[\sqrt{\square_{\partial=1}^{\propto}\left(1-\left((\measuredangle_1^{tb})^2\right)^2\right)^{\jmath_\partial} - \square_{\partial=1}^{\propto}\left(1-\left((\measuredangle_1^{tb}+\varrho_1^{fb})^2\right)^2\right)^{\jmath_\partial}} \right] \end{bmatrix}.$$

If $\partial = \propto +1$,then $\Diamond_{\partial=1}^{\propto}\ \jmath_\partial\ \exists_\partial^2 + \jmath_{\propto+1}\ \exists_{\propto+1}^2 = \Diamond_{\partial=1}^{\propto+1}\ \jmath_\partial\ \exists_\partial^2$.

Now,$\Diamond_{\partial=1}^{\propto}\ \jmath_\partial\ \exists_\partial^2 + \jmath_{\propto+1}\ \exists_{\propto+1}^2 = \jmath_1\ \exists_1^2 \veebar\ \jmath_2\ \exists_2^2 \veebar \ldots \veebar\ \jmath_\propto\ \exists_\propto^2 \veebar\ \jmath_{\propto+1}\ \exists_{\propto+1}^2$

$$= \left[\begin{array}{l} \left[\begin{array}{l} \sqrt{\left(\sqrt{1-\square_{\partial=1}^{\propto}\left(1-\left((\measuredangle_{\partial}^{ta})^2\right)^2\right)^{\jmath_\partial}}\right)^2+\left(\sqrt{1-\left(1-\left((\measuredangle_{\propto+1}^{ta})^2\right)^2\right)^{\jmath_1}}\right)^2}, \\ -\left(\sqrt{1-\square_{\partial=1}^{\propto}\left(1-\left((\measuredangle_{\partial}^{ta})^2\right)^2\right)^{\jmath_\partial}}\right)^2\cdot\left(\sqrt{1-\left(1-\left((\measuredangle_{\propto+1}^{ta})^2\right)^2\right)^{\jmath_1}}\right)^2, \end{array}\right. \\ \left.\left[\begin{array}{l} \sqrt{\left(\sqrt{1-\square_{\partial=1}^{\propto}\left(1-\left((\measuredangle_{\partial}^{tb})^2\right)^2\right)^{\jmath_\partial}}\right)^2+\left(\sqrt{1-\left(1-\left((\measuredangle_{\propto+1}^{tb})^2\right)^2\right)^{\jmath_1}}\right)^2}, \\ -\left(\sqrt{1-\square_{\partial=1}^{\propto}\left(1-\left((\measuredangle_{\partial}^{tb})^2\right)^2\right)^{\jmath_\partial}}\right)^2\cdot\left(\sqrt{1-\left(1-\left((\measuredangle_{\propto+1}^{tb})^2\right)^2\right)^{\jmath_1}}\right)^2 \end{array}\right] \\ \left[\begin{array}{l} \left[\begin{array}{l} \sqrt{\left(\sqrt{1-\square_{\partial=1}^{\propto}\left(1-\left((\sigma_{\partial}^{ia})^2\right)^2\right)^{\jmath_\partial}}\right)^2+\left(\sqrt{1-\left(1-\left((\sigma_{\propto+1}^{ia})^2\right)^2\right)^{\jmath_1}}\right)^2}, \\ -\left(\sqrt{1-\square_{\partial=1}^{\propto}\left(1-\left((\sigma_{\partial}^{ia})^2\right)^2\right)^{\jmath_\partial}}\right)^2\cdot\left(\sqrt{1-\left(1-\left((\sigma_{\propto+1}^{ia})^2\right)^2\right)^{\jmath_1}}\right)^2, \end{array}\right. \\ \left.\left[\begin{array}{l} \sqrt{\left(\sqrt{1-\square_{\partial=1}^{\propto}\left(1-\left((\sigma_{\partial}^{ib})^2\right)^2\right)^{\jmath_\partial}}\right)^2+\left(\sqrt{1-\left(1-\left((\sigma_{\propto+1}^{ib})^2\right)^2\right)^{\jmath_1}}\right)^2}, \\ -\left(\sqrt{1-\square_{\partial=1}^{\propto}\left(1-\left((\sigma_{\partial}^{ib})^2\right)^2\right)^{\jmath_\partial}}\right)^2\cdot\left(\sqrt{1-\left(1-\left((\sigma_{\propto+1}^{ib})^2\right)^2\right)^{\jmath_1}}\right)^2 \end{array}\right] \\ -\left[\begin{array}{l} \left(\sqrt{\square_{\partial=1}^{\propto}\left(1-\left((\measuredangle_{\partial}^{ta})^2\right)^2\right)^{\jmath_\partial}-\square_{\partial=1}^{\propto}\left(1-\left((\measuredangle_{\partial}^{ta}+\varrho_{\partial}^{fa})^2\right)^2\right)^{\jmath_\partial}}\right)^2\cdot \\ \left(\sqrt{\left(1-\left((\measuredangle_{\propto+1}^{ta})^2\right)^2\right)^{\jmath_1}-\left(1-\left((\measuredangle_{\propto+1}^{ta}+\varrho_{\propto+1}^{fa})^2\right)^2\right)^{\jmath_1}}\right)^2, \\ \left(\sqrt{\square_{\partial=1}^{\propto}\left(1-\left((\measuredangle_{\partial}^{tb})^2\right)^2\right)^{\jmath_\partial}-\square_{\partial=1}^{\propto}\left(1-\left((\measuredangle_{\partial}^{tb}+\varrho_{\partial}^{fb})^2\right)^2\right)^{\jmath_\partial}}\right)^2\cdot \\ \left(\sqrt{\left(1-\left((\measuredangle_{\propto+1}^{tb})^2\right)^2\right)^{\jmath_1}-\left(1-\left((\measuredangle_{\propto+1}^{tb}+\varrho_{\propto+1}^{fb})^2\right)^2\right)^{\jmath_1}}\right)^2 \end{array}\right] \end{array}\right]$$

$$= \left[\begin{array}{l} \left[\sqrt{1-\square_{\partial=1}^{\propto+1}\left(1-\left((\measuredangle_{1}^{ta})^2\right)^2\right)^{\jmath_\partial}}, \sqrt{1-\square_{\partial=1}^{\propto+1}\left(1-\left((\measuredangle_{1}^{tb})^2\right)^2\right)^{\jmath_\partial}}\right] \\ \left[\sqrt{1-\square_{\partial=1}^{\propto+1}\left(1-\left((\sigma_{1}^{ia})^2\right)^2\right)^{\jmath_\partial}}, \sqrt{1-\square_{\partial=1}^{\propto+1}\left(1-\left((\sigma_{1}^{ib})^2\right)^2\right)^{\jmath_\partial}}\right] \\ \left[\sqrt{\square_{\partial=1}^{\propto+1}\left(1-\left((\measuredangle_{1}^{ta})^2\right)^2\right)^{\jmath_\partial}-\square_{\partial=1}^{\propto+1}\left(1-\left((\measuredangle_{1}^{ta}+\varrho_{1}^{fa})^2\right)^2\right)^{\jmath_\partial}}, \right. \\ \left.\left[\sqrt{\square_{\partial=1}^{\propto+1}\left(1-\left((\measuredangle_{1}^{tb})^2\right)^2\right)^{\jmath_\partial}-\square_{\partial=1}^{\propto+1}\left(1-\left((\measuredangle_{1}^{tb}+\varrho_{1}^{fb})^2\right)^2\right)^{\jmath_\partial}}\right] \end{array}\right]$$

$$\text{and} \quad \Diamond_{\partial=1}^{\propto+1}\Big(\qquad \beth_\partial \qquad \dashv_\partial^2\Big) = \left[\begin{array}{c} \left[\left(\sqrt{1-\square_{\partial=1}^{\propto+1}\left(1-\left((\measuredangle_\partial^{ta})^2\right)^2\right)^{\beth_\partial}}\right),\left(\sqrt{1-\square_{\partial=1}^{\propto+1}\left(1-\left((\measuredangle_\partial^{tb})^2\right)^2\right)^{\beth_\partial}}\right)\right] \\ \left[\left(\sqrt{1-\square_{\partial=1}^{\propto+1}\left(1-\left((\sigma_\partial^{ia})^2\right)^2\right)^{\beth_\partial}}\right),\left(\sqrt{1-\square_{\partial=1}^{\propto+1}\left(1-\left((\sigma_\partial^{ib})^2\right)^2\right)^{\beth_\partial}}\right)\right] \\ \left[\left(\sqrt{\square_{\partial=1}^{\propto+1}\left(1-\left((\measuredangle_\partial^{ta})^2\right)^2\right)^{\beth_\partial}-\square_{\partial=1}^{\propto+1}\left(1-\left((\measuredangle_\partial^{ta}+\varrho_\partial^{fa})^2\right)^2\right)^{\beth_\partial}}\right), \\ \left(\sqrt{\square_{\partial=1}^{\propto+1}\left(1-\left((\measuredangle_\partial^{tb})^2\right)^2\right)^{\beth_\partial}-\square_{\partial=1}^{\propto+1}\left(1-\left((\measuredangle_\partial^{tb}+\varrho_\partial^{fb})^2\right)^2\right)^{\beth_\partial}}\right)\end{array}\right]. \quad =$$

Corollary 2.10. *Let* $\dashv_\partial = \left\langle(\measuredangle_\partial^{ta},\measuredangle_\partial^{tb}),(\sigma_\partial^{ia},\sigma_\partial^{ib}),(\varrho_\partial^{fa},\varrho_\partial^{fb})\right\rangle$ *be the IPNNs and all are equal. Then GIPNIWAO* $(\dashv_1,\dashv_2,...,\dashv_\propto)=\dashv$.

2.4. *Generalized IPNIWGO (GIPNIWGO)*

Definition 2.11. Let $\dashv_\partial = \left\langle(\measuredangle_\partial^{ta},\measuredangle_\partial^{tb}),(\sigma_\partial^{ia},\sigma_\partial^{ib}),(\varrho_\partial^{fa},\varrho_\partial^{fb})\right\rangle$ be the IPNNs and $\beth_\partial$ be the weight of $\dashv_\partial$,where $\partial=1,2,...,\propto$. Then,the GIPNIWGO$(\dashv_1,\dashv_2,...,\dashv_\propto)=\frac{1}{\zeta}\Big(\square_{\partial=1}^{\propto}(\zeta\dashv_\partial)^{\beth_\partial}\Big)$.

Theorem 2.12. *Let* $\dashv_\partial = \left\langle\left\langle(\measuredangle_\partial^{ta},\measuredangle_\partial^{tb}),(\sigma_\partial^{ia},\sigma_\partial^{ib}),(\varrho_\partial^{fa},\varrho_\partial^{fb})\right\rangle\right\rangle$ *be the collection of IPNNs. Then the GIPNIWGO* $(\dashv_1,\dashv_2,...,\dashv_\propto)=$

$$\left[\begin{array}{c}\left[\sqrt{\square_{\partial=1}^{\propto}\left(1-\left((\varrho_\partial^{fa})^2\right)^2\right)^{\beth_\partial}-\square_{\partial=1}^{\propto}\left(1-\left((\measuredangle_\partial^{ta}+\varrho_\partial^{fa})^2\right)^2\right)^{\beth_\partial}},\right. \\ \left.\sqrt{\square_{\partial=1}^{\propto}\left(1-\left((\varrho_\partial^{fb})^2\right)^2\right)^{\beth_\partial}-\square_{\partial=1}^{\propto}\left(1-\left((\measuredangle_\partial^{tb}+\varrho_\partial^{fb})^2\right)^2\right)^{\beth_\partial}}\right] \\ \left[\sqrt{1-\square_{\partial=1}^{\propto}\left(1-\left((\sigma_\partial^{ia})^2\right)^2\right)^{\beth_\partial}},\sqrt{1-\square_{\partial=1}^{\propto}\left(1-\left((\sigma_\partial^{ib})^2\right)^2\right)^{\beth_\partial}}\right] \\ \left[\sqrt{1-\square_{\partial=1}^{\propto}\left(1-\left((\varrho_\partial^{fa})^2\right)^2\right)^{\beth_\partial}},\sqrt{1-\square_{\partial=1}^{\propto}\left(1-\left((\varrho_\partial^{fb})^2\right)^2\right)^{\beth_\partial}}\right]\end{array}\right]$$

Proof. Using the induction method,

$$\square_{\partial=1}^{\propto}(\zeta\dashv_\partial)^{\beth_\partial} = \left[\begin{array}{c}\left[\sqrt{\square_{\partial=1}^{\propto}\left(1-\left((\varrho_\partial^{fa})^2\right)^2\right)^{\beth_\partial}-\square_{\partial=1}^{\propto}\left(1-\left((\measuredangle_\partial^{ta}+\varrho_\partial^{fa})^2\right)^2\right)^{\beth_\partial}},\right. \\ \left.\sqrt{\square_{\partial=1}^{\propto}\left(1-\left((\varrho_\partial^{fb})^2\right)^2\right)^{\beth_\partial}-\square_{\partial=1}^{\propto}\left(1-\left((\measuredangle_\partial^{tb}+\varrho_\partial^{fb})^2\right)^2\right)^{\beth_\partial}}\right] \\ \left[\sqrt{1-\square_{\partial=1}^{\propto}\left(1-\left((\sigma_\partial^{ia})^2\right)^2\right)^{\beth_\partial}},\sqrt{1-\square_{\partial=1}^{\propto}\left(1-\left((\sigma_\partial^{ib})^2\right)^2\right)^{\beth_\partial}}\right] \\ \left[\sqrt{1-\square_{\partial=1}^{\propto}\left(1-\left((\varrho_\partial^{fa})^2\right)^2\right)^{\beth_\partial}},\sqrt{1-\square_{\partial=1}^{\propto}\left(1-\left((\varrho_\partial^{fb})^2\right)^2\right)^{\beth_\partial}}\right]\end{array}\right]$$

If $\partial = 2$,then

$$(\zeta \dashv_1)^{\jmath_1} = \left[\begin{array}{c} \left[\begin{array}{c} \sqrt{\left(1-\left((\varrho_1^{fa})^2\right)^2\right)^{\jmath_1} - \left(1-\left((\measuredangle_1^{ta}+\varrho_1^{fa})^2\right)^2\right)^{\jmath_1}}, \\ \sqrt{\left(1-\left((\varrho_1^{fb})^2\right)^2\right)^{\jmath_1} - \left(1-\left((\measuredangle_1^{tb}+\varrho_1^{fb})^2\right)^2\right)^{\jmath_1}} \end{array}\right] \\ \left[\sqrt{1-\left(1-\left((\sigma_1^{ia})^2\right)^2\right)^{\jmath_1}}, \sqrt{1-\left(1-\left((\sigma_1^{ib})^2\right)^2\right)^{\jmath_1}}\right] \\ \left[\sqrt{1-\left(1-\left((\varrho_1^{fa})^2\right)^2\right)^{\jmath_1}}, \sqrt{1-\left(1-\left((\varrho_1^{fb})^2\right)^2\right)^{\jmath_1}}\right] \end{array}\right]$$

and

$$(\zeta \dashv_2)^{\jmath_2} = \left[\begin{array}{c} \left[\begin{array}{c} \sqrt{\left(1-\left((\varrho_2^{fa})^2\right)^2\right)^{\jmath_1} - \left(1-\left((\measuredangle_2^{ta}+\varrho_2^{fa})^2\right)^2\right)^{\jmath_1}}, \\ \sqrt{\left(1-\left((\varrho_2^{fb})^2\right)^2\right)^{\jmath_1} - \left(1-\left((\measuredangle_2^{tb}+\varrho_2^{fb})^2\right)^2\right)^{\jmath_1}} \end{array}\right] \\ \left[\sqrt{1-\left(1-\left((\sigma_2^{ia})^2\right)^2\right)^{\jmath_1}}, \sqrt{1-\left(1-\left((\sigma_2^{ib})^2\right)^2\right)^{\jmath_1}}\right] \\ \left[\sqrt{1-\left(1-\left((\varrho_2^{fa})^2\right)^2\right)^{\jmath_1}}, \sqrt{1-\left(1-\left((\varrho_2^{fb})^2\right)^2\right)^{\jmath_1}},\right] \end{array}\right]$$

We get, $(\zeta \dashv_1)^{\jmath_1} \overline{\wedge} (\zeta \dashv_2)^{\jmath_2}$

$$
= \left[\begin{array}{l}
\left[\begin{array}{l}
\sqrt{\begin{array}{l}
\left(\sqrt{1-\left(1-\left((\varrho_1^{fa})^2\right)^2\right)^{\jmath_1}}\right)^2+\left(\sqrt{1-\left(1-\left((\varrho_2^{fa})^2\right)^2\right)^{\jmath_1}}\right)^2 \\
-\left(\sqrt{1-\left(1-\left((\varrho_1^{fa})^2\right)^2\right)^{\jmath_1}}\right)^2\cdot\left(\sqrt{1-\left(1-\left((\varrho_2^{fa})^2\right)^2\right)^{\jmath_1}}\right)^2 \\
-\left[\begin{array}{l}\left(\sqrt{\left(1-\left((\varrho_1^{fa})^2\right)^2\right)^{\jmath_1}-\left(1-\left((\measuredangle_1^{ta}+\measuredangle_1^{ta})^2\right)^2\right)^{\jmath_1}}\right)^2\cdot \\ \left(\sqrt{\left(1-\left((\varrho_2^{fa})^2\right)^2\right)^{\jmath_1}-\left(1-\left((\measuredangle_2^{ta}+\varrho_2^{fa})^2\right)^2\right)^{\jmath_1}}\right)^2\end{array}\right]
\end{array}}, \\
\sqrt{\begin{array}{l}
\left(\sqrt{1-\left(1-\left((\varrho_1^{fb})^2\right)^2\right)^{\jmath_1}}\right)^2+\left(\sqrt{1-\left(1-\left((\varrho_2^{fb})^2\right)^2\right)^{\jmath_1}}\right)^2 \\
-\left(\sqrt{1-\left(1-\left((\varrho_1^{fb})^2\right)^2\right)^{\jmath_1}}\right)^2\cdot\left(\sqrt{1-\left(1-\left((\varrho_2^{fb})^2\right)^2\right)^{\jmath_1}}\right)^2 \\
-\left[\begin{array}{l}\left(\sqrt{\left(1-\left((\varrho_1^{fb})^2\right)^2\right)^{\jmath_1}-\left(1-\left((\measuredangle_1^{tb}+\measuredangle_1^{tb})^2\right)^2\right)^{\jmath_1}}\right)^2\cdot \\ \left(\sqrt{\left(1-\left((\varrho_2^{fb})^2\right)^2\right)^{\jmath_1}-\left(1-\left((\measuredangle_2^{tb}+\varrho_2^{fb})^2\right)^2\right)^{\jmath_1}}\right)^2\end{array}\right]
\end{array}}
\end{array}\right] \\
\left[\begin{array}{l}
\sqrt{\begin{array}{l}
\left(\sqrt{1-\left(1-\left((\sigma_1^{ia})^2\right)^2\right)^{\jmath_1}}\right)^2+\left(\sqrt{1-\left(1-\left((\sigma_2^{ia})^2\right)^2\right)^{\jmath_1}}\right)^2 \\
-\left(\sqrt{1-\left(1-\left((\sigma_1^{ia})^2\right)^2\right)^{\jmath_1}}\right)^2\cdot\left(\sqrt{1-\left(1-\left((\sigma_2^{ia})^2\right)^2\right)^{\jmath_1}}\right)^2
\end{array}}, \\
\sqrt{\begin{array}{l}
\left(\sqrt{1-\left(1-\left((\sigma_1^{ib})^2\right)^2\right)^{\jmath_1}}\right)^2+\left(\sqrt{1-\left(1-\left((\sigma_2^{ib})^2\right)^2\right)^{\jmath_1}}\right)^2 \\
-\left(\sqrt{1-\left(1-\left((\sigma_1^{ib})^2\right)^2\right)^{\jmath_1}}\right)^2\cdot\left(\sqrt{1-\left(1-\left((\sigma_2^{ib})^2\right)^2\right)^{\jmath_1}}\right)^2
\end{array}}
\end{array}\right] \\
\left[\begin{array}{l}
\sqrt{\begin{array}{l}
\left(\sqrt{1-\left(1-\left((\varrho_1^{fa})^2\right)^2\right)^{\jmath_1}}\right)^2+\left(\sqrt{1-\left(1-\left((\varrho_2^{fa})^2\right)^2\right)^{\jmath_1}}\right)^2 \\
-\left(\sqrt{1-\left(1-\left((\varrho_1^{fa})^2\right)^2\right)^{\jmath_1}}\right)^2\cdot\left(\sqrt{1-\left(1-\left((\varrho_2^{fa})^2\right)^2\right)^{\jmath_1}}\right)^2
\end{array}}, \\
\sqrt{\begin{array}{l}
\left(\sqrt{1-\left(1-\left((\varrho_1^{fb})^2\right)^2\right)^{\jmath_1}}\right)^2+\left(\sqrt{1-\left(1-\left((\varrho_2^{fb})^2\right)^2\right)^{\jmath_1}}\right)^2 \\
-\left(\sqrt{1-\left(1-\left((\varrho_1^{fb})^2\right)^2\right)^{\jmath_1}}\right)^2\cdot\left(\sqrt{1-\left(1-\left((\varrho_2^{fb})^2\right)^2\right)^{\jmath_1}}\right)^2
\end{array}}
\end{array}\right]
\end{array}\right]
$$

$$= \left[\begin{array}{c} \left[\begin{array}{c} \sqrt{\square_{\partial=1}^{2}\left(1-\left((\varrho_{\partial}^{fa})^{2}\right)^{2}\right)^{\jmath_{\partial}}-\square_{\partial=1}^{2}\left(1-\left((\measuredangle_{\partial}^{ta}+\varrho_{\partial}^{fa})^{2}\right)^{2}\right)^{\jmath_{\partial}}}, \\ \sqrt{\square_{\partial=1}^{2}\left(1-\left((\varrho_{\partial}^{fb})^{2}\right)^{2}\right)^{\jmath_{\partial}}-\square_{\partial=1}^{2}\left(1-\left((\measuredangle_{\partial}^{tb}+\varrho_{\partial}^{fb})^{2}\right)^{2}\right)^{\jmath_{\partial}}}, \end{array}\right] \\ \left[\sqrt{1-\square_{\partial=1}^{2}\left(1-\left((\sigma_{\partial}^{ia})^{2}\right)^{2}\right)^{\jmath_{\partial}}}, \sqrt{1-\square_{\partial=1}^{2}\left(1-\left((\sigma_{\partial}^{ib})^{2}\right)^{2}\right)^{\jmath_{\partial}}}\right] \\ \left[\sqrt{1-\square_{\partial=1}^{2}\left(1-\left((\varrho_{\partial}^{fa})^{2}\right)^{2}\right)^{\jmath_{\partial}}}, \sqrt{1-\square_{\partial=1}^{2}\left(1-\left((\varrho_{\partial}^{fb})^{2}\right)^{2}\right)^{\jmath_{\partial}}}\right] \end{array}\right]$$

If $\partial = \propto$,then

$$= \left[\begin{array}{c} \left[\begin{array}{c} \sqrt{\square_{\partial=1}^{\propto}\left(1-\left((\varrho_{\partial}^{fa})^{2}\right)^{2}\right)^{\jmath_{\partial}}-\square_{\partial=1}^{\propto}\left(1-\left((\measuredangle_{\partial}^{ta}+\varrho_{\partial}^{fa})^{2}\right)^{2}\right)^{\jmath_{\partial}}}, \\ \sqrt{\square_{\partial=1}^{\propto}\left(1-\left((\varrho_{\partial}^{fb})^{2}\right)^{2}\right)^{\jmath_{\partial}}-\square_{\partial=1}^{\propto}\left(1-\left((\measuredangle_{\partial}^{tb}+\varrho_{\partial}^{fb})^{2}\right)^{2}\right)^{\jmath_{\partial}}} \end{array}\right] \\ \left[\sqrt{1-\square_{\partial=1}^{\propto}\left(1-\left((\sigma_{\partial}^{ia})^{2}\right)^{2}\right)^{\jmath_{\partial}}}, \sqrt{1-\square_{\partial=1}^{\propto}\left(1-\left((\sigma_{\partial}^{ib})^{2}\right)^{2}\right)^{\jmath_{\partial}}}\right] \\ \left[\sqrt{1-\square_{\partial=1}^{\propto}\left(1-\left((\varrho_{\partial}^{fa})^{2}\right)^{2}\right)^{\jmath_{\partial}}}, \sqrt{1-\square_{\partial=1}^{\propto}\left(1-\left((\varrho_{\partial}^{fb})^{2}\right)^{2}\right)^{\jmath_{\partial}}}\right] \end{array}\right]$$

If $\partial = \propto +1$,then $\square_{\partial=1}^{\propto}(\zeta \dashv_{\partial})^{\jmath_{\partial}} \cdot (\zeta \dashv_{\propto+1})^{\jmath_{\propto+1}} = \square_{\partial=1}^{\propto+1}(\zeta \dashv_{\partial})^{\jmath_{\partial}}$.

Now,$\square_{\partial=1}^{\propto}(\zeta \dashv_{\partial})^{\jmath_{\partial}} \cdot (\zeta \dashv_{\propto+1})^{\jmath_{\propto+1}} = (\zeta \dashv_{1})^{\jmath_{1}} \barwedge (\zeta \dashv_{2})^{\jmath_{2}} \barwedge ... \barwedge (\zeta \dashv_{\propto})^{\jmath_{\propto}} \barwedge (\zeta \dashv_{\propto+1})^{\jmath_{\propto+1}}$

$$
= \left[\begin{array}{c}
\left[\begin{array}{c}
\sqrt{\left(\sqrt{1-\square_{\partial=1}^{\propto}\left(1-\left((\varrho_{\partial}^{fa})^2\right)^2\right)^{\jmath_\partial}}\right)^2+\left(\sqrt{1-\left(1-\left((\varrho_{\propto+1}^{fa})^2\right)^2\right)^{\jmath_1}}\right)^2} \\
-\left(\sqrt{1-\square_{\partial=1}^{\propto}\left(1-\left((\varrho_{\partial}^{fa})^2\right)^2\right)^{\jmath_\partial}}\right)^2\cdot\left(\sqrt{1-\left(1-\left((\varrho_{\propto+1}^{fa})^2\right)^2\right)^{\jmath_1}}\right)^2 \\
-\left[\begin{array}{c}\left(\sqrt{\square_{\partial=1}^{\propto}\left(1-\left((\varrho_{\partial}^{fa})^2\right)^2\right)^{\jmath_\partial}-\square_{\partial=1}^{\propto}\left(1-\left((\measuredangle_{\partial}^{ta}+\varrho_{\partial}^{fa})^2\right)^2\right)^{\jmath_\partial}}\right)^2\cdot \\ \left(\sqrt{\left(1-\left((\varrho_{\propto+1}^{fa})^2\right)^2\right)^{\jmath_1}-\left(1-\left((\measuredangle_{\propto+1}^{ta}+\varrho_{\propto+1}^{fa})^2\right)^2\right)^{\jmath_1}}\right)^2\end{array}\right]
\end{array}\right], \\
\left[\begin{array}{c}
\sqrt{\left(\sqrt{1-\square_{\partial=1}^{\propto}\left(1-\left((\varrho_{\partial}^{fb})^2\right)^2\right)^{\jmath_\partial}}\right)^2+\left(\sqrt{1-\left(1-\left((\varrho_{\propto+1}^{fb})^2\right)^2\right)^{\jmath_1}}\right)^2} \\
-\left(\sqrt{1-\square_{\partial=1}^{\propto}\left(1-\left((\varrho_{\partial}^{fb})^2\right)^2\right)^{\jmath_\partial}}\right)^2\cdot\left(\sqrt{1-\left(1-\left((\varrho_{\propto+1}^{fb})^2\right)^2\right)^{\jmath_1}}\right)^2 \\
-\left[\begin{array}{c}\left(\sqrt{\square_{\partial=1}^{\propto}\left(1-\left((\varrho_{\partial}^{fb})^2\right)^2\right)^{\jmath_\partial}-\square_{\partial=1}^{\propto}\left(1-\left((\measuredangle_{\partial}^{tb}+\varrho_{\partial}^{fb})^2\right)^2\right)^{\jmath_\partial}}\right)^2\cdot \\ \left(\sqrt{\left(1-\left((\varrho_{\propto+1}^{fb})^2\right)^2\right)^{\jmath_1}-\left(1-\left((\measuredangle_{\propto+1}^{tb}+\varrho_{\propto+1}^{fb})^2\right)^2\right)^{\jmath_1}}\right)^2\end{array}\right]
\end{array}\right] \\
= \left[\begin{array}{c}
\left[\begin{array}{c}
\sqrt{\left(\sqrt{1-\square_{\partial=1}^{\propto}\left(1-\left((\sigma_{\partial}^{ia})^2\right)^2\right)^{\jmath_\partial}}\right)^2+\left(\sqrt{1-\left(1-\left((\sigma_{\propto+1}^{ia})^2\right)^2\right)^{\jmath_1}}\right)^2} \\
-\left(\sqrt{1-\square_{\partial=1}^{\propto}\left(1-\left((\sigma_{\partial}^{ia})^2\right)^2\right)^{\jmath_\partial}}\right)^2\cdot\left(\sqrt{1-\left(1-\left((\sigma_{\propto+1}^{ia})^2\right)^2\right)^{\jmath_1}}\right)^2
\end{array}\right], \\
\left[\begin{array}{c}
\sqrt{\left(\sqrt{1-\square_{\partial=1}^{\propto}\left(1-\left((\sigma_{\partial}^{ib})^2\right)^2\right)^{\jmath_\partial}}\right)^2+\left(\sqrt{1-\left(1-\left((\sigma_{\propto+1}^{ib})^2\right)^2\right)^{\jmath_1}}\right)^2} \\
-\left(\sqrt{1-\square_{\partial=1}^{\propto}\left(1-\left((\sigma_{\partial}^{ib})^2\right)^2\right)^{\jmath_\partial}}\right)^2\cdot\left(\sqrt{1-\left(1-\left((\sigma_{\propto+1}^{ib})^2\right)^2\right)^{\jmath_1}}\right)^2
\end{array}\right]
\end{array}\right] \\
\left[\begin{array}{c}
\left[\begin{array}{c}
\sqrt{\left(\sqrt{1-\square_{\partial=1}^{\propto}\left(1-\left((\varrho_{\partial}^{fa})^2\right)^2\right)^{\jmath_\partial}}\right)^2+\left(\sqrt{1-\left(1-\left((\varrho_{\propto+1}^{fa})^2\right)^2\right)^{\jmath_1}}\right)^2} \\
-\left(\sqrt{1-\square_{\partial=1}^{\propto}\left(1-\left((\varrho_{\partial}^{fa})^2\right)^2\right)^{\jmath_\partial}}\right)^2\cdot\left(\sqrt{1-\left(1-\left((\varrho_{\propto+1}^{fa})^2\right)^2\right)^{\jmath_1}}\right)^2
\end{array}\right], \\
\left[\begin{array}{c}
\sqrt{\left(\sqrt{1-\square_{\partial=1}^{\propto}\left(1-\left((\varrho_{\partial}^{fb})^2\right)^2\right)^{\jmath_\partial}}\right)^2+\left(\sqrt{1-\left(1-\left((\varrho_{\propto+1}^{fb})^2\right)^2\right)^{\jmath_1}}\right)^2} \\
-\left(\sqrt{1-\square_{\partial=1}^{\propto}\left(1-\left((\varrho_{\partial}^{fb})^2\right)^2\right)^{\jmath_\partial}}\right)^2\cdot\left(\sqrt{1-\left(1-\left((\varrho_{\propto+1}^{fb})^2\right)^2\right)^{\jmath_1}}\right)^2
\end{array}\right]
\end{array}\right]
$$

$$= \begin{bmatrix} \left[\sqrt{\square_{\partial=1}^{\propto+1}\left(1-\left((\varrho_1^{fa})^2\right)^2\right)^{\jmath\partial}} - \square_{\partial=1}^{\propto+1}\left(1-\left((\measuredangle_1^{ta}+\varrho_1^{fa})^2\right)^2\right)^{\jmath\partial}, \right. \\ \left. \sqrt{\square_{\partial=1}^{\propto+1}\left(1-\left((\varrho_1^{fb})^2\right)^2\right)^{\jmath\partial}} - \square_{\partial=1}^{\propto+1}\left(1-\left((\measuredangle_1^{tb}+\varrho_1^{fb})^2\right)^2\right)^{\jmath\partial} \right] \\ \left[\sqrt{1-\square_{\partial=1}^{\propto+1}\left(1-\left((\sigma_1^{ia})^2\right)^2\right)^{\jmath\partial}}, \sqrt{1-\square_{\partial=1}^{\propto+1}\left(1-\left((\sigma_1^{ib})^2\right)^2\right)^{\jmath\partial}} \right] \\ \left[\sqrt{1-\square_{\partial=1}^{\propto+1}\left(1-\left((\varrho_1^{fa})^2\right)^2\right)^{\jmath\partial}}, \sqrt{1-\square_{\partial=1}^{\propto+1}\left(1-\left((\varrho_1^{fb})^2\right)^2\right)^{\jmath\partial}} \right] \end{bmatrix}$$

Hence

$$\frac{1}{\zeta}\left(\square_{\partial=1}^{\propto+1}(\zeta \beth_\partial)^{\jmath\partial}\right) = \begin{bmatrix} \left(\sqrt{\square_{\partial=1}^{\propto+1}\left(1-\left((\varrho_\partial^{fa})^2\right)^2\right)^{\jmath\partial}} - \square_{\partial=1}^{\propto+1}\left(1-\left((\measuredangle_\partial^{ta}+\varrho_\partial^{fa})^2\right)^2\right)^{\jmath\partial} \right), \\ \left(\sqrt{1-\square_{\partial=1}^{\propto+1}\left(1-\left(\sigma_\partial^{ia}\right)^2\right)^2\right)^{\jmath\partial}} \right)\left(\sqrt{1-\square_{\partial=1}^{\propto+1}\left(1-\left((\varrho_\partial^{fa})^2\right)^2\right)^{\jmath\partial}} \right) \end{bmatrix}$$

Corollary 2.13. *Let* $\beth_\partial = \left\langle (\measuredangle_\partial^{ta}, \measuredangle_\partial^{tb}), (\sigma_\partial^{ia}, \sigma_\partial^{ib}), (\varrho_\partial^{fa}, \varrho_\partial^{fb}) \right\rangle$ *be the collection of IPNNs and all are equal. Then the GIPNIWGO* $(\beth_1, \beth_2, ..., \beth_\propto) = \beth$.

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Optimizing Horizontal Collaboration in Logistics with Neutrosophic Z-Number Decision Models

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Abstract: Horizontal collaboration (HC) has emerged as a strategic approach to improve efficiency, sustainability, and competitive advantage while reducing CO2 emissions in supply chain operations. Despite its potential, HC implementation faces significant challenges, including trust management, strategic alignment, and decision-making complexities. This study proposes an integrated methodology that combines the Delphi technique and the DEMATEL framework, enhanced by Neutrosophic Z-number (NZN) theories, to identify and prioritize critical success factors (CSFs) influencing HC. A case study in Vietnam validates the robustness and practical applicability of the proposed model. The findings highlight economic factors—such as financial stability, data sharing, and innovation adoption—as key drivers of successful HC. Additionally, social factors, including organizational culture and conflict resolution, are found to be intricately linked to economic performance. Environmental factors emphasize the critical role of green practices and resource optimization. This research provides actionable insights for logistics companies and policymakers aiming to promote effective HC, thereby advancing logistics efficiency and sustainability in Vietnam and other developing economies. The proposed framework also offers a valuable foundation for future research and practical innovations in optimizing logistics collaboration.

Keywords: Neutrosophic sets, Z-number, MCDM, Delphi, DEMATEL, sustainable horizontal collaboration, logistics, Vietnam

1. Introduction

The logistics industry is vital to the economic framework of many countries, closely connected to economic expansion and the scale of goods movement [1]. This industry is a significant employer and facilitates the movement of products essential for daily economic activities. In the United States, for instance, the logistics sector moves an enormous amount of freight daily, serving numerous manufacturing sites, warehouses, and businesses across the country. The industry also employs millions of workers annually, with substantial job growth. Likewise, in the European Union, the logistics sector accounts for a significant market share and employs a considerable part of the labor force. Globally, the volume of freight movement is expected to continue expanding, driven by factors such as increased international trade and the growth of e-commerce [2].

A significant reason, however, is that even though logistics is of critical importance, inefficiencies abound. For example, whereas most freight is delivered by truck in the U.S., trailers are only 43% full by weight capacity on average. Additionally, 25% of all miles driven have either empty or nearly empty trailers, while the trailers are, on average, only 57% complete during the remaining 75% of miles traveled [3]. The same inefficiencies exist in the European Union, where country-specific data exist for empty truck miles, but studies estimate a range of 15% to 20% [4]. In addition to raising freight transportation costs, these inefficiencies contribute to adverse environmental effects, notably by increasing CO2 emissions from burning fossil fuels. Whereas transportation contributes approximately 29% to CO2 emissions in the United States, freight transportation accounts for about 10% annually [5]. Moreover, increasing congestion on roads and highways aggravates matters and reduces the likelihood of delivery on time, very much so in urban areas; in such a scenario, the implications for the quality of life of a citizen are huge.

At this time, optimizing individual supply chains has reached a point where further improvements yield diminishing returns. Addressing these inefficiencies requires a significant shift in logistics practices; collaborative logistics, precisely HC, offers a promising solution. HC in logistics is defined as partnerships among companies operating at similar levels of the supply chain, focusing on resource-sharing to achieve greater efficiency and lessen ecological effects. This approach differs from vertical collaboration (VC), where coordination occurs across various supply chain tiers, like manufacturers and retailers. HC has demonstrated potential for substantial cost savings and CO2 emissions reductions. Additionally, HC helps firms reduce duplication of efforts and increase load factors, enhancing overall operational efficiency.

The necessity of conducting this research in Vietnam is underscored by the unique challenges and opportunities the country's logistics sector presents. Vietnam's logistics sector is notably fragmented, with a significant presence of small and medium-sized enterprises (SMEs) that frequently face limited resources and capabilities, impacting their ability to function effectively and sustainably [6]. As Vietnam continues to integrate economically globally, the demand for efficient logistics services grows. However, these SMEs face significant hurdles, such as inadequate infrastructure, a lack of robust legal frameworks, and deeply ingrained competitive mindsets that hinder collaborative efforts. Furthermore, Vietnam's rapid urbanization and increasing trade volumes exacerbate congestion and environmental concerns, necessitating innovative solutions to enhance logistics efficiency and sustainability. HC presents a promising strategy to address these issues by enabling resource sharing and reducing operational redundancies. However, the implementation of HC in Vietnam is fraught with challenges, including managing trust among partners and aligning strategic goals, which are particularly complex when partners are competitors. Identifying and prioritizing the CSFs for effective HC is crucial to overcoming these barriers and fostering a more collaborative and efficient logistics

environment in Vietnam. This study addresses existing knowledge gaps and offers practical guidance to Vietnamese logistics companies, policymakers, and stakeholders for effective HC adoption. By doing so, it aims to enhance Vietnam's logistics efficiency and strengthen its competitive position in the global market. The objectives of the study are:

i) Identification and prioritization of CSF contributing to efficiency in LHC.

ii) Analysis of the complex inter-relationships and interdependencies between these variables.

The following research questions are proposed by this current study with respect to the above objectives:

i) What CSFs lead to efficiency in LHC?

ii) What is the nature of interaction and interdependence among these factors in logistic companies?

To achieve the objectives of this study, which focuses on identifying and prioritizing CSFs in LHC, it is essential to adopt methodologies capable of handling the logistics domain's inherent uncertainty, complexity, and variability[7]. This study employs an MCDM-based Neutrosophic Z-Number (NZN) approach, which integrates Neutrosophic Sets (NS) with Z-Numbers to form a robust framework for decision-making. NS represents a significant advancement over traditional fuzzy sets (FS) by addressing the limitations of existing uncertainty modeling techniques [8]. While FS relies on a single membership grade to quantify uncertainty, NS introduces three independent components: truth, indeterminacy, and falsity. This multifaceted structure enables a more comprehensive representation of real-world scenarios, accommodating not only the degree of membership but also the levels of hesitation and contradiction inherent in uncertain information [9], [10].

Compared to other fuzzy set extensions, such as Triangular Fuzzy Numbers (TFNs) [11], Intuitionistic Fuzzy Sets (IFS) [12], Pythagorean Fuzzy Sets (PFS) [13], and newer variants like Picture Fuzzy Sets [14] and Spherical Fuzzy Sets [15], NS offers unparalleled flexibility. For instance, TFNs are constrained to defining membership degrees within a triangular distribution, making them inadequate for capturing non-membership or hesitation. IFS extends FS by incorporating non-membership degrees but fails to capture indeterminacy explicitly. PFS allows the squared sum of membership and non-membership degrees to total within the range of 0 to 1, which increases flexibility but imposes mathematical constraints that limit real-world applicability [16]. Similarly, Picture and Spherical Fuzzy Sets introduce hesitation degrees but restrict their components (membership, non-membership, and hesitation) to summation or squared constraints within fixed ranges [17]. NS overcomes these limitations by treating truth, indeterminacy, and falsity as independent variables, each ranging freely between 0 and 1, with their combined total spanning from 0 to 3. This independence provides unparalleled flexibility in modeling complex systems, making NS uniquely suited for analyzing the multifaceted challenges of LHC.

While NS provides a robust foundation for handling uncertainty, its effectiveness can be further enhanced by integrating Z-Numbers, which Zadeh [18] introduced to include two components: a fuzzy representation of an uncertain variable (M) and the degree of reliability or confidence in the information source (N). This dual-layer representation of uncertainty addresses the uncertain variable and the trustworthiness of the data or expert judgment. Combining NS and Z-Numbers into the NZN framework creates a robust method for managing ambiguity and assessing source reliability. For example, in Vietnam's logistics sector, diverse stakeholders often provide information with varying levels of reliability [19]. Z-numbers allow for explicitly evaluating these variations in data trustworthiness, complementing NS's ability to handle indeterminacy.

The NZN framework provides a nuanced representation of stakeholder hesitation and conflicting priorities, which are common in LHC. NS captures these complexities through its indeterminacy component, while Z-numbers enhance the analysis by evaluating the reliability of conflicting information. This dual-layer approach ensures that decisions are not only comprehensive but also grounded in the most reliable data [9]. Furthermore, the NZN framework systematically prioritizes CSFs by integrating truth, indeterminacy, and falsity dimensions with reliability evaluations. For example, green practices may have a high truth value (importance), moderate indeterminacy (uncertainty about implementation), and low falsity (minimal contradiction among stakeholders), with a high-reliability score reinforcing its prioritization [8].

The NZN framework is particularly suitable for Vietnam's logistics landscape, characterized by fragmented infrastructure, trust deficits, and conflicting stakeholder objectives. These issues necessitate a framework that can handle the indeterminate and ambiguous nature of stakeholder inputs, accommodate varying levels of data reliability, and systematically prioritize factors influencing collaboration. Combining NS's flexibility in uncertainty modeling with Z-Numbers' ability to evaluate data reliability, the NZN framework offers a comprehensive approach to addressing these challenges. This hybrid approach generates actionable insights and robust prioritizations, fostering successful collaboration in Vietnam's logistics sector and serving as a template for similar developing economies.

This research conducted a comprehensive literature review and expert judgment involving logistics professionals and academics to analyze the interconnections and interdependencies among the identified CSFs. Together, these MCDM tools enable a rigorous evaluation of strategies, enhance decision-making by quantifying qualitative data, and provide actionable recommendations tailored to enhance efficiency and sustainability in Vietnam's logistics sector. Among various MCDM models, the Delphi method facilitates expert consensus-building, crucial for understanding qualitative insights and ensuring comprehensive coverage of factors impacting HC implementation [20]. DEMATEL, on the other hand, provides a structured way to analyze the causal relationships and interdependencies among identified CSFs, offering insights into which factors are pivotal and how they interact [19]. By leveraging Delphi and DEMATEL, this research aims to uncover nuanced insights and foster collaborative frameworks that can effectively address Vietnamese logistics firms' unique challenges, paving the way for improved competitiveness and environmental stewardship in the global logistics landscape.

It can be concluded that a comprehensive understanding of the current operational and strategic CSFs is crucial for the successful implementation of LHC within Vietnam's logistics sector. Additionally, the study proposes practical frameworks and models tailored to the unique challenges faced by Vietnamese logistics firms, particularly SMEs operating in a fragmented market. This research aims to thoroughly analyze LHC's benefits, challenges, and key success factors, offering actionable insights and strategies for logistics firms, policymakers, and relevant stakeholders. These findings can support effective collaboration, boost the efficiency and sustainability of logistics operations in Vietnam, and strengthen the nation's competitive edge within the global logistics market. Integrating NS and Z-Number theory with MCDM techniques, such as Delphi and DEMATEL, will increase the comprehensiveness of expert assessments. This combination allows for the inclusion of zero-level certainty factors in the calculation process, leading to more accurate and reliable results and improving model efficiency. Additionally, incorporating expert weighting and multiplying these weights

by the ratings ensures that the outcomes are more comprehensive and realistic. This approach considers experts' varying knowledge and experience across different industries.

The structure of the study is elaborated upon in the subsequent section. Section 2 of this paper provides an in-depth review of the current literature. Section 3 of the document furnishes a comprehensive elucidation of the study process and the methodology employed. This study will explore the discussions and insights from the empirical analysis, mainly focusing on Sections 4 and 5. Section 6 summarizes the key conclusions, implications, limitations, and potential directions for future research.

2. Literature Review

2.1. Theoretical Foundation

Mason et al. [20] delineated two primary forms of logistics collaboration: Vertical Collaboration (VC) and Horizontal Collaboration (HC). VC emphasizes fostering advantageous relationships across different supply chain tiers, such as manufacturers, distributors, and retailers, promoting integration and efficiency across stages [21]. Conversely, HC pertains to cooperative arrangements between organizations operating at the same level within the supply chain or within the same industry [21]. Unlike VC, HC unites peers or competitors to address shared challenges and pursue mutual objectives jointly. By leveraging complementary resources and capabilities, HC enables organizations to achieve superior outcomes collectively compared to operating in isolation [22]. HC initiatives encompass various practices, including knowledge sharing, co-developing industry standards, joint research and development (R&D), and sector-wide sustainability programs.

The potential benefits of HC are substantial, encompassing cost reductions, operational efficiencies, innovation facilitation, and the development of industry-wide solutions. For example, a collaborative effort in Spain’s automotive sector successfully reduced the number of freight journeys and associated emissions without compromising delivery lead times [23]. However, implementing HC faces critical challenges, such as fostering participant trust, aligning goals—especially among competitors—and establishing strategic frameworks to guide collaboration. These challenges are particularly pronounced when adopting sustainable HC practices, which are crucial for transitioning the logistics sector towards "green logistics," given the environmental impact of freight transport. Globally, freight transport contributes approximately 5.5% to total emissions, with the need for sustainable practices growing increasingly urgent [24]. In Vietnam, logistics contributes approximately 25% to the national GDP, with transport costs accounting for 50-60% of total logistics expenses, further intensifying emission concerns. These challenges are magnified by Vietnam's vulnerabilities to climate change and its rapidly expanding import-export market. Sustainable HC offers a viable pathway for Vietnamese logistics enterprises—many SMEs—to collaboratively drive green innovations, establish sustainability standards, and achieve economies of scale. Such an approach allows for joint investments in initiatives like route optimization, alternative fuels, and environmental stewardship, accelerating the adoption of green logistics practices while enhancing competitiveness in the global market.

HC can be conceptualized as a strategic supply chain when partnerships exhibit dynamic, operational, and technological integration and maintain long-term adaptability [25]. In this context, HC fosters a structured supply chain environment where collaboration amplifies value creation. Integrating Sustainable Development Theory (SDT) and the Triple Bottom Line (TBL) framework provides a comprehensive lens for understanding

and enhancing sustainable HC practices in Vietnam's logistics sector. This theoretical synergy offers valuable insights into the CSFs necessary to facilitate effective partnerships among logistics enterprises while balancing economic, social, and environmental dimensions. Supply chain partnerships often present a social dilemma characterized by a tension between individual short-term gains and collective long-term benefits [26]. A social dilemma arises when individuals must choose between actions that benefit themselves versus those that benefit the group. In fragmented logistics markets like Vietnam, companies might prioritize immediate cost savings over investments in sustainable practices that benefit the broader network. SDT addresses these dilemmas by offering strategies to resolve conflicts, such as fostering trust, enhancing communication, and aligning partner incentives. The TBL framework complements this by advocating for a broader definition of success—encompassing environmental, social, and economic impacts—encapsulated by the principles of "People, Planet, and Profit" [27]. This holistic perspective encourages businesses to align their operations with sustainable development goals, ensuring resource preservation for future generations while uncovering opportunities for growth and innovation beyond purely financial considerations. By integrating SDT and TBL, a robust foundation emerges for identifying CSFs essential for sustainable HC.

2.2. Literature Review on CSFs in LHC

Previous studies employed diverse methodologies to analyze LHC strategies and CSFs. Qualitative techniques like case studies and in-depth interviews provided rich insights into companies' motivations, collaboration processes, and experiences. Quantitative methods were also employed, with analytical hierarchy process (AHP) models and fuzzy logic frameworks used to assess partner compatibility and strategic fit. Process models and conceptual frameworks systematically mapped the stages of collaboration formation, operation, and outcome evaluation. Literature reviews and cross-case analyses synthesized findings across multiple studies to identify common themes, enablers, and barriers to successful collaboration [28]. Quantitative techniques like game and optimization models further analyzed dynamics such as gain/cost-sharing mechanisms, resource allocation strategies, and joint decision-making processes in horizontal alliances. Across these studies, critical factors like achieving monetary gains, future earnings potential, responding to regulatory pressures and customer demands, establishing trust between partners, ensuring strategic alignment and cultural compatibility, developing robust legal and governance frameworks, designing fair gain/risk sharing models, and enabling efficient information sharing have been identified as critical drivers influencing the successful adoption of HC initiatives [29], [30].

Previous research has identified various critical factors that drive companies to participate in HC initiatives, ultimately determining their success or failure. Internal motivations like achieving monetary gains, future earnings potential, and enhancing reputation play a significant role. External factors such as regulatory pressures, customer demands, and competitive forces also act as catalysts [31]. Establishing trust between partners is pivotal, as collaboration involves sharing resources and sensitive information between competing firms with diverse backgrounds. Trust mitigates potential conflicts and opportunistic behavior. Careful partner selection based on strategic alignment, compatible cultures, financial stability, and proven track records is crucial in the formation stage [32]. Other key factors include developing robust legal frameworks, governance mechanisms for joint decision-making, fair gain/risk-sharing models, and efficient information-sharing protocols. The involvement of neutral third parties or "network brokers" can facilitate partner identification,

negotiation, and coordination, especially in early collaboration stages. By identifying and prioritizing CSFs, logistics companies can develop strategies that enhance collaboration efficiency and sustainability. These insights are invaluable for policymakers and industry stakeholders aiming to foster a more collaborative and competitive logistics environment. Below are CSF categories encompassing the economic, environmental, and social dimensions that integrate social dilemma theory, which organizations must consider, providing a holistic perspective on ensuring the financial viability, ecological sustainability, and societal impact of HC initiatives, as presented in **Table 1**.

Table 1. Potential Factors Affecting HC

Main Dimensions	Code	Factors	References
Economic	EC1	Financial Stability	Bahinipati et al. [28]
	EC2	Data Sharing and Integration	Malviya et al. [33]
	EC3	Collaborative Distribution Centers	Malviya et al. [33]
	EC4	Innovation and Technology Adoption Ability	Malviya et al. [33]
	EC5	General Governance Agreements & Commitment to Responsibilities	Karam et al. [34]
	EC6	Consensus on Cost/Profit Sharing Policies	Karam et al. [34]
	EC7	Trust coordination	Malviya et al. [33]
	EC8	Governmental Support	Malviya et al. [33]
	EC9	Cross-Functional Collaboration	Karam et al. [34]
Social	SC1	The consensus on organizational culture	Weare et al. [35]
	SC2	Labor Relations	Chen et al. [36]
	SC3	Supplier Relationship Management	Büyüközkan et al. [37]
	SC4	Community Engagement	Head et al. [38]
	SC5	Coordinate employee training and development programs	Yousuf et al. [39]
	SC6	Clear Conflict Resolution Mechanisms	Karam et al. [34]
Environmental	EN1	General Green Transportation & Packaging Practices	Pomponi et al. [21]
	EN2	Environmental efficiency optimizes coordination	Turki et al. [40]

EN3	General Waste Reduction and Recycling Mechanisms	Vlajic et al. [41]
EN4	Circular Economy Practices	Sudusinghe et al. [42]

For the economic dimension, the collaborating partners' financial stability (EC1)s essential, as HC often involves pooling resources, making joint investments in shared assets like distribution centers, and bearing coordination costs [28]. Substantial data sharing and integration capabilities (EC2) enable a seamless exchange of operational information for activities like consolidated shipment planning and order fulfillment coordination. Establishing collaborative distribution centers (EC3) allows partners to consolidate logistics activities, reduce redundancies, and achieve economies of scale. Innovation and technology adoption ability (EC4) ensures collaborating firms can implement process improvements and adopt digital solutions for transparency and efficiency gains. Robust governance agreements (EC5) that clearly define roles and decision rights and ensure commitment to assumed responsibilities provide a solid foundation for collaboration. Reaching consensus on equitable and mutually accepted cost and profit-sharing policies (EC6) is critical to prevent conflicts over the distribution of partnership gains. Building trust and coordination mechanisms (EC7) mitigates risks like free-riding and facilitates alignment of potentially divergent partner interests. Governmental support (EC8) through enabling policies, financial incentives, and investments in logistics infrastructure can greatly facilitate HC initiatives. Finally, effective cross-functional collaboration (EC9) helps align different business units like procurement, operations, and finance within each partner organization behind the collaboration goals. In the social dimension, critical factors include consensus on organizational culture (SC1), supplier relationship management (SC2), coordinated employee training and development programs (SC3), and precise conflict resolution mechanisms (SC4).

In the social dimension, achieving a consensus on organizational cultures (SC1) across the collaborating partners is vital to bridging potential differences in values, management styles, and working methods [39]. Cultural compatibility facilitates effective communication, cooperation, and integration of operations. Strong supplier relationship management capabilities (SC2) enable the extension of collaboration beyond the core partners to upstream suppliers in the supply chain. This unlocks opportunities for broader process integration and sustainability initiatives spanning the end-to-end supply network. Coordinating employee training and development programs (SC3) across partners helps build a skilled joint workforce aligned with the collaboration objectives and protocols. Conflicts and disagreements are inevitable in any collaboration, so precise conflict resolution mechanisms (SC4) are vital for addressing issues through prescribed negotiation, mediation, or arbitration channels. Robust governance and defined escalation paths for conflict resolution prevent disputes from derailing the partnership. By proactively addressing these social factors, collaborating organizations can establish an environment conducive to long-term cooperation, knowledge sharing, and realizing collective goals.

The environmental dimension focuses on factors that enable collaborating organizations to jointly drive sustainability initiatives and reduce their ecological footprint [40]. Adopting general green transportation and packaging practices (EN1), such as using cleaner vehicles/fuels, optimizing routes, and sustainable packaging materials, helps lower carbon emissions and waste from logistics activities. Optimizing coordination between partners to improve environmental efficiency (EN2) minimizes unnecessary product movements and redundant

assets like facilities and vehicles, resulting in lower energy use and emissions. Implementing robust waste reduction and recycling mechanisms (EN3) allows collaborating firms to manage reverse logistics flows and reuse/recycle packaging and products in a closed-loop system. Taking this circular approach further, adopting comprehensive circular economy practices (EN4) enables partners to move toward a restorative industrial model by redesigning products and processes for longevity, reuse, refurbishing, and recycling across multiple life cycles. This extends resource productivity and eliminates waste. By collectively focusing on these environmental factors, HC empowers organizations to amplify their sustainability efforts, achieve scale in green initiatives, and transition towards a low-carbon, circular logistics model aligned with environmental stewardship goals.

2.3. Literature Review on Established Methods

Evaluating CSFs for sustainable LHC in Vietnam's logistics sector requires addressing various stakeholders, accommodating diverse perspectives, and managing inherent uncertainties. These complexities demand methodological approaches capable of effectively modeling ambiguity, quantifying uncertainty, and capturing the intricate interdependencies among various success factors. In this context, MCDM techniques have emerged as a valuable tool for tackling such challenges. This section reviews established methods, highlighting their applications, strengths, and limitations in evaluating CSFs for sustainable HC.

MCDM techniques, including widely used methods like the AHP [43] and the Technique for Order Preference by Similarity to Ideal Solution (TOPSIS) [44], have gained broad acceptance in logistics and supply chain management. These methods effectively address complex decision-making scenarios, such as supply chain design, sustainable manufacturing, and logistics network planning. For instance, Musumba et al. [45] employed a hybrid Fuzzy AHP-TOPSIS approach to facilitate partner selection for virtual enterprises, showcasing the method's ability to synthesize diverse stakeholder preferences. Similarly, Zhou et al. [46] developed a hybrid model combining fuzzy DEMATEL, AEW, and FVIKOR techniques to evaluate and select sustainable recycling partners, highlighting the utility of integrating multiple MCDM tools to address multidimensional problems. Prakash et al.[47] applied a fuzzy MCDM framework to assess third-party providers for reverse logistics collaboration, while Lo et al. used a Fuzzy-rough FARE-MABAC approach to incorporate stakeholder insights for selecting sustainable logistics providers [48]. **Table 2** provides a comprehensive summary of these studies and their application areas.

Table 1. Related Works

Techniques	Application Area	References
Fuzzy AHP-TOPSIS	Partner selection in establishing virtual enterprises	Musumba et al. [45]
Fuzzy DEMATEL-AEW-FVIKOR	Select a sustainable recycling partner	Zhou et al. [46]
Fuzzy MCDM	Evaluating third-party reverse logistics providers	Prakash et al. [47]

Fuzzy-rough FARE-MABAC	Sustainable logistics provider selection	Lo et al. [48]
Fuzzy Delphi-AHP	Assess critical criteria in food supply chains	Gupta et al. [49]
BWM–VIKOR	Selecting and evaluating sustainable outsourcing partner	Garg et al. [50]
Neutrosophic CRITIC	Selecting warehouse management software in sustainable logistics systems	Kara et al. [51]
SVNFNPBM	Selecting third-party logistics providers	Ji et al. [52]
Neutrosophic TOPSIS	Fourth-party logistics firm assessment	Aydin et al. [53]
SVN-CRITIC-CoCoSo	Assessment of sustainable third-party reverse logistic provider	Mishra et al. [54]

Despite the evident utility of these methods, they exhibit certain limitations when applied to the complexities of sustainable HC in Vietnam's fragmented logistics market. While techniques such as Fuzzy Delphi-AHP and Fuzzy DEMATEL-AEW-FVIKOR have effectively addressed stakeholder divergence and ambiguity, they have primarily relied on fuzzy logic [46]. Integrating NZN theory remains underexplored, representing a critical gap in managing uncertainties more comprehensively [8], [19]. Neutrosophic Sets extend traditional fuzzy logic by accounting for indeterminacy, and Z-number, which incorporates reliability through confidence levels, offers significant potential for enhancing decision-making reliability in HC evaluations [10], [52].

Secondly, existing studies have not sufficiently utilized two-stage models combining the Delphi method and DEMATEL. While the Delphi method excels at achieving expert consensus, DEMATEL is particularly useful for analyzing causal relationships among CSFs. However, the absence of their combined application limits the ability to refine input data through consensus-building and systematically analyze the complex interdependencies among factors [19]. Such an integrated approach would provide a more nuanced understanding of the causal dynamics that underpin successful HC in logistics.

Lastly, the practical application of Neutrosophic-based methods in real-world HC scenarios, particularly in logistics, remains limited. Although approaches like Neutrosophic CRITIC and SVN-CRITIC-CoCoSo have demonstrated utility in evaluating logistics providers, their application to sustainable HC has not been explored in depth [54]. The lack of integration of Neutrosophic Sets with MCDM frameworks leaves a significant gap in understanding how to prioritize and evaluate factors effectively under uncertainty, especially in collaborative logistics settings.

The proposed two-stage approach combines the Delphi method for achieving expert consensus with DEMATEL for analyzing causal relationships among CSFs. By leveraging the strengths of these methodologies, the framework aims to manage uncertainties more effectively, capture the complex interdependencies among factors, and provide actionable insights for logistics enterprises. This approach supports adopting sustainable HC practices in Vietnam's logistics sector, enabling companies to balance economic, environmental, and social considerations while addressing the unique challenges of fragmented market dynamics.

3. Proposed Methods

3.1 Preliminaries

Traditional fuzzy sets, while groundbreaking in handling uncertainty, have limitations. They express membership through a single value in [0,1], which fails to capture non-membership or hesitancy often encountered in real-world decisions [55]. Intuitionistic Fuzzy Sets (IFS) extended fuzzy sets by adding non-membership values, but the constraint that their sum must not exceed 1 limits their flexibility in complex scenarios [56]. Type-2 fuzzy sets addressed this by modeling higher uncertainty through secondary membership functions, but their computational complexity made them impractical for real-time use [57].

Z-numbers introduced reliability measures, combining truth values with reliability assessments, but struggled with contradictory information [57]. NS independently model truth, indeterminacy, and falsity, excelling in handling vague or contradictory data, though they lacked a mechanism for expressing the reliability of these values [58]. Picture Fuzzy Sets added a neutral component but did not adequately address reliability [59]. The NZN combines uncertainty and reliability, overcoming the limitations of previous methods and providing a robust tool for complex decision-making scenarios [60].

Definition 1 [19], [61], [62], [63], [64] A NZN set is defined as Equation (1):

$$\mathrm{NZN_z} = \{[x, a(M,N)(x), b(M,N)(x), c(M,N)(x)] | x \epsilon X\} \quad (1)$$

Where

$\alpha(M,N)(x) = \left(a_M(x), a_N(x)\right);\ b(M,N)(x) = \left(b_M(x), b_N(x)\right);\ c(M,N)(x) = \left(c_M(x), c_N(x)\right): X \rightarrow [0,1]^2$.

M represents neutrosophic values corresponding to the universal set X, while N signifies the neutrosophic reliability measures associated with M. Both components follow specific predefined conditions:

$0 \leq a_M(x) + b_M(x) + c_M(x) \leq 3$ and $0 \leq a_N(x) + b_N(x) + c_N(x) \leq 3$

$[x, a(M,N)(x), b(M,N)(x), c(M,N)(x)]$ in N_Z is succinctly represented as $N_Z = [a(M,N), b(M,N), c(M,N)] = [(a_M, a_N), (b_M, b_N), (c_M, c_N)]$

Definition 2 [19], [61], [63], [64], [65]

$\mathrm{NZN_{Z1}} = [a_1(M,N), b_1(M,N), c_1(M,N)] = [(a_{M1}, a_{N1}), (b_{M1}, b_{N1}), (c_{M1}, c_{N1})$ and

$\mathrm{NZN_{Z2}} = [a_2(M,N), b_2(M,N), c_2(M,N)] = [(a_{M2}, a_{N2}), (b_{M2}, b_{N2}), (c_{M2}, c_{N2})$ are two NZNs and $\vartheta > 0$. Then, we give the following relations (Equations (2) – (10)).

1. $\mathrm{NZN_{Z1}} \supseteq \mathrm{NZN_{Z2}} \Leftrightarrow a_{M1} \geq a_{M2}, a_{N1} \geq a_{N2}, b_{M1} \leq b_{M2}, b_{N1} \leq b_{N2}, c_{M1} \leq c_{M2}$ and $c_{N1} \leq c_{N2}$ (2)

2. $\mathrm{NZN_{Z1}} = \mathrm{NZN_{Z2}} \iff \mathrm{NZN_{Z1}} \supseteq \mathrm{NZN_{Z2}}$ and $\mathrm{NZN_{Z2}} \supseteq \mathrm{NZN_{Z1}}$ (3)

3. $\mathrm{NZN_{Z1}} \cup \mathrm{NZN_{Z2}} \iff [(a_{M1} \vee a_{M2},\ a_{N1} \vee a_{N2}),\ (b_{M1} \wedge b_{M2},\ b_{N1} \wedge b_{N2}),\ (c_{M1} \wedge c_{M2}, c_{N1} \wedge c_{N2})]$ (4)

3. $\mathrm{NZN_{Z1}} \cap \mathrm{NZN_{Z2}} \iff [(a_{M1} \wedge a_{M2},\ a_{N1} \wedge a_{N2}),\ (b_{M1} \vee b_{M2},\ b_{N1} \vee b_{N2}),\ (c_{M1} \vee c_{M2}, c_{N1} \vee c_{N2})]$ (5)

4. $(\mathrm{NZN_{Z1}})^{C} = [(c_{M1}, c_{N1}), (1 - b_{M1}, 1 - b_{N1}), (a_{M1}, a_{N1})]$ (Complement of $\mathrm{NZN_{Z1}}$) (6)

5. $\mathrm{NZN_{Z1}} \oplus \mathrm{NZN_{Z2}} = [(a_{M1} + a_{M2} - a_{M1}a_{M2}, a_{N1} + a_{N2} -$ (7)

$a_{N1}a_{N2}), (b_{M1}b_{M2}, b_{N1}b_{N2}), (c_{M1}c_{M2}, c_{N1}c_{N2})$

6. $\mathrm{NZN_{Z1}} \otimes \mathrm{NZN_{Z2}} = [(a_{M1}a_{M2}, a_{N1}\alpha_{N2}), (b_{M1} + b_{M2} - b_{M1}b_{M2}, b_{N1} + b_{N2} - b_{N1}b_{N2}), (c_{M1} +$ (8)

$c_{M2} - c_{M1}c_{M2}, c_{N1} + c_{N2} - c_{N1}c_{N2})]$

7. $\vartheta\mathrm{NZN_{Z1}} = [(1 - (1 - a_{M1})^{\vartheta}, 1 - (1 - a_{N1})^{\vartheta}), (\beta_{M1}^{\vartheta}, \beta_{N1}^{\vartheta}), (\gamma_{M1}^{\vartheta}, \gamma_{N1}^{\vartheta})]$ (9)

8. $(\mathrm{NZN_{Z1}})^{\vartheta} = [\ (a_{M1}^{\vartheta}, a_{N1}^{\vartheta}), (1 - (1 - b_{M1})^{\vartheta}, 1 - (1 - b_{N1})^{\vartheta}), (1 - (1 - c_{M1})^{\vartheta}, 1 - (1 - c_{N1})^{\vartheta})]$ (10)

Defuzzy $\mathrm{NZN_{Z1}} = [a_1(M,N), b_1(M,N), c_1(M,N)] = [(a_{M1}, a_{N1}), (b_{M1}, b_{N1}), (c_{M1}, c_{N1})$ Using Equation (11).

$$\mathrm{DEF}(\mathrm{NZN_{Z1}}) = \frac{2 + a_{M1}\, a_{N1} - b_{M1}\, b_{N1} - c_{M1}\, c_{N1}}{3}, \mathrm{DEF}(\mathrm{NZN_{Z1}}) \in [0,1] \quad (11)$$

Illustrative example 1:

With two NZN: NZN_{Z1} =[(0,2;0,6);(0,85;0,35);(0,8;0,4)], NZN_{Z2} =[(0,4;0,2);(0,65;0,85);(0,6;0,8)] and $\vartheta = 3$, the results of calculation by use operations (7) - (11) are shown below:

$\mathrm{NZN_{Z1}} \oplus \mathrm{NZN_{Z2}} = [(0{,}2; 0{,}6); (0{,}85; 0{,}35); (0{,}8; 0{,}4)] \oplus [(0{,}4; 0{,}2); (0{,}65; 0{,}85); (0{,}6; 0{,}8)]$

$= [(0{,}52; 0{,}68); (0{,}5525; 0{,}2975); (0{,}48; 0{,}32)]$

$\mathrm{NZN_{Z1}} \otimes \mathrm{NZN_{Z2}} = [(0{,}2; 0{,}6); (0{,}85; 0{,}35); (0{,}8; 0{,}4)] \otimes [(0{,}4; 0{,}2); (0{,}65; 0{,}85); (0{,}6; 0{,}8)]$

$= [(0{,}08; 0{,}12); (0{,}9475; 0{,}9025); (0{,}92; 0{,}88)]$

$3 \times \mathrm{NZN_{Z1}} = 3 \times [(0{,}2; 0{,}6); (0{,}85; 0{,}35); (0{,}8; 0{,}4)]$

$= [(0{,}488; 0{,}936); (0{,}6141; 0{,}0429); (0{,}512; 0{,}064)]$

$(\mathrm{NZN_{Z1}})^3 = ([(0{,}2; 0{,}6); (0{,}85; 0{,}35); (0{,}8; 0{,}4)]\ )^3$

$= [(0{,}008; 0{,}216); (0{,}9966; 0{,}7254); (0{,}992; 0{,}784)]$

$$DEF(\mathrm{NZN_{Z1}})^3 = \frac{2 + 0{,}2 * 0{,}6 - 0{,}85 * 0{,}35 - 0{,}8 * 0{,}4}{3} = 0{,}5008$$

Definition 3 [19], [61], [62], [63]:

Using Equations (7) and (9) in Definition 2, the equation for the Weighted Aggregation Arithmetic Mean (NZNWAA) for NZNs is derived. Let $\mathrm{NZN_{Zi}} = [a_i(M,N), b_i(M,N), c_i(M,N)] = [(a_{Mi}, a_{Ni}), (b_{Mi}, b_{Ni}), (c_{Mi}, c_{Ni})], (i = 1, 2, \ldots n)$ be a group of NZN and NZNWAA: $\Omega^n \longrightarrow \Omega$. Subsequently, the NZNWAA equation is formally defined as Equation (12).

$$NZNWAA(\mathrm{NZN_{Z1}}, \mathrm{NZN_{Z2}}, \dots, \mathrm{NZN_{Zn}}) = \sum_{i=1}^{n} \vartheta_i \, \mathrm{NZN_{Zi}} \tag{12}$$
$$= \left[1 - \prod_{i=1}^{n}(1 - a_{Mi})^{\vartheta_i}, 1 - \prod_{i=1}^{n}(1 - a_{Ni})^{\vartheta_i}\right), \left(\prod_{i=1}^{n} b_{Mi}{}^{\vartheta_i}, \prod_{i=1}^{n} b_{Ni}{}^{\vartheta_i}\right), \left(\prod_{i=1}^{n} c_{Mi}{}^{\vartheta_i}, \prod_{i=1}^{n} c_{Ni}{}^{\vartheta_i}\right)]$$

where $\vartheta_i (i = 1, 2..n)$ is the weight of $\mathrm{NZN_{Zi}}$, $0 \le \vartheta_i \le 1$ and $\sum_{i=1}^{n} \vartheta_i = 1$

The equation for calculating the weighted aggregation through the geometric mean (NZNWAGM) is as follows: $\mathrm{NZN_{Zi}} = [a_i(M,N), b_i(M,N), c_i(M,N)] = [(a_{Mi}, a_{Ni}), (b_{Mi}, b_{Ni}), (c_{Mi}, c_{Ni})], (i = 1, 2, \dots n)$ be a group of NZN and NZNWGA: $\Omega^n \longrightarrow \Omega$. The NZNWGA equation is formally defined as shown in Equation (13).

$$NZNWGA(\mathrm{NZN_{Z1}}, \mathrm{NZN_{Z2}}, \dots, \mathrm{NZN_{Zn}}) = \prod_{i=1}^{n} (\mathrm{NZN_{Zi}})^{\vartheta_i} \tag{13}$$
$$= [\left(\prod_{i=1}^{n} (a_{Mi})^{\vartheta_i}, \prod_{i=1}^{n} (a_{Ni})^{\vartheta_i}\right), \left(1 - \prod_{i=1}^{n}(1 - b_{Mi})^{\vartheta_i}, 1 - \prod_{i=1}^{n}(1 - b_{Ni})^{\vartheta_i}\right), 1$$
$$- \prod_{i=1}^{n}(1 - c_{Mi})^{\vartheta_i}, 1 - \prod_{i=1}^{n}(1 - c_{Ni})^{\vartheta_i})$$

where $\vartheta_i (i = 1, 2..n)$ is the weight of $\mathrm{NZN_{Zi}}$ with $0 \le \vartheta_i \le 1$ and $\sum_{i=1}^{n} \vartheta_i = 1$

Illustrative example 2:

With set S include 3 NZN number S = {[(0,2;0,6);(0,85;0,35);(0,8;0,4)], [(0,4;0,2);(0,65;0,85);(0,6;0,8)], [(0,4;0,4);(0,65;0,65);(0,6;0,6)]} and corresponding weights $\vartheta_i = \{0{,}5; 0{,}4; 0{,}1\}$, the agreed results obtained using the NZNWAA and NZNWGA methods are summarized as follows:

NZNWAA (S) = [(0,3072;0,4503);(0,7433;0,531);(0,6928;0,5497)]

NZNWGA (S) = [(0,2828;0,3713);(0,7709;0,6601);(0,7172;0,6287)]

Definition 4 [19], [66]: Metrics for assessing distance and similarity within NZN ts

Let $\mathrm{NZN_{Z1}} = \{\mathrm{NZN_{Z11}}, \mathrm{NZN_{Z12}} \dots, \mathrm{NZN_{Z1n}}\}$ and $\mathrm{NZN_{Z2}} = \{\mathrm{NZN_{Z21}}, \mathrm{NZN_{Z22}} \dots, \mathrm{NZN_{Z2n}}\}$, where $\mathrm{NZN_{Z1k}} = [a_{1k}(M,N), b_{1k}(M,N), c_{1k}(M,N)] = [(a_{M1k}, a_{N1k}), (b_{M1k}, b_{N1k}), (c_{M1k}, c_{N1k})]$, and $\mathrm{NZN_{Z2k}} = [a_{2k}(M,N), b_{2k}(M,N), c_{2k}(M,N)] = [(a_{M2k}, a_{N2k}), (b_{M2k}, b_{N2k}), (c_{M2k}, c_{N2k})]$ are two NZNs, set $\theta \ge 1$ as any integer, and the corresponding weights of n pairs of NZN $\vartheta_k = (\vartheta_1, \vartheta_2, \dots \vartheta_n), \sum_{k=1}^{n} \vartheta_k = 1$. Subsequently, the generalized distance between $\mathrm{NZN_{Z1}}$ and $\mathrm{NZN_{Z2}}$ is calculated by the Equation (14):

$$\mathrm{DIS_{w_\theta}}(\mathrm{NZN_{Z1}}, \mathrm{NZN_{Z2}}) \tag{14}$$
$$= \frac{1}{2}\left\{\sqrt[\theta]{\frac{1}{3}\sum_{k=1}^{n} \vartheta_k (|a_{M1k} - a_{M2k}|^{\vartheta} + |b_{M1k} - b_{M2k}|^{\vartheta} + |c_{M1k} - c_{M2k}|^{\vartheta})}\right.$$
$$\left. + \sqrt[\theta]{\frac{1}{3}\sum_{k=1}^{n} \vartheta_k (|a_{N1k} - a_{N2k}|^{\vartheta} + |b_{N1k} - b_{N2k}|^{\vartheta} + |c_{N1k} - c_{N2k}|^{\vartheta})}\right\}$$

When $\theta = 1$, the generalized distance formula reduces to the Hamming distance formula DIS_{w1}, as shown in Equation (15).

$$\begin{aligned}&\mathrm{DIS}_{w1}(\mathrm{NZN}_{Z1}, \mathrm{NZN}_{Z2})\\&= \frac{1}{6}\left\{\sum_{k=1}^{n} \vartheta_k(|a_{M1k} - a_{M2k}| + |b_{M1k} - b_{M2k}| + |c_{M1k} - c_{M2k}|)\right.\\&\left.+ \sum_{k=1}^{n} \vartheta_k(|a_{N1k} - a_{N2k}| + |b_{N1k} - b_{N2k}| + |c_{N1k} - c_{N2k}|)\right\}\end{aligned} \tag{15}$$

When $\theta = 2$, this generalized distance formula simplifies to the Euclidean distance formula DIS_{w2}, shown in Equation (16).

$$\begin{aligned}&\mathrm{DIS}_{w2}(\mathrm{NZN}_{Z1}, \mathrm{NZN}_{Z2})\\&= \frac{1}{2}\left\{\sqrt{\frac{1}{3}\sum_{k=1}^{n} \vartheta_k(|a_{M1k} - a_{M2k}|^2 + |b_{M1k} - b_{M2k}|^2 + |c_{M1k} - c_{M2k}|^2)}\right.\\&\left.+ \sqrt{\frac{1}{3}\sum_{k=1}^{n} \vartheta_k(|a_{N1k} - a_{N2k}|^2 + |b_{N1k} - b_{N2k}|^2 + |c_{N1k} - c_{N2k}|^2)}\right\}\end{aligned} \tag{16}$$

Illustrative example 3

With two sets NZN: $SNZN_{Z1}$ ={[(0,2;0,6);(0,85;0,35);(0,8;0,4)], [(0,4;0,2);(0,65;0,85);(0,6;0,8)]} and $SNZN_{Z2}$ ={[(0,8;0);(0,15;1);(0,2;1)]; [(0,4;0,6);(0,65;0,35);(0,6;0,4)]}with corresponding weights $\vartheta_2 = (0{,}7; 0{,}3)$, the Euclidean distance between these two NZN sets is calculated using Equation (16):

$$\begin{aligned}&DIS_{w2}(SNZN_{Z1}, SNZN_{Z2})\\&= \frac{1}{2}\left\{\sqrt{\frac{1}{3}\left[\begin{matrix}0{,}7(|0{,}2 - 0{,}8|^2 + |0{,}85 - 0{,}15|^2 + |0{,}8 - 0{,}2|^2)\\+ 0{,}3(|0{,}4 - 0{,}4|^2 + |0{,}65 - 0{,}4|^2 + |0{,}6 - 0{,}6|^2)\end{matrix}\right]}\right.\\&\left.+ \sqrt{\frac{1}{3}\left[\begin{matrix}0{,}7(|0{,}6 - 0|^2 + |0{,}35 - 1|^2 + |0{,}4 - 1|^2)\\+ 0{,}3(|0{,}2 - 0{,}6|^2 + |0{,}85 - 0{,}35|^2 + |0{,}8 - 0{,}4|^2)\end{matrix}\right]}\right\} = 0{,}5501\end{aligned}$$

3.2 Research Procedure

This study presents a two-stage MCDM approach combining NZN with the Delphi and DEMATEL methods, as shown in **Figure 1**. In the first phase, the NZN-Delphi method gathers expert opinions via a structured Delphi survey, where logistics experts identify and refine the CSFs for LHC. The iterative process ensures consensus and clarity on these factors, while NZN captures uncertainties and improves the reliability of expert judgments. In the second phase, the NZN-DEMATEL approach explores the causal relationships between the identified factors. DEMATEL creates influence relation maps to visualize the cause-and-effect dynamics and interdependencies among CSFs. By integrating NZN, the approach handles data uncertainty and

provides a more accurate analysis, enabling a reliable prioritization of the success factors based on their impact and interconnections.

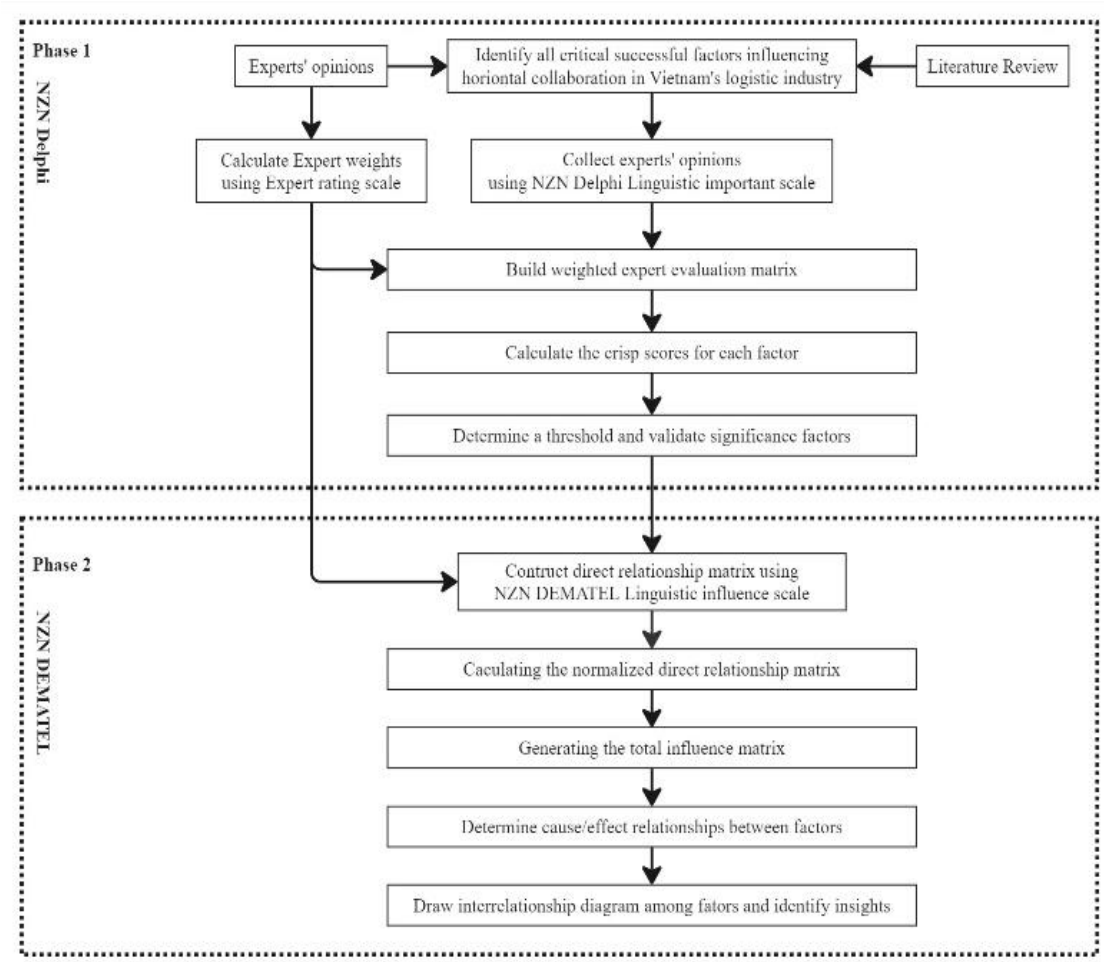
Phase 1
NZN Delphi
Experts' opinions
Identify all critical successful factors influencing horiontal collaboration in Vietnam's logistic industry
Literature Review
Calculate Expert weights using Expert rating scale
Collect experts' opinions using NZN Delphi Linguistic important scale
Build weighted expert evaluation matrix
Calculate the crisp scores for each factor
Determine a threshold and validate significance factors
Phase 2
NZN DEMATEL
Contruct direct relationship matrix using NZN DEMATEL Linguistic influence scale
Caculating the normalized direct relationship matrix
Generating the total influence matrix
Determine cause/effect relationships between factors
Draw interrelationship diagram among fators and identify insights

Figure 1. Proposed research framework

3.3 NZN Delphi Model

Suppose k experts provide evaluations for n factors. Each expert assesses the significance of these factors using a linguistic scale, which is subsequently converted to NZN numbers using NZN sets. Furthermore, experts are weighted according to their education and years of experience.

Step 1: Compute the experts' weights.

Expert weights will be determined using NZN numbers, comprising two key components: MM, representing the evaluation level based on the expert's experience and educational background, and NN, indicating the certainty level derived from the research team's understanding of the expert. These NZN values, reflecting the expert's rating, will be combined using Equation (7) and converted into a precise score via Equation (11). **Table 3** presents the expert assessments and corresponding linguistic scales [64].

Table 3. Expert rating scale

Education (A)	Experience (A)	Linguistic scale (C)	NZN	Code
Doctor	Over 20 years	Very high	(0.8,0.15,0,2)	VH
Master	10 – 20 years	High	(0.6,0.35,0.4)	H
Bachelor	5 – 10 years	Medium	(0.4,0.65,0.6)	M
Under Bachelor	Under five years	Low	(0.2,0.85,0.8)	L
		Very low	(0,1,1)	VL

Calculate the evaluation value for k experts, obtaining k values EXK: $\text{exk}_j = \{\text{exk}_1, \text{exk}_2, \ldots \text{exk}_k\}$. The weight of expert EXW: $\text{exw}_j = \{\text{exw}_1, \text{exw}_2, \ldots \text{exw}_k\}$ is calculated as Equation (17) below:

$$\text{exw}_j = \frac{\text{exk}_j}{\sum_{j=1}^{k} \text{exk}_j} \tag{17}$$

Step 2: Construct a matrix that incorporates weighted evaluations from experts.

Experts will assess the importance of n factors, initially presented in linguistic form and then converted to NZN numbers. This process forms a matrix $\otimes \text{EM} = \left[\text{em}_{ij}\right]_{nxk}$, where n is a number of factors and k is several experts. The evaluation linguistic scale and corresponding NZN are detailed in **Table 4**.

Table 4. NZN Delphi linguistic significant scale

Important level	Code	Membership		
		α	**β**	**γ**
Very low	VL	0	1	1
Low	L	0.2	0.85	0.8
Medium	M	0.4	0.65	0.6
High	H	0.6	0.35	0.4
Very high	VH	0.8	0.15	0.2

The weighted expert evaluation matrix $\otimes \text{EMW} = \left[\text{emw}_{ij}\right]_{nxk}$ is created using Equation (18) below.

$$\text{emw}_{ij} = \text{em}_{ij} \otimes \text{exw}_j \tag{18}$$

with $i = 1, 2, \ldots n$ and $j = 1, 2, \ldots k$; $\text{exw}_j = \{\text{exw}_1, \text{exw}_2, \ldots \text{exw}_k\}$ is the expert's j of weight.

Step 3: Determine the threshold and verify the variables.

A panel of k experts evaluates each factor, with Equation (12) consolidating these assessments into aggregated results for n factors in NZN format. These ratings are then transformed into precise crisp scores using Equation (11), producing \(n \) EV values: $\text{epv}_i = \{\text{epv}_1, \text{epv}_2, \ldots \text{evp}_n\}$. The threshold value is calculated using Equation (19):

$$\delta = \frac{\sum_{i=1}^{n} \text{epv}_i}{n} \tag{19}$$

If value $\text{epv}_i \geq \delta$, then factor i is accepted. If value $\text{epv}_i < \delta$, then factor i is rejected.

3.4 NZN DELMATEL Model

Assume there are k experts, each assigned a specific weigh ew , assessing the reciprocal influence among n factors. These assessments are initially provided in linguistic terms, then translated into NZN. Table 5 displays the rating scale along with its associated NZN values [64].

Table 5. NZN DEMATEL linguistic influence scale

Influence level	Code	Membership		
		a	**b**	**c**
Equal influence	EI	0	1	1
Week influence	WI	0.2	0.85	0.8
Fair influence	FI	0.4	0.65	0.6
Very influence	VI	0.6	0.35	0.4
Absolute influence	AI	0.8	0.15	0.2

Once the assessments are transformed into NZN numbers, the data undergoes analysis through the DEMATEL method, with the calculation steps outlined below.

Step 1: Calculating direct relationship matrix $\otimes DR$

The assessments of the mutual influence among n factors - where factor i impacts factor j - as provided by k experts and denoted asdr_{ij}^{k}, are transformed into NZN - based on the corresponding expert weights exw_t. These transformed evaluations are then aggregated using equation (20), yielding the direct influence matrix $\otimes \mathrm{DR} = \left[\otimes \mathrm{dr}_{ij}\right]_{nxn}$, where:

$$\mathrm{dr}_{ij} = \mathrm{NZNWAA}\left(\mathrm{dr}_{ij}^{1}, \mathrm{dr}_{ij}^{2}, \ldots, \mathrm{dr}_{ij}^{k}\right) = \sum_{t=1}^{k} \mathrm{exw}_t \mathrm{dr}_{ij}^{k} \tag{20}$$

where $i = 1, 2, \ldots n,\ \ j = 1, 2, \ldots n, t = 1, 2.. k;$

$\otimes \mathrm{d}_{ij} = \left[\left(\mathrm{dr}_{ij}^{a_E}, \mathrm{dr}_{ij}^{a_R}\right), \left(\mathrm{dr}_{ij}^{b_E}, \mathrm{dr}_{ij}^{b_R}\right), \left(\mathrm{dr}_{ij}^{c_E}, \mathrm{dr}_{ij}^{c_R}\right)\right]$. The diagonal elements in the matrix are 0, i.e, $\otimes \mathrm{dr}_{ij} = 0$ (when i = j).

Step 2:Determining the normalized form of the direct relationship matrix $\otimes DR^{*}$

Matrix $\otimes \mathrm{DR} = \left[\otimes \mathrm{dr}_{ij}\right]_{nxn}$ is transformed into a normalized form, represented as $\otimes \mathrm{DR}^{*} = \left[\otimes \mathrm{dr}_{ij}^{*}\right]_{nxn}$ by applying Equation (21) below:

$$\mathrm{dr}_{ij}^{*} = (\theta_E, \theta_R).\mathrm{dr}_{ij} = \tag{21}$$

$$\left[\left(\theta_E \mathrm{dr}_{ij}^{a_E}, \theta_R \mathrm{dr}_{ij}^{a_R}\right), \left(\theta_E \mathrm{dr}_{ij}^{b_E}, \theta_R \mathrm{dr}_{ij}^{b_R}\right), \left(\theta_E \mathrm{dr}_{ij}^{c_E}, \theta_R \mathrm{dr}_{ij}^{c_R}\right)\right]$$

where $\otimes \mathrm{dr}_{ij}^{*} = \left[\left(\mathrm{dr}_{ij}^{*a_E}, \mathrm{dr}_{ij}^{*a_R}\right), \left(\mathrm{dr}_{ij}^{*b_E}, \mathrm{dr}_{ij}^{*b_R}\right), \left(\mathrm{dr}_{ij}^{*c_E}, \mathrm{dr}_{ij}^{*c_R}\right)\right]$

$$\theta_A = \mathrm{Min}\left\{\frac{1}{\sum_{j=1}^{n} \mathrm{dr}_{ij}^{a_E}}; \frac{1}{\sum_{j=1}^{n} \mathrm{dr}_{ij}^{b_E}}; \frac{1}{\sum_{j=1}^{n} \mathrm{dr}_{ij}^{c_E}}\right\}$$

$$\theta_C = \mathrm{Min}\left\{\frac{1}{\sum_{j=1}^{n} \mathrm{dr}_{ij}^{a_R}}; \frac{1}{\sum_{j=1}^{n} \mathrm{dr}_{ij}^{b_R}}; \frac{1}{\sum_{j=1}^{n} \mathrm{dr}_{ij}^{c_R}}\right\} \tag{22}$$

Step 3: Constructing the comprehensive influence matrix $\otimes O$

The normalized direct relationship matrix $\otimes O$ is calculated to form the total influence matrix, incorporating both direct and indirect connections aggregated from minimal to maximal impact over an infinite range. The steps are detailed in Equations (23)-(24).

$$\otimes O = \left[\otimes o_{ij}\right]_{nxn}, \; i = j = 1, 2, \ldots n \tag{23}$$

where $\otimes o_{ij} = \left[\left(o_{ij}^{a_E}, o_{ij}^{a_R}\right), \left(o_{ij}^{b_E}, o_{ij}^{b_R}\right), \left(o_{ij}^{c_E}, o_{ij}^{c_R}\right)\right]$

$$\otimes O = \otimes DR^{*} + \otimes DR^{*2} + \cdots + \otimes DR^{*\infty}$$
$$= \otimes O(I + \otimes DR^{*} + \otimes DR^{*2} + \cdots + \otimes DR^{*\infty-1})$$
$$= \otimes DR^{*}(I - \otimes DR^{*\infty})(I - \otimes DR^{*})^{-1} = \otimes D^{*}(I - \otimes DR^{*})^{-1} \tag{24}$$

where $\otimes O^{\infty} = [0]_{nxn}$ and I is the identity matrix

Equation (11) is used to convert the NZN into crisp values, forming matrix $\otimes O^{*} = \left[\otimes o_{ij}^{*}\right]_{nxn}$

Step 4: Establish cause-effect relationships between factors and compute weights as given in Equations (25)-(28)

$\otimes r, \otimes c$ is calculated as below, with "superscript T" is the transpose of the matrix:

$$\otimes r = [\otimes r_i]_{nx1} = (\otimes r_1, \otimes r_2, \ldots, \otimes r_i, \ldots, \otimes r_n) \tag{25}$$

$$[\otimes r_i]_{nx1} = \left[\sum_{j=1}^{n} \otimes o_{ij}^{*}\right]_{nx1} \tag{26}$$

$$\otimes c = [\otimes c_i]_{1xn} = (\otimes c_1, \otimes c_2, \ldots, \otimes c_j, \ldots, \otimes c_n)^{T} \tag{27}$$

$$\left[\otimes c_j\right]_{1xn} = \left[\sum_{i=1}^{n} \otimes o_{ij}^{*}\right]_{1xn} = [\otimes c_i]_{nx1}^{T} \tag{28}$$

The index of the strength of influences imparted and received is $\otimes r_i + \otimes c_i$. The net influence is represented by $\otimes r_i - \otimes c_i$. A higher $\otimes r_i + \otimes c_i$ indicates that criterion i has a more significant influence on the evaluation system. If $\otimes r_i - \otimes c_i > 0$, indicator i significantly influences others. If $\otimes r_i - \otimes c_i < 0$, indicator other indicators influence me. In general, it denotes the overall effect of the indicator on the assessment system. The indicator's impact weight is calculated using Equation (29).

$$w_i = \frac{(r_i + c_i)}{\sum_{i=1}^{n}(r_i + c_i)} \tag{29}$$

4. Results and Discussions

The research was conducted within the Vietnamese context to address specific issues in the country's logistics industry. Vietnam's logistics sector, a critical component of its economy, faces significant challenges related to efficiency, sustainability, and collaboration. This study focused on identifying and prioritizing CSFs for sustainable HC among logistics companies to investigate these challenges.

4.1. Experts Panel Selection

In this study, purposive sampling was utilized to carefully select experts with substantial expertise in logistics and supply chain management, as this approach is well-suited to the specialized requirements of the research. This non-probability sampling method allows for the deliberate selection of individuals who meet specific criteria, ensuring that the data collected is relevant and insightful. Experts were chosen based on their significant professional experience (at least ten years), educational background (holding at least a bachelor's

degree, with many having master's or doctoral degrees), and their professional roles (lecturers, researchers from renowned universities, or leaders from prominent logistics corporations in Vietnam). The experts were all 35 years and above, ensuring mature and experienced viewpoints. Potential experts were identified through professional networks, industry associations, and academic institutions, with personalized invitations to encourage participation. Both online methods (JotForm surveys and virtual interviews via Zoom and Microsoft Teams) and offline methods (face-to-face interviews and paper-based surveys) were used to ensure comprehensive data collection. Before the final distribution, a pilot test with five logistics professionals who were not part of the final panel was conducted to refine the questionnaire based on clarity, relevance, and comprehensiveness feedback.

According to Nguyen et al. [65], a minimum of 10 to 18 experts is sufficient for this type of research to ensure diverse and reliable data collection. In this study, 35 survey invitations were sent, and 30 valid responses were received. All participating experts had at least ten years of experience, including lecturers, researchers from renowned universities, and leaders from prominent logistics corporations. The expert panel comprised 30 individuals, including five aged 35-40 years (16.6%), 19 aged 40-60 years (63.3%), and six aged over 60 years (20%). Gender representation included 13 males (43.3%), 14 females (46.6%), and three individuals identifying as other (10%). Educationally, one expert held a bachelor's degree (3.3%), 22 had a master's degree (73.3%), and 7 had a doctorate (23.3%). Professionally, the panel included seven government officials (23.3%), 12 researchers/professors (40%), and 11 managers/directors (36.7%). Regarding experience, 24 experts had 10-20 years (80%), while 6 had more than 20 years (20%). This meticulous selection process and data collection method underscore the robustness and validity of the research findings, offering valuable insights into the CSFs for sustainable HC in the Vietnamese logistics industry. **Table 6** provides detailed information about the experts' profiles.

Table 6. Experts's Profiles

	Criteria	Number of experts	Percentage
Age	35 – 40	5	16.6 %
(years old)	40 – 60	19	63.3 %
	More than 60	6	20 %
Gender	Male	13	43.3 %
	Female	14	46.6 %
	Other	3	10 %
Education	Bachelor	1	3.3%
	Master	22	73.3 %
	Doctor	7	23.3 %
Work	Government Officials	7	23.3 %

	Researchers/Professors	12	40 %
	Managers/Directors	11	36.7 %
Experience (years)	10 - 20	24	80 %
	More than 20	6	20 %

4.2. Results of NZN Delphi

In this study, the practical HC factors identified through the literature review encompass 20 factors distributed across three dimensions. Experts have judged these factors for their suitability in the form of linguistic scales, which were converted into Z numbers in Table 4. Expert evaluations on each factor were synthesized, and relevant indexes were calculated. Results of NZN Delphi analysis are given in Table 7 below, indicating whether the factor is retained or rejected. It compares each (***Vc***) value obtained for every factor with a given threshold value and selects three factors that did not meet the threshold criteria and were therefore rejected: SC2, SC4, and EN5. The comprehensive evaluation ensures that only the most relevant and suitable factors are carried forward for further analysis in Phase 2, enhancing the robustness and validity of the study's findings on effective HC in the logistics sector.

Table 7. The NZN Delphi method results

Factor	**Aggregate**	**Score**	**Validate**
EC1	[(0.7397;0.6093);(0.218;0.3499);(0.2603;0.3907)]	0.7576	Accepted
EC2	[(0.7021;0.5536);(0.2614;0.4271);(0.2979;0.4464)]	0.7147	Accepted
EC3	[(0.69;0.5235);(0.2737;0.4613);(0.31;0.4765)]	0.6957	Accepted
EC4	[(0.6987;0.5222);(0.2651;0.462);(0.3013;0.4778)]	0.6995	Accepted
EC5	[(0.729;0.6338);(0.2287;0.331);(0.271;0.3662)]	0.7624	Accepted
EC6	[(0.6663;0.5748);(0.3006;0.3917);(0.3337;0.4252)]	0.7078	Accepted
EC7	[(0.6735;0.7102);(0.2898;0.2448);(0.3265;0.2898)]	0.7709	Accepted
EC8	[(0.6967;0.5109);(0.2629;0.4744);(0.3033;0.4891)]	0.6943	Accepted
EC9	[(0.731;0.6119);(0.2299;0.3518);(0.269;0.3881)]	0.754	Accepted
SC1	[(0.7762;0.562);(0.1779;0.4207);(0.2238;0.438)]	0.7545	Accepted
SC2	[(0.2323;0.5091);(0.8114;0.47);(0.7677;0.4909)]	0.4533	**Rejected**
SC3	[(0.6971;0.6271);(0.2683;0.335);(0.3029;0.3729)]	0.7448	Accepted
SC4	[(0.2268;0.5621);(0.8218;0.4147);(0.7732;0.4379)]	0.4827	**Rejected**

SC5	[(0.7579;0.6027);(0.1987;0.3588);(0.2421;0.3973)]	0.7631	Accepted
SC6	[(0.6978;0.5553);(0.2601;0.4159);(0.3022;0.4447)]	0.715	Accepted
EN1	[(0.7097;0.5034);(0.2475;0.4675);(0.2903;0.4966)]	0.6991	Accepted
EN2	[(0.7083;0.496);(0.2509;0.4843);(0.2917;0.504)]	0.6943	Accepted
EN3	[(0.6544;0.5646);(0.3112;0.3984);(0.3456;0.4354)]	0.6983	Accepted
EN4	[(0.7277;0.6198);(0.2324;0.3441);(0.2723;0.3802)]	0.7558	Accepted
EN5	[(0.2747;0.4627);(0.756;0.5062);(0.7253;0.5373)]	0.4516	**Rejected**
	Threshold (*Vc*)	**0.68847**	

4.3 Results of NZN DEMATEL

After the validation and acceptance of factors during the NZN DEMATEL phase, an analysis of the interrelationships among these factors shall be undertaken. The cohort of thirty experts will participate in the second phase of the investigation, wherein they will evaluate the impact relationships between factors pairwise. The evaluation results obtained from the thirty experts will be synthesized through a process involving the calculation of averages, standardization, and defuzzification. **Table 8** shows the total-influence matrix, elucidating the interrelationships among the factors under consideration.

Table 8. Total influence matrix

	EC1	**EC2**	**EC3**	**EC4**	**EC5**	**EC6**	**EC7**	**EC8**	**EC9**	**SC1**	**SC3**	**SC5**	**SC6**	**EN1**	**EN2**	**EN3**	**EN4**
EC1	0.621	0.598	0.635	0.615	0.625	0.635	0.621	0.615	0.621	0.593	0.637	0.617	0.626	0.634	0.619	0.630	0.634
EC2	0.576	0.625	0.627	0.621	0.587	0.588	0.612	0.576	0.636	0.584	0.629	0.592	0.589	0.625	0.612	0.610	0.637
EC3	0.590	0.642	0.642	0.632	0.639	0.617	0.624	0.607	0.646	0.615	0.651	0.605	0.632	0.638	0.622	0.623	0.609

EC4	0.5 79	0.6 24	0.6 41	0.6 31	0.6 08	0.5 92	0.5 84	0.5 80	0.6 29	0.6 04	0.6 20	0.6 26	0.5 95	0.6 31	0.6 28	0.6 27	0.6 37
EC5	0.5 90	0.6 22	0.6 27	0.6 21	0.6 36	0.6 31	0.6 26	0.6 33	0.6 38	0.6 27	0.6 52	0.6 45	0.6 43	0.6 27	0.6 24	0.6 09	0.6 39
EC6	0.6 11	0.5 98	0.6 06	0.6 13	0.6 11	0.6 29	0.6 19	0.6 01	0.6 01	0.5 92	0.6 36	0.6 16	0.6 26	0.6 32	0.6 41	0.6 41	0.6 43
EC7	0.5 90	0.6 46	0.6 51	0.6 34	0.6 42	0.6 41	0.6 32	0.5 93	0.6 49	0.6 16	0.6 52	0.6 35	0.6 33	0.6 09	0.6 39	0.6 09	0.6 13
EC8	0.6 31	0.6 23	0.6 27	0.6 35	0.6 31	0.6 18	0.5 98	0.6 29	0.6 25	0.6 40	0.6 30	0.6 35	0.6 33	0.6 39	0.6 38	0.6 24	0.6 50
EC9	0.5 81	0.6 27	0.6 43	0.6 27	0.5 95	0.5 95	0.5 88	0.5 82	0.6 36	0.6 07	0.6 34	0.6 17	0.6 13	0.6 42	0.6 18	0.6 16	0.6 32
SC1	0.5 59	0.5 79	0.6 03	0.5 77	0.5 73	0.5 74	0.5 65	0.5 61	0.6 13	0.6 06	0.6 05	0.6 08	0.5 94	0.5 85	0.5 84	0.5 81	0.5 85
SC3	0.5 43	0.5 68	0.5 75	0.5 66	0.5 64	0.5 62	0.5 55	0.5 45	0.5 74	0.5 55	0.6 14	0.5 69	0.5 64	0.5 75	0.5 73	0.5 72	0.5 76
SC5	0.5 62	0.5 85	0.6 07	0.5 99	0.5 77	0.5 77	0.5 71	0.5 67	0.6 28	0.6 03	0.6 10	0.6 18	0.5 97	0.5 91	0.5 88	0.5 85	0.5 92

SC6	0.601	0.630	0.645	0.628	0.613	0.614	0.621	0.605	0.620	0.611	0.624	0.618	0.634	0.623	0.622	0.620	0.621
ENI1	0.548	0.570	0.578	0.573	0.567	0.567	0.562	0.551	0.576	0.563	0.583	0.573	0.569	0.515	0.517	0.574	0.579
ENI2	0.564	0.583	0.590	0.583	0.578	0.577	0.574	0.566	0.590	0.576	0.594	0.587	0.582	0.517	0.522	0.523	0.529
ENI3	0.552	0.575	0.580	0.574	0.569	0.569	0.562	0.557	0.577	0.563	0.584	0.575	0.572	0.597	0.595	0.511	0.580
ENI4	0.605	0.621	0.626	0.619	0.616	0.616	0.612	0.606	0.623	0.613	0.627	0.621	0.620	0.649	0.646	0.645	0.641

Table 9 presents $\otimes$ r, $\otimes$ c, $\otimes$ r $+\otimes$ c, and $\otimes$ r, $-\otimes$ c scores and weights of each CFS after defuzzy. If $\otimes$ r $-\otimes$ c connection is positive, then a factor is causal, that is, the cause factor; on the other hand, when the connection is negative, the factor is affected, that is, the effect factor. Following defuzzification, each factor is assigned a weight representing its relative significance within the causal framework. These weights quantify the magnitude of influence exerted by each factor on the overall system dynamics. Figure 2 illustrates significant causal relationships among factors.

Table 9. NZN DEMATEL Results

Criteria	**$\otimes$ r**	**$\otimes$ c**	**$\otimes$ r $+\otimes$ c**	**w**	**Rank**	**$\otimes$ r $-\otimes$ c**	**Relation**
EC1	10.57	9.90	20.47	0.05836	12	0.6739	**Cause**
EC2	10.33	10.32	20.64	0.05884	10	0.0118	**Cause**
EC3	10.63	10.50	21.14	0.06024	1	0.1317	**Cause**
EC4	10.44	10.35	20.78	0.05924	7	0.0882	**Cause**
EC5	10.69	10.23	20.92	0.05963	4	0.4575	**Cause**

EC6	10.51	10.20	20.72	0.05905	8	0.3109	**Cause**
EC7	10.68	10.12	20.81	0.05931	6	0.5578	**Cause**
EC8	10.71	9.97	20.68	0.05894	9	0.7326	**Cause**
EC9	10.45	10.48	20.93	0.05966	3	-0.0303	Effect
SC1	9.95	10.17	20.12	0.05734	17	-0.2162	Effect
SC3	9.65	10.58	20.23	0.05767	15	-0.9294	Effect
SC5	10.06	10.36	20.42	0.05819	13	-0.2995	Effect
SC6	10.55	10.32	20.87	0.05948	5	0.2264	**Cause**
EN1	9.76	10.53	20.29	0.05784	14	-0.7643	Effect
EN2	10.03	10.49	20.52	0.05849	11	-0.4522	Effect
EN3	9.79	10.40	20.19	0.05754	16	-0.6076	Effect
EN4	10.61	10.50	21.10	0.06015	2	0.1087	**Cause**

Most economic factors denoted as EC1 to EC8, are identified as primary causes within the logistics system, exerting significant influence on other factors. Collaborative Distribution Centers (EC3), Financial Stability EC1), and General Governance Agreements (EC5) play pivotal roles in shaping logistics firms' strategic decisions and overall performance. Specifically in HC, these economic factors are essential for aligning goals, reducing operational costs, and realizing economies of scale. The centrality of economic factors in strategic planning underscores their role as key drivers within the system. Effective HC implementation necessitates substantial investments in technology, infrastructure, and process optimization, all inherently linked to economic considerations. As such, economic factors assume a causative role, influencing both the adoption and success of HC initiatives. Furthermore, economic factors directly impact operational efficiency and productivity. For example, enhancements in financial stability or market competitiveness can result in more streamlined logistics operations, influencing other interconnected factors within the system.

Unlike other economic factors, cross-functional collaboration (EC9) is influenced by the organization's effectiveness in implementing existing economic strategies and social dynamics. Its success often depends on the foundational economic conditions and the quality of interdepartmental relationships.

Social factors within the logistics sector, consensus on organizational culture (SC1), supplier relationship management (SC3), and Coordinated employee training and development programs (SC5) are often influenced by economic stability and performance. Economic optimization sets the stage for positive social outcomes. Broader economic conditions and policies shape these social factors. For example, cost efficiency and market competitiveness improvements can lead to better working conditions and increased stakeholder engagement. In other instances, organizations achieving financial stability are more likely to invest in employee development and conflict resolution mechanisms.

One exception is SC6 (Clear Conflict Resolution Mechanisms), categorized as a cause factor. Effective conflict resolution mechanisms can enhance organizational efficiency and cohesion, impacting social and economic outcomes. When managed effectively, conflicts contribute to smoother operations and optimal resource utilization, thus causatively influencing other factors.

This study analyzes environmental sustainability factors through a cause-effect lens to clarify their significance within the logistics ecosystem. Circular Economy Practices (EN4) is considered as a cause factor. By adopting circular economy principles, circular Economy Practices involve proactive strategies such as resource optimization and waste minimization, making them critical drivers in logistics sustainability. Logistics entities initiate transformative changes to align with sustainability goals, positioning themselves as proactive agents in environmental stewardship.

General Green Transportation and Packaging Practices (EN1) and Environmental Efficiency Optimization Coordination (EN2) are shown as Effect Factors. These factors emerge as outcomes influenced by broader environmental initiatives. They represent tangible manifestations of sustainability efforts, reflecting the downstream impacts of proactive interventions. General Waste Reduction and Recycling Mechanisms (EN3) is also an Effect Factor. These mechanisms epitomize reactive measures enacted in response to sustainability goals. As logistics enterprises embrace circular economy paradigms, waste reduction and recycling mechanisms become integral components of environmental management frameworks.

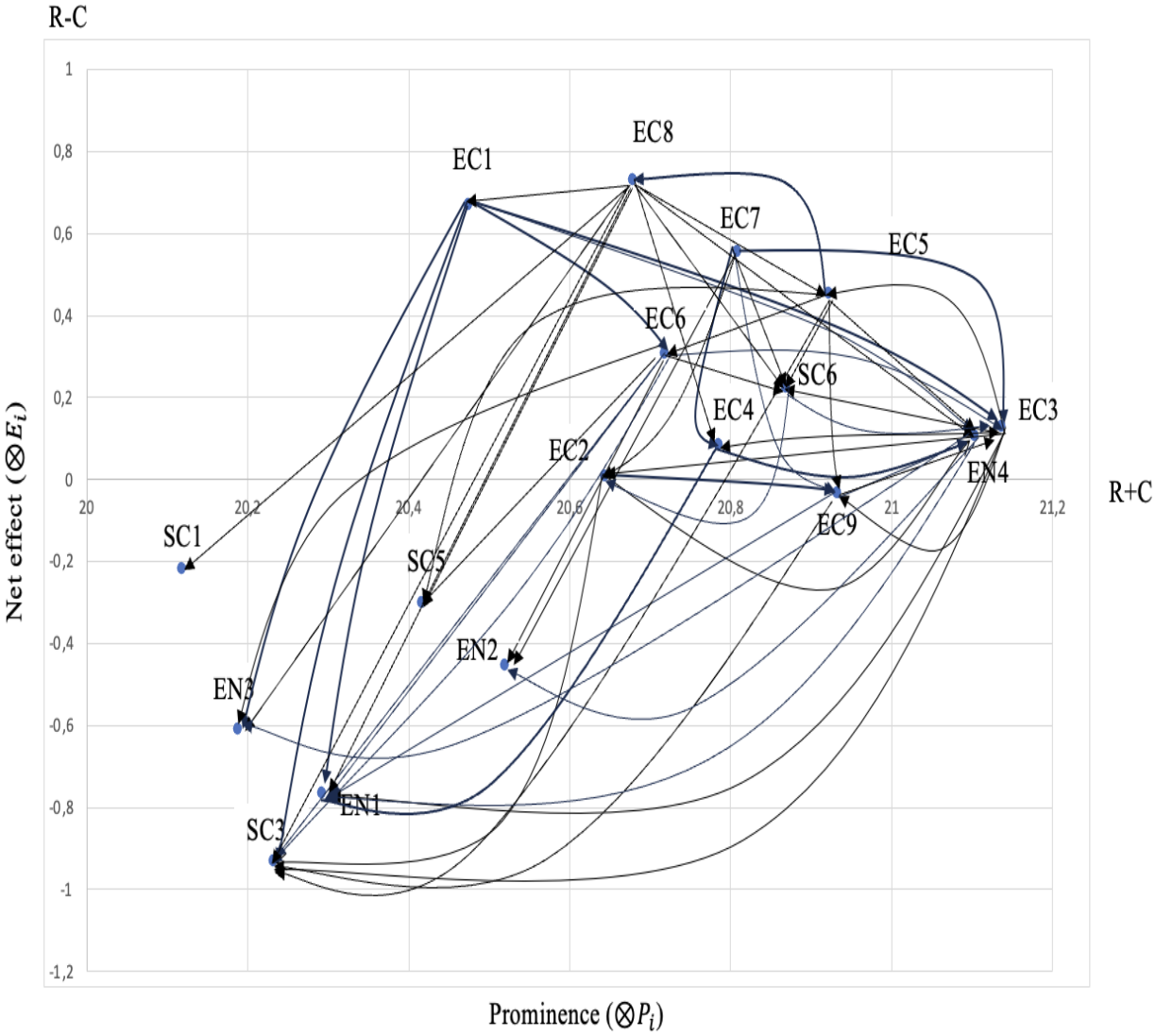

Figure 2. Influential Maps of significant relationships among factors

4.4 Discussions

The hybrid application of neutrosophic sets and Z-numbers in conjunction with Delphi and DEMATEL methods creates an advanced decision-making framework. This integration successfully addresses the complexities and uncertainties in HC, providing a multi-faceted analytical lens that traditional methods often overlook. The dual application enhances reliability in expert assessments and decision criteria, significantly advancing over previous studies. Our findings underscore critical factors for successful horizontal logistics collaboration, which are aligned with the previous studies. Comparatively, similar studies have often employed traditional fuzzy set theories or singular methodological approaches, lacking the hybrid analytical depth

provided by our framework. For instance, Ferrell et al.[66] explored opportunities for improved logistics planning through HC using standard fuzzy set theory and conventional survey methods, highlighting the potential but needing to address the complexities of uncertainty and indeterminacy in the decision-making process.

Conversely, our study leverages neutrosophic sets to manage the indeterminacy, thus providing a more robust and flexible evaluation framework. This approach aligns with the recent advancements Zadeh [67] suggested, enhancing the decision-making framework's credibility and robustness. However, they needed to integrate advanced fuzzy logic techniques, thereby limiting the analytical precision regarding the uncertainties inherent in logistics collaborations. Our methodology surpasses this limitation by incorporating Z-numbers, which offer a more nuanced handling of uncertainty and reliability in expert judgments.

After the NZN-Delphi phase, several factors were eliminated based on their aggregated scores not meeting the predetermined threshold. Factors such as SC2, SC4, and EN5 were rejected. This elimination process involved rigorous expert evaluations, converted into NZN to handle uncertainty and ensure reliability. The exclusion of these factors does not necessarily imply their irrelevance within the context of Vietnam. Instead, their exclusion reflects that, based on the Delphi method and subsequent evaluations using the NZN Set MCDM model, these factors did not meet the predetermined threshold for criticality in driving the effectiveness and sustainability of HC in the logistics industry. The rigorous expert evaluations and the use of NZN were employed to handle uncertainty and ensure the reliability of the decision-making process regarding which factors to prioritize. Factors that did not meet the threshold criteria were deemed less critical than those that were retained. This selective process aims to concentrate efforts and resources on the most influential factors likely to have the highest impact on the success of HC. Therefore, while SC2, SC4, and EN5 may still be relevant to the logistics industry in Vietnam, their exclusion suggests that they are perceived to have less immediate influence or may not be as pivotal in driving the desired outcomes of HC compared to other retained factors. Thus, the study focuses on what are believed to be the most significant drivers of success, ensuring a targeted approach to enhancing collaboration effectiveness and sustainability in the logistics sector in Vietnam.

The DEMATEL analysis offers significant insights into HTC's CSF. Among the most prominent factors identified are those related to economic aspects and firm-level concerns, underscoring the crucial role that individual companies and financial considerations play in HTC initiatives. This finding aligns with existing literature emphasizing the importance of firm-level dynamics and economic motivations in collaborative logistics efforts [23], [68]. The most critical factor identified is EC3, which pertains to Collaborative Distribution Centers as Partners. This highlights the complexity of establishing leadership and management structures within HTC, as partners may harbor distrust regarding equitable treatment. The challenge of selecting a neutral, trustworthy party to lead the collaboration emerges as a primary concern, resonating with findings from various studies [4], [68]. Closely following in importance is EN4, focusing on Circular Economy Practices. This factor's high ranking suggests a growing recognition among companies of the need to adopt sustainable practices, potentially driving the push toward HTC. The prominence of this environmental factor amidst predominantly economic considerations indicates an emerging shift towards sustainability in the logistics sector. This aligns with research by Torres et al.[69] on the importance of green logistics practices. Governance and trust-related factors also emerged as significant factors, including General Governance Agreements and Commitment to Responsibilities (EC9) and Clear Conflict Resolution Mechanisms (EC5), which rank high,

underscoring the need for robust contractual frameworks and conflict management strategies. These findings resonate with research by Raue et al. [70], which stresses the importance of formal agreements in ensuring partner commitment and accountability.

Trust coordination presents a critical challenge, given the inherent competitive nature of potential HTC partners. The reluctance to share sensitive information and concerns about the fairness of benefit/cost-sharing methods can significantly impede collaboration. This aligns with studies by Islam et al. [71], highlighting trust issues in competitive collaborations. Innovation and technology adoption ability are also prominent among the top CSFs. This factor reflects the dual challenge of overcoming misconceptions about information security risks and addressing smaller companies' financial barriers in adopting collaborative decision support systems (CDSSs). The findings align with Basso et al.[72], which identifies technology-related concerns as significant hurdles in HTC implementation. Interestingly, while factors related to cost/profit-sharing policies and data sharing and integration are recognized as necessary, they are not among the top-ranked CSFs. This suggests that while these aspects pose challenges, they may be more manageable than trust, governance, and technology adoption. The existence of ride-sharing applications in Vietnam demonstrates that technological solutions for logistics optimization are feasible when there is a determined effort to implement HTC.

In conclusion, the DEMATEL analysis highlights the interplay between economic, technological, governance, and trust-related aspects. The results suggest that the successful implementation of HTC requires a multifaceted approach that addresses firm-level challenges, particularly in trust-building, governance structuring, and technology adoption. While economic factors remain paramount, the growing importance of sustainability practices indicates a potential shift towards more environmentally conscious collaboration in the logistics sector. Future HTC initiatives should focus on developing robust governance frameworks, fostering trust among partners, and leveraging technology to overcome collaboration barriers while integrating circular economy practices to enhance overall sustainability.

5. Conclusions

This study utilized the combined NZN Delphi and NZN DEMATEL model to identify and analyze the CSFs for logistics HC. The findings highlight key factors influencing the HC process and their interrelationships, offering valuable strategies and priorities for enhancing HC in the logistics industry. Methodologically, the study demonstrates the benefits of integrating neutrosophic and Z-number theories, effectively addressing real-world challenges both mathematically and linguistically. By combining Delphi and DEMATEL techniques with advanced fuzzy set theories, this research provides a comprehensive framework for understanding and optimizing logistics HC.

6. Implications

6.1 Theoretical implications

This study on the CSFs for sustainable LHC in Vietnam's logistics industry using the NZN Set MCDM model offers significant theoretical implications that extend beyond its immediate context. Firstly, the use of Neutrosophic and Z-number theories addresses the inherent uncertainty and imprecision in human judgment, providing a more nuanced approach to decision-making processes in logistics by simultaneously capturing truth,

indeterminacy, and falsity degrees while incorporating both restriction and reliability components. This methodological advancement could be extended to other fields where decision-making under uncertainty is critical, such as healthcare (treatment protocol evaluation), finance (risk assessment), and urban planning (infrastructure development). Secondly, combining Delphi and DEMATEL techniques with these contemporary fuzzy set theories provides a structured approach to expert consensus-building and causal relationship analysis through multiple rounds of expert consultation and systematic convergence of opinions. The Delphi method ensures comprehensive coverage of factors impacting HC through iterative rounds of expert consultation, while DEMATEL offers insights into the interdependencies and causal relationships among these factors, enabling the development of impact-relation maps and the identification of critical nodes in the factor network. This integrated approach can inform future studies aiming to explore complex, interrelated phenomena in various domains, promoting a deeper understanding of underlying mechanisms and enhancing the rigor of MCDM models. The findings also underscore the importance of adopting a holistic perspective when examining CSFs for HC, incorporating economic dimensions (cost efficiency, resource optimization), social aspects (stakeholder engagement, community impact), and environmental considerations (carbon footprint reduction, waste management). This aligns with the principles of the TBL framework, which broadens the scope of organizational success beyond financial metrics to include social responsibility and environmental sustainability. By integrating TBL with SDT, the study provides a comprehensive theoretical framework for analyzing the dynamics of collaborative efforts among competing firms, addressing both short-term competitive advantages and long-term collaboration benefits. SDT highlights the conflict between immediate self-interest and longer-term collective interest, a common challenge in HC while offering insights into partner selection criteria and resource-sharing mechanisms. The identification and prioritization of CSFs using this integrated framework contribute to the theoretical understanding of collaboration dynamics in the logistics sector, emphasizing the importance of trust-building mechanisms (information sharing protocols, joint problem-solving approaches), robust governance structures (formal contracts and informal relationship management), and technology adoption (integration platforms, data sharing systems). These findings support existing theories on collaboration and governance, reinforcing the significance of trust and transparency in cooperative ventures, while the emphasis on sustainability practices reflects a growing theoretical shift towards integrating environmental considerations into logistics and supply chain management, aligning with the broader agenda of green logistics and sustainable development through eco-friendly transportation, green warehousing, and circular economy initiatives.

6.2 Practical implications

The practical implications of this study on the CSFs for sustainable HC in Vietnam's logistics industry are substantial and multifaceted, offering actionable insights for various stakeholders across the logistics ecosystem. Firstly, logistics companies can leverage the identified CSFs to design and implement more effective collaboration strategies, with a particular emphasis on trust-building mechanisms that serve as foundational elements for successful collaboration through the implementation of comprehensive trust-development programs, including regular partner satisfaction surveys, joint problem-solving sessions, and transparent performance reporting systems. Companies should invest in developing transparent communication channels through multiple synchronized touchpoints (such as daily operational briefings, weekly coordination meetings,

monthly strategic reviews, and quarterly performance assessments), establishing clear and fair governance structures (including detailed standard operating procedures, multi-level conflict resolution mechanisms, escalation matrices, and joint decision-making protocols), and fostering a culture of mutual respect and reliability among partners through structured joint training programs, cross-functional teams, shared innovation initiatives, and regular cultural alignment workshops. Practical steps include implementing regular joint meetings with structured agendas covering operational updates, performance reviews, and strategic planning, supported by detailed documentation and follow-up action items; developing shared performance metrics encompassing both individual and collective KPIs (such as on-time delivery rates, cost savings achieved, carbon footprint reduction, customer satisfaction scores, and sustainability targets met), with automated reporting systems and real-time dashboards; and deploying integrated technology platforms to facilitate real-time information sharing across partners, including blockchain-based transaction tracking, IoT-enabled asset monitoring, and AI-powered analytics for optimization. The study's emphasis on robust governance frameworks translates into specific, actionable measures, including the establishment of joint oversight committees with clearly defined roles, responsibilities, and authority matrices, the implementation of standardized decision-making processes using weighted scoring models and multi-criteria evaluation frameworks, and the development of comprehensive partnership agreements covering aspects such as resource allocation, risk sharing, profit distribution, intellectual property rights, data ownership, and exit strategies. These governance structures should be supported by detailed documentation, including service level agreements (SLAs), operating manuals, compliance guidelines, and contingency plans, while incorporating flexibility mechanisms to adapt to changing market conditions through regular review and revision processes. Technology adoption emerges as a CSF, with specific recommendations for implementing blockchain solutions for transparent transaction tracking, automated smart contracts, and immutable audit trails; deploying IoT sensors for real-time asset monitoring, predictive maintenance, and environmental condition tracking; and utilizing AI algorithms for route optimization, demand forecasting, resource allocation, and predictive analytics, supported by machine learning models for continuous improvement. Companies should also consider implementing cloud-based collaboration platforms that integrate various technological solutions, enabling seamless data sharing and analysis while maintaining robust security protocols and privacy standards through encrypted communications and role-based access controls. The sustainability dimension requires practical initiatives such as joint investment in electric vehicle fleets with shared charging infrastructure, energy-efficient warehousing facilities equipped with solar panels and smart energy management systems, and collaborative last-mile delivery solutions that optimize routes and reduce environmental impact through consolidated shipments and reverse logistics programs. Companies should establish specific environmental targets (e.g., 20% reduction in carbon emissions within two years, 50% reduction in packaging waste, 100% renewable energy usage), implement joint recycling and waste management programs with circular economy principles, and develop shared green logistics infrastructure including eco-friendly packaging systems and carbon offset programs. For SMEs, the study recommends specific collaboration models such as forming regional logistics clusters with shared resources and capabilities, participating in shared technology platforms to reduce individual investment costs through pay-per-use models, and establishing joint training programs for capability development in areas such as digital skills, sustainability practices, and operational excellence. These initiatives should be supported by detailed implementation roadmaps with clear milestones and deliverables, resource allocation plans including financial and human

resource requirements, and performance monitoring systems with regular audits and improvement cycles. Additionally, the study provides comprehensive guidance for policymakers to create enabling environments through specific policy measures such as tax incentives for collaborative logistics initiatives, streamlined regulations for data sharing and joint operations, investment in digital infrastructure to support logistics collaboration, and development of industry standards for sustainable logistics practices. This includes developing standardized frameworks for logistics partnerships with legal templates and guidelines, establishing industry-wide data-sharing protocols with cybersecurity standards, and creating financial support mechanisms for sustainable logistics initiatives through green bonds and sustainability-linked loans, while also addressing change management aspects through stakeholder engagement strategies, comprehensive communication plans, and training programs to ensure successful adoption of collaborative practices across the entire logistics ecosystem.

6.3 Limitations and Future Work

This study on the CSFs for sustainable HC in Vietnam's logistics industry, utilizing the NZN Set MCDM model, offers valuable insights but also has several limitations that should be addressed in future research. A key limitation is the potential subjectivity in expert evaluations. While expert judgment is crucial in Delphi and DEMATEL techniques, it may introduce biases that influence the findings. Experts' perspectives and experiences can shape the consensus and impact the analysis. The study's sample size is limited, mainly focusing on experts within Vietnam's logistics sector. This geographic and industry-specific focus may not fully reflect the diversity needed to generalize the findings across other regions and industries. Future research should aim for a broader and more diverse sample to enhance the generalizability of the results. Another limitation is the dynamic nature of CSFs. The factors identified are based on the current logistics industry landscape and prevailing environmental, economic, and social conditions. However, these factors can change over time due to technological advancements, market shifts, regulatory updates, and evolving sustainability norms. Therefore, periodic reevaluation of the identified CSFs will be necessary to ensure their relevance.

Additionally, while this study integrates qualitative insights with quantitative data through NZN theories, the quantitative aspect remains limited by the precision and reliability of the initial expert inputs. These inputs' inherent fuzziness and uncertainty pose challenges in achieving absolute accuracy.

Future research could enhance the methodological approach by incorporating more advanced statistical techniques and machine learning algorithms to analyze expert inputs better and validate findings. Combining traditional MCDM methods with advanced analytics would offer deeper insights and improve result reliability. Expanding the geographic scope of the study is also crucial. Including a broader range of countries and regions will provide a more comprehensive understanding of how CSFs for HC vary across logistics markets and cultural contexts, potentially identifying universal factors and region-specific nuances.

Longitudinal studies would also be valuable in capturing the evolving nature of CSFs over time. Tracking the development of collaboration practices and success factors in logistics could provide more up-to-date recommendations and allow for observing long-term impacts on sustainability and performance. Exploring HC in other sectors, such as healthcare, manufacturing, and finance, would also offer insights into the similarities and differences in CSFs across industries, supporting the development of more targeted strategies for collaboration and sustainability.

As technology advances, future research should investigate the role of emerging technologies like blockchain, AI, and IoT in facilitating HC. Understanding how these technologies can be integrated into collaboration strategies would provide practical insights for logistics companies seeking to improve their operations and sustainability. Finally, further research is needed on the role of policy and regulatory frameworks in supporting HC. Analyzing the effectiveness of current policies and proposing new regulatory measures could help promote collaboration and sustainability in the logistics industry, addressing incentive structures, compliance mechanisms, and the impact of international regulations on local practices.

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Expanded Triangular Fuzzy Neutrosophic Numbers for Safety Risk Evaluation of Urban Rail Transit Engineering: An Innovative Method for Better Risk Assessment

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Abstract
Safety assessment of urban rail transit (URT) operations plays a crucial role not only in shaping passenger travel choices but also in urban development planning. The findings of such assessments are valuable for both passengers and transport operators. In this study, we propose an integrated multi-stage evaluation framework to assess URT operations from a safety perspective. This framework considers decision-makers' (DMs') risk preferences and the uncertainty of available information. We employ two key methods: the CRITIC method to determine the weights of evaluation criteria and the MOORA method to rank the alternatives. Both methods operate within the framework of triangular fuzzy neutrosophic sets (TFNSs) to handle uncertainty and imprecise data. Our evaluation includes nine criteria and six alternatives, assessed by four experts and decision-makers. The results indicate that Structural Integrity is the most critical criterion, having the highest weight in the evaluation process.

Keywords: Triangular Fuzzy Neutrosophic Numbers; Urban Rail Transit Engineering; Uncertainty; Safety Risk.

1. Introduction and Literature Review

Subway and light rail subsystems are only two examples of the various components that make up the present urban rail transit (URT), which is a complicated system. URT has steadily grown to be one of the most well-liked forms of transportation among locals because of its many benefits, which include excellent timeliness, affordability, and time savings. As a result, URT, one essential part of the transportation system, is becoming increasingly significant in fostering the steady growth of the entire city. However, because of the peculiarities of URT, such as its fast speed, frequent stops and starts, dense passenger flow, and difficult emergency evacuation, there are numerous security concerns[1], [2].

Additionally, as public areas with a large population density, trains and stations are crucial components of the URT system, which is crucial to maintaining the regular operation of public transit. URT has begun putting networked operations into practice. The operating state of URT is

comparatively steady thanks to the coordinated efforts of the pertinent departments[3], [4]. However, safety issues with URT operation are becoming more noticeable, and corresponding safe accidents are continuously increasing, because of the recent rapid increase in operational length and ongoing expansion of the urban rail network scale.

Considering these hazards and obstacles, how to guarantee URT operation safety in the current rapid operation environment has emerged as a ticklish issue that all cities are dealing with collectively. Therefore, this study's primary goal is to talk about how to perform safety assessments, manage risks, and enhance operational safety for URT systems[5], [6]s. That is, to thoroughly examine the various aspects of parameters influencing URT operational safety. Evaluation of this problem contains uncertainty and vague data.

The fuzziness and complexity of the alternatives make it difficult to represent the criteria values with precise values in many real-world MCGDM problems; therefore, it may be more beneficial and efficient to describe the criteria values with fuzzy information. For MCGDM difficulties, fuzzy set theory has been employed as a workable solution. The neutrosophic set (NS) was proposed by Smarandache [7], [8]. The single-valued neutrosophic sets (SVNSs) and interval neutrosophic sets (INSs) were then defined by Wang et al. Wang et al. expanded the SVNS to a 2-tuple linguistic neutrosophic number environment and investigated a few aggregation operators of SVNNs[9], [10].

SVNNs with Hamy operators were investigated by Wu et al. under 2-tuple linguistic neutrosophic numbers[11], [12]. Triangular fuzzy neutrosophic numbers (TFNNs), which represent the degree of truth-membership (MD), indeterminacy-membership (IMD), and falsity-membership (FMD), were defined by Biswas et al. Sahin et al. investigated centroid single-valued triangular neutrosophic numbers in multiple attribute decision-making (MADM) situations[13], [14].

Our paper is organized as follows. The ideas, operation formulas, and distance calculation method of TFNNs are introduced in Section 2. Section 3 presents the necessary computation steps of TFNNs CRITIC-MOORA method. In Section 4, the results and sensitivity analysis were conducted. Conclusions are summarized in Section 5.

1.1. Motivation

Urban rail transit (URT) plays a crucial role in modern city transportation due to its efficiency, affordability, and capacity to reduce road congestion. However, the rapid expansion of URT systems has brought about increasing safety concerns. The high-speed nature of URT operations, dense passenger flows, and frequent stops pose significant risks that need to be addressed. Ensuring safety in URT is not only important for passenger well-being but also for maintaining public confidence and the sustainable development of urban infrastructure. This study is motivated by the need to provide a structured and systematic safety assessment framework that can handle uncertainty in risk evaluation while improving decision-making for urban rail safety management.

1.2. Research Gap

While many studies have focused on urban rail safety assessment, several gaps remain unaddressed. Traditional safety evaluation methods often fail to capture the uncertainty and vagueness inherent in risk assessment. Furthermore, existing models lack a comprehensive integration of decision-makers' risk preferences. Many approaches rely on deterministic values, which do not adequately reflect real-world complexities. This research fills these gaps by introducing a hybrid multi-phase evaluation model that incorporates triangular fuzzy neutrosophic sets (TFNSs) to manage uncertainty more effectively. Additionally, the study applies both the CRITIC and MOORA methods to improve accuracy in criteria weighting and alternative ranking.

1.3. Contribution of the Study

This study makes several key contributions to the field of urban rail transit safety evaluation. First, it introduces a hybrid evaluation framework that integrates triangular fuzzy neutrosophic sets (TFNSs) with the CRITIC and MOORA methods. This integration enhances decision-making accuracy by systematically addressing uncertainty and imprecise data. Second, the study improves risk management by offering a structured approach to quantify safety risks using fuzzy logic and neutrosophic principles, which allows for more flexible and realistic assessments. Third, the research incorporates a detailed sensitivity analysis to evaluate the stability of ranking outcomes under varying weight scenarios, ensuring the robustness of the proposed methodology. Finally, the framework is applied to a real-world case study involving nine criteria and six alternatives, evaluated by four experts. This practical implementation highlights its effectiveness and applicability in real urban rail transit safety assessments.

1.4. Justification for Method Selection

The selection of TFNSs, CRITIC, and MOORA methods in this study is based on their ability to handle uncertainty and improve decision-making precision. Triangular fuzzy neutrosophic sets (TFNSs) are particularly suitable for capturing uncertainty and vagueness in safety risk data, providing a nuanced and flexible representation of expert opinions. The CRITIC method is employed to determine the criteria weights objectively, ensuring that the weighting process remains free from subjective biases. The MOORA method is chosen for its efficiency in ranking alternatives while maintaining computational simplicity, making it ideal for real-world urban rail safety evaluations. Together, these methods create a powerful and adaptable decision-making framework that enhances accuracy in safety risk assessment.

1.5 Literature Review

Urban rail transit (URT) safety has been extensively studied due to the increasing demand for efficient and secure public transportation systems. Researchers have explored various methodologies for evaluating and mitigating safety risks in URT operations, with a particular focus on multi-criteria decision-making (MCDM) approaches, probabilistic risk assessments, and fuzzy logic-based models. Despite these efforts, challenges related to uncertainty, data incompleteness, and subjective expert evaluations remain. This section provides a detailed

literature review covering different techniques and approaches used in URT safety assessment, highlighting their strengths and limitations.

1.5.1 Traditional Risk Assessment Methods in URT Safety

Early studies on URT safety relied on traditional risk assessment methodologies such as probabilistic risk analysis (PRA), fault tree analysis (FTA), and event tree analysis (ETA). These methods aim to quantify risks based on historical accident data and failure probabilities. For instance, Zhang et al. applied PRA to identify key risk factors in urban rail transit systems, emphasizing the need for a structured approach to hazard identification and mitigation [15]. Similarly, Kumar and Singh used FTA to model failure pathways in railway systems, concluding that human error and technical malfunctions were the leading causes of safety incidents [16].

Although these methodologies provided valuable insights, they were limited by their reliance on precise numerical data, which is often unavailable in real-world safety assessments. Moreover, traditional risk assessment methods do not adequately address the complexity of modern URT systems, where multiple interdependent factors contribute to overall safety performance.

1.5.2 Integration of Multi-Criteria Decision-Making (MCDM) Methods

To overcome the limitations of traditional risk assessments, researchers have increasingly turned to MCDM methods, which allow for the systematic evaluation of multiple safety criteria. The Analytic Hierarchy Process (AHP) and the Technique for Order Preference by Similarity to Ideal Solution (TOPSIS) have been widely used in this context. Lee and Park employed AHP to rank URT safety risks based on expert judgments, demonstrating that station crowding, signal failures, and emergency response time were among the most critical factors [17]. Meanwhile, Wei and Zhao utilized TOPSIS to prioritize safety measures, showing that infrastructure resilience and real-time monitoring systems significantly enhance URT safety [18].

Despite their effectiveness, MCDM approaches such as AHP and TOPSIS require precise numerical inputs, which may not always be available. Moreover, these methods often assume that expert opinions are entirely consistent, overlooking the inherent uncertainty and vagueness present in real-world decision-making.

1.5.3 Application of Fuzzy Logic in URT Safety Assessment

Fuzzy logic has been introduced as a solution to handle the uncertainty associated with expert judgments and incomplete data. Traditional fuzzy set theory (FST) enables the representation of linguistic variables, allowing for more flexible safety evaluations. Gupta and Sharma applied fuzzy AHP to assess the risk levels of various URT components, demonstrating that integrating fuzziness into safety assessments improves decision-making accuracy [19].

An extension of fuzzy logic, neutrosophic sets, has been proposed to further enhance safety evaluations. Neutrosophic logic incorporates degrees of truth, indeterminacy, and falsity, making it more effective in dealing with ambiguous and contradictory information. Li and Sun explored

the application of triangular fuzzy neutrosophic sets (TFNSs) in URT safety analysis, highlighting their ability to incorporate uncertainty while maintaining computational efficiency [20].

1.5.3 Weight Determination and Ranking Methods in URT Safety

Determining the relative importance of safety criteria is crucial for effective risk management. Several methodologies have been explored to optimize weight determination and alternative ranking. The CRITIC method, for example, has been used to objectively compute criteria weights based on contrast intensity. Studies by Zhang et al. demonstrated that CRITIC outperforms traditional weighting techniques by minimizing subjectivity in expert evaluations [21].

Furthermore, the MOORA method has been widely adopted for alternative ranking in URT safety assessments. Sun et al. applied MOORA to prioritize safety improvements across different urban transit networks, concluding that structural integrity and emergency response systems were the most influential factors in ensuring passenger safety [22].

1.5.4 Hybrid Models for URT Safety Evaluation

Recent research has emphasized the need for hybrid models that integrate multiple methodologies to enhance safety evaluations. A combination of TFNSs, CRITIC, and MOORA methods has shown promising results in improving decision-making accuracy under uncertainty. Wang et al. introduced a hybrid framework that utilized TFNSs for data representation, CRITIC for weight computation, and MOORA for ranking alternatives. Their findings demonstrated that hybrid models offer greater flexibility and precision compared to standalone MCDM techniques [23].

15.5 Future Directions in URT Safety Research

Despite significant advancements, several research gaps remain in URT safety evaluations. Future studies should focus on integrating real-time monitoring systems using Internet of Things (IoT) technologies to enhance dynamic risk assessment. Additionally, exploring alternative MCDM methods such as VIKOR and ELECTRE could provide further insights into optimizing safety strategies. Expanding case studies across diverse urban environments will also be crucial in validating the generalizability of proposed methodologies.

2. Triangular Fuzzy Neutrosophic Numbers (TFNNs)

The TFNNs are built based on the theory of single-valued neutrosophic numbers and traditional triangular fuzzy sets [13].

Definition 1

The TFNNs can be defined as:

$$B = \{(x, T_B(x))|, I_B(x), F_B(x) x \in X\} \quad (1)$$

These values present the truth, indeterminacy, and falsity functions

$$T_B(x) = (T_B^L(x), T_B^M(x), T_B^U(x), 0 \leq T_B^L(x), T_B^M(x), T_B^U(x) \leq 1) \quad (2)$$

$$I_B(x) = (I_B^L(x), I_B^M(x), I_B^U(x), 0 \le I_B^L(x), I_B^M(x), I_B^U(x) \le 1) \tag{3}$$

$$F_B(x) = (F_B^L(x), F_B^M(x), F_B^U(x), 0 \le F_B^L(x), F_B^M(x), F_B^U(x) \le 1) \tag{4}$$

$$T_B(x) = \begin{cases} \frac{x - T_B^L(x)}{T_B^M(x) - T_B^L(x)}, & T_B^L(x) \le x \le T_B^M(x) \\ \frac{x - T_B^U(x)}{T_B^M(x) - T_B^U(x)}, & T_B^M(x) \le x \le T_B^U(x) \\ 0, & otherwise \end{cases} \tag{5}$$

Definition 2

Assume two TFNNs such as:

$$B_1 = \left\{\left(T_{B_1}^L(x), T_{B_1}^M(x), T_{B_1}^U(x)\right), \left(I_{B_1}^L(x), I_{B_1}^M(x), I_{B_1}^U(x)\right), \left(F_{B_1}^L(x), F_{B_1}^M(x), F_{B_1}^U(x)\right)\right\}$$

$$B_2 = \left\{\left(T_{B_2}^L(x), T_{B_2}^M(x), T_{B_2}^U(x)\right), \left(I_{B_2}^L(x), I_{B_2}^M(x), I_{B_2}^U(x)\right), \left(F_{B_2}^L(x), F_{B_2}^M(x), F_{B_2}^U(x)\right)\right\}$$

$$B_1 \oplus B_2 = \begin{pmatrix} \begin{pmatrix} T_{B_1}^L(x) + T_{B_2}^L(x) - T_{B_1}^L(x)T_{B_2}^L(x), \\ T_{B_1}^M(x) + T_{B_2}^M(x) - T_{B_1}^M(x)T_{B_2}^M(x), \\ T_{B_1}^U(x) + T_{B_2}^U(x) - T_{B_1}^U(x)T_{B_2}^U(x) \end{pmatrix}, \\ \left(I_{B_1}^L(x)I_{B_2}^L(x), I_{B_1}^M(x)I_{B_2}^M(x), I_{B_1}^U(x)I_{B_2}^U(x)\right), \\ \left(F_{B_1}^L(x)F_{B_2}^L(x), F_{B_1}^M(x)F_{B_2}^M(x), F_{B_1}^U(x)F_{B_2}^U(x)\right) \end{pmatrix} \tag{6}$$

$$B_1 \otimes B_2 = \begin{pmatrix} \left(T_{B_1}^L(x)T_{B_2}^L(x), T_{B_1}^M(x)T_{B_2}^M(x), T_{B_1}^U(x)T_{B_2}^U(x)\right), \\ \begin{pmatrix} I_{B_1}^L(x) + I_{B_2}^L(x) - I_{B_1}^L(x)I_{B_2}^L(x), \\ I_{B_1}^M(x) + I_{B_2}^M(x) - I_{B_1}^M(x)I_{B_2}^M(x), \\ I_{B_1}^U(x) + I_{B_2}^U(x) - I_{B_1}^U(x)I_{B_2}^U(x) \end{pmatrix}, \\ \begin{pmatrix} F_{B_1}^L(x) + F_{B_2}^L(x) - F_{B_1}^L(x)F_{B_2}^L(x), \\ F_{B_1}^M(x) + F_{B_2}^M(x) - F_{B_1}^M(x)F_{B_2}^M(x), \\ F_{B_1}^U(x) + F_{B_2}^U(x) - F_{B_1}^U(x)F_{B_2}^U(x) \end{pmatrix} \end{pmatrix} \tag{7}$$

$$\lambda B_1 = \begin{pmatrix} \left(1 - \left(1 - T_{B_1}^L(x)\right)^\lambda, 1 - \left(1 - T_{B_1}^M(x)\right)^\lambda, 1 - \left(1 - T_{B_1}^U(x)\right)^\lambda\right), \\ \left(\left(I_{B_1}^L(x)\right)^\lambda, \left(I_{B_1}^M(x)\right)^\lambda, \left(I_{B_1}^U(x)\right)^\lambda\right), \\ \left(\left(F_{B_1}^L(x)\right)^\lambda, \left(F_{B_1}^M(x)\right)^\lambda, \left(F_{B_1}^U(x)\right)^\lambda\right) \end{pmatrix} \tag{8}$$

$$B_1^\lambda = \begin{pmatrix} \left(\left(T_{B_1}^L(x)\right)^\lambda, \left(T_{B_1}^M(x)\right)^\lambda, \left(T_{B_1}^U(x)\right)^\lambda\right), \\ \left(1 - \left(1 - I_{B_1}^L(x)\right)^\lambda, 1 - \left(1 - I_{B_1}^M(x)\right)^\lambda, 1 - \left(1 - I_{B_1}^U(x)\right)^\lambda\right), \\ \left(1 - \left(1 - F_{B_1}^L(x)\right)^\lambda, 1 - \left(1 - F_{B_1}^M(x)\right)^\lambda, 1 - \left(1 - F_{B_1}^U(x)\right)^\lambda\right) \end{pmatrix} \tag{9}$$

Definition 3

The score function can be computed as:

$$S(B_1) = \frac{1}{12}\begin{bmatrix} 8 + \left(T_{B_1}^L(x) + 2 * T_{B_1}^M(x) + T_{B_1}^U(x)\right) - \\ \left(I_{B_1}^L(x) + 2 * I_{B_1}^M(x) + I_{B_1}^U(x)\right) - \\ \left(F_{B_1}^L(x) + 2 * F_{B_1}^M(x) + F_{B_1}^U(x)\right) \end{bmatrix} \tag{10}$$

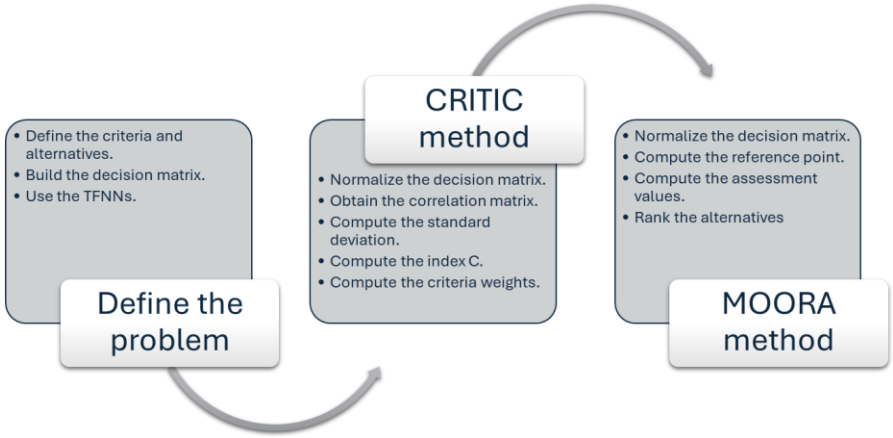
Define the problem
• Define the criteria and alternatives.
• Build the decision matrix.
• Use the TFNNs.
CRITIC method
• Normalize the decision matrix.
• Obtain the correlation matrix.
• Compute the standard deviation.
• Compute the index C.
• Compute the criteria weights.
MOORA method
• Normalize the decision matrix.
• Compute the reference point.
• Compute the assessment values.
• Rank the alternatives

Fig 1. The steps of the proposed approach.

2.1 TFNS-CRITIC Method

This section shows the steps of the CRITIC method to compute the criteria weights. Fig 1. Shows the steps of the proposed approach.

1. Build the decision matrix.

2. Normalize the decision matrix.

$$p_{ij} = \frac{x_{ij} - \min x_i}{\max x_i - \min x_i} \text{ positive criteria} \tag{11}$$

$$p_{ij} = \frac{x_{ij} - \max x_i}{\min x_i - \max x_i} \text{ negative criteria} \tag{12}$$

3. Obtain the correlation matrix

$$g_{jk} = \frac{\sum_{i=1}^{m}\left(p_{ij} - p_j^-\right)\left(p_{ij} - p_k^-\right)}{\sqrt{\sum_{i=1}^{m}\left(p_{ij} - p_j^-\right)^2 \left(p_{ij} - p_k^-\right)^2}} \tag{13}$$

$$p_j^- = \frac{1}{n}\sum_{j=1}^{n} p_{ij} \tag{14}$$

4. Compute the standard deviation

5. Compute the index C

$$C_j = stand_j \sum_{k=1}^{n}(1 - g_{jk}) \tag{15}$$

6. Compute the criteria weights.

$$w_j = \frac{C_j}{\sum_{j=1}^{n} C_j} \tag{16}$$

2.2 TFNS-MOORA Method

This section shows the steps of the MOORA method to rank the alternatives.

1. Normalize the decision matrix

$$y_{ij} = \frac{x_{ij}}{\sqrt{\sum_{i=1}^{m} x_{ij}^2}} \tag{17}$$

2. Compute the reference point

We obtain the maximum value for positive criteria and the minimum value for negative criteria.

3. Compute the assessment values

$$u_i = \sum_{j=1}^{g} y_{ij} w_j - \sum_{j=g+1}^{n} y_{ij} w_j \tag{18}$$

4. Rank the alternatives.

3. Application and Analysis

This section shows the results of the proposed approach under the TFNNs to deal with uncertain information. We evaluated the Safety Risk Evaluation of Urban Rail Transit Engineering to compute the criteria weights and rank the alternatives. We want to evaluate the Key Safety Risk Areas in Urban Rail Transit Engineering and select the best one. Four experts and decision-makers have evaluated the criteria and alternatives. Nine criteria and six alternatives are shown in Fig 2. We used the TFNNs to evaluate the criteria and alternatives as shown in Tables 1-4.

Fig 2. The Key Safety Risk Areas in Urban Rail Transit Engineering.

Table 1. The TFNNs matrix one.

	A_1	A_2	A_3	A_4	A_5	A_6
C_1	{(0.5,0.6,0.8),(0.3,0.4,0.5),(0.4,0.5,0.6)}	{(0.4,0.7,0.9),(0.5,0.6,0.8),(0.2,0.4,0.6)}	{(0.1,0.3,0.5),(0.2,0.4,0.7),(0.5,0.8,0.9)}	{(0.2,0.3,0.4),(0.4,0.5,0.6),(0.6,0.7,0.8)}	{(0.7,0.8,0.9),(0.2,0.3,0.4),(0.2,0.3,0.4)}	{(0.7,0.8,0.9),(0.2,0.3,0.4),(0.2,0.3,0.4)}
C_2	{(0.5,0.6,0.7),(0.4,0.5,0.6),(0.3,0.4,0.5)}	{(0.5,0.6,0.8),(0.3,0.4,0.5),(0.4,0.5,0.6)}	{(0.4,0.7,0.9),(0.5,0.6,0.8),(0.2,0.4,0.6)}	{(0.1,0.3,0.5),(0.2,0.4,0.7),(0.5,0.8,0.9)}	{(0.2,0.5,0.7),(0.3,0.6,0.8),(0.1,0.2,0.3)}	{(0.2,0.3,0.4),(0.4,0.5,0.6),(0.6,0.7,0.8)}
C_3	{(0.5,0.6,0.8),(0.3,0.4,0.5),(0.4,0.5,0.6)}	{(0.4,0.7,0.9),(0.5,0.6,0.8),(0.2,0.4,0.6)}	{(0.1,0.3,0.5),(0.2,0.4,0.7),(0.5,0.8,0.9)}	{(0.2,0.3,0.4),(0.4,0.5,0.6),(0.6,0.7,0.8)}	{(0.3,0.5,0.6),(0.2,0.4,0.5),(0.5,0.6,0.8)}	{(0.4,0.7,0.9),(0.5,0.6,0.8),(0.2,0.4,0.6)}
C_4	{(0.2,0.5,0.7),(0.3,0.6,0.8),(0.1,0.2,0.3)}	{(0.5,0.6,0.8),(0.3,0.4,0.5),(0.4,0.5,0.6)}	{(0.4,0.7,0.9),(0.5,0.6,0.8),(0.2,0.4,0.6)}	{(0.1,0.3,0.5),(0.2,0.4,0.7),(0.5,0.8,0.9)}	{(0.5,0.6,0.7),(0.4,0.5,0.6),(0.3,0.4,0.5)}	{(0.6,0.8,0.9),(0.2,0.3,0.5),(0.1,0.3,0.4)}
C_5	{(0.7,0.8,0.9),(0.2,0.3,0.4),(0.2,0.3,0.4)}	{(0.3,0.5,0.6),(0.2,0.4,0.5),(0.5,0.6,0.8)}	{(0.5,0.6,0.8),(0.3,0.4,0.5),(0.4,0.5,0.6)}	{(0.4,0.7,0.9),(0.5,0.6,0.8),(0.2,0.4,0.6)}	{(0.1,0.3,0.5),(0.2,0.4,0.7),(0.5,0.8,0.9)}	{(0.5,0.6,0.7),(0.4,0.5,0.6),(0.3,0.4,0.5)}
C_6	{(0.2,0.3,0.4),(0.4,0.5,0.6),(0.6,0.7,0.8)}	{(0.5,0.6,0.8),(0.3,0.4,0.5),(0.4,0.5,0.6)}	{(0.4,0.7,0.9),(0.5,0.6,0.8),(0.2,0.4,0.6)}	{(0.1,0.3,0.5),(0.2,0.4,0.7),(0.5,0.8,0.9)}	{(0.5,0.6,0.8),(0.3,0.4,0.5),(0.4,0.5,0.6)}	{(0.2,0.3,0.4),(0.4,0.5,0.6),(0.6,0.7,0.8)}
C_7	{(0.5,0.6,0.8),(0.3,0.4,0.5),(0.4,0.5,0.6)}	{(0.4,0.7,0.9),(0.5,0.6,0.8),(0.2,0.4,0.6)}	{(0.1,0.3,0.5),(0.2,0.4,0.7),(0.5,0.8,0.9)}	{(0.5,0.6,0.7),(0.4,0.5,0.6),(0.3,0.4,0.5)}	{(0.1,0.3,0.5),(0.2,0.4,0.7),(0.5,0.8,0.9)}	{(0.2,0.5,0.7),(0.3,0.6,0.8),(0.1,0.2,0.3)}
C_8	{(0.1,0.3,0.5),(0.2,0.4,0.7),(0.5,0.8,0.9)}	{(0.5,0.6,0.8),(0.3,0.4,0.5),(0.4,0.5,0.6)}	{(0.4,0.7,0.9),(0.5,0.6,0.8),(0.2,0.4,0.6)}	{(0.1,0.3,0.5),(0.2,0.4,0.7),(0.5,0.8,0.9)}	{(0.4,0.7,0.9),(0.5,0.6,0.8),(0.2,0.4,0.6)}	{(0.7,0.8,0.9),(0.2,0.3,0.4),(0.2,0.3,0.4)}
C_9	{(0.5,0.6,0.8),(0.3,0.4,0.5),(0.4,0.5,0.6)}	{(0.4,0.7,0.9),(0.5,0.6,0.8),(0.2,0.4,0.6)}	{(0.1,0.3,0.5),(0.2,0.4,0.7),(0.5,0.8,0.9)}	{(0.5,0.6,0.8),(0.3,0.4,0.5),(0.4,0.5,0.6)}	{(0.2,0.3,0.4),(0.4,0.5,0.6),(0.6,0.7,0.8)}	{(0.6,0.8,0.9),(0.2,0.3,0.5),(0.1,0.3,0.4)}

Table two. The TFNNs matrix two.

	A_1	A_2	A_3	A_4	A_5	A_6
C_1	{(0.1,0.3,0.5),(0.2,0.4,0.7),(0.5,0.8,0.9)}	{(0.2,0.3,0.4),(0.4,0.5,0.6),(0.6,0.7,0.8)}	{(0.2,0.5,0.7),(0.3,0.6,0.8),(0.1,0.2,0.3)}	{(0.2,0.3,0.4),(0.4,0.5,0.6),(0.6,0.7,0.8)}	{(0.7,0.8,0.9),(0.2,0.3,0.4),(0.2,0.3,0.4)}	{(0.6,0.8,0.9),(0.2,0.3,0.5),(0.1,0.3,0.4)}
C_2	{(0.5,0.6,0.7),(0.4,0.5,0.6),(0.3,0.4,0.5)}	{(0.1,0.3,0.5),(0.2,0.4,0.7),(0.5,0.8,0.9)}	{(0.2,0.3,0.4),(0.4,0.5,0.6),(0.6,0.7,0.8)}	{(0.2,0.5,0.7),(0.3,0.6,0.8),(0.1,0.2,0.3)}	{(0.2,0.5,0.7),(0.3,0.6,0.8),(0.1,0.2,0.3)}	{(0.5,0.6,0.7),(0.4,0.5,0.6),(0.3,0.4,0.5)}
C_3	{(0.3,0.5,0.6),(0.2,0.4,0.5),(0.5,0.6,0.8)}	{(0.6,0.8,0.9),(0.2,0.3,0.5),(0.1,0.3,0.4)}	{(0.1,0.3,0.5),(0.2,0.4,0.7),(0.5,0.8,0.9)}	{(0.2,0.3,0.4),(0.4,0.5,0.6),(0.6,0.7,0.8)}	{(0.2,0.5,0.7),(0.3,0.6,0.8),(0.1,0.2,0.3)}	{(0.1,0.3,0.5),(0.2,0.4,0.7),(0.5,0.8,0.9)}
C_4	{(0.2,0.5,0.7),(0.3,0.6,0.8),(0.1,0.2,0.3)}	{(0.1,0.3,0.5),(0.2,0.4,0.7),(0.5,0.8,0.9)}	{(0.2,0.3,0.4),(0.4,0.5,0.6),(0.6,0.7,0.8)}	{(0.2,0.5,0.7),(0.3,0.6,0.8),(0.1,0.2,0.3)}	{(0.5,0.6,0.7),(0.4,0.5,0.6),(0.3,0.4,0.5)}	{(0.2,0.5,0.7),(0.3,0.6,0.8),(0.1,0.2,0.3)}
C_5	{(0.7,0.8,0.9),(0.2,0.3,0.4),(0.2,0.3,0.4)}	{(0.3,0.5,0.6),(0.2,0.4,0.5),(0.5,0.6,0.8)}	{(0.1,0.3,0.5),(0.2,0.4,0.7),(0.5,0.8,0.9)}	{(0.2,0.3,0.4),(0.4,0.5,0.6),(0.6,0.7,0.8)}	{(0.2,0.5,0.7),(0.3,0.6,0.8),(0.1,0.2,0.3)}	{(0.7,0.8,0.9),(0.2,0.3,0.4),(0.2,0.3,0.4)}
C_6	{(0.2,0.3,0.4),(0.4,0.5,0.6),(0.6,0.7,0.8)}	{(0.2,0.5,0.7),(0.3,0.6,0.8),(0.1,0.2,0.3)}	{(0.1,0.3,0.5),(0.2,0.4,0.7),(0.5,0.8,0.9)}	{(0.2,0.3,0.4),(0.4,0.5,0.6),(0.6,0.7,0.8)}	{(0.2,0.5,0.7),(0.3,0.6,0.8),(0.1,0.2,0.3)}	{(0.7,0.8,0.9),(0.2,0.3,0.4),(0.2,0.3,0.4)}
C_7	{(0.4,0.7,0.9),(0.5,0.6,0.8),(0.2,0.4,0.6)}	{(0.1,0.3,0.5),(0.2,0.4,0.7),(0.5,0.8,0.9)}	{(0.2,0.3,0.4),(0.4,0.5,0.6),(0.6,0.7,0.8)}	{(0.2,0.5,0.7),(0.3,0.6,0.8),(0.1,0.2,0.3)}	{(0.1,0.3,0.5),(0.2,0.4,0.7),(0.5,0.8,0.9)}	{(0.2,0.3,0.4),(0.4,0.5,0.6),(0.6,0.7,0.8)}
C_8	{(0.1,0.3,0.5),(0.2,0.4,0.7),(0.5,0.8,0.9)}	{(0.2,0.3,0.4),(0.4,0.5,0.6),(0.6,0.7,0.8)}	{(0.2,0.5,0.7),(0.3,0.6,0.8),(0.1,0.2,0.3)}	{(0.6,0.8,0.9),(0.2,0.3,0.5),(0.1,0.3,0.4)}	{(0.4,0.7,0.9),(0.5,0.6,0.8),(0.2,0.4,0.6)}	{(0.2,0.5,0.7),(0.3,0.6,0.8),(0.1,0.2,0.3)}
C_9	{(0.5,0.6,0.8),(0.3,0.4,0.5),(0.4,0.5,0.6)}	{(0.4,0.7,0.9),(0.5,0.6,0.8),(0.2,0.4,0.6)}	{(0.1,0.3,0.5),(0.2,0.4,0.7),(0.5,0.8,0.9)}	{(0.5,0.6,0.8),(0.3,0.4,0.5),(0.4,0.5,0.6)}	{(0.2,0.3,0.4),(0.4,0.5,0.6),(0.6,0.7,0.8)}	{(0.7,0.8,0.9),(0.2,0.3,0.4),(0.2,0.3,0.4)}

Table 3. The TFNNs matrix three.

	A_1	A_2	A_3	A_4	A_5	A_6
C_1	{(0.6,0.8,0.9),(0.2,0.3,0.5),(0.1,0.3,0.4)}	{(0.1,0.3,0.5),(0.2,0.4,0.7),(0.5,0.8,0.9)}	{(0.4,0.7,0.9),(0.5,0.6,0.8),(0.2,0.4,0.6)}	{(0.2,0.3,0.4),(0.4,0.5,0.6),(0.6,0.7,0.8)}	{(0.7,0.8,0.9),(0.2,0.3,0.4),(0.2,0.3,0.4)}	{(0.6,0.8,0.9),(0.2,0.3,0.5),(0.1,0.3,0.4)}
C_2	{(0.6,0.8,0.9),(0.2,0.3,0.5),(0.1,0.3,0.4)}	{(0.5,0.6,0.7),(0.4,0.5,0.6),(0.3,0.4,0.5)}	{(0.7,0.8,0.9),(0.2,0.3,0.4),(0.2,0.3,0.4)}	{(0.4,0.7,0.9),(0.5,0.6,0.8),(0.2,0.4,0.6)}	{(0.2,0.5,0.7),(0.3,0.6,0.8),(0.1,0.2,0.3)}	{(0.5,0.6,0.7),(0.4,0.5,0.6),(0.3,0.4,0.5)}
C_3	{(0.3,0.5,0.6),(0.2,0.4,0.5),(0.5,0.6,0.8)}	{(0.6,0.8,0.9),(0.2,0.3,0.5),(0.1,0.3,0.4)}	{(0.5,0.6,0.7),(0.4,0.5,0.6),(0.3,0.4,0.5)}	{(0.7,0.8,0.9),(0.2,0.3,0.4),(0.2,0.3,0.4)}	{(0.3,0.5,0.6),(0.2,0.4,0.5),(0.5,0.6,0.8)}	{(0.3,0.5,0.6),(0.2,0.4,0.5),(0.5,0.6,0.8)}
C_4	{(0.2,0.5,0.7),(0.3,0.6,0.8),(0.1,0.2,0.3)}	{(0.5,0.6,0.7),(0.4,0.5,0.6),(0.3,0.4,0.5)}	{(0.6,0.8,0.9),(0.2,0.3,0.5),(0.1,0.3,0.4)}	{(0.5,0.6,0.7),(0.4,0.5,0.6),(0.3,0.4,0.5)}	{(0.7,0.8,0.9),(0.2,0.3,0.4),(0.2,0.3,0.4)}	{(0.2,0.5,0.7),(0.3,0.6,0.8),(0.1,0.2,0.3)}
C_5	{(0.7,0.8,0.9),(0.2,0.3,0.4),(0.2,0.3,0.4)}	{(0.6,0.8,0.9),(0.2,0.3,0.5),(0.1,0.3,0.4)}	{(0.5,0.6,0.7),(0.4,0.5,0.6),(0.3,0.4,0.5)}	{(0.7,0.8,0.9),(0.2,0.3,0.4),(0.2,0.3,0.4)}	{(0.6,0.8,0.9),(0.2,0.3,0.5),(0.1,0.3,0.4)}	{(0.6,0.8,0.9),(0.2,0.3,0.5),(0.1,0.3,0.4)}
C_6	{(0.2,0.3,0.4),(0.4,0.5,0.6),(0.6,0.7,0.8)}	{(0.2,0.5,0.7),(0.3,0.6,0.8),(0.1,0.2,0.3)}	{(0.6,0.8,0.9),(0.2,0.3,0.5),(0.1,0.3,0.4)}	{(0.5,0.6,0.7),(0.4,0.5,0.6),(0.3,0.4,0.5)}	{(0.7,0.8,0.9),(0.2,0.3,0.4),(0.2,0.3,0.4)}	{(0.5,0.6,0.7),(0.4,0.5,0.6),(0.3,0.4,0.5)}
C_7	{(0.4,0.7,0.9),(0.5,0.6,0.8),(0.2,0.4,0.6)}	{(0.6,0.8,0.9),(0.2,0.3,0.5),(0.1,0.3,0.4)}	{(0.5,0.6,0.7),(0.4,0.5,0.6),(0.3,0.4,0.5)}	{(0.7,0.8,0.9),(0.2,0.3,0.4),(0.2,0.3,0.4)}	{(0.1,0.3,0.5),(0.2,0.4,0.7),(0.5,0.8,0.9)}	{(0.7,0.8,0.9),(0.2,0.3,0.4),(0.2,0.3,0.4)}
C_8	{(0.6,0.8,0.9),(0.2,0.3,0.5),(0.1,0.3,0.4)}	{(0.5,0.6,0.7),(0.4,0.5,0.6),(0.3,0.4,0.5)}	{(0.7,0.8,0.9),(0.2,0.3,0.4),(0.2,0.3,0.4)}	{(0.6,0.8,0.9),(0.2,0.3,0.5),(0.1,0.3,0.4)}	{(0.4,0.7,0.9),(0.5,0.6,0.8),(0.2,0.4,0.6)}	{(0.2,0.5,0.7),(0.3,0.6,0.8),(0.1,0.2,0.3)}
C_9	{(0.5,0.6,0.8),(0.3,0.4,0.5),(0.4,0.5,0.6)}	{(0.6,0.8,0.9),(0.2,0.3,0.5),(0.1,0.3,0.4)}	{(0.5,0.6,0.7),(0.4,0.5,0.6),(0.3,0.4,0.5)}	{(0.7,0.8,0.9),(0.2,0.3,0.4),(0.2,0.3,0.4)}	{(0.2,0.3,0.4),(0.4,0.5,0.6),(0.6,0.7,0.8)}	{(0.7,0.8,0.9),(0.2,0.3,0.4),(0.2,0.3,0.4)}

Table 4. The TFNNs matrix four.

	A_1	A_2	A_3	A_4	A_5	A_6
C_1	{(0.6,0.8,0.9),(0.2,0.3,0.5),(0.1,0.3,0.4)}	{(0.1,0.3,0.5),(0.2,0.4,0.7),(0.5,0.8,0.9)}	{(0.4,0.7,0.9),(0.5,0.6,0.8),(0.2,0.4,0.6)}	{(0.2,0.3,0.4),(0.4,0.5,0.6),(0.6,0.7,0.8)}	{(0.7,0.8,0.9),(0.2,0.3,0.4),(0.2,0.3,0.4)}	{(0.6,0.8,0.9),(0.2,0.3,0.5),(0.1,0.3,0.4)}
C_2	{(0.5,0.6,0.7),(0.4,0.5,0.6),(0.3,0.4,0.5)}	{(0.5,0.6,0.8),(0.3,0.4,0.5),(0.4,0.5,0.6)}	{(0.1,0.3,0.5),(0.2,0.4,0.7),(0.5,0.8,0.9)}	{(0.4,0.7,0.9),(0.5,0.6,0.8),(0.2,0.4,0.6)}	{(0.2,0.5,0.7),(0.3,0.6,0.8),(0.1,0.2,0.3)}	{(0.5,0.6,0.7),(0.4,0.5,0.6),(0.3,0.4,0.5)}
C_3	{(0.3,0.5,0.6),(0.2,0.4,0.5),(0.5,0.6,0.8)}	{(0.6,0.8,0.9),(0.2,0.3,0.5),(0.1,0.3,0.4)}	{(0.4,0.7,0.9),(0.5,0.6,0.8),(0.2,0.4,0.6)}	{(0.2,0.3,0.4),(0.4,0.5,0.6),(0.6,0.7,0.8)}	{(0.3,0.5,0.6),(0.2,0.4,0.5),(0.5,0.6,0.8)}	{(0.1,0.3,0.5),(0.2,0.4,0.7),(0.5,0.8,0.9)}
C_4	{(0.2,0.5,0.7),(0.3,0.6,0.8),(0.1,0.2,0.3)}	{(0.5,0.6,0.7),(0.4,0.5,0.6),(0.3,0.4,0.5)}	{(0.5,0.6,0.8),(0.3,0.4,0.5),(0.4,0.5,0.6)}	{(0.7,0.8,0.9),(0.2,0.3,0.4),(0.2,0.3,0.4)}	{(0.5,0.6,0.7),(0.4,0.5,0.6),(0.3,0.4,0.5)}	{(0.2,0.3,0.4),(0.4,0.5,0.6),(0.6,0.7,0.8)}
C_5	{(0.7,0.8,0.9),(0.2,0.3,0.4),(0.2,0.3,0.4)}	{(0.3,0.5,0.6),(0.2,0.4,0.5),(0.5,0.6,0.8)}	{(0.6,0.8,0.9),(0.2,0.3,0.5),(0.1,0.3,0.4)}	{(0.2,0.5,0.7),(0.3,0.6,0.8),(0.1,0.2,0.3)}	{(0.6,0.8,0.9),(0.2,0.3,0.5),(0.1,0.3,0.4)}	{(0.2,0.5,0.7),(0.3,0.6,0.8),(0.1,0.2,0.3)}
C_6	{(0.2,0.3,0.4),(0.4,0.5,0.6),(0.6,0.7,0.8)}	{(0.2,0.5,0.7),(0.3,0.6,0.8),(0.1,0.2,0.3)}	{(0.5,0.6,0.7),(0.4,0.5,0.6),(0.3,0.4,0.5)}	{(0.5,0.6,0.7),(0.4,0.5,0.6),(0.3,0.4,0.5)}	{(0.5,0.6,0.7),(0.4,0.5,0.6),(0.3,0.4,0.5)}	{(0.5,0.6,0.8),(0.3,0.4,0.5),(0.4,0.5,0.6)}
C_7	{(0.4,0.7,0.9),(0.5,0.6,0.8),(0.2,0.4,0.6)}	{(0.7,0.8,0.9),(0.2,0.3,0.4),(0.2,0.3,0.4)}	{(0.3,0.5,0.6),(0.2,0.4,0.5),(0.5,0.6,0.8)}	{(0.5,0.6,0.7),(0.4,0.5,0.6),(0.3,0.4,0.5)}	{(0.1,0.3,0.5),(0.2,0.4,0.7),(0.5,0.8,0.9)}	{(0.5,0.6,0.7),(0.4,0.5,0.6),(0.3,0.4,0.5)}
C_8	{(0.1,0.3,0.5),(0.2,0.4,0.7),(0.5,0.8,0.9)}	{(0.2,0.3,0.4),(0.4,0.5,0.6),(0.6,0.7,0.8)}	{(0.2,0.5,0.7),(0.3,0.6,0.8),(0.1,0.2,0.3)}	{(0.6,0.8,0.9),(0.2,0.3,0.5),(0.1,0.3,0.4)}	{(0.4,0.7,0.9),(0.5,0.6,0.8),(0.2,0.4,0.6)}	{(0.6,0.8,0.9),(0.2,0.3,0.5),(0.1,0.3,0.4)}
C_9	{(0.5,0.6,0.8),(0.3,0.4,0.5),(0.4,0.5,0.6)}	{(0.4,0.7,0.9),(0.5,0.6,0.8),(0.2,0.4,0.6)}	{(0.1,0.3,0.5),(0.2,0.4,0.7),(0.5,0.8,0.9)}	{(0.5,0.6,0.8),(0.3,0.4,0.5),(0.4,0.5,0.6)}	{(0.2,0.3,0.4),(0.4,0.5,0.6),(0.6,0.7,0.8)}	{(0.5,0.6,0.8),(0.3,0.4,0.5),(0.4,0.5,0.6)}

Then we applied the steps of the CRITIC method to compute the criteria weights. We normalize the decision matrix using Eq. (11) as shown in Table 5. Then we obtain the correlation matrix as shown in Table 6. Then we compute the standard deviation values. Then we compute the index C and criteria weights using Eq. (16) as shown in Fig 3.

Table 5. The TFNNs normalization matrix by CRITIC method.

	A_1	A_2	A_3	A_4	A_5	A_6
C_1	0.636364	0.136364	0.392045	0	1	0.982955
C_2	1	0.166667	0	0.041667	0.604167	0.104167
C_3	0.276786	1	0.080357	0.044643	0.267857	0
C_4	0.52381	0	0.380952	0.452381	1	0.404762
C_5	1	0	0.056818	0.022727	0.261364	0.534091
C_6	0	0.822034	0.762712	0.338983	1	0.70339
C_7	0.776786	0.946429	0.321429	1	0	0.785714
C_8	0	0.033708	0.764045	0.94382	0.47191	1
C_9	0.641026	0.698718	0.173077	0.762821	0	1

Table 6. The correlation matrix by the CRITIC method.

	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8	C_9
C_1	1	0.428136	-0.33605	0.635742	0.556123	0.257385	-0.58681	0.078331	-0.20482
C_2	0.428136	1	0.099583	0.479154	0.786955	-0.45133	-0.21123	-0.67276	-0.20204
C_3	-0.33605	0.099583	1	-0.50989	-0.23181	0.229749	0.226737	-0.76668	0.01779
C_4	0.635742	0.479154	-0.50989	1	0.269586	0.099925	-0.73263	0.162518	-0.58308
C_5	0.556123	0.786955	-0.23181	0.269586	1	-0.6175	0.05231	-0.3247	0.239277
C_6	0.257385	-0.45133	0.229749	0.099925	-0.6175	1	-0.55444	0.200864	-0.44666
C_7	-0.58681	-0.21123	0.226737	-0.73263	0.05231	-0.55444	1	-0.05108	0.898556
C_8	0.078331	-0.67276	-0.76668	0.162518	-0.3247	0.200864	-0.05108	1	0.144439
C_9	-0.20482	-0.20204	0.01779	-0.58308	0.239277	-0.44666	0.898556	0.144439	1

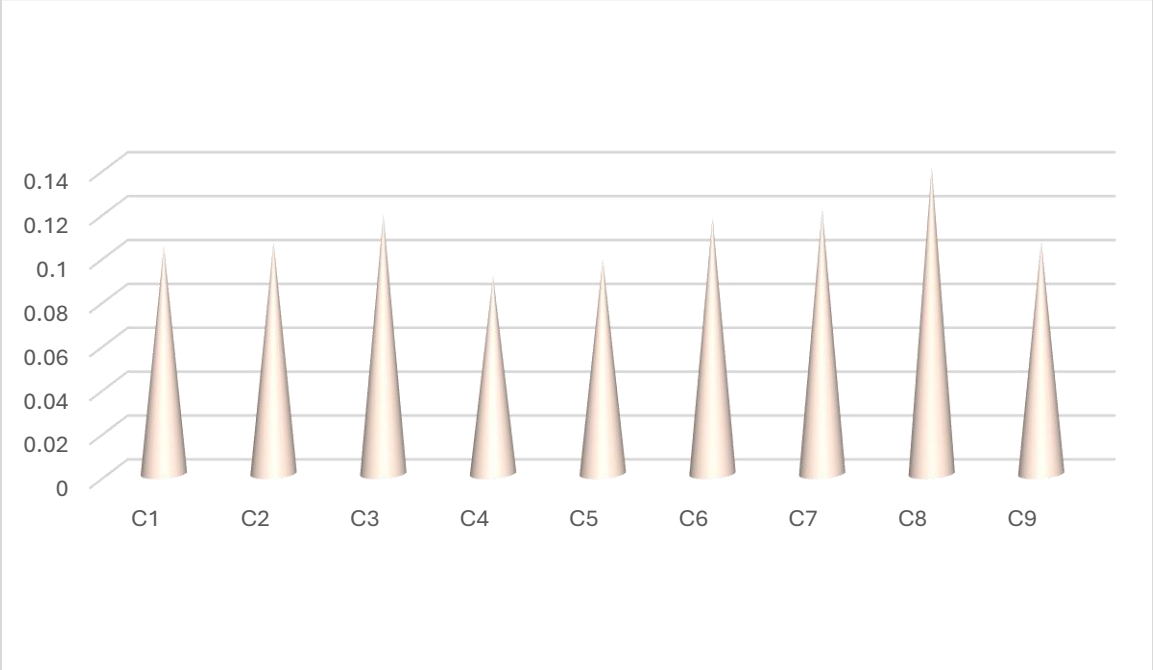

Fig 3. The criteria weights.

Results of the TFNS-MOORA Method

Eq. (17) is used to normalize the decision matrix by the MOORA method as shown in Table 7. Then we compute the reference points. Then we compute the assessment values using Eq. (18) as shown in Table 8. Then we rank the alternatives as shown in Table 9 and Fig 4.

Table 7. The TFNNs normalization matrix by MOORA method.

	A_1	A_2	A_3	A_4	A_5	A_6
C_1	0.300751	0.208855	0.255847	0.183792	0.367584	0.364451
C_2	0.347252	0.29952	0.289973	0.29236	0.324579	0.29594
C_3	0.317188	0.421628	0.288822	0.283664	0.315899	0.277217
C_4	0.299344	0.275132	0.292741	0.296042	0.321354	0.293841

C_5	0.327564	0.245673	0.250326	0.247534	0.267077	0.289411
C_6	0.224422	0.348109	0.339183	0.275427	0.374886	0.330257
C_7	0.329266	0.352697	0.266373	0.360096	0.221977	0.3305
C_8	0.239966	0.243209	0.31347	0.330765	0.285366	0.336169
C_9	0.310378	0.320499	0.228285	0.331744	0.197922	0.373353

Table 8. The TFNNs weighted normalization matrix by MOORA method.

	A_1	**A_2**	**A_3**	**A_4**	**A_5**	**A_6**
C_1	0.031249	0.021701	0.026584	0.019097	0.038194	0.037868
C_2	0.036732	0.031683	0.030673	0.030925	0.034333	0.031304
C_3	0.037526	0.049882	0.03417	0.033559	0.037373	0.032797
C_4	0.026992	0.024809	0.026397	0.026694	0.028977	0.026496
C_5	0.03208	0.02406	0.024516	0.024243	0.026156	0.028344
C_6	0.026241	0.040703	0.03966	0.032205	0.043834	0.038616
C_7	0.039819	0.042653	0.032213	0.043548	0.026844	0.039968
C_8	0.033583	0.034037	0.043869	0.04629	0.039936	0.047046
C_9	0.03293	0.034004	0.02422	0.035197	0.020999	0.039611

Table 9. The values of MOORA method.

	Score of positive criteria	**Score of negative criteria**	**Final score**
A_1	0.237981	0.059171	0.17881
A_2	0.228824	0.074707	0.154117
A_3	0.218421	0.06388	0.154542
A_4	0.224356	0.067402	0.156954
A_5	0.231814	0.064833	0.166981
A_6	0.243823	0.078227	0.165596

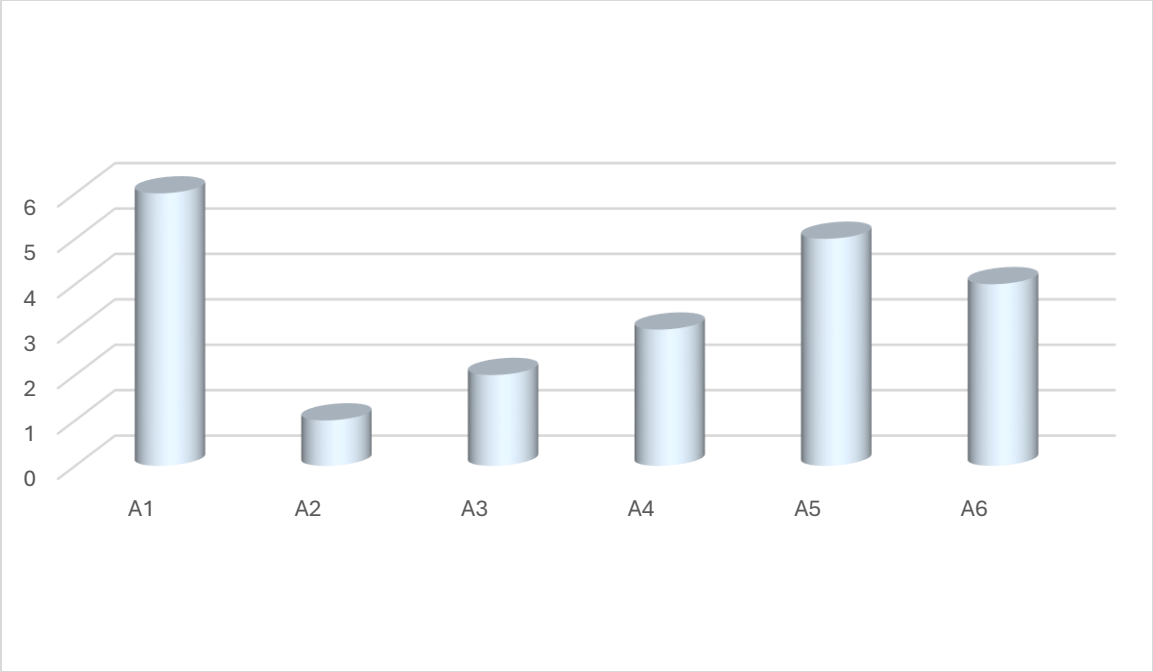

Fig 4. The rank of alternatives.

4. Discussion

This section presents a comprehensive sensitivity analysis conducted under ten different scenarios to examine how variations in criterion weights influence the ranking of alternatives. The purpose of this analysis is to assess the robustness and stability of the proposed evaluation framework and to determine how sensitive the ranking results are to changes in the weighting of criteria.

As illustrated in Figure 5, different weight distributions were considered to evaluate their impact on the final rankings. In the first case, all criteria were assigned equal weights to establish a baseline for comparison. In the second case, the weight of the first criterion was increased by 14%, while the remaining criteria retained equal weights. Similarly, in the third case, the weight of the second criterion was increased by 14%, keeping all other criteria unchanged. This pattern was systematically repeated for each of the ten cases, ensuring that each criterion was individually tested for its impact on the final ranking outcomes.

Following this, the MOORA method was applied under each weight configuration. This involved computing distinct normalization matrices and corresponding weighted normalization matrices for each scenario. Once these matrices were established, we calculated the assessment values for each alternative, ultimately deriving the final scores as presented in Figure 6.

Upon ranking the alternatives across all cases, the results consistently indicated that Alternative 1 emerged as the best-performing option, while Alternative 2 was ranked as the least favorable

choice. These rankings remained stable across various weighting scenarios, as depicted in Figure 7, demonstrating the reliability of the proposed approach in decision-making under uncertainty.

This sensitivity analysis highlights the robustness of our methodology in handling uncertain and imprecise data. The consistency in rankings suggests that Structural Integrity, which received the highest weight in the initial assessment, plays a dominant role in the evaluation process. Additionally, the findings confirm that the framework effectively accommodates variations in expert opinions and weighting schemes without significantly altering the overall conclusions.

4.1 Managerial Implications

The findings of this study offer valuable insights for transport authorities, policymakers, and urban planners. By identifying the most critical safety risk factors, decision-makers can implement targeted strategies to improve URT safety. The proposed framework allows for optimized resource allocation, ensuring that safety investments are directed toward the most significant risk areas, thus maximizing their effectiveness. Strengthening safety measures based on this model can also increase passenger confidence, leading to higher ridership and improved urban mobility. Moreover, the structured methodology supports policymakers in formulating well-informed regulations and operational guidelines aimed at enhancing overall URT safety standards.

4.2. Limitations of the Study

Although the proposed methodology provides a structured and effective approach for evaluating URT safety risks, certain limitations should be acknowledged. The study relies on expert opinions for assigning weights to different criteria, which may introduce subjective biases despite the use of the CRITIC method. Additionally, the research focuses primarily on safety risk evaluation, while other operational factors such as environmental impact and financial feasibility are not explicitly addressed. Furthermore, the study is based on a limited set of criteria and alternatives; expanding these in future research could provide a more holistic perspective.

4.3 Practical Recommendations

Based on the study's findings, several practical recommendations can be made to enhance URT safety. First, transport operators should prioritize improvements in structural integrity and operational safety, as these were identified as key risk factors. Second, implementing real-time monitoring systems using IoT technologies can help detect potential safety threats early. Third, training programs for staff should be enhanced to ensure proper response to emergencies and risk mitigation. Finally, collaboration between policymakers, transport authorities, and urban planners is essential to ensure a comprehensive and well-regulated safety management system for URT.

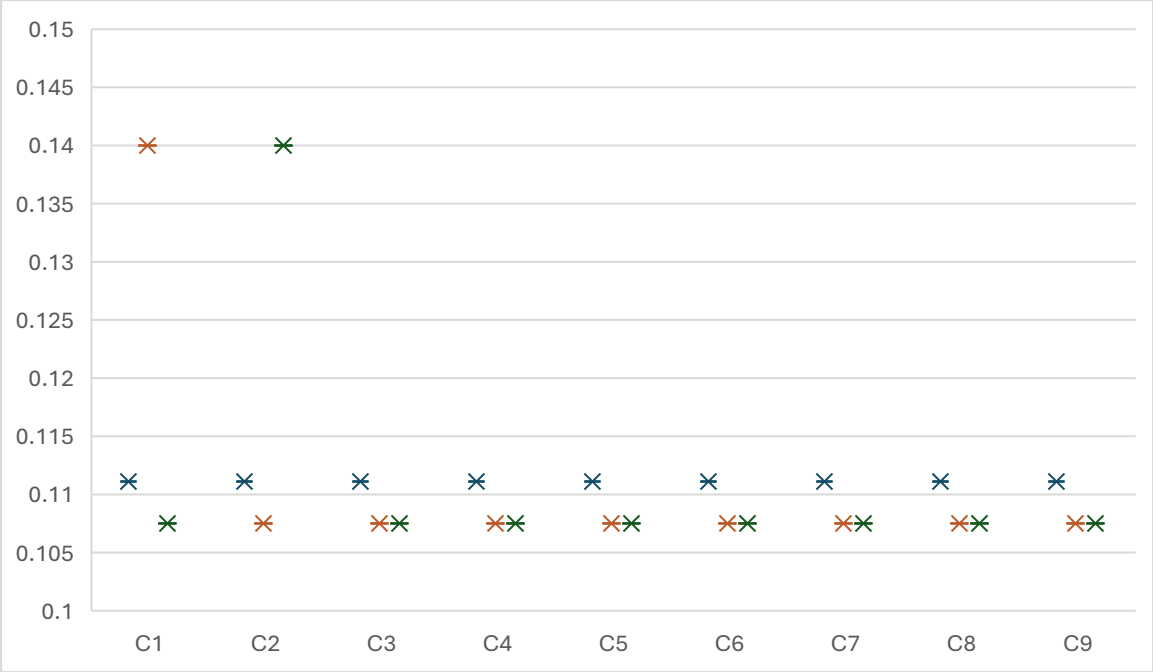

Fig 5. The different criteria weights.

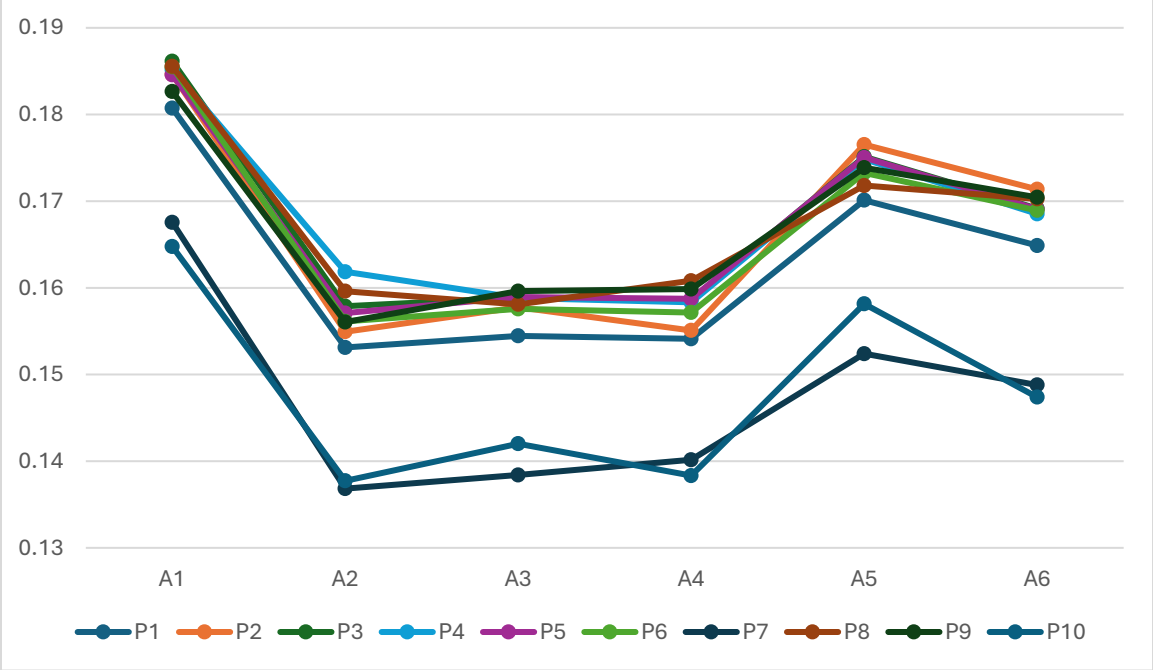

Fig 6. The different assessment values by the MOORA method.

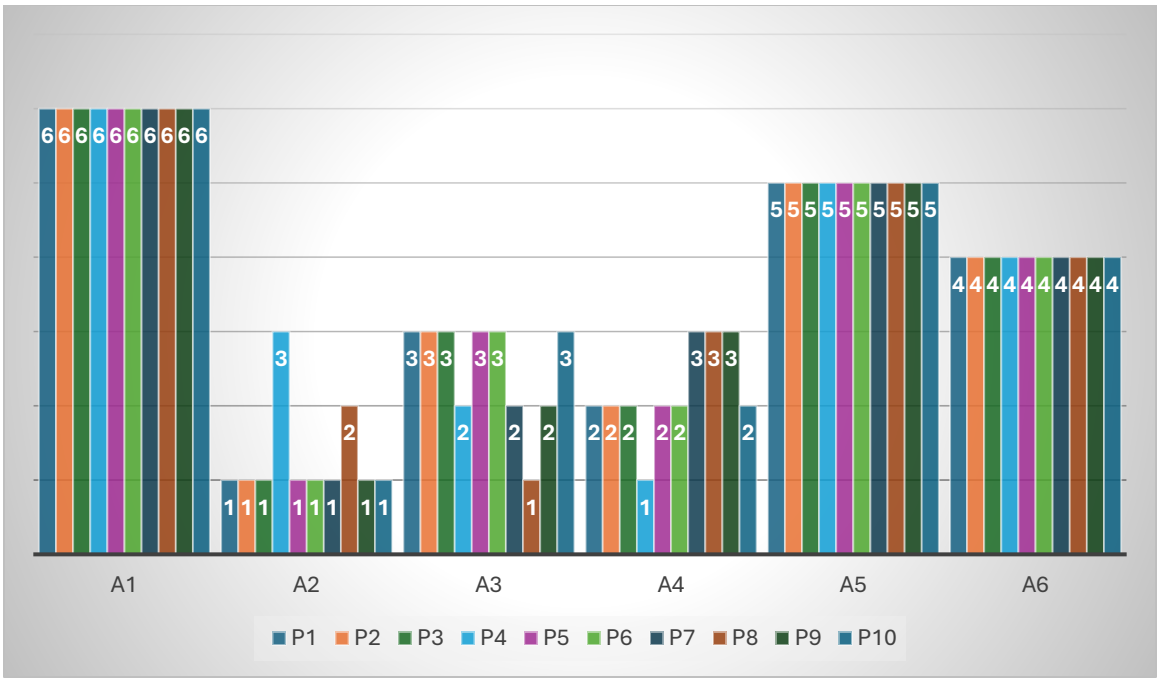

Fig 7. The different ranks of alternatives

5. Conclusions

This study introduces a structured methodology for evaluating urban rail transit safety by integrating triangular fuzzy neutrosophic sets (TFNSs) with the CRITIC and MOORA methods. The proposed model effectively captures uncertainty in safety assessments, allowing for more precise decision-making. The research highlights the importance of considering decision-makers' risk preferences, ensuring flexibility and adaptability in safety evaluations. By applying this framework to nine criteria and six alternatives, the study provides practical insights for transport authorities and urban planners.

The sensitivity analysis confirms the stability and reliability of the ranking process, demonstrating that the model is robust across different weighting scenarios. Furthermore, the findings emphasize the critical role of structural integrity in urban rail transit safety, supporting policymakers in prioritizing essential risk factors. The study's contributions extend beyond theoretical advancements by offering actionable recommendations for safety improvements.

6. Future Work

Although this study presents a comprehensive safety evaluation framework, there are several directions for future research. One promising avenue is the integration of real-time monitoring data from IoT-based sensors and surveillance systems, which would enable dynamic safety risk assessments. Additionally, the methodology could be expanded to evaluate other forms of public transportation, such as buses and high-speed rail, to explore its broader applicability. Further studies can also investigate alternative multi-criteria decision-making (MCDM) methods, such as

TOPSIS or VIKOR, to compare their effectiveness in safety evaluations. Finally, conducting more extensive case studies in different urban settings would help validate the generalizability and effectiveness of the proposed model.

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Classification of Organizational Behavior at the National Agrarian University of La Selva Based on Plithogenic Sets

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Abstract. In human organizations, there are group ways of conducting themselves, the interaction between members, and internal dynamics, among others that make the organization maintain a specific behavior as if it were an individual. This is the so-called Organizational Behavior, which is important to identify within a particular organization to improve the performance and efficiency of the entity. There are four types of organizational behavior models, namely, autocratic, custodial, supportive, and collegial. This paper aims to classify the organization that belongs to the office of the Academic Vice-Rector of the National Agrarian University of La Selva in Peru. To do this we use the theory of Plithogenic Sets. This theory is flexible enough to allow the study of complex phenomena such as the one we are dealing with, where there are influences from various aspects. In the research, we surveyed 30 members of these offices to carry out the evaluation. This will contribute to improving the performance of the organization.

Keywords: Organizational Development, Organizational Behavior, Plithogeny, plithogenic sets, plithogenic decision-making, Single-Valued Neutrosophic Number.

1 Introduction

Organizational Development (OD) emerged in the late 1950s and early 1960s as a dynamic, continuous process adopted to achieve planned changes in the organization, based on realistic diagnoses of the situation. This change focuses on values, attitudes, relationships, and organizational climate; it uses strategies, methods, and instruments aimed at optimizing the interaction between people and groups. Its main objectives include generating a higher quality of work life, and productivity, increasing the effectiveness and health of the organization, as well as the necessary capacity to adapt to an increasingly competitive world, ensuring its survival and the mutual development of the company-employee binomial.

Achieving this requires a global vision of the organization, an open systems approach, compatibility with environmental conditions, a responsible and conscious contract of management, developing the potential of people, groups, subsystems, and their relationships, the institutionalization of the process, and the self-sustainability of changes.

OD is not something new and although it is based on discoveries from the human and administrative sciences, it is a different art. Thus, the great and new contribution of OD is in the systemic, integrated, and flexible treatment of what already existed in those sciences but was dispersed, fragmented, or stagnant. The novelty brought by the new OD practice consists of going beyond the classic objectives of greater efficiency and productivity in the company, beyond the maximization of profits or optimization of services, beyond the search for effectiveness, and also the need to ensure organizational health.

OD is at the same time a new art of perfecting the organization, integrating the needs of the company and of the people. It is a new philosophy of administration whose style is renewing and revitalizing; it is a new technology, a practical set of techniques and instruments for planned changes in which psychological and behavioral aspects

must be given due attention.

The DO should not be confused with certain purposes contrary to its nature, however on many occasions, it has been taken wrongly, thus, the DO is not a training course, an emergency solution for moments of crisis, an isolated intervention, and therefore disconnected from the managerial processes, an initiative without continuity, an effort of specialists and other well-intentioned people but without the commitment of the responsible executives, a series of diagnostic meetings without generating solutions and actions, a maneuver to obtain or preserve power, prestige or advantages at the expense of other people, a process to manipulate, harm or punish workers.

OD is ultimately a planned, systematic process that incorporates the principles and practices of organizational behavior (OB) sciences to implement cultural and structural changes in the organization to increase individual and organizational effectiveness and to constantly optimize its functioning at all levels. Finally, OD focuses on organizations, and on making them work better through a total system change.

Organizational Behavior is an academic discipline that emerged from an interdisciplinary body of knowledge to study human behavior in organizations and is part of the integrated structure of OD theories and practices. Thus, OB is a field of study that investigates the behavior, attitudes, and performance of individuals and groups in the organizational environment, as well as the processes and practices that influence the effectiveness of individuals, groups, and organizations.

OB portrays the continuous interaction and reciprocal influence between people and the organization to apply this knowledge to improve the effectiveness of organizations. This involves relying on theories, methods, and principles drawn from various disciplines.

OB is a field that is based primarily on the contributions of psychology, a discipline that deals with the behavior of individuals and its most widely applied branches such as social psychology, which studies the concepts related to group behavior, the interactions between individuals and society to understand the social properties of behavior and clinical psychology, which investigates the elements that harm mental health or cause discomfort to individuals to treat and prevent them. OB is also based on the contributions of sociology, anthropology, organizational sociology, and other disciplines.

OB is an interdisciplinary field that allows us to learn about individuals, groups, structures, and processes and examine the relationships between them, as well as to understand the performance of the organization as an integrated system.

OB is an academic discipline that is highly sensitive to some of the particularities of the organization and its environment and is largely dependent on the people who participate in the organization. OB is essential to understand the organization, whose intrinsic value resides mainly in its intangible assets (which cannot be seen) since these constitute its true wealth and provide the basis for its success. Intangible assets depend on human capital, that is, on the set of talents that act in the organizational context; the effectiveness of organizations is directly dependent on the effectiveness of the people who make them up, and this is where OB arises since it establishes the foundations and general particularities of its operation.

As an academic discipline on what individuals do in organizations and how their behavior impacts their dynamics, OB:

- It is multidisciplinary,
- It is a method to encourage the transmission of ideas, information, and emotions clearly and openly,
- It creates a feedback mechanism to improve working conditions,
- It is a means of applying technologies and methods that support effective decision-making,
- It is a way to evaluate performance at different organizational levels and promote corrective actions.

Plithogeny is one of the most recent theories developed by F. Smarandache [1, 2]. It was created to mathematically model the generalization of classical dialectics where a concept, phenomenon, idea, etc. denoted by <A> interacts with its opposite denoted by <antiA>. Plithogeny takes into account the dynamic interaction between <A>, <antiA>, and <NeutA>, the latter being neither <A> nor <antiA>. Additionally, it includes the interaction with other concepts, phenomena, ideas, etc., with their opposites or their neuters, <B>, <NeutB> and <AntiB>; <C> or <NeutC> or <AntiC>, and so on.

Plithogeny allows the modeling of situations closer to the real world. This is why it allows the association of different concepts taken from various cognitive origins, as is the case in this article. Organizational behavior consists of different hierarchical levels such as individual, group, and organizational. Each of them is related to each other in a complex way. So far, Plithogeny theory has been applied mainly to decision-making and Statistics, [3-15].

In this article, we propose to classify the Organizational Behavior within the offices dependent on the Academic Vice-Rectorate of the National Agrarian University of La Selva in Peru. However, we will not make a usual classification, rather we will calculate a degree of belonging to each of the possible classifications. To do so, we use the theory of Plithogenic Sets and the survey carried out on 30 workers in these offices. This is an alternative of the Indeterminate Likert Scale applied to process surveys [16, 17].

The structure of the paper is, the Materials and Methods section follows, where the basic concepts of

organizational behavior and plithogenic sets are explained. The Results section summarizes the results obtained from the study and the procedures followed. At the end, the conclusions of the article are given.

2 Materials and Methods

This section contains the basic and theoretical elements discussed in this article. The sub-section below summarizes the most important definitions of Organizational Behavior.

2.1 Basics of Organizational Behavior

The OB operates on 3 hierarchical levels or planes: the individual level, the group level or level of groups and teams, and the organizational level or system of the organization, so that when studying the OB, the level of analysis used must be identified: behavior at the individual, group, or organizational level.

The levels are described below:

a. Macroperspective of OB. Also called macroorganizational behavior, it refers to the study of the behavior of the organizational system as a whole (Organizational level).
b. Meso-organizational Behavior focuses on the behavior of groups and teams in the organization and acts as a link between the other perspectives (Group level).
c. Microperspective of OB. Also called microorganizational behavior, it analyzes the behavior of the individual (individual level).

OB studies what individuals do in the organization and how their behavior influences the effectiveness of the organization, which is based on the systematic research procedure characteristic of the scientific method. Thus, the three levels operate as independent variables of OB. The study of OB is important for which two approaches are defined:

- The systematic approach is based on the belief that behavior is not circumstantial, on the contrary, it is possible to verify certain consistencies on which people's behavior is based, which allows it to be predictable. The systematic study of behavior means to develop reasonable and accurate predictions by searching for relationships, causes, and effects, basing the conclusions on scientific evidence.
- Evidence-based management (EBM) complements systematic study by basing managerial decisions and organizational practices on the best available scientific certainty, taking management difficulties more scientifically. The manager would pose an administrative problem, then investigate and obtain the best possible evidence, critically assess the information found, and apply the relevant information to the question or case under study. Unfortunately, most managerial provisions are made "on the fly" based on few available studies.

A decision made based on intuition or gut feeling alone is similar to making an investment decision based on only half the data. The effects of relying on intuition are compounded by the fact that we tend to overestimate the accuracy of what we think we know. This is evidenced in various organizational studies, which reveal, for example, that 86% of executives assumed that their organization served its workers well, while only 55% of the latter believed that it did. Many managers hold certain opinions about the state of their organization that are strongly refuted by the reality of empirical evidence.

OB refers to everything related to people in organizations, acting alone or in groups, so one of the most important reasons for studying OB is that certain problems are common among organizations. Therefore, studying and understanding them helps to know the best way to address them. This study includes not only a catalog of situations and problems but also specific support material for managers.

Just as organizations differ from one another, practices within an organization can also vary, whether from one department to another or from one branch to another. Therefore, no OB model is sufficient to represent everything that happens in an organization, but the identification of a particular model can help distinguish the different ways of life of organizations. The choice of a particular model must take into account the philosophy, vision, mission, values, and goals of the organization. In addition, in the face of a changing environment, the selected model must not be static or immutable but must be adapted over time. The four models are the following:

1. Autocratic Model. It obeys power; this is held by those who occupy positions of command to order workers what to do and whoever does not comply will be sanctioned. The dominant orientation is authority, management believes it knows what is best and that workers are obliged to comply with its orders, their salary is low and their performance is equally low or minimal. Later, the combined force of new knowledge about the needs of workers and changes in social values gave rise to the recognition that there were better ways to manage organizational systems.
2. Custody model. In a formal study of workers carried out by managers, it was found that, in a custodial administration, although they did not rebel against their bosses, they held a very poor opinion of them and this opinion was expressed in crises. The bosses caused them insecurity, frustration, and aggression, and, as it was impossible to give free rein to these feelings, they transferred them to their homes and took them out on their

families and neighbors in such a way that the community suffered the consequences of this relationship. It was necessary to make the workers have notable complacency, as well as a higher level of security, thus increasing their enjoyment of their work and generally improving their working life quality. The custody model depends on economic resources to be able to pay salaries and benefits.

3. Supportive model. It depends on leadership rather than power and money. Through leadership, management provides an environment that helps workers grow and provide the organization with what they are capable of doing. Management's orientation is not aimed at simply supporting workers' benefits but at supporting their work performance. As a result, workers have a feeling of involvement and collaboration in the organization's work, abandoning the vision of "them" to adopt that of "us." Supportive behavior does not require financial resources but rather is part of the work lifestyle of managers, of their way of treating others, since their function is to help workers fulfill their work and solve their problems.
4. Collegial model. It is a useful extension of the support model; the term collegial refers to a group of people with a common purpose. This model embodies the concept of a team, it depends on the generation of a feeling of camaraderie in the workers by the management, resulting in them feeling useful and necessary, it is also easy for them to accept and respect the role of the administrators and instead of being seen as bosses, they are considered as collaborators. The administrative orientation is directed towards teamwork, having as a response the responsibility and discipline of the workers who offer quality work and not because of management's demands, on the contrary, they feel obliged to provide high quality.

The collegial and support models are more compatible with the real needs of workers, but it must be taken into account that adopting one model does not mean discarding the others; each organization is unique and therefore has a particular way of management. Managers must evaluate the model used and establish its suitability, as well as remain flexible in the use of new and different models. Making a "revolution" seeking to implement models for which the organization is not prepared would be a very bad decision; on the contrary, helping to improve through a change of culture would be a better goal. Table 1 summarizes the main characteristics of the four models analyzed.

Criterion/Model	Autocratic	Custody	Supportive	Collegial
Model base	Power	Economic resources	Leadership	Association
Administrative guidance	Authority	Money	Support	Teamwork
Orientation of workers	Obedience	Safety and performance	Job performance	Responsible Conduct
Psychological outcome in workers	Dependence on the boss	Organizational dependency	Stake	Self-discipline
Workers' needs met	Subsistence	Security	Category and recognition	Self-realization
Performance result	Minimum	Passive cooperation	Pulse animation	Moderate enthusiasm

Table 1. Adaptation of Keith Davis's of the four OB models.

The motivation theory, also known as the hierarchy of needs expressed by Abraham Maslow, integrates a hierarchy of five needs grouped into lower order needs: physiological, which include physical needs such as hunger, thirst, sex, shelter, among others; security, which encompass care and protection against emotional and physical harm; and higher order needs: social, which include affection, sense of belonging, acceptance and friendship; esteem, with internal factors such as self-respect, autonomy and achievement and external factors such as status, recognition and attention; self-actualization, drive to become what one is capable of being, includes growth and development of one's potential. When a need is sufficiently satisfied, the next in the hierarchy becomes dominant. The four OB models described are closely connected to human needs; the improvement of new models arose from the urgency to solve different needs whose significance dominates a certain moment as observed in Table 2.

	Autocratic	Custody	Supportive	Collegial
Maslow's hierarchy of needs	In order lower	In order lower	In order superior	In order superior
	Physiological (subsistence)	About security	Social and esteem	Self-realization

Table 2. Coherence matrix between Davis's four OB models and Maslow's hierarchy of needs.

2.2 Plithogenic Sets

For plithogenic sets, we have the following concepts ([1, 2, 8, 18]):

Let U be a universe of discourse, and P a non-empty set of elements, $P \subseteq U$. Let A be a non-empty set of *one-dimensional attributes* $A = \{\alpha_1, \alpha_2, \dots, \alpha_m\}$, $m \geq 1$; and let $\alpha \in A$ be a given attribute whose spectrum of all possible values (or states) is the non-empty set S, where S can be a discrete finite set $S = \{s_1, s_2, \dots, s_l\}$, $1 \leq l < \infty$, or infinitely countable set $S = \{s_1, s_2, \dots, s_\infty\}$, or infinitely uncountable (continuous) set $S =]a, b[$, $a < b$ where] ... [is an open, semi-open, or closed interval of the set of real numbers or another general set.

Let V be a non-empty subset of S, where V is the range of all attribute values needed by experts for their application. Each element $x \in P$ is characterized by the values of all attributes in $V = \{v_1, v_2, \dots, v_n\}$, for $n \geq 1$.

In the set of attribute values V, in general, there is a dominant attribute value, which is determined by experts in its application. The dominant attribute value means the most important attribute value that experts are interested in.

Each attribute value $v \in V$ has a corresponding *degree of appurtenance* $d(x, v)$ of element x, to the set P, for some given criteria.

The degree of appurtenance can be a *fuzzy degree of appurtenance*, an *intuitionistic degree of appurtenance*, or a *neutrosophic degree of appurtenance* to the plithogenic set.

Therefore, the *degree of appurtenance function* is*:*

$\forall x \in P, d: P \times V \to P\ ([0,1]^z)$ (1)

So $d(x, v)$ is a subset of $[0,1]^z$, where $\mathcal{P}([0,1]^z)$ is the power set of $[0,1]^z$, where z = 1 (fuzzy degree of appurtenance), z = 2 (for the intuitionistic degree of appurtenance), or z = 3 (for the neutrosophic degree of appurtenance).

Given the cardinal $|V| \geq 1$. Let $c: V \times V \to [0,1]$ be the *fuzzy attribute value contradiction degree function* between any two attribute values v_1 and v_2, denoted by $c(v_1, v_2)$, and satisfying the following axioms:

1. $c(v_1, v_1) = 0$, the degree of contradiction between the same attribute value is zero;
2. $c(v_1, v_2) = c(v_2, v_1)$, commutativity.

One can define a *fuzzy attribute value contradiction degree function* (c as before, which we can denoted by c_F to distinguish it from the next two), an *intuitionistic attribute value contradiction degree function* ($c_{IF}: V \times V \to [0,1]^2$), or more generally, a *neutrosophic attribute value contradiction degree function* ($c_N: V \times V \to [0,1]^3$) can be used by increasing the complexity of the calculation, but also increasing the precision.

We mainly calculate the degree of contradiction between the values of uni-dimensional attributes. For multi-dimensional attribute values, we divide them into corresponding one-dimensional attribute values.

The *attribute value contradiction degree function* helps the plithogenic aggregation operators and the plithogenic inclusion relation (partial order) to obtain a more accurate result.

The attribute value contradiction degree function is designed in each field where a plithogenic set is used according to the application to be solved. If ignored, aggregations still work, but the result may lose precision.

Then (P, a, V, d, c) is called a *plithogenic set*, ([1, 2, 8, 18]):

1. Where "P" is a set, "a" is an attribute (multidimensional in general), "V" is the range of the attribute values, "d" is the degree of appurtenance of the attribute value of each element x to the set P to some given criteria ($x \in P$), and "d" means "d_F" or " d_{IF}" or " d_N", when it is a fuzzy degree of appurtenance, an intuitionistic degree of appurtenance, or a neutrosophic degree of appurtenance respectively of an element x to the plithogenic set P;
2. "c" means either "c_F" or "c_{IF}" or " c_N", when it comes to the fuzzy degree of contradiction, intuitionistic degree of contradiction, or neutrosophic degree of contradiction between the attribute values respectively.

The functions $d(\cdot,\cdot)$ and $c(\cdot,\cdot)$ are defined according to the applications that the experts need to solve.

The notation below is used:

$x(d(x, V))$, $d(x, V) = \{d(x, v), \text{for all } v \in V\}, \forall x \in P$.

The attribute value contradiction degree function is calculated between each attribute value concerning the dominant attribute value (denoted by v_D) in particular, and with concerning attribute values as well.

The attribute value contradiction degree function c between the attribute values is used in the definition of plithogenic aggregation operators (intersection (AND), union (OR), implication ($\Longrightarrow$), equivalence ($\Longleftrightarrow$), inclusion relation (partial order), and other plithogenic aggregation operators that combine two or more degrees of attribute value acting on the t-norm and the t-conorm, ([1, 2, 8, 18]).

Most plithogenic aggregation operators are linear combinations of the fuzzy t-norm (denoted by $\wedge_F$), and the fuzzy t-conorm (denoted by $\vee_F$), but nonlinear combinations can also be constructed.

If the t-norm is applied on the dominant attribute value denoted by v_D, and the contradiction between v_D and

v_2 is $c(v_D, v_2)$, then on the attribute value v_2 it is applied:

$$[1 - c(v_D, v_2)] \cdot t_{norm}(v_D, v_2) + c(v_D, v_2) \cdot t_{conorm}(v_D, v_2) \quad (2)$$

Or, by using symbols:

$$[1 - c(v_D, v_2)] \cdot (v_D \wedge_D v_2) + c(v_D, v_2) \cdot (v_D \vee_D v_2) \quad (3)$$

Similarly, if the t- conorm is applied on the dominant attribute value denoted by v_D, and the contradiction between v_D and v_2 is $c(v_D, v_2)$, then on the attribute value v_2 it is applied:

$$[1 - c(v_D, v_2)] \cdot t_{conorm}(v_D, v_2) + c(v_D, v_2) \cdot t_{norm}(v_D, v_2) \quad (4)$$

Or, by using symbols:

$$[1 - c(v_D, v_2)] \cdot (v_D \vee_D v_2) + c(v_D, v_2) \cdot (v_D \wedge_D v_2) \quad (5)$$

The *Plithogenic Intersection* is defined as:

$$(a_1, a_2, a_3) \wedge_P (b_1, b_2, b_3) = \left(a_1 \wedge_D b_1, \frac{1}{2}[(a_2 \wedge_D b_2) + (a_2 \vee_D b_2)], a_3 \vee_D b_3\right) \quad (6)$$

The *Plithogenic Union* is defined as:

$$(a_1, a_2, a_3) \vee_P (b_1, b_2, b_3) = \left(a_1 \vee_D b_1, \frac{1}{2}[(a_2 \wedge_D b_2) + (a_2 \vee_D b_2)], a_3 \wedge_D b_3\right) \quad (7)$$

In other words, for what applies to the appurtenance, the opposite applies to the non-appurtenance, while in indeterminacy the average between them applies[19,20].

3 Results

Thirty administrative workers from the group of offices dependent on the Academic Vice-Rectorate of the National Agrarian University of La Selva participated. Each of them was asked to give a series of opinions as shown below:

A. Basis of your institution's model:
   1. Power,
   2. Economic resources,
   3. Leadership,
   4. Association,

B. Administrative guidance at your institution:
   5. Authority,
   6. Money,
   7. Support,
   8. Teamwork,

C. Orientation of workers in your institution:
   9. Obedience,
   10. Safety and performance,
   11. Job performance,
   12. Responsible conduct,

D. Psychological outcome in your institution's workers:
   13. Dependence on the boss,
   14. Organizational dependency,
   15. Stake,
   16. Self-discipline,

E. The employee needs to find in your institution:
   17. Subsistence,
   18. Security,
   19. Category and recognition,
   20. Self-realization,

F. Performance results at your institution:
   21. Minimum,
   22. Passive cooperation,
   23. Pulse animation,
   24. Moderate enthusiasm.

The 30 respondents were asked to rate their institution on the scale below:

Linguistic Value	Associated Single-Valued Neutrosophic Number (SVNN)
Disagree	(0.1,0.1,0.8)
More agree than disagree	(0.55,0.1,0.35)

Agree	(0.8,0.1,0.1)

Table 3. The linguistic scale and the Single-Valued Neutrosophic Numbers were used in the survey of the study carried out.

Each of the aspects to be evaluated denoted by the letters A, B, C, D, E, and F are evaluated according to the answers given denoted by 1, 2, 3,…, 24. They are asked to use the evaluations "Disagree", "More agree than disagree" and "Agree", as proposed in Table 3. Let us note that each of the possible responses will have an evaluation.

Each linguistic value is replaced by its corresponding associated Single-Valued Neutrosophic Number in Table 3. Let a_i be the evaluation given by the ith respondent (i = 1, 2,…, 30) where the 24 responses in SVNN form are denoted by $a_i = (a_{i1}, a_{i2}, \cdots, a_{i24})$ and are aggregated using Equation 6 with the t-norm minimum and the t-conorm maximum. This way we obtain the SVNN for each interviewee, see Equation 8:

$$\bar{a}_{ik} = \Lambda_{\mathrm{P}}{}_{j=0}^{5}\, a_{i(4j+k)} \qquad (8)$$

Equation 8 aggregates the respondent's results concerning the autocratic style for k = 1, the custodial style for k = 2, the supportive style for k = 3, and the collegial style for k = 4.

The arithmetic mean of these values is then calculated for all respondents, as shown in Equation 9:

$$\bar{\bar{a}}_k = \frac{\sum_{i=1}^{30} \bar{a}_{ik}}{30} \qquad (9)$$

To determine the k index that is closest to the maximum option, the following distance formula is calculated:

$$d_k(\bar{\bar{a}}_k) = \sqrt{\left(T_{\bar{\bar{a}}_k} - 1\right)^2 + \left(I_{\bar{\bar{a}}_k}\right)^2 + \left(F_{\bar{\bar{a}}_k}\right)^2} \qquad (10)$$

In this way, the k with the smallest distance is considered the best approximation to the organization model.

Following the steps of the previous algorithm we have the following results that appear in Table 4:

$\bar{\bar{a}}_k$	$d_k(\bar{\bar{a}}_k)$	Ranking
(0.4831,0.1941,0.5831)	0.8030355098001583	4
(0.4382,0.1916,0.5351)	0.7991631935969024	3
(0.5053,0.1147,0.3244)	0.602594009263285	1
(0.5430,0.1403,0.3840)	0.613179492481606	2

Table 4. Results of the evaluations regarding $\bar{\bar{a}}_k$ (Equation 9), $d_k(\bar{\bar{a}}_k)$ (Equation 10), and the ranking.

The organizational model of this entity is evaluated as predominantly Supportive, although it presents characteristics of the other models with little difference, which are, viz. Collegial, Custody, and Autocratic, in that order.

Conclusion

This paper presented a study carried out in the offices of the Academic Vice-Rectorate of the National Agrarian University of La Selva, Peru, regarding the organizational behavior model. We used the Plithogenic Sets tool because it allows us to easily measure components of a system with varied characteristics. According to the results obtained from a survey carried out on 30 workers in these offices, we have that in the first place there is a Supportive Model behavior, followed by a Collegial Model, a Custodial Model, and finally an Autocratic Model. We recommend that managers analyze how to overcome the deficiencies that the presence of lower-order styles such as autocratic and custodial models can bring into the organization's functioning to a considerable degree.

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Quality Evaluation of Immersive Virtual Reality Interactive Art Design under Cultural Heritage Digitization under Interval Valued Neutrosophic Numbers

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Abstract: The integration of disciplines has become a crucial issue that the entire society must deal with due to the economy's and society's rapid development. Particularly for teaching art and design, the new features of quickly iterative technology and the developing theory of the digital age presented new difficulties. Simultaneously, as computer hardware continues to advance, new simulation technologies are continually emerging, opening new avenues for creative design work. As a methodical endeavor, teaching art design should incorporate cutting-edge instructional strategies with scientific theories. This study proposes a multi-criteria decision-making (MCDM) approach to evaluate the immersive virtual reality interactive art design under cultural heritage digitization. This study uses two MCDM methods such as the CRITIC method to compute the criteria weights and the ARAS method to rank the alternatives. Two MCDMs are used under the interval-valued neutrosophic sets (IVNSs) to deal with vague and uncertain information. We collect ten criteria and five alternatives to be evaluated by four experts. The sensitivity analysis is conducted to show the stability of the rank under different cases.

Keywords: Interval Valued Neutrosophic Numbers; Virtual Reality; Cultural Heritage Digitization.

1. Introduction

Since science and technology have advanced so quickly in recent years, the virtual reality sector has grown steadily. Advanced technologies like artificial intelligence, 5G, IoT (the Internet of Things), and big data are continuously integrating into virtual reality. The industrial use of virtual reality is facilitated by science and technology, which also creates new businesses and their ecosystems, like, Digital Twins, and other innovative technological ideas[1], [2]. Richer mixed virtual reality applications are emerging because of the widespread use of 5G and the expansion of numerous virtualization technologies into other industries[3], [4]. One area that needs new technology like these to make virtual reality more intelligent is education.

In recent years, virtual reality technology has gained popularity in the network technology sector. Other names for it include "virtual environment," "spiritual environment technology," and "cyberspace." Numerous technological domains are involved in virtual reality technology, which is a synthesis of numerous technologies[5], [6]. It is under the broad category of simulation technology, which also includes network technology, computer perception technology, multimedia technology, and graphics technology.

The technology behind virtual reality consists of four parts. The first is the virtual scene, a dynamic 3D image created by a computer[7], [8]. It has a very lifelike effect that creates an immersive experience. The second component is perception, which encompasses all the senses that people ought to possess, such as vision, touch, movement, and so forth.

Virtual reality can produce a sensation of reality thanks to these perceptual simulations. The third section consists of interactive features such as handle operation, hand-and-foot position movement, and angle of view movement. Information about the user's behavior will be obtained by the computer, which will then process and react to it before providing the user with virtual reality feedback. Currently, it mostly uses the handle and helmet to interact with the virtual environment. The handle controls how hands interact with virtual things, while the head theft creates the three-dimensional display effect[9], [10]. There will be more gadgets in the future that can mimic different interactive functions as virtual reality technology advances. As a result, virtual reality technology aligns with the idea of contemporary educational advancement.

This study evaluates Immersive Virtual Reality Interactive Art Design under Cultural Heritage Digitization. This problem is a multi-criteria decision-making (MCDM) issue. In the framework of the MCDM issue, management science and decision analysis have made significant contributions. The MCDM approaches offer appropriate models for supporting intricate and interrelated decision issues with significant tactics, concepts, and outcomes in contrast to other types of decision-making methods. A collection of multiple criterion approaches that evaluate and choose options based on several factors directly characterize it[11], [12]. It discusses a methodical quantitative strategy that attempts to assist DMs in making logical and effective judgments by taking significant goals and criteria into account.

Therefore, setting up models that can assist DMs in dealing with unclear situations is the goal of MCDM researchers. The fuzzy set (FS) approach is the basic idea behind overcoming uncertainty in decision-making processes[13], [14]. In several applications, classical fuzzy decision-making models have received the attention they deserve. In these circumstances, interval numbers, intuitionistic models, rough sets (RS), and FS are frequently used[15], [16]. Neutrosophic set (NS) notions are the most modern method in this category.

When addressing real-time issues, NSs consider elemental uncertainties and inconsistencies in addition to the membership and non-membership degrees of elements[17], [18]. NSs represent the incompleteness and inconsistency of an element to set, which makes them more relevant for solving complicated problems than FSs and intuitionistic FSs (IFSs).

1.1 Questions of this Study

In essence, this essay aims to respond to the following questions:

A. Determining the key factors for choosing an immersive Virtual Reality Interactive Art Design type in the face of uncertainty.
B. Examining whether the factors taken into consideration are appropriate for immersive Virtual Reality Interactive Art Design issues.
C. Examining the level of uncertainty surrounding these standards.
D. Developing strategies to deal with ambiguity and inaccurate information.
E. Examining which kind of uncertainty model would be best for enabling experts to compare and make decisions when assessing criteria and alternatives.

F. Confirming that the best-performing alternative may be found by applying the suggested neutrosophic model.
G. When addressing real-time issues, NSs consider elemental uncertainties and inconsistencies in addition to the membership and non-membership degrees of elements. NSs convey an element's inconsistency and incompleteness.

1.2 Contributions of this study

The following goals and contemporary issues that this paper aims to address can be identified based on the questions raised above:

1) This study presents an integrated approach to find, assess, and choose the best types of immersive Virtual Reality Interactive Art Design.
2) Assisting DMs in obtaining dependable solutions using the suggested neutrosophic MCDM model while taking information imprecision, inconsistency, and indeterminacy into consideration.
3) Use the interval-valued neutrosophic numbers to evaluate the criteria and alternatives.
4) Two MCDM methods are used in this study such as the CRITIC method to compute the criteria weights and the ARAS method to rank the alternatives.
5) The sensitivity analysis is conducted to show the stability of the results.

1.3 Organization of this study

The study is organized as follows: Section 2 presents the necessary mathematical formulations for the suggested methods. This section shows the steps of the CRITIC method to compute the criteria weights and the ARAS method to rank the alternatives. Deduced results, case study implementation, and performance sensitivity analysis are shown in section 3. Section 4 shows the conclusions.

2. IVN-CRITIC-ARAS Approach

In this section, an integrated CRITIC-ARAS method under the neutrosophic environment to rank the different types of Immersive Virtual Reality Interactive Art Design. This approach has two main stages. In the first stage, the criteria weights are computed using the IVN-CRITIC method. In the second stage, the rank of alternatives is computed using the IVN-ARAS approach. Figure 1 shows the two main stages. Types of Immersive Virtual Reality Interactive Art Design are determined such as $(A_1, \dots, A_m)$ and n $criteria$ $(C_1, \dots, C_2)$.

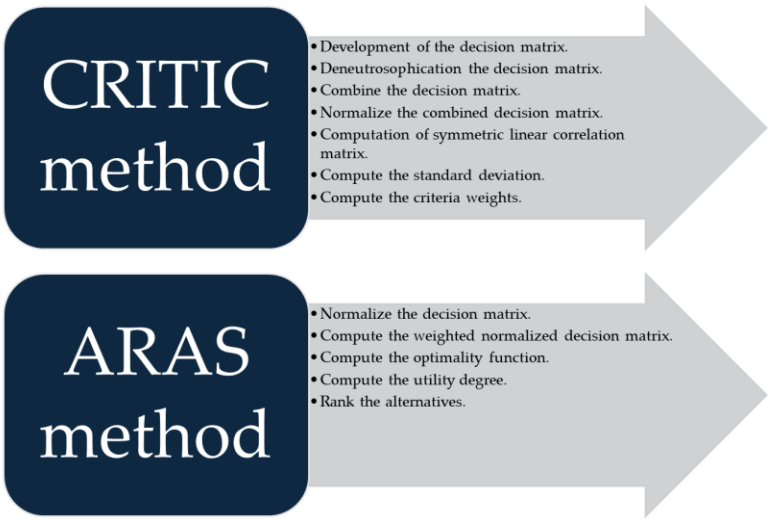
CRITIC method
• Development of the decision matrix.
• Deneutrosophication the decision matrix.
• Combine the decision matrix.
• Normalize the combined decision matrix.
• Computation of symmetric linear correlation matrix.
• Compute the standard deviation.
• Compute the criteria weights.
ARAS method
• Normalize the decision matrix.
• Compute the weighted normalized decision matrix.
• Compute the optimality function.
• Compute the utility degree.
• Rank the alternatives.

Fig 1. Methodological procedure.

2.1 Neutrosophic Approach

In this section, some definitions of the IVNs are described[19].

Definition 1.

The neutrosophic sets have three membership functions such as Truth. $T_A(x)$, Indeterminacy $I_A(x)$, and Falsity $F_A(x)$.

$$0-\leq \sup T_A(x) + \sup T_A(x) + \sup T_A(x) \leq 3 + \tag{1}$$

Definition 2.

Let two intervals valued neutrosophic numbers (IVNNs) with operations such as:

$$\pi A = \begin{pmatrix} \left[1-\left(1-T_A^L(x)\right)^\pi, 1-\left(1-T_A^U(x)\right)^\pi\right], \\ \left[\left(I_A^L(x)\right)^\pi, \left(I_A^U(x)\right)^\pi\right], \\ \left[\left(F_A^L(x)\right)^\pi, \left(F_A^U(x)\right)^\pi\right] \end{pmatrix} \tag{2}$$

$$A^\pi = \begin{pmatrix} \left[\left(T_A^L(x)\right)^\pi, \left(T_A^U(x)\right)^\pi\right], \\ \left[1-\left(1-I_A^L(x)\right)^\pi, 1-\left(1-I_A^U(x)\right)^\pi\right], \\ \left[1-\left(1-F_A^L(x)\right)^\pi, 1-\left(1-F_A^U(x)\right)^\pi\right] \end{pmatrix} \tag{3}$$

$$A+B = \begin{pmatrix} [T_A^L(x)+T_B^L(x)-T_A^L(x)T_B^L(x), T_A^U(x)+T_B^U(x)-T_A^U(x)T_B^U(x)], \\ [I_A^L(x)I_B^L(x), I_A^U(x)I_B^U(x)], \\ [F_A^L(x)F_B^L(x), F_A^U(x)F_B^U(x)] \end{pmatrix} \tag{4}$$

$$AB = \begin{pmatrix} [T_A^L(x)T_B^L(x), T_A^U(x)T_B^U(x)], \\ [I_A^L(x)+I_B^L(x)-I_A^L(x)I_B^L(x), I_A^U(x)+I_B^U(x)-I_A^U(x)I_B^U(x)], \\ [F_A^L(x)+F_B^L(x)-F_A^L(x)F_B^L(x), F_A^U(x)+F_B^U(x)-F_A^U(x)T_B^U(x)] \end{pmatrix} \tag{5}$$

$$A-B=\begin{pmatrix}[T_A^L(x)-T_B^U(x),T_A^U(x)-T_B^L(x)],\\ \left[\max\left(I_A^L(x),I_B^L(x)\right),\max\left(I_A^U(x),I_B^U(x)\right)\right],\\ [F_A^L(x)-F_B^U(x),F_A^U(x)-F_B^L(x)]\end{pmatrix} \tag{6}$$

$$\frac{A}{\pi}=\begin{pmatrix}\left[\min\left(\frac{T_A^L(x)}{\pi},1\right),\min\left(\frac{T_A^U(x)}{\pi},1\right)\right],\\ \left[\min\left(\frac{I_A^L(x)}{\pi},1\right),\min\left(\frac{I_A^U(x)}{\pi},1\right)\right]\\ \left[\min\left(\frac{F_A^L(x)}{\pi},1\right),\min\left(\frac{F_A^U(x)}{\pi},1\right)\right]\end{pmatrix} \tag{7}$$

$$\frac{A}{B}=\begin{pmatrix}\left[\min\left\{\frac{T_A^L(x)}{T_B^L(x)},\frac{T_A^L(x)}{T_B^U(x)},\frac{T_A^U(x)}{T_B^L(x)},\frac{T_A^U(x)}{T_B^U(x)}\right\},\right.\\ \left.\max\left\{\frac{T_A^L(x)}{T_B^L(x)},\frac{T_A^L(x)}{T_B^U(x)},\frac{T_A^U(x)}{T_B^L(x)},\frac{T_A^U(x)}{T_B^U(x)}\right\}\right],\\ \left[\min\left\{\frac{I_A^L(x)}{I_B^L(x)},\frac{I_A^L(x)}{I_B^U(x)},\frac{I_A^U(x)}{I_B^L(x)},\frac{I_A^U(x)}{I_B^U(x)}\right\},\right.\\ \left.\max\left\{\frac{I_A^L(x)}{I_B^L(x)},\frac{I_A^L(x)}{I_B^U(x)},\frac{I_A^U(x)}{I_B^L(x)},\frac{I_A^U(x)}{I_B^U(x)}\right\}\right],\\ \left[\min\left\{\frac{F_A^L(x)}{F_B^L(x)},\frac{F_A^L(x)}{F_B^U(x)},\frac{F_A^U(x)}{F_B^L(x)},\frac{F_A^U(x)}{F_B^U(x)}\right\},\right.\\ \left.\max\left\{\frac{F_A^L(x)}{F_B^L(x)},\frac{F_A^L(x)}{F_B^U(x)},\frac{F_A^U(x)}{F_B^L(x)},\frac{F_A^U(x)}{F_B^U(x)}\right\}\right]\end{pmatrix} \tag{8}$$

$$A^{-1}=\begin{pmatrix}\left[\left(T_A^L(x)\right)^{-1},\left(T_A^U(x)\right)^{-1}\right],\\ \left[\left(I_A^L(x)\right)^{-1},\left(I_A^U(x)\right)^{-1}\right],\\ \left[\left(F_A^L(x)\right)^{-1},\left(F_A^U(x)\right)^{-1}\right]\end{pmatrix} \tag{9}$$

Definition 3.

We can compute the de-neutrosophication function such as:

$$S(A)=\begin{pmatrix}\frac{T_A^L(x)+T_A^U(x)}{2}+\\ \left(1-\frac{\left(I_A^L(x)+I_A^U(x)\right)}{2}\right)\left(I_A^U(x)\right)-\\ \left(\frac{\left(F_A^L(x)+F_A^U(x)\right)}{2}\right)\left(1-F_A^U(x)\right)\end{pmatrix} \tag{10}$$

2.2 CRITIC Method

This method is used to compute the criteria weights.

1. Development of the decision matrix such as:

$$y_{ij}=\begin{pmatrix}y_{11}&\cdots&y_{1n}\\ \vdots&\ddots&\vdots\\ y_{m1}&\cdots&y_{mn}\end{pmatrix};i=1,\dots,m;j=1,\dots,n \tag{11}$$

2. Deneutrosophication of the decision matrix

3. Combine the decision matrix.

The decision matrix is combined using the average method.

4. Normalize the combined decision matrix.

$$u_{ij} = \frac{y_{ij} - \min y_{ij}}{\max y_{ij} - \min y_{ij}} \text{ for positive criteria} \quad (12)$$

$$u_{ij} = \frac{\min y_{ij} - y_{ij}}{\max y_{ij} - \min y_{ij}} \text{ for cost criteria} \quad (13)$$

5. Computation of symmetric linear correlation matrix.

We compute the correlation values between the criteria. Then we build the correlation matrix between the main criteria. Then we subtract one from the correlation matrix.

6. Compute the standard deviation.

7. Then we multiply the standard deviation by the correlation matrix by subtracting one.

$$C_j = Stand * \sum_{j=1}^{n} 1 - m_{ij} \quad (14)$$

8. Compute the criteria weights.

$$w_j = \frac{c_j}{\sum_{j=1}^{n} c_j} \quad (15)$$

2.3 IVN-ARAS Method

This part shows the steps of the ARAS method under neutrosophic sets to rank the alternatives.

9. Normalize the decision matrix.

$$r_{ij} = \frac{y_{ij}}{\sum_{i=0}^{m} y_{ij}} \quad (16)$$

10. Compute the weighted normalized decision matrix.

$$T_{ij} = w_j * r_{ij} \quad (17)$$

11. Compute the optimality function.

$$O_i = \sum_{j=1}^{n} T_{ij} \quad (18)$$

12. Compute the utility degree

$$D_i = \frac{O_i}{K} \quad (19)$$

Where k is the optimality value of O_i

13. Rank the alternatives.

3. Illustrative Example and Results

This section shows the results and discussion.

3.1 Case Study

This study uses the MCDM approach to rank the Immersive Virtual Reality Interactive Art Design under Cultural Heritage Digitization. This study uses types of Immersive Virtual Reality Interactive Art Design

to select the best one based on a set of criteria. We collected ten criteria and five alternatives as shown in Table 1. Four experts have evaluated the criteria and alternatives based on their experience. Table 2 shows the details of four experts.

Table 1. The criteria and alternatives.

Criteria	Alternatives
Technical Performance	Interactive AR/VR Installations
Narrative and Storytelling	Educational VR Experiences
Sustainability and Preservation	Virtual Museum Exhibits
User Experience (UX) and Interface Design	Immersive Storytelling and Roleplay
Accessibility and Inclusivity	
Immersion and Presence	
Historical and Cultural Accuracy	
Visual and Aesthetic Quality	
Interactivity and User Engagement	
Emotional and Educational Impact	VR Heritage Site Reconstructions

Table 2. Sets of experts.

Expert	Gender	Work experience	Job position
1	Male	18 years of experience	Virtual Reality Interactive
2	Male	20 years of experience	Virtual Reality Interactive
3	Male	23 years of experience	Virtual Reality Interactive
4	Female	15 years of experience	Virtual Reality Interactive

3.2 Model Implementation, Results Analysis and Discussion

After the determination of the criteria and alternatives, the next step is to obtain the data and information. The data and information are obtained using four experts. Four experts evaluated the criteria and alternatives using the terms of IVNSs as shown in Table 3. Table 4 shows the IVNNs for the criteria and alternatives. Then we apply the score function to obtain one matrix. Then we combined the decision matrix into one matrix.

Table 3. IVN values.

A	([0.1,0.2],[0.1,0.2],[0.8,0.9])
B	([0.2,0.3],[0.3,0.4],[0.7,0.8])
C	([0.3,0.4],[0.4,0.5],[0.6,0.7])
D	([0.4,0.5],[0.5,0.6],[0.5,0.6])
E	([0.5,0.5],[0.6,0.7],[0.4,0.5])
F	([0.5,0.6],[0.5,0.6],[0.4,0.5])
G	([0.6,0.7],[0.4,0.5],[0.3,0.4])
H	([0.7,0.8],[0.2,0.3],[0.2,0.3])
I	([0.8,0.9],[0.1,0.2],[0.1,0.2])

Table 4. The decision matrix.

E_1	A_1	A_2	A_3	A_4	A_5
C_1	([0.1,0.2],[0.1,0.2],[0.8,0.9])	([0.7,0.8],[0.2,0.3],[0.2,0.3])	([0.6,0.7],[0.4,0.5],[0.3,0.4])	([0.5,0.6],[0.5,0.6],[0.4,0.5])	([0.5,0.5],[0.6,0.7],[0.4,0.5])
C_2	([0.2,0.3],[0.3,0.4],[0.7,0.8])	([0.8,0.9],[0.1,0.2],[0.1,0.2])	([0.7,0.8],[0.2,0.3],[0.2,0.3])	([0.6,0.7],[0.4,0.5],[0.3,0.4])	([0.4,0.5],[0.5,0.6],[0.5,0.6])
C_3	([0.3,0.4],[0.4,0.5],[0.6,0.7])	([0.1,0.2],[0.1,0.2],[0.8,0.9])	([0.6,0.7],[0.4,0.5],[0.3,0.4])	([0.5,0.6],[0.5,0.6],[0.4,0.5])	([0.3,0.4],[0.4,0.5],[0.6,0.7])
C_4	([0.4,0.5],[0.5,0.6],[0.5,0.6])	([0.2,0.3],[0.3,0.4],[0.7,0.8])	([0.8,0.9],[0.1,0.2],[0.1,0.2])	([0.5,0.5],[0.6,0.7],[0.4,0.5])	([0.2,0.3],[0.3,0.4],[0.7,0.8])
C_5	([0.5,0.5],[0.6,0.7],[0.4,0.5])	([0.3,0.4],[0.4,0.5],[0.6,0.7])	([0.1,0.2],[0.1,0.2],[0.8,0.9])	([0.4,0.5],[0.5,0.6],[0.5,0.6])	([0.1,0.2],[0.1,0.2],[0.8,0.9])
C_6	([0.5,0.6],[0.5,0.6],[0.4,0.5])	([0.4,0.5],[0.5,0.6],[0.5,0.6])	([0.2,0.3],[0.3,0.4],[0.7,0.8])	([0.3,0.4],[0.4,0.5],[0.6,0.7])	([0.8,0.9],[0.1,0.2],[0.1,0.2])
C_7	([0.6,0.7],[0.4,0.5],[0.3,0.4])	([0.5,0.5],[0.6,0.7],[0.4,0.5])	([0.3,0.4],[0.4,0.5],[0.6,0.7])	([0.2,0.3],[0.3,0.4],[0.7,0.8])	([0.7,0.8],[0.2,0.3],[0.2,0.3])
C_8	([0.7,0.8],[0.2,0.3],[0.2,0.3])	([0.5,0.6],[0.5,0.6],[0.4,0.5])	([0.4,0.5],[0.5,0.6],[0.5,0.6])	([0.1,0.2],[0.1,0.2],[0.8,0.9])	([0.6,0.7],[0.4,0.5],[0.3,0.4])
C_9	([0.8,0.9],[0.1,0.2],[0.1,0.2])	([0.6,0.7],[0.4,0.5],[0.3,0.4])	([0.7,0.8],[0.2,0.3],[0.2,0.3])	([0.8,0.9],[0.1,0.2],[0.1,0.2])	([0.5,0.6],[0.5,0.6],[0.4,0.5])
C_{10}	([0.1,0.2],[0.1,0.2],[0.8,0.9])	([0.2,0.3],[0.3,0.4],[0.7,0.8])	([0.7,0.8],[0.2,0.3],[0.2,0.3])	([0.4,0.5],[0.5,0.6],[0.5,0.6])	([0.5,0.5],[0.6,0.7],[0.4,0.5])
E_2	A_1	A_2	A_3	A_4	A_5
C_1	([0.1,0.2],[0.1,0.2],[0.8,0.9])	([0.7,0.8],[0.2,0.3],[0.2,0.3])	([0.6,0.7],[0.4,0.5],[0.3,0.4])	([0.5,0.6],[0.5,0.6],[0.4,0.5])	([0.5,0.5],[0.6,0.7],[0.4,0.5])
C_2	([0.1,0.2],[0.1,0.2],[0.8,0.9])	([0.2,0.3],[0.3,0.4],[0.7,0.8])	([0.5,0.5],[0.6,0.7],[0.4,0.5])	([0.6,0.7],[0.4,0.5],[0.3,0.4])	([0.4,0.5],[0.5,0.6],[0.5,0.6])
C_3	([0.3,0.4],[0.4,0.5],[0.6,0.7])	([0.1,0.2],[0.1,0.2],[0.8,0.9])	([0.2,0.3],[0.3,0.4],[0.7,0.8])	([0.5,0.5],[0.6,0.7],[0.4,0.5])	([0.3,0.4],[0.4,0.5],[0.6,0.7])
C_4	([0.4,0.5],[0.5,0.6],[0.5,0.6])	([0.2,0.3],[0.3,0.4],[0.7,0.8])	([0.1,0.2],[0.1,0.2],[0.8,0.9])	([0.2,0.3],[0.3,0.4],[0.7,0.8])	([0.5,0.5],[0.6,0.7],[0.4,0.5])
C_5	([0.5,0.5],[0.6,0.7],[0.4,0.5])	([0.1,0.2],[0.1,0.2],[0.8,0.9])	([0.2,0.3],[0.3,0.4],[0.7,0.8])	([0.5,0.5],[0.6,0.7],[0.4,0.5])	([0.1,0.2],[0.1,0.2],[0.8,0.9])
C_6	([0.5,0.6],[0.5,0.6],[0.4,0.5])	([0.4,0.5],[0.5,0.6],[0.5,0.6])	([0.1,0.2],[0.1,0.2],[0.8,0.9])	([0.2,0.3],[0.3,0.4],[0.7,0.8])	([0.5,0.5],[0.6,0.7],[0.4,0.5])
C_7	([0.6,0.7],[0.4,0.5],[0.3,0.4])	([0.1,0.2],[0.1,0.2],[0.8,0.9])	([0.2,0.3],[0.3,0.4],[0.7,0.8])	([0.5,0.5],[0.6,0.7],[0.4,0.5])	([0.7,0.8],[0.2,0.3],[0.2,0.3])
C_8	([0.1,0.2],[0.1,0.2],[0.8,0.9])	([0.2,0.3],[0.3,0.4],[0.7,0.8])	([0.5,0.5],[0.6,0.7],[0.4,0.5])	([0.1,0.2],[0.1,0.2],[0.8,0.9])	([0.6,0.7],[0.4,0.5],[0.3,0.4])
C_9	([0.8,0.9],[0.1,0.2],[0.1,0.2])	([0.1,0.2],[0.1,0.2],[0.8,0.9])	([0.2,0.3],[0.3,0.4],[0.7,0.8])	([0.5,0.5],[0.6,0.7],[0.4,0.5])	([0.5,0.6],[0.5,0.6],[0.4,0.5])
C_{10}	([0.1,0.2],[0.1,0.2],[0.8,0.9])	([0.2,0.3],[0.3,0.4],[0.7,0.8])	([0.5,0.5],[0.6,0.7],[0.4,0.5])	([0.4,0.5],[0.5,0.6],[0.5,0.6])	([0.5,0.5],[0.6,0.7],[0.4,0.5])
E_3	A_1	A_2	A_3	A_4	A_5
C_1	([0.7,0.8],[0.2,0.3],[0.2,0.3])	([0.5,0.6],[0.5,0.6],[0.4,0.5])	([0.4,0.5],[0.5,0.6],[0.5,0.6])	([0.5,0.6],[0.5,0.6],[0.4,0.5])	([0.5,0.5],[0.6,0.7],[0.4,0.5])
C_2	([0.2,0.3],[0.3,0.4],[0.7,0.8])	([0.7,0.8],[0.2,0.3],[0.2,0.3])	([0.5,0.6],[0.5,0.6],[0.4,0.5])	([0.4,0.5],[0.5,0.6],[0.5,0.6])	([0.4,0.5],[0.5,0.6],[0.5,0.6])
C_3	([0.3,0.4],[0.4,0.5],[0.6,0.7])	([0.1,0.2],[0.1,0.2],[0.8,0.9])	([0.7,0.8],[0.2,0.3],[0.2,0.3])	([0.5,0.6],[0.5,0.6],[0.4,0.5])	([0.4,0.5],[0.5,0.6],[0.5,0.6])
C_4	([0.4,0.5],[0.5,0.6],[0.5,0.6])	([0.7,0.8],[0.2,0.3],[0.2,0.3])	([0.5,0.6],[0.5,0.6],[0.4,0.5])	([0.4,0.5],[0.5,0.6],[0.5,0.6])	([0.2,0.3],[0.3,0.4],[0.7,0.8])
C_5	([0.5,0.5],[0.6,0.7],[0.4,0.5])	([0.3,0.4],[0.4,0.5],[0.6,0.7])	([0.7,0.8],[0.2,0.3],[0.2,0.3])	([0.5,0.6],[0.5,0.6],[0.4,0.5])	([0.4,0.5],[0.5,0.6],[0.5,0.6])
C_6	([0.5,0.6],[0.5,0.6],[0.4,0.5])	([0.4,0.5],[0.5,0.6],[0.5,0.6])	([0.7,0.8],[0.2,0.3],[0.2,0.3])	([0.5,0.6],[0.5,0.6],[0.4,0.5])	([0.4,0.5],[0.5,0.6],[0.5,0.6])
C_7	([0.6,0.7],[0.4,0.5],[0.3,0.4])	([0.7,0.8],[0.2,0.3],[0.2,0.3])	([0.5,0.6],[0.5,0.6],[0.4,0.5])	([0.4,0.5],[0.5,0.6],[0.5,0.6])	([0.7,0.8],[0.2,0.3],[0.2,0.3])
C_8	([0.7,0.8],[0.2,0.3],[0.2,0.3])	([0.5,0.6],[0.5,0.6],[0.4,0.5])	([0.4,0.5],[0.5,0.6],[0.5,0.6])	([0.1,0.2],[0.1,0.2],[0.8,0.9])	([0.6,0.7],[0.4,0.5],[0.3,0.4])
C_9	([0.8,0.9],[0.1,0.2],[0.1,0.2])	([0.6,0.7],[0.4,0.5],[0.3,0.4])	([0.7,0.8],[0.2,0.3],[0.2,0.3])	([0.8,0.9],[0.1,0.2],[0.1,0.2])	([0.5,0.6],[0.5,0.6],[0.4,0.5])
C_{10}	([0.1,0.2],[0.1,0.2],[0.8,0.9])	([0.2,0.3],[0.3,0.4],[0.7,0.8])	([0.7,0.8],[0.2,0.3],[0.2,0.3])	([0.5,0.6],[0.5,0.6],[0.4,0.5])	([0.4,0.5],[0.5,0.6],[0.5,0.6])
E_4	A_1	A_2	A_3	A_4	A_5
C_1	([0.8,0.9],[0.1,0.2],[0.1,0.2])	([0.6,0.7],[0.4,0.5],[0.3,0.4])	([0.7,0.8],[0.2,0.3],[0.2,0.3])	([0.5,0.6],[0.5,0.6],[0.4,0.5])	([0.5,0.5],[0.6,0.7],[0.4,0.5])
C_2	([0.2,0.3],[0.3,0.4],[0.7,0.8])	([0.8,0.9],[0.1,0.2],[0.1,0.2])	([0.6,0.7],[0.4,0.5],[0.3,0.4])	([0.7,0.8],[0.2,0.3],[0.2,0.3])	([0.4,0.5],[0.5,0.6],[0.5,0.6])
C_3	([0.8,0.9],[0.1,0.2],[0.1,0.2])	([0.6,0.7],[0.4,0.5],[0.3,0.4])	([0.7,0.8],[0.2,0.3],[0.2,0.3])	([0.5,0.6],[0.5,0.6],[0.4,0.5])	([0.3,0.4],[0.4,0.5],[0.6,0.7])
C_4	([0.4,0.5],[0.5,0.6],[0.5,0.6])	([0.8,0.9],[0.1,0.2],[0.1,0.2])	([0.6,0.7],[0.4,0.5],[0.3,0.4])	([0.7,0.8],[0.2,0.3],[0.2,0.3])	([0.2,0.3],[0.3,0.4],[0.7,0.8])
C_5	([0.5,0.5],[0.6,0.7],[0.4,0.5])	([0.3,0.4],[0.4,0.5],[0.6,0.7])	([0.8,0.9],[0.1,0.2],[0.1,0.2])	([0.6,0.7],[0.4,0.5],[0.3,0.4])	([0.7,0.8],[0.2,0.3],[0.2,0.3])
C_6	([0.5,0.6],[0.5,0.6],[0.4,0.5])	([0.8,0.9],[0.1,0.2],[0.1,0.2])	([0.6,0.7],[0.4,0.5],[0.3,0.4])	([0.7,0.8],[0.2,0.3],[0.2,0.3])	([0.8,0.9],[0.1,0.2],[0.1,0.2])
C_7	([0.8,0.9],[0.1,0.2],[0.1,0.2])	([0.6,0.7],[0.4,0.5],[0.3,0.4])	([0.7,0.8],[0.2,0.3],[0.2,0.3])	([0.2,0.3],[0.3,0.4],[0.7,0.8])	([0.7,0.8],[0.2,0.3],[0.2,0.3])
C_8	([0.7,0.8],[0.2,0.3],[0.2,0.3])	([0.8,0.9],[0.1,0.2],[0.1,0.2])	([0.6,0.7],[0.4,0.5],[0.3,0.4])	([0.7,0.8],[0.2,0.3],[0.2,0.3])	([0.6,0.7],[0.4,0.5],[0.3,0.4])
C_9	([0.8,0.9],[0.1,0.2],[0.1,0.2])	([0.6,0.7],[0.4,0.5],[0.3,0.4])	([0.7,0.8],[0.2,0.3],[0.2,0.3])	([0.8,0.9],[0.1,0.2],[0.1,0.2])	([0.5,0.6],[0.5,0.6],[0.4,0.5])
C_{10}	([0.1,0.2],[0.1,0.2],[0.8,0.9])	([0.2,0.3],[0.3,0.4],[0.7,0.8])	([0.5,0.6],[0.5,0.6],[0.4,0.5])	([0.4,0.5],[0.5,0.6],[0.5,0.6])	([0.5,0.5],[0.6,0.7],[0.4,0.5])

3.3 Determination of Criteria Weights Using IVN-CRITIC Method

The IVN-CRITIC method is used to compute the criteria weights. We normalize the decision matrix as shown in Table 5. Then we Compute the correlation between criteria as shown in table 6. The correlation matrix shows a high correlation between criteria and a low correlation between criteria. There are positive and negative correlations between the criteria. Then we subtract one from the correlation matrix. Then we obtain the sum of each row and compute the criteria weights as shown in Figure 2.

Table 5. The normalization matrix by the IVN-CRITIC method.

	A_1	A_2	A_3	A_4	A_5
C_1	0.108434	1	0.783133	0.361446	0
C_2	0	1	0.799392	0.860182	0.416413
C_3	0.613546	0	1	0.705179	0.294821
C_4	0.473373	0.970414	1	0.686391	0
C_5	0.689441	0	0.956522	1	0.304348

C_6	0.380282	0.408451	0	0.105634	1
C_7	0.893836	0.363014	0.304795	0	1
C_8	0.833948	0.697417	0.538745	0	1
C_9	1	0	0.311475	0.688525	0
C_{10}	0	0.28169	1	0.650704	0.630986

Table 6. The correlation matrix.

	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8	C_9	C_{10}
C_1	1	0.788017	-0.11021	0.90724	-0.19444	-0.54649	-0.62788	-0.26338	-0.39699	0.219911
C_2	0.788017	1	-0.12628	0.649938	-0.0907	-0.39517	-0.80691	-0.55304	-0.5455	0.524795
C_3	-0.11021	-0.12628	1	0.217932	0.951961	-0.6463	-0.29369	-0.43935	0.539292	0.5164
C_4	0.90724	0.649938	0.217932	1	0.172479	-0.83905	-0.75425	-0.49434	0.007436	0.182784
C_5	-0.19444	-0.0907	0.951961	0.172479	1	-0.66517	-0.41213	-0.63174	0.655427	0.434529
C_6	-0.54649	-0.39517	-0.6463	-0.83905	-0.66517	1	0.789216	0.736426	-0.44621	-0.26256
C_7	-0.62788	-0.80691	-0.29369	-0.75425	-0.41213	0.789216	1	0.912958	-0.04846	-0.42611
C_8	-0.26338	-0.55304	-0.43935	-0.49434	-0.63174	0.736426	0.912958	1	-0.33636	-0.35406
C_9	-0.39699	-0.5455	0.539292	0.007436	0.655427	-0.44621	-0.04846	-0.33636	1	-0.39525
C_{10}	0.219911	0.524795	0.5164	0.182784	0.434529	-0.26256	-0.42611	-0.35406	-0.39525	1

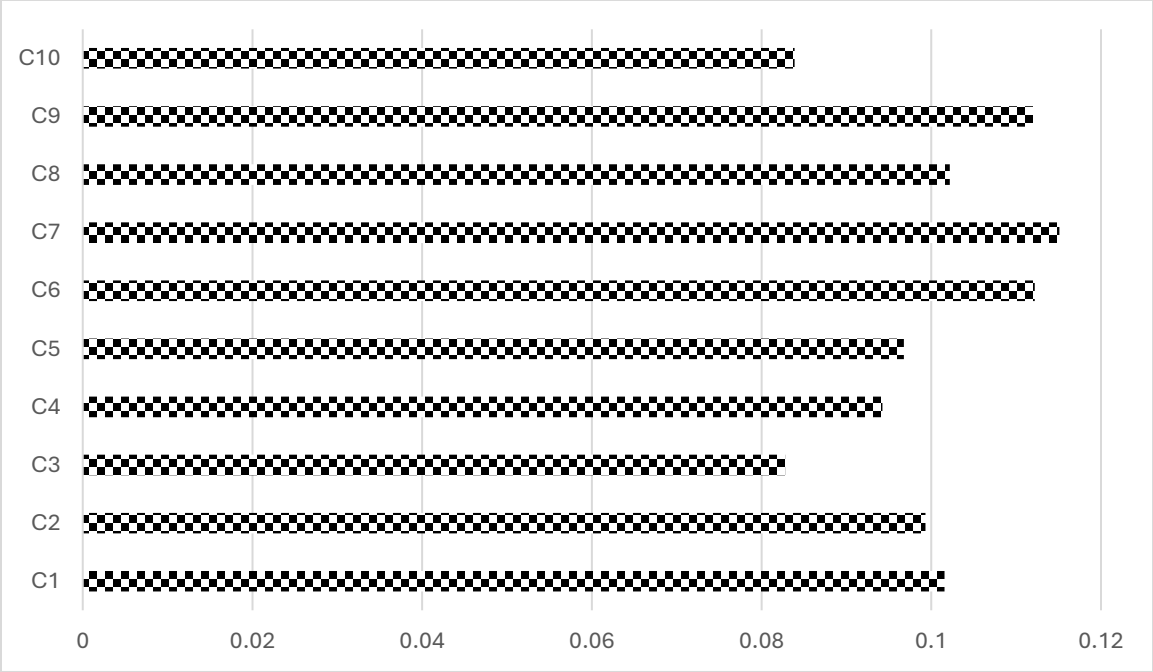

Fig. 2. Criteria weights.

3.4 Rank the alternatives Using the IVN-ARAS Method

The IVN-ARAS method is used to rank the alternatives. We start with the combined decision matrix between the criteria and alternatives. Eq. (16) is used to compute the normalized decision matrix as shown in Table 7. Then we compute the weighted normalized decision matrix using Eq. (17) as shown in Table 8.

Then we compute the optimal function using Eq. (18). Then we compute the utility degree using Eq. (19). Then we rank the alternatives as shown in Table 9.

Table 7. The normalization matrix by the IVN-ARAS method.

	A_1	A_2	A_3	A_4	A_5
C_1	0.176854	0.237164	0.222494	0.193969	0.169519
C_2	0.113021	0.254405	0.226042	0.234637	0.171895
C_3	0.210983	0.136802	0.257707	0.222062	0.172447
C_4	0.187882	0.227337	0.229685	0.204791	0.150305
C_5	0.208	0.1525	0.2295	0.233	0.177
C_6	0.200084	0.201765	0.177385	0.183691	0.237074
C_7	0.244775	0.182476	0.175643	0.139871	0.257235
C_8	0.225503	0.209671	0.191271	0.128798	0.244758
C_9	0.251046	0.165969	0.192469	0.224547	0.165969
C_{10}	0.101622	0.155676	0.293514	0.226486	0.222703

Table 8. The weighted normalized decision matrix.

	A_1	A_2	A_3	A_4	A_5
C_1	0.017961	0.024086	0.022596	0.019699	0.017216
C_2	0.011225	0.025267	0.02245	0.023304	0.017072
C_3	0.017464	0.011323	0.021331	0.018381	0.014274
C_4	0.017706	0.021425	0.021646	0.0193	0.014165
C_5	0.020127	0.014757	0.022208	0.022547	0.017128
C_6	0.02245	0.022638	0.019903	0.02061	0.0266
C_7	0.028174	0.021003	0.020217	0.0161	0.029608
C_8	0.02304	0.021423	0.019543	0.01316	0.025008
C_9	0.028111	0.018585	0.021552	0.025144	0.018585
C_{10}	0.008525	0.013059	0.024622	0.019	0.018682

Table 9. The rank of alternatives.

Alternatives	O_i	D_i	Rank
A_1	0.194784	0.901493	2
A_2	0.193567	0.895859	1
A_3	0.216068	1	5
A_4	0.197243	0.912875	3
A_5	0.198338	0.917942	4

4. Discussion, Validation and Sensitivity Analysis

This section shows the discussion of the results. The results show the criterion Historical and Cultural Accuracy has the highest weights and the Sustainability and Preservation criterion has the lowest weights. Then we applied the ARAS method to rank the alternatives. The results show the Virtual Museum Exhibits has the highest rank and the Educational VR Experiences has the lowest rank.

Then we applied the sensitivity analysis results to show different ranks of alternatives. We conducted 11 cases in the weight criteria. In the first case, we put all criteria with the same weights. In the second case, we increase the first criterion by 25% and other criteria have the same weights. In the third case, we increase the sconed criterion by 25% and other criteria have the same weights. Figure 3 shows the different cases in criteria weights.

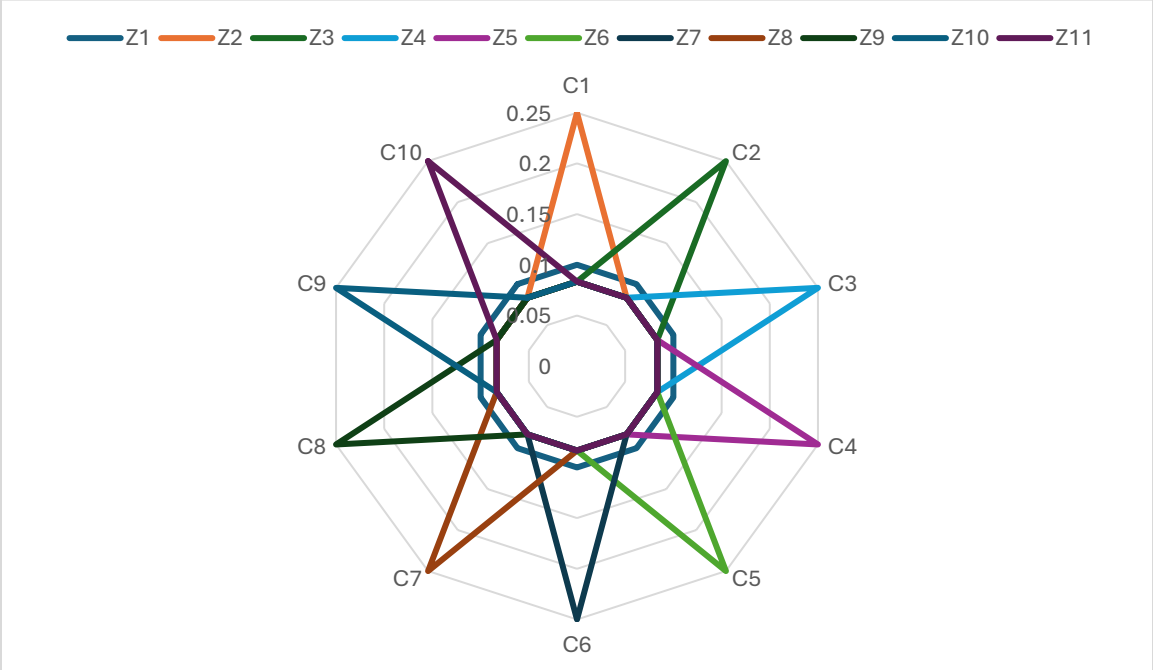

Fig. 3. Different criteria weights.

Then we rank the alternatives under these cases to show the stability of the results. The results show that alternative 3 is the best alternative in all cases. We show that cases 1,2,3,7,11 obtained alternative 1 is the worst alternative. Case 5 shows that alternative 5 is the worst alternative. The cases 4,6,10 show that alternative 2 is the worst. Cases 8 and 9 show that alternative 4 is the worst. We show the rank of alternative is stable under different cases. Table 10 shows the rank of alternatives.

Table 10. The rank of alternatives under different cases.

	Z_1	Z_2	Z_3	Z_4	Z_5	Z_6	Z_7	Z_8	Z_9	Z_{10}	Z_{11}
A_1	1	1	1	3	2	3	1	3	3	3	1
A_2	2	4	3	1	3	1	2	2	2	1	2
A_3	5	5	5	5	5	5	5	5	5	5	5
A_4	4	3	4	4	4	4	3	1	1	4	4
A_5	3	2	2	2	1	2	4	4	4	2	3

5. Conclusions

This study assesses the relationship between the criteria to establish criteria weights using IVN-CRITIC. Then, using the IVN-ARAS approach, alternatives are evaluated and ranked concerning the established

criteria. According to the study's specialists and research team, the suggested decision-making model would enable policymakers to reconsider comparable procedures and make sensible choices in real-world scenarios. The suggested model helps the organization create a knowledge-based window. The model's results can be used and compared to determine the strengths and weaknesses of various MCDM issues. The results show that the Historical and Cultural Accuracy criterion has the highest weights and Sustainability and Preservation have the lowest weights. Then the rank of alternatives is computed. The Virtual Museum Exhibits have the highest rank, and the Educational VR Experiences have the lowest rank.

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A Neutrosophic Model for Assessing the Impact of Artificial Intelligence on Civil Liability

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Abstract. The creation of Artificial Intelligence (AI) tools has grown exponentially in recent years. The number of these tools, their capabilities to solve problems and the number of people who use them are increasing rapidly. This constitutes a positive advance for humanity; however, there is also a negative side that can arise from its inappropriate use. To avoid the problems that may arise with the improper use of AI, it is necessary to have a regulatory legal framework that offers the guidelines to follow for evaluating the AI tools that arise and the civil and criminal consequences for those who fail to comply with what is established. This is an extremely complex problem since AI is constantly changing at a faster pace than lawyers understand the subject and the changes in the laws that must be applied. In our case, Ecuadorian legislation is not sufficiently prepared to face the problem, there is indeterminacy that manifests itself in the legal loopholes where some individuals can get away with it or be unfairly convicted. This paper proposes a model to help calculate the impact of AI on civil liability, especially in the legal system. Neutrosophic logic and other tools from this discipline are specifically used because they allow the explicit representation and calculation of the indeterminacy that is part of this problem that occurs in real life.

Keywords: Artificial Intelligence (AI), Civil Liability**,** Neutrosophic Logic, n- norm, n-conorm, neutrosophic existential quantifier, neutrosophic universal quantifier.

1 Introduction

Artificial Intelligence is mainly made up of computer programs that perform tasks that solve problems as natural intelligence such as human intelligence would do. Although this is a discipline with several decades of development, its use has now become popular, to the point of becoming part of our daily lives. The advantages that this discipline offers for the development of humanity and the personal support it provides to people in their daily and professional lives are undeniable.

However, there are dangers in its use that cannot be ignored. Artificial Intelligence can create advantages for some individuals over others because when it is misused, someone can be harmed. It can also be used to violate ethical principles or lend itself to criminal acts. That is why national and international regulatory frameworks are needed to guarantee the proper use of this tool. This regulatory framework is a challenge for the legislation of each country and international legislation since these are not crimes that are easy to establish.

Formally, the regulation of Artificial Intelligence consists of developing public sector policies and laws that promote and regulate the use of AI. This is an emerging issue, even in the so-called developed countries, where this type of tool is most developed, which usually requires technology and trained personnel for its generation. In particular, we have regulations developed in countries such as the United States and the members of the European Union. In addition, with the expansion of the Internet worldwide and other technologies that allow communication between individuals from different countries, it is essential to create laws within international organizations such as the IEEE or the OECD.

Regulation is seen as necessary to foster AI and also to manage the associated risks. Regulators should play a leading role in creating AI tools that adhere to trustworthiness principles, as well as take responsibility for

mitigating the risks associated with these tools. Regulation of AI through mechanisms such as review boards can also be seen as a social means of addressing the problem of AI governance.

There have been proposals for both "hard law" and "soft law" approaches to regulating AI. Some legal scholars have pointed out that "hard law" approaches to AI regulation have substantial challenges. Among the challenges is that AI technology is evolving rapidly, leading to a problem where traditional laws and regulations often cannot keep up with emerging applications, their associated risks and benefits. Similarly, the diversity of AI applications challenges existing regulatory agencies, which often have limited jurisdictional scope.

Alternatively, some legal scholars argue that soft law approaches to AI regulation hold promise because these laws can be more flexibly tailored to meet the needs of emerging and evolving AI technology and nascent applications. However, soft law approaches often lack substantial enforcement potential.

AI regulation is a positive social means of managing the problem of AI control in such a way as to ensure long-term beneficial AI. On the other hand, responses such as doing nothing or prohibiting it are impractical and even counterproductive. A useful strategy is the creation of national and international review committees, involving members of society, academics, scientists, and ethics committees specialized in the subject, among other groups of people, where the appropriate use of each AI tool is discussed.

There are currently regulatory gaps and legal uncertainties in Ecuador regarding the use of Artificial Intelligence. There is no totally clear legislation in our country where this problem is addressed effectively. Several factors cause this, including that it is a very new topic to deal with, it represents a challenge for Ecuadorian legal specialists because it has technological and scientific foundations that are not easy to understand, it has an ethical basis of considerable complexity, and there is considerable uncertainty about how each tool created can behave, which grows every day in quantity and complexity. However, Ecuador must prepare itself legally for this avalanche to guarantee a regulation that is fair for all.

The objective of this article is to design a model that measures how specific regulations in the Ecuadorian regulatory framework influence the attribution of civil liability derived from the use of Artificial Intelligence, considering technological advances and their potential impact on compliance with current legal rights and obligations. This model can be a way to support our legal representatives to organize the approach to this problem effectively. Likewise, the model is susceptible to automation and therefore to socialization, improvement, and interaction with users and society in general.

Neutrosophic Logic is an ideal tool [1-10]. It provides a proposition with the propositional calculation of a triad of values, one of truth, one of indeterminacy, and one of falsehood. This evaluation explicitly incorporates a degree of indeterminacy that exists in the legal loopholes that are seen in Ecuadorian laws. This is completed by the proven certainties of the capacity for justice that these legislations present, as well as the certainties of inability to resolve others.

This paper is divided into a Materials and Methods section below, where the main concepts of Neutrosophic Logic are explained. The section that follows the second one contains the presentation of the model and an illustrative example. The last section is devoted to giving conclusions.

2 Materials and Methods

This section is dedicated to exposing the main concepts of neutrosophic logic [1, 4].

Definition 1 ([1, 4, 11])**.** Given X, a universe of discourse containing elements or objects. A is a *neutrosophic set* if it has the form: $A = \{(x: T_A(x), I_A(x), F_A(x)),\ x \in X\}$, where $T_A(x), I_A(x), F_A(x) \subseteq]^-0, 1^+[$, i.e., they are three functions over either the standard or nonstandard subsets of $]^-0, 1^+[$. $T_A(x)$ represents the degree of membership of x to A, $I_A(x)$ represents its degree of indeterminacy, and $F_A(x)$ its degree of non-membership. They do not satisfy any restriction, i.e., $\forall x \in X,\ ^-0 \leq \inf T_A(x) + \inf I_A(x) + \inf F_A(x) \leq \sup T_A(x) + \sup I_A(x) + \sup F_A(x) \leq 3^+$.

Another particular definition is that of a *single-valued neutrosophic set*, which is formally defined as follows:

Definition 2 ([1, 4, 11])**.** Given X, a universe of discourse that contains elements or objects. A is a *single-valued neutrosophic set* (SVNS) if it has the form: $A = \{(x: T_A(x), I_A(x), F_A(x)), x \in X\}$, where $T_A(x), I_A(x), F_A(x) \in [0, 1]$. $T_A(x)$ represents the degree of membership of x to A, $I_A(x)$ represents its degree of indeterminacy, and $F_A(x)$ its degree of non-membership. $\forall x \in X, 0 \leq T_A(x) + I_A(x) + F_A(x) \leq 3$.

So, SVNS is derived from the definition of neutrosophic sets.

In neutrosophic set theory, a lattice can be defined as follows:

Given the universe of discourse X and $x(T_x, I_x, F_x)$, $y(T_y, I_y, F_y)$ two SVNS, we say that $x \leq_N y$ if and only if $T_x \leq T_y$, $I_x \geq I_y$ and $F_x \geq F_y$, $(X, \leq_N)$ is a poset. Whereas, $(L, \wedge, \vee)$ is a lattice, because it is a triple direct product of lattices [11, 12]. $.x \wedge y = (\min\{T_x, T_y\},\ \max\{I_x, I_y\}, \max\{F_x, F_y\})$ and $x \vee y = (\max\{T_x, T_y\}, \min\{I_x, I_y\}, \min\{F_x, F_y\})$. Moreover, it is easy to prove that it is complete.

Let us remark that this definition is valid for interval-valued neutrosophic sets when we substitute their operators with interval-valued operators.

See also that there exist two special elements, viz., $0_N = (0, 1, 1)$ and $1_N = (1, 0, 0)$, which are the infimum and the supremum, respectively, of every SVNS concerning $\leq_N$.

Given two neutrosophic sets, A and B, three basic operations over them are the following, [1, 4 ,11]:

1. $A \cap B = A \wedge B$ (Conjunction),

2. $A \cup B = A \vee B$ (Disjunction),

3. $\overline{A} = (F_A, 1 - I_A, T_A)$ (Complement).

Definition 3 ([11, 13, 14])**.** A *neutrosophic norm* or *n-norm* N_n, is a mapping N_n: $]^-0, 1^+[\times]^-0, 1^+[\times]^-0, 1^+[^2 \rightarrow]^-0, 1^+[\times]^-0, 1^+[\times]^-0, 1^+[$, such that $N_n\left(x(T_x, I_x, F_x), y(T_y, I_y, F_y)\right) = (N_nT(x, y), N_nI(x, y), N_nF(x, y))$, where N_nT means the degree of membership, N_nI the degree of indeterminacy, and N_nF the degree of non-membership of the conjunction of both, x and y.

For every x, y, and z belonging to $]^-0, 1^+[\times]^-0, 1^+[\times]^-0, 1^+[$, N_n must satisfy the following axioms:

1. $N_n(x, 0_N) = 0_N$ and $N_n(x, 1_N) = x$ (Boundary conditions),
2. $N_n(x, y) = N_n(y, x)$ (Commutativity),
3. If $x \leq_N y$, then $N_n(x, z) \leq_N N_n(y, z)$ (Monotonicity),
4. $N_n(N_n(x, y), z) = N_n(x, N_n(y, z))$ (Associativity).

Definition 4 ([11, 13, 14])**.** A *neutrosophic conorm* or *n- conorm* N_c, is a mapping N_c: $]^-0, 1^+[\times]^-0, 1^+[\times]^-0, 1^+[^2 \rightarrow]^-0, 1^+[\times]^-0, 1^+[\times]^-0, 1^+[$, such that $N_c\left(x(T_x, I_x, F_x), y(T_y, I_y, F_y)\right) = (N_cT(x, y), N_cI(x, y), N_cF(x, y))$, where N_cT means the degree of membership, N_cI the degree of indeterminacy and N_cF the degree of non-membership of the disjunction of x with y.

For every x, y, and z belonging to $]^-0, 1^+[\times]^-0, 1^+[\times]^-0, 1^+[$, N_c must satisfy the following axioms:

1. $N_c(x, 0_N) = x$ and $N_c(x, 1_N) = 1_N$ (Boundary conditions).
2. $N_c(x, y) = N_c(y, x)$(Commutativity).
3. If $x \leq_N y$, then $N_c(x, z) \leq_N N_c(y, z)$(Monotonicity).
4. $N_c(N_c(x, y), z) = N_c(x, N_c(y, z))$(Associativity).

A Singled-valued neutrosophic negator is defined as follows:

Definition 5 ([11, 13, 14])**.** A *single-valued Neutrosophic negator* is a decreasing unary neutrosophic operator N_N: $[0, 1]^3 \longrightarrow [0, 1]^3$, satisfying the following boundary conditions:

1. $N_N(0_N) = 1_N$.
2. $N_N(1_N) = 0_N$.

It is called *involutive* if and only if $N_N(N_N(x)) = x$ for every $x \in [0, 1]^3$.

In the following, we show the neutrosophic negators that we shall consider here, extracted from the literature [11]. Given an SVNS A (T_A, I_A, F_A), we have:

1. $N_N((T_A, I_A, F_A)) = (1 - T_A, 1 - I_A, 1 - F_A)$, $N_N((T_A, I_A, F_A)) = (1 - T_A, I_A, 1 - F_A)$, $N_N((T_A, I_A, F_A)) = (F_A, I_A, T_A)$ and $N_N((T_A, I_A, F_A)) = (F_A, 1 - I_A, T_A)$ (Involutive negators).
2. $N_N((T_A, I_A, F_A)) = \left(F_A, \frac{F_A + I_A + T_A}{3}, T_A\right)$ and $N_N((T_A, I_A, F_A)) = \left(1 - T_A, \frac{F_A + I_A + T_A}{3}, 1 - F_A\right)$ (Non-involutive deniers).

Definition 6 ([11, 15, 16]). A *single-valued neutrosophic implicator* is an operator I_N: $[0, 1]^3 \times [0, 1]^3 \rightarrow [0, 1]^3$ which satisfies the following conditions, for all $x, x', y, y' \in [0, 1]^3$:

1. If $x' \leq_N x$, then $I_N(x, y) \leq_N I_N(x', y)$,
2. If $y \leq_N y'$, then $I_N(x, y) \leq_N I_N(x, y')$,
3. $I_N(0_N, 0_N) = I_N(0_N, 1_N) = I_N(1_N, 1_N) = 1_N$,

4. $I_N(1_N, 0_N) = 0_N$.

Herein we use the term neutrosophic implicator or n- implicator to mean single-valued Neutrosophic implicator.

It can satisfy the following properties for every $x, y, z \in [0,1]^3$:

1. $I_N(1_N, x) = x$ (Neutrality principle),
2. $I_N(x, y) = I_N(N_{IN}(y), N_{IN}(x))$, where $N_{IN}(x) = I_N(x, 0_N)$ is an n-negator (Contrapositivity),
3. $I_N(x, I_N(y, z)) = I_N(y, I_N(x, z))$ (Interchangeability principle),
4. $x \leq_N y$ if and only if $I_N(x, y) = 1_N$ (Confinement principle),
5. I_N is a continuous mapping (Continuity).

Given the triple $n = (t, i, f)$ we could need to convert n into a real value using the following formula of the *Score function* [17]:

$$\mathcal{S}(n) = \frac{2+t-i-f}{3} \qquad (1)$$

3 The Model

This section is dedicated to presenting the basis of the proposed model. To do this, the three basic variables for carrying out the measurements are defined, which are summarized in Table 1:

Variable	Dimension
Legal regulation	Clear laws
	Regulatory ambiguities
	Legal loopholes
Civil liability	AI-based cases solved
	Areas where there are no precedents
	Cases where liability could not be established
Social and ethical impact	Recognized benefits
	Ethical or moral dilemmas still without consensus
	Potential damages

Table 1. Variables used in the model and their dimensions.

The variable "Legal regulation" denoted by *R* corresponds to the triad $NR = (T_R, I_R, F_R)$ defined as follows:

T_R: Proportion of clear laws,

I_R: Normative ambiguities,

F_R: Legal loopholes.

The variable "Civil liability" denoted by *C* corresponds to the triad $NC = (T_C, I_C, F_C)$ defined as follows:

T_C: Proportion of cases solved based on AI,

I_C: Areas where there are no precedents,

F_C: Proportion of cases in which liability could not be established.

The variable "Social and ethical impact" denoted by *S* corresponds to the triad $NS = (T_S, I_S, F_S)$ defined as follows:

T_S: Recognized benefits,

I_S: Ethical or moral dilemmas still without consensus,

F_S: Potential damage.

These variables are further broken down as follows:

Suppose *n* is the number of civil laws in Ecuador, let us denote each of them by l_i with $i = 1, 2, \dots, n$. Then let us denote by $NR_i = (T_{R_i}, I_{R_i}, F_{R_i})$ the variable "Legal regulation" for the ith civil law; $NC_i = (T_{C_i}, I_{C_i}, F_{C_i})$ the results of "Civil liability" for the ith law and finally $NS_i = (T_{S_i}, I_{S_i}, F_{S_i})$ is the result of measuring the "Social and ethical impact" for the ith civil law.

Note that with the help of the benefits of neutrosophy, we can have triads such as $(1,0,0)$ as meaning true, $(0,0,1)$ meaning false, and also $(0,1,0)$ meaning indeterminate, and so on.

One of the challenges of the proposed model is the initialization and updating of the values of the proposed triads. The variable *C* is measured objectively by the cases obtained; however, it is necessary to update it. For this reason, the solution of conflicts related to this variable in each of the civil laws should be included in a database.

For *R* and *S*, they are initialized and updated subjectively, with the help of a group of expert lawyers specialized in civil law. Let us denote these experts by e_j with $j = 1, 2, \dots, m$. Then, each of them must give an

evaluation at a given time on these variables, let us denote it by $NR_{ij} = \left(T_{R_{ij}}, I_{R_{ij}}, F_{R_{ij}}\right)$ and $NS_{ij} = \left(T_{S_{ij}}, I_{S_{ij}}, F_{S_{ij}}\right)$. Let us also consider that the weight or level of expertise of each of them is measured by the following table:

Level of Expertise	**Associated weight (w_j)**
Very Low	0.1
Low	0.3
Middle	0.5
High	0.7
Very High	0.9

Table 2. Relationship between the level of expertise on a linguistic scale and its associated weight in a real numerical value.

That is, each of the experts is evaluated in terms of knowledge and work experience concerning the variables *R* and *S* at the time the model is updated. It is preferred to consult experts with "High" and "Very High" levels of expertise. Then the following result is obtained for each of the laws calculated as the weighted arithmetic means:

$$NR_i = \left(\frac{\sum_{j=1}^{m} w_j T_{R_{ij}}}{\sum_{j=1}^{m} w_j}, \frac{\sum_{j=1}^{m} w_j I_{R_{ij}}}{\sum_{j=1}^{m} w_j}, \frac{\sum_{j=1}^{m} w_j F_{R_{ij}}}{\sum_{j=1}^{m} w_j}\right) \quad (2)$$

$$NS_i = \left(\frac{\sum_{j=1}^{m} w_j T_{S_{ij}}}{\sum_{j=1}^{m} w_j}, \frac{\sum_{j=1}^{m} w_j I_{S_{ij}}}{\sum_{j=1}^{m} w_j}, \frac{\sum_{j=1}^{m} w_j F_{S_{ij}}}{\sum_{j=1}^{m} w_j}\right) \quad (3)$$

For greater convenience in automating the model, these results are placed in a neutrosophic matrix (which can be a database) denoted by NM of order $n \times 3$, such that the ith component corresponds to the ith law evaluated in each column by the values NR_i, NS_i and NC_i in that order.

To perform the evaluation, two logical predicates are used, which are defined in formulas 4 and 5.

$$N = \forall_{i \in V}(NR_i \wedge_N NS_i) \wedge_N (NR_i \wedge_N NS_i \rightarrow_N NC_i) \quad (4)$$

$$P = \exists_{i \in V}(NR_i \wedge_N NS_i) \wedge_N (NR_i \wedge_N NS_i \rightarrow_N NC_i) \quad (5)$$

Let us explain each of these equations:

Both equations are defined for laws belonging to *V* which is the set of laws applicable to cases involving the use of AI. This prevents the predicate in Equation 4 from being evaluated too lowly. On the other hand, the predicate in Equation 4 is interpreted as a *modus ponens* variant for all laws in the set *V* and which are necessarily useful in "Civil Liability". The predicate defined in Equation 5 is a measure of the possibility that a law exists that beneficially influences "Civil Liability".

Aside from that, let us remember that $\wedge_N$ is the neutrosophic conjunction, $\rightarrow_N$ is the neutrosophic implication, $\forall_{i \in V} x_i = \wedge_{N\, i \in V}\, x_i$ and $\exists_{i \in V} x_i = \vee_{N\, i \in V}\, x_i$, where each x_i is a triad.

Now let us illustrate the model with an example.

Example 1. Let us suppose that Ecuadorian civil laws numbered 10, 15, and 24 are of interest to be applied in litigation related to the use of AI. After a thorough study of cases, the following numbers have been established:

$T_{C10} = 0.4$, $I_{C10} = 0.4$, $F_{C10} = 0.2$;

$T_{C15} = 0.6$, $I_{C15} = 0.2$, $F_{C15} = 0.2$;

$T_{C24} = 0.3$, $I_{C24} = 0.1$, $F_{C24} = 0.6$.

That is, we have $NC_{10} = (0.4,0.4,0.2)$, $NC_{15} = (0.6,0.2,0.2)$ and $NC_{24} = (0.3,0.1,0.6)$. As noted above, these values are obtained objectively.

Suppose then that the set of laws relating to the use of AI is $V = \{10, 15, 24\}$.

Now, let us take the set $E = \{e_1, e_2, e_3\}$, and each of them is evaluated as "High", "Very High", and "High" in their level of expertise, respectively. So, according to Table 2, the following weights are obtained, $w_1 = w_3 = 0.7$ and $w_2 = 0.9$.

The evaluations of each expert for the variables NR_i and NS_i are shown in Table 3.

Dimension/Expert	$\mathbf{e_1}$	$\mathbf{e_2}$	$\mathbf{e_3}$	Aggregated values
NR_{10}	(0.33, 0.25, 0.42)	(0.06, 0.73, 0.21)	(0.74, 0.09, 0.18)	(0.35, 0.39, 0.26)
NR_{15}	(0.36, 0.44, 0.2)	(0.29, 0.5, 0.21)	(0.37, 0.24, 0.4)	(0.34, 0.40, 0.26)
NR_{24}	(0.29, 0.45, 0.26)	(0.29, 0.09, 0.62)	(0.44, 0.2, 0.36)	(0.34, 0.23, 0.43)
NS_{10}	(0.6, 0.01, 0.4)	(0.13, 0.62, 0.25)	(0.47, 0.08, 0.45)	(0.38, 0.27, 0.36)
NS_{15}	(0.46, 0.12, 0.43)	(0.51, 0.27, 0.22)	(0.1, 0.52, 0.38)	(0.37, 0.30, 0.33)
NS_{24}	(0.17, 0.38, 0.45)	(0.59, 0.12, 0.29)	(0.07, 0.35, 0.58)	(0.30, 0.27, 0.42)

Table 3. Expert evaluation table for the variables "Legal regulation" and "Social and ethical impact". The last column shows the aggregated values using Equations 2 and 3.

Therefore, the resulting neutrosophic matrix NM is as follows:

$$NM = \begin{bmatrix} \vdots & \vdots & \vdots \\ (0.35, 0.39, 0.26) & (0.38, 0.27, 0.36) & (0.4,0.4,0.2) \\ \vdots & \vdots & \vdots \\ (0.34, 0.40, 0.26) & (0.37, 0.30, 0.33) & (0.6,0.2,0.2) \\ \vdots & \vdots & \vdots \\ (0.34, 0.23, 0.43) & (0.30, 0.27, 0.42) & (0.3,0.1,0.6) \\ \vdots & \vdots & \vdots \end{bmatrix}$$

The three components of Equations 4 and 5 to each of the laws are as follows as shown in Table 4:

Law/Predicate	$(\boldsymbol{NR_i} \wedge_N \boldsymbol{NS_i})$	$(\boldsymbol{NR_i} \wedge_N \boldsymbol{NS_i} \rightarrow_N \boldsymbol{NC_i})$	$(\boldsymbol{NR_i} \wedge_N \boldsymbol{NS_i}) \wedge_N (\boldsymbol{NR_i} \wedge_N \boldsymbol{NS_i} \rightarrow_N \boldsymbol{NC_i})$
l_{10}	(0.35, 0.39, 0.36)	(0.4, 0.39, 0.2)	(0.35, 0.39, 0.36)
l_{15}	(0.34, 0.40, 0.33)	(0.6, 0.2, 0.2)	(0.34, 0.40, 0.33)
l_{24}	(0.34, 0.27, 0.43)	(0.43, 0.1, 0.34)	(0.34, 0.27, 0.43)

Table 4. Components of Equations 3 and 4 are broken down.

Let us observe that the norm $(T_1, I_1, F_1) \wedge_N (T_2, I_2, F_2) = (\min(T_1, T_2), \max(I_1, I_2), \max(F_1, F_2))$ was used, the n-conorm $(T_1, I_1, F_1) \vee_N (T_2, I_2, F_2) = (\max(T_1, T_2), \min(I_1, I_2), \min(F_1, F_2))$, the neutrosophic negation $\neg_N(\mathrm{T}, \mathrm{I}, \mathrm{F}) = (\mathrm{F}, \mathrm{I}, \mathrm{T})$, and the neutrosophic implication $(T_1, I_1, F_1) \rightarrow_N (T_2, I_2, F_2) = \neg_N(T_1, I_1, F_1) \vee_N (T_2, I_2, F_2)$.

Finally, formulas 4 and 5 yield to:

$N = (0.34, 0.40, 0.43)$ and $P = (0.35, 0.27, 0.33)$. Both formulas can be converted to real numerical values using the score function (1), resulting in $\mathcal{S}(N) = 0.503$ and $\mathcal{S}(P) = 0.583$. It means that the possibility and need for having laws to resolve conflicts related to AI is more or less medium, neither low nor high.

Conclusion

This paper is dedicated to proposing a neutrosophic logic model to deal with the problem of legal disputes related to the issue of the inappropriate use of AI. The case of Ecuador is discussed. Specifically, two indices are calculated to measure how "Legal Regulation" and "Social and Ethical Impact" influence "Civil Liability". The indices indicate the need for this predicate and the existence of laws for this. We base ourselves on neutrosophic logic because it extends fuzzy logic to types of knowledge that are semantically impossible to represent in the preceding extensions of fuzzy logic. Specifically, the indeterminacy component is treated independently of the other components. This is a simple model to implement in software and if its automation is carried out, it will allow its extensive use, in addition to its improvement.

In future works, we will consider complesizing the modelization of this issue, where neutrosophic measures for legal reasoning are used [18, 19, 20]. This will allow users for easier updating of the matrix data NM, although, on the other hand, it will require the definition of neutrosophic measures, which will allow capturing the opinions of experts, including the evaluation of laws viewed as systems and not as isolated components. However, this may make the model more difficult to implement.

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Adaptive Triangular Linguistic Neutrosophic Cubic Fuzzy Sets for College English Blended Teaching Mode Evaluation Method

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Abstract

This paper presents a proactive strategy that employs a multiple-criteria decision-making (MCDM) model as an evaluation tool. To assess online and offline blended teaching modes for college English, we propose an evaluation framework called CIMAS-COBRA, which integrates the Criteria Importance Assessment (CIMAS) method for determining criteria weights and the Cost Estimation, Benchmarking, and Risk Assessment (COBRA) method for ranking alternatives. Both methods operate under the Triangular Linguistic Neutrosophic Cubic Fuzzy Sets framework to handle uncertain and vague data. Seven criteria and ten alternatives are examined in this study. The Triangular Linguistic Neutrosophic Cubic Fuzzy Number approach is employed to evaluate these criteria and alternatives.

Keywords: Triangular Linguistic Neutrosophic Cubic Fuzzy Sets; CIMAS Method; COBRA Method; Uncertainty.

1. Introduction and Related Work

One of the main challenges in evaluating the blended teaching mode for college English is the inherent uncertainty and complexity of the decision-making process. Multiple factors such as teaching quality, learner satisfaction, resource availability, and technological support—must be considered simultaneously, making it difficult to arrive at an objective and comprehensive assessment. To address these challenges, researchers have turned to advanced fuzzy set theories, including Triangular Linguistic Neutrosophic Cubic Fuzzy Sets, which provide sophisticated tools for handling vagueness, indeterminacy, and inconsistent data.

The intuitionistic fuzzy set (IFS), introduced by Atanassov, extends the traditional fuzzy set by considering both membership and non-membership degrees, thereby handling hesitancy and vagueness more effectively. Jianhua et al. noted that several water techniques, including membrane-assisted technologies, have been proposed and successfully applied in various locations in recent years [1], [2]. Moreover, a new hybrid fuzzy multiple-criteria decision-making (MCDM) model incorporating the work of Gülçin and Çifçi has also been introduced.

Cubic sets, which generalize both fuzzy sets and intuitionistic fuzzy sets, were later introduced by Jun et al. These sets feature two representations: one for the degree of membership and another for the degree of non-membership [3], [4]. In this approach, the non-membership aspect is considered alongside a membership function that is expressed in the form of an interval.

Building on these developments, Wang et al. recently proposed a single-valued neutrosophic set—a subclass of the neutrosophic set introduced by Smarandache. This set not only handles inconsistent,

indeterminate, and incomplete data but also independently expresses truth-membership, indeterminacy-membership, and falsity-membership degrees [5], [6]. Given the imperfections in human knowledge and perception, the elements defined by the single-valued neutrosophic set are particularly well-suited for modeling human reasoning.

Since indeterminacy reflects the uncertainty between truth and untruth, the human brain often cannot produce definitive "yes" or "no" answers. For example, when considering a statement like "Movie X would be a hit," the intuitionistic fuzzy set falls short in adequately addressing indeterminacy and inconsistent information, whereas neutrosophic components offer a more effective solution [7], [8]. Consequently, the single-valued neutrosophic set has rapidly gained popularity and found numerous applications.

Interval-valued fuzzy sets further enhance decision-making by providing a broader range of uncertain data compared to conventional fuzzy sets, as both membership and non-membership can be evaluated more flexibly. The concept of interval-valued triangular linguistic neutrosophic fuzzy sets was introduced because, in real-world decision-making, it is crucial to consider the degree of membership and non-membership that an option meets for a specific criterion—across all interval values [9], [10].

In recent years, blended teaching has become a popular approach in college English courses by integrating traditional classroom instruction with online learning. This mode offers flexibility and increased interactivity; however, it also presents challenges when evaluating overall effectiveness. Conventional evaluation models often rely on precise, crisp data and may not fully capture the dynamic and uncertain nature of blended learning environments [16]. Factors such as varying student engagement, diverse teaching practices, and the unpredictable performance of online components contribute to significant uncertainty in these assessments [17].

To better manage this complexity, researchers have adopted neutrosophic sets—a mathematical framework that extends traditional fuzzy set theory by explicitly incorporating a measure for indeterminacy. Unlike classical fuzzy sets, which only account for membership and non-membership degrees, neutrosophic sets enable evaluators to quantify the uncertainty and incomplete information that frequently arise in educational settings [18]. This additional component makes neutrosophic approaches particularly suitable for evaluating blended teaching modes, where both quantitative and qualitative factors must be integrated.

Several studies have applied neutrosophic set theory to improve educational assessments. For example, Chen and Zhao [19] proposed a neutrosophic-based evaluation model specifically for college English courses, demonstrating that this method yields more reliable assessments by capturing nuances that conventional models overlook. Li and Zhao [20] developed a hybrid framework that integrates neutrosophic concepts to effectively process diverse student feedback and instructor performance data. In another study, Zhang and Xu [21] employed single-valued neutrosophic sets to distinguish between different levels of teaching effectiveness, thereby providing clearer insights into course quality.

Additional research has extended these findings to broader decision-making processes in education. Garcia and Patel [22] explored advanced decision-making models incorporating neutrosophic sets, which revealed their ability to manage uncertainty in higher education evaluations. Kim and Park [23] offered a comprehensive review of fuzzy logic applications in blended learning assessment, emphasizing the advantages of including neutrosophic approaches. Similarly, Nour and Ali [24] demonstrated the efficacy of neutrosophic methods in evaluating both the online and offline components of blended courses, highlighting a reduction in the negative effects of ambiguous data.

Other studies have also made significant contributions. Hassan and Ibrahim [25] provided a neutrosophic perspective on blended learning evaluation, showing that this approach enhances the accuracy of assessments. Santos and Costa [26] introduced a decision-making model based on neutrosophic theory to address inconsistencies in teaching evaluations, while Lee and Wang [27] demonstrated that using neutrosophic fuzzy sets can significantly improve the analysis of complex behavioral data in blended classroom settings.

A systematic review by Miller and Davis [28] confirmed the growing trend of applying neutrosophic logic in educational research and its effectiveness in handling indeterminate information. Gupta and Verma [29] offered evidence that advanced neutrosophic models provide clearer insights into the effectiveness of blended learning. Kim and Choi [30] further refined evaluation methods by integrating neutrosophic set theory, resulting in significant improvements in data interpretation.

The decision-making aspect of blended teaching evaluation has also received considerable attention. El-Hawary and Fouad [31] examined how neutrosophic logic can reduce uncertainty in educational decisions, while Peters and Singh [32] applied a neutrosophic Multiple Criteria Decision Making (MCDM) framework to integrate various performance indicators. Oliveira and Mendes [33] provided empirical support for the use of neutrosophic set theory in capturing the multifaceted nature of blended teaching environments. Ahmed and Qureshi [34] conducted a comparative study that found evaluation models based on neutrosophic sets offer a more robust assessment of blended teaching. Finally, Rodriguez and Thompson [35] discussed recent advancements in neutrosophic set-based evaluation models and emphasized their potential to transform teaching assessments in higher education.

In summary, the literature indicates that incorporating neutrosophic sets into the evaluation of college English blended teaching modes provides a comprehensive approach that effectively addresses uncertainty and indeterminacy. This advanced methodology not only improves the reliability of assessments but also offers valuable insights for the continuous enhancement of teaching practices in modern educational situations.

1.1. Research Objectives and Contributions

The primary objective of this study is to develop a robust evaluation framework for assessing online and offline blended teaching modes in college English education. The study aims to introduce an advanced decision-making approach that integrates the CIMAS method for computing criteria weights and the COBRA method for ranking alternatives, all within the framework of TLNCFSs. This research contributes to the field by offering a novel hybrid model that effectively captures the uncertainty and vagueness inherent in educational assessments. Furthermore, by incorporating expert evaluations, the study ensures that the proposed method is both practical and applicable in real-world educational settings. The findings provide valuable insights for educators and administrators in making informed decisions about blended teaching methodologies, ultimately leading to improved teaching strategies and student learning outcomes.

1.2. Identified Research Gaps

While several methods have been proposed for evaluating blended teaching models, many existing approaches rely on conventional decision-making techniques that do not fully address the challenges of uncertainty and subjective assessment. Most previous studies have used traditional fuzzy logic or intuitionistic fuzzy methods, which cannot explicitly handle indeterminacy. Additionally, few studies have

explored the integration of neutrosophic sets with multiple-criteria decision-making (MCDM) techniques to enhance the accuracy and reliability of evaluations. Another gap in the literature is the limited focus on practical implementations of advanced fuzzy models in education. Many studies remain theoretical without real-world validation through expert-based evaluations. This study fills these gaps by introducing a comprehensive hybrid model that not only applies neutrosophic principles but also integrates expert decision-making, ensuring a more practical and effective assessment of blended teaching environments.

2. Preliminaries

In this section, we introduce the fundamental concepts behind Triangular Linguistic Neutrosophic Cubic Fuzzy Sets (TLNCFSs) [11], [12], [13]. TLNCFSs extend traditional fuzzy set theory by incorporating both linguistic variables and neutrosophic logic, which helps in capturing uncertainty, indeterminacy, and conflicting information more effectively. We will explain how these sets are constructed, discuss their main components, and highlight their advantages in handling complex decision-making scenarios. The ideas presented here are based on the key studies cited in references [11], [12], and [13], which provide a detailed foundation for understanding TLNCFSs.

Definition 1

The interval-valued triangular linguistic neutrosophic fuzzy sets can be defined as:

$$x_b(h) = \begin{cases} S_\theta, \left[\frac{(h-r)}{(s-r)} F_b^-, F_b^+\right], & (r \le h < s) \\ [F_b^-, F_b^+], & h = s, s \\ S_\theta, \left[\frac{(t-h)}{(t-s)} F_b^-, F_b^+\right], & (s \le h < t) \\ 0, & otherwise \end{cases} \tag{1}$$

$$y_b(h) = \begin{cases} S_\theta, \left[\frac{(h-r)}{(s-r)} G_b^-, G_b^+\right], & (r \le h < s) \\ [G_b^-, G_b^+], & h = s, s \\ S_\theta, \left[\frac{(t-h)}{(t-s)} G_b^-, G_b^+\right], & (s \le h < t) \\ 0, & otherwise \end{cases} \tag{2}$$

$$x_b(h) = \begin{cases} S_\theta, \left[\frac{(h-r)}{(s-r)} I_b^-, I_b^+\right], & (r \le h < s) \\ [I_b^-, I_b^+], & h = s, s \\ S_\theta, \left[\frac{(t-h)}{(t-s)} I_b^-, I_b^+\right], & (s \le h < t) \\ 0, & otherwise \end{cases} \tag{3}$$

The triangular linguistic neutrosophic fuzzy sets can be defined as:

$$c_b(h) = \begin{cases} S_\theta, \left[\frac{(h-r)}{(s-r)} F_b\right], & (r \le h < s) \\ F_b, & h = s, s \\ S_\theta, \left[\frac{(t-h)}{(t-s)} F_b\right], & (s \le h < t) \\ 0, & otherwise \end{cases} \tag{4}$$

$$d_b(h) = \begin{cases} S_\theta, \left[\frac{(h-r)}{(s-r)} G_b\right], & (r \le h < s) \\ G_b, \quad h = s, s \\ S_\theta, \left[\frac{(t-h)}{(t-s)} G_b\right], & (s \le h < t) \\ 0, \quad otherwise \end{cases} \tag{5}$$

$$e_b(h) = \begin{cases} S_\theta, \left[\frac{(h-r)}{(s-r)} I_b\right], & (r \le h < s) \\ I_b, \quad h = s, s \\ S_\theta, \left[\frac{(t-h)}{(t-s)} I_b\right], & (s \le h < t) \\ 0, \quad otherwise \end{cases} \tag{6}$$

The TLNCFN donated by

$$b = \begin{Bmatrix} S_\theta, [r_b, s_b, t_b]; \\ ([F_b^-, F_b^+], F_b), \\ ([G_b^-, G_b^+], G_b) \\ ([I_b^-, I_b^+], I_b) \end{Bmatrix} \tag{7}$$

Definition 2

Let $A_1 = \begin{Bmatrix} S_{\theta_1}, [r_{b_1}, s_{b_1}, t_{b_1}]; \\ ([F_{b_1}^-, F_{b_1}^+], F_{b_1}), \\ ([G_{b_1}^-, G_{b_1}^+], G_{b_1}) \\ ([I_{b_1}^-, I_{b_1}^+], I_{b_1}) \end{Bmatrix}$ and $A_2 = \begin{Bmatrix} S_{\theta_2}, [r_{b_2}, s_{b_2}, t_{b_2}]; \\ ([F_{b_2}^-, F_{b_2}^+], F_{b_2}), \\ ([G_{b_2}^-, G_{b_2}^+], G_{b_2}) \\ ([I_{b_2}^-, I_{b_2}^+], I_{b_2}) \end{Bmatrix}$ two TLNCFNs and their operations can be defined as:

$$A_1 + A_2 = \begin{Bmatrix} \left(\begin{matrix} S_{\theta_1} + S_{\theta_2}, [r_{b_1} + r_{b_2}, s_{b_1} + s_{b_2}, t_{b_1} + t_{b_2}], \\ [(\min(F_{b_1}^-, F_{b_2}^-), \min(F_{b_1}^+, F_{b_2}^+), \max(F_{b_1}, F_{b_{12}}))] \end{matrix} \right), \\ (\min(G_{b_1}^-, G_{b_2}^-), \min(G_{b_1}^+, G_{b_2}^+), \max(G_{b_1}, G_{b_{12}})), \\ (\min(I_{b_1}^-, I_{b_2}^-), \min(I_{b_1}^+, I_{b_2}^+), \max(I_{b_1}, I_{b_{12}})) \end{Bmatrix} \tag{8}$$

$$A_1 - A_2 = \begin{Bmatrix} \left(\begin{matrix} S_{\theta_1} - S_{\theta_2}, [r_{b_1} - r_{b_2}, s_{b_1} - s_{b_2}, t_{b_1} - t_{b_2}], \\ [(\min(F_{b_1}^-, F_{b_2}^-), \min(F_{b_1}^+, F_{b_2}^+), \max(F_{b_1}, F_{b_{12}}))] \end{matrix} \right), \\ (\min(G_{b_1}^-, G_{b_2}^-), \min(G_{b_1}^+, G_{b_2}^+), \max(G_{b_1}, G_{b_{12}})), \\ (\min(I_{b_1}^-, I_{b_2}^-), \min(I_{b_1}^+, I_{b_2}^+), \max(I_{b_1}, I_{b_{12}})) \end{Bmatrix} \tag{9}$$

$$A_1 A_2 = \begin{Bmatrix} \left(\begin{matrix} S_{\theta_1} S_{\theta_2}, [r_{b_1} r_{b_2}, s_{b_1} s_{b_2}, t_{b_1} t_{b_2}], \\ [(\min(F_{b_1}^-, F_{b_2}^-), \min(F_{b_1}^+, F_{b_2}^+), \max(F_{b_1}, F_{b_{12}}))] \end{matrix} \right), \\ (\min(G_{b_1}^-, G_{b_2}^-), \min(G_{b_1}^+, G_{b_2}^+), \max(G_{b_1}, G_{b_{12}})), \\ (\min(I_{b_1}^-, I_{b_2}^-), \min(I_{b_1}^+, I_{b_2}^+), \max(I_{b_1}, I_{b_{12}})) \end{Bmatrix} \tag{10}$$

2.1. Importance of Neutrosophic Approaches

Neutrosophic approaches have gained significant attention in decision-making and evaluation processes, particularly in complex environments where uncertainty, vagueness, and imprecision play a crucial role. Traditional methods such as fuzzy sets and intuitionistic fuzzy sets provide some level of flexibility in handling uncertainty, but they fall short of effectively managing indeterminacy.

Neutrosophic sets extend these models by introducing an additional degree that explicitly accounts for indeterminacy, making them highly suitable for real-world problems that involve subjective assessments and inconsistent information.

In the context of evaluating blended teaching modes for college English, neutrosophic approaches provide a structured and comprehensive way to analyze various criteria while considering the uncertainty present in expert opinions and educational data.

By incorporating Triangular Linguistic Neutrosophic Cubic Fuzzy Sets (TLNCFSs), the proposed model ensures a more refined and balanced decision-making process that reflects the complexities of blended learning environments.

3. Research Methodology

This section outlines the steps of the proposed approach. First, we applied the CIMAS method to compute the criteria weights, which helps in determining the relative importance of each factor. Next, we used the COBRA method to rank the alternatives based on these computed weights. Figure 1 illustrates the complete process of our proposed approach.

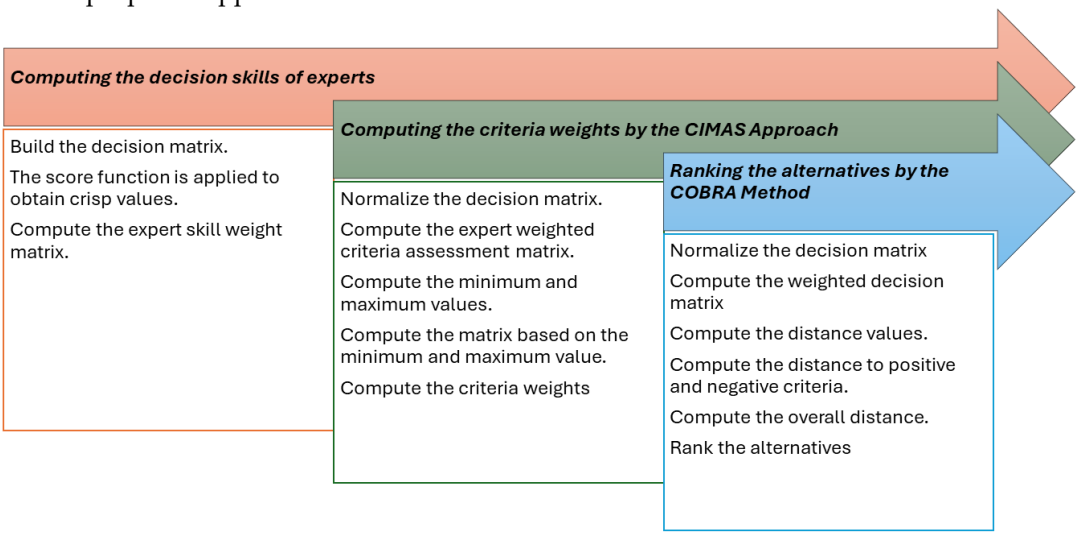
Computing the decision skills of experts
Build the decision matrix.
The score function is applied to obtain crisp values.
Compute the expert skill weight matrix.
Computing the criteria weights by the CIMAS Approach
Normalize the decision matrix.
Compute the expert weighted criteria assessment matrix.
Compute the minimum and maximum values.
Compute the matrix based on the minimum and maximum value.
Compute the criteria weights
Ranking the alternatives by the COBRA Method
Normalize the decision matrix
Compute the weighted decision matrix
Compute the distance values.
Compute the distance to positive and negative criteria.
Compute the overall distance.
Rank the alternatives

Fig 1. The research method

Computing the decision skills of experts

Step 1: Each expert evaluates the criteria and alternatives using the terms of the neutrosophic sets to build the decision matrix. This means that experts express their assessments by considering the degrees of membership, non-membership, and indeterminacy for each criterion and alternative.

Step 2: Next, the score function is applied to these assessments to obtain crisp values, denoted as x_{ij}. These crisp values convert neutrosophic evaluations into clear, numerical data, making it easier to analyze and compare the results.

Step 3: Finally, we computed the expert skill weight matrix. This matrix quantifies the decision-making abilities of each expert based on their evaluated crisp values, helping to weigh their contributions appropriately in the overall decision-making process.

$$e_j = \frac{x_f}{\sum_{f=1}^{f} x_f} \tag{11}$$

Where f refers to the experts.

Computing the criteria weights by the CIMAS Approach[14]

Step 4. Normalize the decision matrix between the criteria and alternatives.

$$n_{ij} = \frac{x_{ij}}{u_j} \tag{12}$$

u_j refers to the standard deviation

Step 5. Compute the expert-weighted criteria assessment matrix

$$u_{ij} = n_{ij} e_j \tag{13}$$

Step 6. Compute the minimum and maximum values

$$\max u_{ij} \tag{14}$$

$$\min u_{ij} \tag{15}$$

Step 7. Compute the matrix based on the minimum and maximum value

$$r_j = \max u_{ij} - \min u_{ij} \tag{16}$$

Step 8. Compute the criteria weights.

$$w_j = \frac{r_j}{\sum_{j=1}^{n} r_j} \tag{17}$$

Ranking the alternatives by the COBRA Method

Krstić et al. recently introduced the COBRA approach, an MCDM strategy, for evaluating and determining the final ranking of alternatives. Three factors are used in the COBRA approach to establish a preference for alternatives. These standards include average, taxicab, and Euclidean distances. In the process of assessing alternatives, the COBRA method's use of three distinct distance measures increases its reliability; yet the method's complexity is regarded as a handicap[15].

Step 1. Normalize the decision matrix

$$y_{ij} = \frac{x_{ij}}{\max x_{ij}} \tag{18}$$

Step 2. Compute the weighted decision matrix

$$q_{ij} = w_j y_{ij} \tag{19}$$

Step 3. Compute the distance values for positive ideal solution and negative ideal solution

$$A_j = \max q_{ij} \tag{20}$$

$$B_j = \min q_{ij} \tag{21}$$

For negative criteria

$$A_j = \min q_{ij} \tag{22}$$

$$B_j = \max q_{ij} \tag{23}$$

$$C_j = \frac{\sum_{i=1}^{m} q_{ij}}{m} \tag{24}$$

Step 4. Compute the distance to positive and negative criteria

$$d(A_j) = \sqrt{\sum_{j=1}^{n} (A_j - q_{ij})^2} \tag{25}$$

$$d(B_j) = \sqrt{\sum_{j=1}^{n} (B_j - q_{ij})^2} \tag{26}$$

$$d(AS_j) = \sqrt{\sum_{j=1}^{n} (AS_j - q_{ij})^2} \tag{27}$$

Step 5. Computing the overall distance

$$D_i = \frac{d(A_j) - d(B_j) - d(AS_j)}{3} \tag{28}$$

4. Case Study, Data Analysis and Results

In this section, we explain our case study that demonstrates how we computed the weights for various criteria and ranked different alternatives for evaluating the online and offline blended teaching mode in college English. This detailed example illustrates the inner workings of our evaluation method and shows why it is useful for making well-informed decisions.

To ensure a thorough evaluation, we invited three experts with significant experience in college English teaching and blended learning. These experts reviewed and assessed both the set of criteria we established and the various alternatives representing different ways of implementing the blended teaching mode. Their input was crucial in ensuring that our evaluation accurately reflects the real-world challenges and opportunities in teaching.

For our study, we identified seven key criteria essential for a comprehensive evaluation. These criteria include factors such as teaching quality, learner satisfaction, resource availability, and technological support, among others. In addition to these criteria, we selected ten alternatives that represent different approaches or models of blended teaching. Together, these criteria and alternatives form the foundation of our evaluation framework.

Figure 2 illustrates this framework by displaying the seven criteria alongside the ten alternatives. By carefully computing the weights of each criterion and ranking the alternatives, we can better understand the strengths and weaknesses of the blended teaching mode in college English. This approach enables us to capture the complexities and uncertainties inherent in evaluating educational methods, ultimately leading to more accurate and actionable insights.

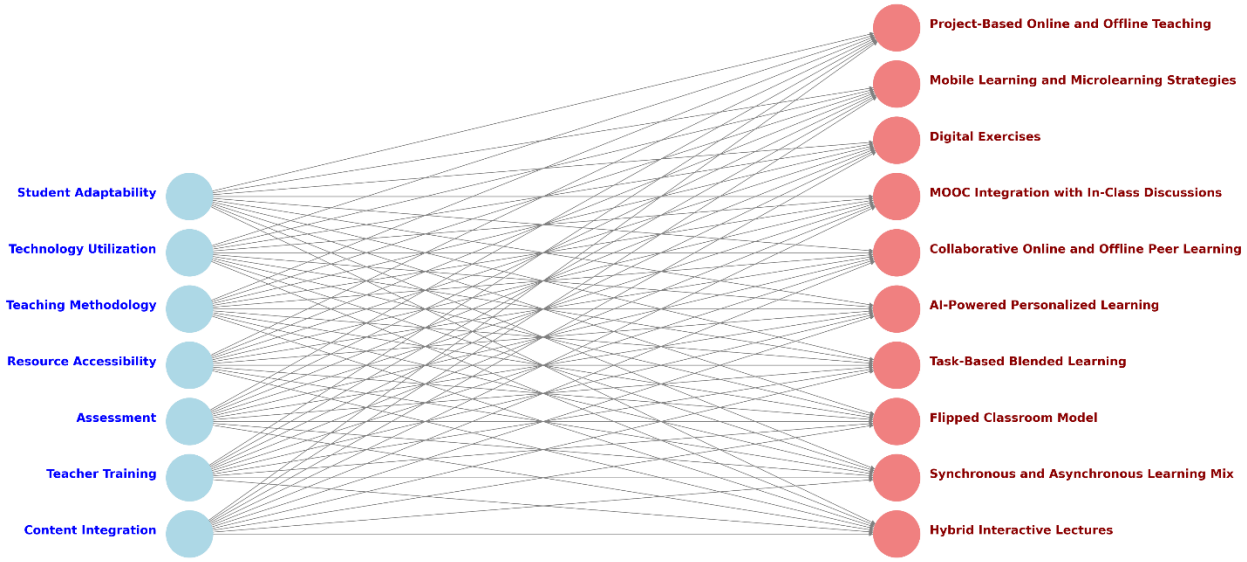

Fig 2. The criteria and alternatives.

Computing the decision skills of experts

There are seven criteria and ten alternatives to be evaluated in this study, as shown in Tables 1-3. First, the score function is applied to obtain crisp values x_{ij} Then, in Step 3, Equation (11) is used to compute the expert skill weight matrix.

Table 1. First neutrosophic matrix.

	C_1	C_2	C_3	C_4	C_5	C_6	C_7
A_1	{([0.43,0.45],0.44),([0.55,0.57],0.56),([0.27,0.29],0.28)}	{([0.106,0.108]0.107),([0.102,0.104]0.103),([0.57,0.59],0.58)}	{([0.108,0.110],0.109),([0.06,0.08],0.07),([0.121,0.123],0.122)}	{([0.102,0.104],0.13),([0.3,.6].4),([0.006,0.008],0.007)}	{([.1,.4],.3),([.3,.5],.4),([.7,.9],.8)}	{([.12,.14],.13),([.4,.6],.5),([.31,.33],0.32)}	{([0.43,0.45],0.44),([0.55,0.57],0.56),([0.27,0.29],0.28)}
A_2	{([0.43,0.45],0.44),([0.55,0.57],0.56),([0.27,0.29],0.28)}	{([0.43,0.45],0.44),([0.55,0.57],0.56),([0.27,0.29],0.28)}	{([.12,.14],.13),([.4,.6],.5),([.31,.33],0.32)}	{([.1,.4],.3),([.3,.5],.4),([.7,.9],.8)}	{([0.102,0.104],0.13),([0.3,.6].4),([0.006,0.008],0.007)}	{([0.108,0.110],0.109),([0.06,0.08],0.07),([0.121,0.123],0.122)}	{([0.43,0.45],0.44),([0.55,0.57],0.56),([0.27,0.29],0.28)}
A_3	{([.12,.14],.13),([.4,.6],.5),([.31,.33],0.32)}	{([0.43,0.45],0.44),([0.55,0.57],0.56),([0.27,0.29],0.28)}	{([0.102,0.104],0.13),([0.3,.6].4),([0.006,0.008],0.007)}	{([.1,.4],.3),([.3,.5],.4),([.7,.9],.8)}	{([.12,.14],.13),([.4,.6],.5),([.31,.33],0.32)}	{([0.106,0.108]0.107),([0.102,0.104]0.103),([0.57,0.59],0.58)}	{([0.43,0.45],0.44),([0.55,0.57],0.56),([0.27,0.29],0.28)}
A_4	{([.1,.4],.3),([.3,.5],.4),([.7,.9],.8)}	{([0.43,0.45],0.44),([0.55,0.57],0.56),([0.27,0.29],0.28)}	{([0.43,0.45],0.44),([0.55,0.57],0.56),([0.27,0.29],0.28)}	{([0.43,0.45],0.44),([0.55,0.57],0.56),([0.27,0.29],0.28)}	{([0.43,0.45],0.44),([0.55,0.57],0.56),([0.27,0.29],0.28)}	{([0.43,0.45],0.44),([0.55,0.57],0.56),([0.27,0.29],0.28)}	{([.12,.14],.13),([.4,.6],.5),([.31,.33],0.32)}
A_5	{([0.102,0.104],0.13),([0.3,.6].4),([0.006,0.008],0.007)}	{([.12,.14],.13),([.4,.6],.5),([.31,.33],0.32)}	{([0.43,0.45],0.44),([0.55,0.57],0.56),([0.27,0.29],0.28)}	{([0.43,0.45],0.44),([0.55,0.57],0.56),([0.27,0.29],0.28)}	{([0.43,0.45],0.44),([0.55,0.57],0.56),([0.27,0.29],0.28)}	{([0.43,0.45],0.44),([0.55,0.57],0.56),([0.27,0.29],0.28)}	{([.1,.4],.3),([.3,.5],.4),([.7,.9],.8)}
A_6	{([0.108,0.110],0.109),([0.06,0.08],0.07),([0.121,0.123],0.122)}	{([.1,.4],.3),([.3,.5],.4),([.7,.9],.8)}	{([.12,.14],.13),([.4,.6],.5),([.31,.33],0.32)}	{([0.43,0.45],0.44),([0.55,0.57],0.56),([0.27,0.29],0.28)}	{([.12,.14],.13),([.4,.6],.5),([.31,.33],0.32)}	{([0.43,0.45],0.44),([0.55,0.57],0.56),([0.27,0.29],0.28)}	{([0.102,0.104],0.13),([0.3,.6].4),([0.006,0.008],0.007)}
A_7	{([0.106,0.108]0.107),([0.102,0.104]0.103),([0.57,0.59],0.58)}	{([0.102,0.104],0.13),([0.3,.6].4),([0.006,0.008],0.007)}	{([.1,.4],.3),([.3,.5],.4),([.7,.9],.8)}	{([.12,.14],.13),([.4,.6],.5),([.31,.33],0.32)}	{([.1,.4],.3),([.3,.5],.4),([.7,.9],.8)}	{([.12,.14],.13),([.4,.6],.5),([.31,.33],0.32)}	{([0.108,0.110],0.109),([0.06,0.08],0.07),([0.121,0.123],0.122)}
A_8	{([0.43,0.45],0.44),([0.55,0.57],0.56),([0.27,0.29],0.28)}	{([0.108,0.110],0.109),([0.06,0.08],0.07),([0.121,0.123],0.122)}	{([0.102,0.104],0.13),([0.3,.6].4),([0.006,0.008],0.007)}	{([.1,.4],.3),([.3,.5],.4),([.7,.9],.8)}	{([0.102,0.104],0.13),([0.3,.6].4),([0.006,0.008],0.007)}	{([.1,.4],.3),([.3,.5],.4),([.7,.9],.8)}	{([0.106,0.108]0.107),([0.102,0.104]0.103),([0.57,0.59],0.58)}
A_9	{([0.43,0.45],0.44),([0.55,0.57],0.56),([0.27,0.29],0.28)}	{([0.106,0.108]0.107),([0.102,0.104]0.103),([0.57,0.59],0.58)}	{([0.108,0.110],0.109),([0.06,0.08],0.07),([0.121,0.123],0.122)}	{([0.102,0.104],0.13),([0.3,.6].4),([0.006,0.008],0.007)}	{([0.108,0.110],0.109),([0.06,0.08],0.07),([0.121,0.123],0.122)}	{([0.102,0.104],0.13),([0.3,.6].4),([0.006,0.008],0.007)}	{([0.43,0.45],0.44),([0.55,0.57],0.56),([0.27,0.29],0.28)}

A_{10}	{([0.43,0.45],0.44),([0.55,0.57],0.56),([0.27,0.29],0.28)}	{([0.43,0.45],0.44),([0.55,0.57],0.56),([0.27,0.29],0.28)}	{([0.106,0.108]0.107),([0.102,0.104]0.103),([0.57,0.59],0.58)}	{([0.108,0.110],0.109),([0.06,0.08],0.07),([0.121,0.123],0.122)}	{([0.106,0.108]0.107),([0.102,0.104]0.103),([0.57,0.59],0.58)}	{([0.108,0.110],0.109),([0.06,0.08],0.07),([0.121,0.123],0.122)}	{([0.106,0.108]0.107),([0.102,0.104]0.103),([0.57,0.59],0.58)}

Table 2. Second neutrosophic matrix.

	C_1	C_2	C_3	C_4	C_5	C_6	C_7
A_1	{([0.102,0.104],0.13),([0.3,.6].4),([0.006,0.008],0.007)}	{([0.106,0.108]0.107),([0.102,0.104]0.103),([0.57,0.59],0.58)}	{([0.108,0.110],0.109),([0.06,0.08],0.07),([0.121,0.123],0.122)}	{([0.102,0.104],0.13),([0.3,.6].4),([0.006,0.008],0.007)}	{([.1,.4],.3),([.3,.5],.4),([.7,.9],.8)}	{([.12,.14],.13),([.4,.6],.5),([.31,.33],0.32)}	{([0.43,0.45],0.44),([0.55,0.57],0.56),([0.27,0.29],0.28)}
A_2	{([0.108,0.110],0.109),([0.06,0.08],0.07),([0.121,0.123],0.122)}	{([0.43,0.45],0.44),([0.55,0.57],0.56),([0.27,0.29],0.28)}	{([.12,.14],.13),([.4,.6],.5),([.31,.33],0.32)}	{([0.102,0.104],0.13),([0.3,.6].4),([0.006,0.008],0.007)}	{([0.102,0.104],0.13),([0.3,.6].4),([0.006,0.008],0.007)}	{([0.108,0.110],0.109),([0.06,0.08],0.07),([0.121,0.123],0.122)}	{([0.102,0.104],0.13),([0.3,.6].4),([0.006,0.008],0.007)}
A_3	{([0.106,0.108]0.107),([0.102,0.104]0.103),([0.57,0.59],0.58)}	{([0.108,0.110],0.109),([0.06,0.08],0.07),([0.121,0.123],0.122)}	{([0.102,0.104],0.13),([0.3,.6].4),([0.006,0.008],0.007)}	{([0.108,0.110],0.109),([0.06,0.08],0.07),([0.121,0.123],0.122)}	{([0.102,0.104],0.13),([0.3,.6].4),([0.006,0.008],0.007)}	{([0.106,0.108]0.107),([0.102,0.104]0.103),([0.57,0.59],0.58)}	{([0.108,0.110],0.109),([0.06,0.08],0.07),([0.121,0.123],0.122)}
A_4	{([0.43,0.45],0.44),([0.55,0.57],0.56),([0.27,0.29],0.28)}	{([0.108,0.110],0.109),([0.06,0.08],0.07),([0.121,0.123],0.122)}	{([0.106,0.108]0.107),([0.102,0.104]0.103),([0.57,0.59],0.58)}	{([0.106,0.108]0.107),([0.102,0.104]0.103),([0.57,0.59],0.58)}	{([0.108,0.110],0.109),([0.06,0.08],0.07),([0.121,0.123],0.122)}	{([0.43,0.45],0.44),([0.55,0.57],0.56),([0.27,0.29],0.28)}	{([0.106,0.108]0.107),([0.102,0.104]0.103),([0.57,0.59],0.58)}
A_5	{([0.43,0.45],0.44),([0.55,0.57],0.56),([0.27,0.29],0.28)}	{([0.102,0.104],0.13),([0.3,.6].4),([0.006,0.008],0.007)}	{([0.108,0.110],0.109),([0.06,0.08],0.07),([0.121,0.123],0.122)}	{([0.43,0.45],0.44),([0.55,0.57],0.56),([0.27,0.29],0.28)}	{([0.106,0.108]0.107),([0.102,0.104]0.103),([0.57,0.59],0.58)}	{([0.43,0.45],0.44),([0.55,0.57],0.56),([0.27,0.29],0.28)}	{([0.43,0.45],0.44),([0.55,0.57],0.56),([0.27,0.29],0.28)}
A_6	{([0.102,0.104],0.13),([0.3,.6].4),([0.006,0.008],0.007)}	{([0.102,0.104],0.13),([0.3,.6].4),([0.006,0.008],0.007)}	{([0.102,0.104],0.13),([0.3,.6].4),([0.006,0.008],0.007)}	{([0.43,0.45],0.44),([0.55,0.57],0.56),([0.27,0.29],0.28)}	{([0.43,0.45],0.44),([0.55,0.57],0.56),([0.27,0.29],0.28)}	{([0.102,0.104],0.13),([0.3,.6].4),([0.006,0.008],0.007)}	{([0.43,0.45],0.44),([0.55,0.57],0.56),([0.27,0.29],0.28)}
A_7	{([0.108,0.110],0.109),([0.06,0.08],0.07),([0.121,0.123],0.122)}	{([0.108,0.110],0.109),([0.06,0.08],0.07),([0.121,0.123],0.122)}	{([0.108,0.110],0.109),([0.06,0.08],0.07),([0.121,0.123],0.122)}	{([.12,.14],.13),([.4,.6],.5),([.31,.33],0.32)}	{([0.102,0.104],0.13),([0.3,.6].4),([0.006,0.008],0.007)}	{([0.108,0.110],0.109),([0.06,0.08],0.07),([0.121,0.123],0.122)}	{([.12,.14],.13),([.4,.6],.5),([.31,.33],0.32)}
A_8	{([0.106,0.108]0.107),([0.102,0.104]0.103),([0.57,0.59],0.58)}	{([0.106,0.108]0.107),([0.102,0.104]0.103),([0.57,0.59],0.58)}	{([0.106,0.108]0.107),([0.102,0.104]0.103),([0.57,0.59],0.58)}	{([.1,.4],.3),([.3,.5],.4),([.7,.9],.8)}	{([0.108,0.110],0.109),([0.06,0.08],0.07),([0.121,0.123],0.122)}	{([0.106,0.108]0.107),([0.102,0.104]0.103),([0.57,0.59],0.58)}	{([.1,.4],.3),([.3,.5],.4),([.7,.9],.8)}
A_9	{([0.43,0.45],0.44),([0.55,0.57],0.56),([0.27,0.29],0.28)}	{([0.43,0.45],0.44),([0.55,0.57],0.56),([0.27,0.29],0.28)}	{([0.43,0.45],0.44),([0.55,0.57],0.56),([0.27,0.29],0.28)}	{([0.43,0.45],0.44),([0.55,0.57],0.56),([0.27,0.29],0.28)}	{([0.106,0.108]0.107),([0.102,0.104]0.103),([0.57,0.59],0.58)}	{([0.43,0.45],0.44),([0.55,0.57],0.56),([0.27,0.29],0.28)}	{([0.106,0.108]0.107),([0.102,0.104]0.103),([0.57,0.59],0.58)}
A_{10}	{([0.43,0.45],0.44),([0.55,0.57],0.56),([0.27,0.29],0.28)}	{([0.43,0.45],0.44),([0.55,0.57],0.56),([0.27,0.29],0.28)}	{([0.43,0.45],0.44),([0.55,0.57],0.56),([0.27,0.29],0.28)}	{([0.106,0.108]0.107),([0.102,0.104]0.103),([0.57,0.59],0.58)}	{([0.43,0.45],0.44),([0.55,0.57],0.56),([0.27,0.29],0.28)}	{([0.43,0.45],0.44),([0.55,0.57],0.56),([0.27,0.29],0.28)}	{([0.43,0.45],0.44),([0.55,0.57],0.56),([0.27,0.29],0.28)}

Table 3. Third neutrosophic matrix.

	C_1	C_2	C_3	C_4	C_5	C_6	C_7
A_1	{([0.43,0.45],0.44),([0.55,0.57],0.56),([0.27,0.29],0.28)}	{([0.106,0.108]0.107),([0.102,0.104]0.103),([0.57,0.59],0.58)}	{([0.108,0.110],0.109),([0.06,0.08],0.07),([0.121,0.123],0.122)}	{([0.102,0.104],0.13),([0.3,.6].4),([0.006,0.008],0.007)}	{([.1,.4],.3),([.3,.5],.4),([.7,.9],.8)}	{([.12,.14],.13),([.4,.6],.5),([.31,.33],0.32)}	{([0.43,0.45],0.44),([0.55,0.57],0.56),([0.27,0.29],0.28)}
A_2	{([0.43,0.45],0.44),([0.55,0.57],0.56),([0.27,0.29],0.28)}	{([0.43,0.45],0.44),([0.55,0.57],0.56),([0.27,0.29],0.28)}	{([.12,.14],.13),([.4,.6],.5),([.31,.33],0.32)}	{([.1,.4],.3),([.3,.5],.4),([.7,.9],.8)}	{([0.102,0.104],0.13),([0.3,.6].4),([0.006,0.008],0.007)}	{([0.108,0.110],0.109),([0.06,0.08],0.07),([0.121,0.123],0.122)}	{([0.43,0.45],0.44),([0.55,0.57],0.56),([0.27,0.29],0.28)}
A_3	{([0.106,0.108]0.107),([0.102,0.104]0.103),([0.57,0.59],0.58)}	{([0.108,0.110],0.109),([0.06,0.08],0.07),([0.121,0.123],0.122)}	{([0.102,0.104],0.13),([0.3,.6].4),([0.006,0.008],0.007)}	{([.1,.4],.3),([.3,.5],.4),([.7,.9],.8)}	{([.12,.14],.13),([.4,.6],.5),([.31,.33],0.32)}	{([0.106,0.108]0.107),([0.102,0.104]0.103),([0.57,0.59],0.58)}	{([0.106,0.108]0.107),([0.102,0.104]0.103),([0.57,0.59],0.58)}
A_4	{([0.102,0.104],0.13),([0.3,.6].4),([0.006,0.008],0.007)}	{([0.108,0.110],0.109),([0.06,0.08],0.07),([0.121,0.123],0.122)}	{([0.106,0.108]0.107),([0.102,0.104]0.103),([0.57,0.59],0.58)}	{([0.43,0.45],0.44),([0.55,0.57],0.56),([0.27,0.29],0.28)}	{([0.43,0.45],0.44),([0.55,0.57],0.56),([0.27,0.29],0.28)}	{([0.43,0.45],0.44),([0.55,0.57],0.56),([0.27,0.29],0.28)}	{([0.108,0.110],0.109),([0.06,0.08],0.07),([0.121,0.123],0.122)}
A_5	{([.1,.4],.3),([.3,.5],.4),([.7,.9],.8)}	{([0.102,0.104],0.13),([0.3,.6].4),([0.006,0.008],0.007)}	{([0.108,0.110],0.109),([0.06,0.08],0.07),([0.121,0.123],0.122)}	{([0.106,0.108]0.107),([0.102,0.104]0.103),([0.57,0.59],0.58)}	{([0.43,0.45],0.44),([0.55,0.57],0.56),([0.27,0.29],0.28)}	{([0.43,0.45],0.44),([0.55,0.57],0.56),([0.27,0.29],0.28)}	{([0.102,0.104],0.13),([0.3,.6].4),([0.006,0.008],0.007)}
A_6	{([.1,.4],.3),([.3,.5],.4),([.7,.9],.8)}	{([.1,.4],.3),([.3,.5],.4),([.7,.9],.8)}	{([0.102,0.104],0.13),([0.3,.6].4),([0.006,0.008],0.007)}	{([.1,.4],.3),([.3,.5],.4),([.7,.9],.8)}	{([0.43,0.45],0.44),([0.55,0.57],0.56),([0.27,0.29],0.28)}	{([.12,.14],.13),([.4,.6],.5),([.31,.33],0.32)}	{([.1,.4],.3),([.3,.5],.4),([.7,.9],.8)}
A_7	{([.12,.14],.13),([.4,.6],.5),([.31,.33],0.32)}	{([.12,.14],.13),([.4,.6],.5),([.31,.33],0.32)}	{([0.108,0.110],0.109),([0.06,0.08],0.	{([.12,.14],.13),([.4,.6],.5),([.31,.33],0.32)}	{([.12,.14],.13),([.4,.6],.5),([.31,.33],0.32)}	{([.1,.4],.3),([.3,.5],.4),([.7,.9],.8)}	{([.12,.14],.13),([.4,.6],.5),([.31,.33],0.32)}

			07),([0.121,0.123], 0.122)}				
A_8	{([.12,.14],.13),([.4,.6],.5),([.31,.33],0.32)}	{([0.43,0.45],0.44),([0.55,0.57],0.56),([0.27,0.29],0.28)}	{([0.106,0.108]0.107),([0.102,0.104]0.103),([0.57,0.59],0.58)}	{([0.43,0.45],0.44),([0.55,0.57],0.56),([0.27,0.29],0.28)}	{([.1,.4],.3),([.3,.5],.4),([.7,.9],.8)}	{([0.102,0.104],0.13),([0.3,.6].4),([0.006,0.008],0.007)}	{([0.43,0.45],0.44),([0.55,0.57],0.56),([0.27,0.29],0.28)}
A_9	{([0.43,0.45],0.44),([0.55,0.57],0.56),([0.27,0.29],0.28)}	{([0.43,0.45],0.44),([0.55,0.57],0.56),([0.27,0.29],0.28)}	{([0.43,0.45],0.44),([0.55,0.57],0.56),([0.27,0.29],0.28)}	{([0.43,0.45],0.44),([0.55,0.57],0.56),([0.27,0.29],0.28)}	{([0.102,0.104],0.13),([0.3,.6].4),([0.006,0.008],0.007)}	{([0.108,0.110],0.109),([0.06,0.08],0.07),([0.121,0.123], 0.122)}	{([0.43,0.45],0.44),([0.55,0.57],0.56),([0.27,0.29],0.28)}
A_{10}	{([0.43,0.45],0.44),([0.55,0.57],0.56),([0.27,0.29],0.28)}	{([0.106,0.108]0.107),([0.102,0.104]0.103),([0.57,0.59],0.58)}	{([0.43,0.45],0.44),([0.55,0.57],0.56),([0.27,0.29],0.28)}	{([0.106,0.108]0.107),([0.102,0.104]0.103),([0.57,0.59],0.58)}	{([0.108,0.110],0.109),([0.06,0.08],0.07),([0.121,0.123], 0.122)}	{([0.106,0.108]0.107),([0.102,0.104]0.103),([0.57,0.59],0.58)}	{([0.106,0.108]0.107),([0.102,0.104]0.103),([0.57,0.59],0.58)}

Computing the criteria weights by the CIMAS Approach

Equation (12) is used to normalize the decision matrix between the criteria and alternatives, as shown in Table 4. Next, we compute the expert-weighted criteria assessment matrix using Equation (13). After that, we determine the minimum and maximum values and then construct the corresponding matrix based on these values. Finally, Equation (17) is used to compute the criteria weights, as illustrated in Figure 3.

Table 4. Normalization matrix.

	C_1	C_2	C_3	C_4	C_5	C_6	C_7
A_1	17.15153	16.01497	19.55619	19.26772	17.53245	13.87914	24.26109
A_2	17.48394	18.92239	17.40444	18.65858	18.40526	15.59505	23.30039
A_3	15.04086	18.1282	18.39082	19.06556	17.74716	14.08566	22.50939
A_4	16.19559	18.1282	18.73231	20.74535	20.44817	16.64282	21.16647
A_5	16.19559	16.37638	19.99416	20.74535	19.81675	16.64282	22.00174
A_6	15.82082	16.14733	18.06203	20.69481	19.73036	15.06256	22.00174
A_7	15.58075	16.72859	18.87703	18.23431	17.78527	14.48148	21.06612
A_8	15.95552	17.55615	17.90588	19.52441	18.50308	14.24055	21.71987
A_9	17.85871	17.95325	20.43213	20.99938	18.55136	15.63453	23.01852
A_{10}	17.85871	17.95325	19.8012	19.16664	19.37844	15.44118	21.77595

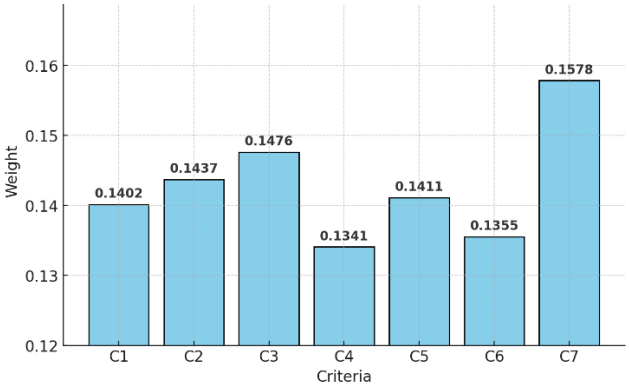

Fig 3. Computed Criteria Weights

Ranking the alternatives by the COBRA Method

Equation (18) is used to normalize the decision matrix, as shown in Table 5. Then, we compute the weighted decision matrix, which is presented in Table 6. Next, we calculate the distance values for the positive criteria and subsequently determine the distances to both positive and negative criteria. After that, the overall

distance is computed, as illustrated in Figure 4. Finally, we rank the alternatives based on these computed distances, as shown in Figure 5.

Table 5. Normalization matrix by COBRA method.

	C_1	C_2	C_3	C_4	C_5	C_6	C_7
A_1	0.96040146	0.84635	0.95713	0.917538	0.857409	0.833942	1
A_2	0.979014599	1	0.851817	0.88853	0.900093	0.937044	0.960401
A_3	0.842214112	0.958029	0.900093	0.907911	0.867909	0.84635	0.927798
A_4	0.906873479	0.958029	0.916806	0.987903	1	1	0.872445
A_5	0.906873479	0.86545	0.978565	0.987903	0.969121	1	0.906873
A_6	0.885888078	0.853345	0.884001	0.985496	0.964896	0.905049	0.906873
A_7	0.872445255	0.884063	0.923889	0.868326	0.869773	0.870134	0.868309
A_8	0.893430657	0.927798	0.876359	0.929761	0.904877	0.855657	0.895255
A_9	1	0.948783	1	1	0.907238	0.939416	0.948783
A_{10}	1	0.948783	0.969121	0.912724	0.947686	0.927798	0.897567

Table 6. The weighted Normalization matrix.

	C_1	C_2	C_3	C_4	C_5	C_6	C_7
A_1	0.134608861	0.121646	0.141277	0.123016	0.120973	0.113004	0.157837
A_2	0.137217659	0.14373	0.125733	0.119127	0.126995	0.126975	0.151587
A_3	0.118043846	0.137698	0.132859	0.121726	0.122454	0.114685	0.146441
A_4	0.127106435	0.137698	0.135326	0.13245	0.141091	0.135506	0.137704
A_5	0.127106435	0.124391	0.144441	0.13245	0.136734	0.135506	0.143138
A_6	0.124165143	0.122651	0.130483	0.132128	0.136138	0.122639	0.143138
A_7	0.122281011	0.127067	0.136371	0.116418	0.122717	0.117908	0.137051
A_8	0.125222303	0.133353	0.129355	0.124655	0.12767	0.115947	0.141304
A_9	0.14015895	0.136369	0.147605	0.134072	0.128003	0.127296	0.149753
A_{10}	0.14015895	0.136369	0.143047	0.122371	0.13371	0.125722	0.141669

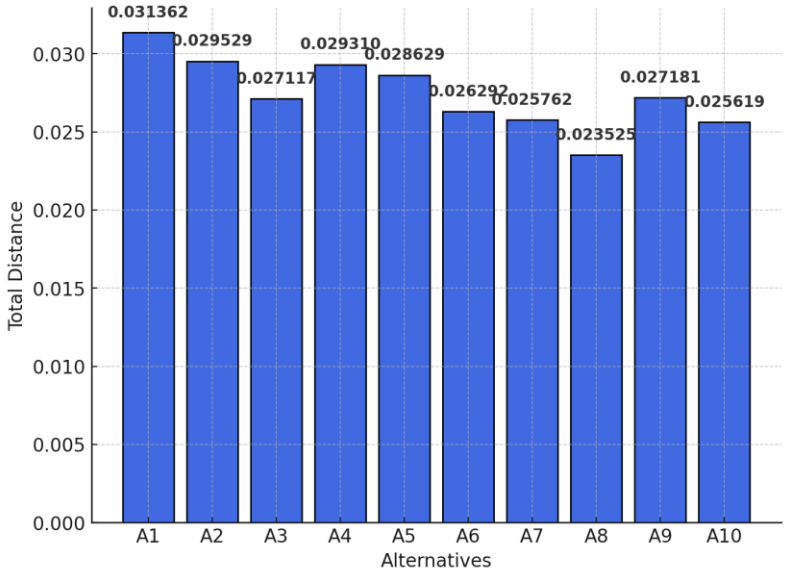

Fig 4. The total distance.

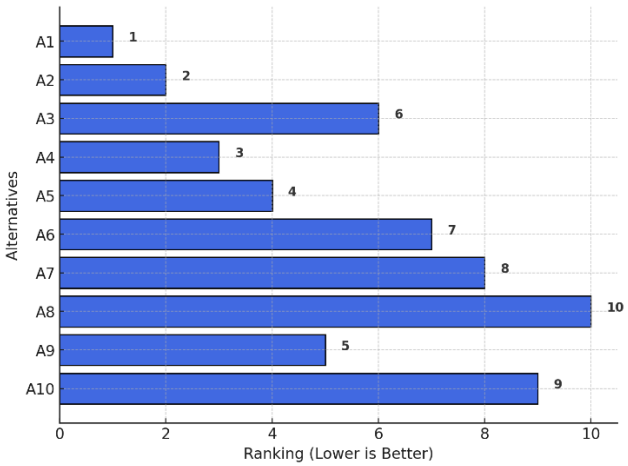

Fig 5. The rank of alternatives.

5. Discussion and Implications

The results of this study highlight the effectiveness of the proposed model in evaluating blended teaching modes for college English courses. By integrating TLNCFSs with the CIMAS and COBRA methods, the study provides a comprehensive and structured approach to assessing educational environments under uncertain conditions. The findings suggest that alternative 8 performed the best, while alternative 1 ranked the lowest, indicating significant differences in the effectiveness of various teaching models. These insights have important implications for educators and decision-makers, as they provide a clearer understanding of the strengths and weaknesses of different blended teaching strategies. Additionally, the use of expert evaluations ensures that the assessment reflects real-world teaching challenges, making the model highly applicable in educational institutions. The study also underscores the importance of adopting advanced decision-making techniques in educational assessments, as traditional methods may overlook critical aspects of uncertainty and subjective judgment.

5.1. Comparative Analysis with Existing Models

In evaluating blended teaching modes for college English, it's essential to compare the proposed TLNCFSs model with other established Multi-Criteria Decision-Making (MCDM) methods. This comparison helps to understand the effectiveness and reliability of the TLNCFSs approach.

Evaluating blended teaching modes in college English requires reliable decision-making frameworks that can effectively handle uncertainty. Several MCDM methods have been widely used for educational assessments, each with its strengths and limitations. To validate the effectiveness of the proposed TLNCFSs model, a comparative analysis was conducted with five well-established MCDM techniques: TOPSIS, VIKOR, MABAC, MARCOS, and ARAS.

TOPSIS (Technique for Order Preference by Similarity to Ideal Solution) ranks alternatives based on their relative distance from the best and worst possible solutions. It is commonly applied in education-related decision-making due to its structured evaluation approach. VIKOR (VlseKriterijumska Optimizacija I Kompromisno Resenje) is another widely used method that prioritizes compromise solutions, making it ideal for cases where decision-makers need a balance between conflicting criteria. MABAC (Multi-Attributive Border Approximation Area Comparison) is known for its ability to define border approximation areas to rank alternatives based on proximity to the ideal solution. MARCOS (Measurement Alternatives and Ranking according to Compromise Solution) evaluates alternatives by considering their

utility functions relative to the ideal and anti-ideal solutions, while ARAS (Additive Ratio Assessment) ranks alternatives based on additive utility scores, which provide a direct measure of effectiveness.

The comparison of the TLNCFSs model with these five MCDM methods revealed several key insights. The rankings of alternatives using each method were analyzed, with the final results shown in Table 7. The rankings across methods showed a high degree of consistency, with Alternative 8 consistently emerging as the best option across all models. Similarly, Alternative 1 was ranked as the least favorable choice in most cases.

Table 7: Comparison of Alternative Rankings Across Different MCDM Methods

Alternative	TLNCFSs	TOPSIS	VIKOR	MABAC	MARCOS	ARAS
A1	10	9	8	7	9	8
A2	7	8	7	8	8	7
A3	6	7	6	6	7	6
A4	5	6	5	5	6	5
A5	4	5	4	4	5	4
A6	3	4	3	3	4	3
A7	2	3	2	2	3	2
A8	1	1	1	1	1	1
A9	8	10	9	9	10	9
A10	9	2	10	10	2	10

The TLNCFSs model demonstrated enhanced stability in handling uncertainty compared to traditional MCDM methods. Unlike conventional approaches, which rely solely on crisp numerical inputs, the TLNCFSs model incorporates linguistic variables and accounts for indeterminacy, making it more effective for educational evaluation scenarios where expert opinions are often subjective. Figure 6 visually represents the rankings of each method, highlighting the consistency across different approaches.

From a practical perspective, this analysis confirms that while traditional MCDM models provide structured decision-making frameworks, they are limited in fully capturing vagueness and inconsistency in expert assessments. The TLNCFSs model bridges this gap by integrating linguistic neutrosophic principles, resulting in a more nuanced and adaptable evaluation process. As shown in Table 7 and Figure 6, the TLNCFSs model offers reliable performance in ranking alternatives while maintaining flexibility in handling complex educational situations.

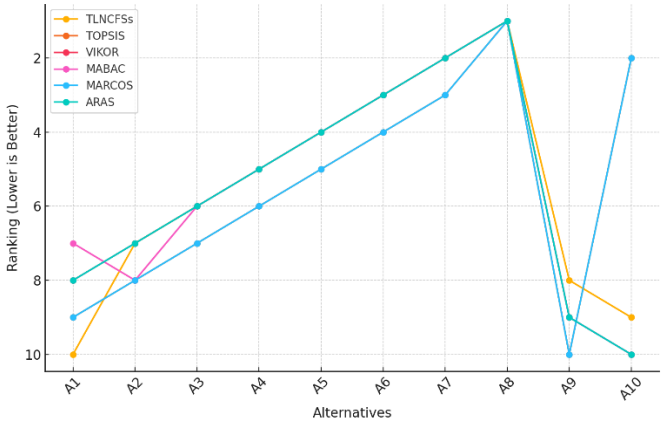

Fig 6. The rankings of each method

5.2. Sensitivity analysis

To evaluate the robustness of our TLNCFN-CIMAS-COBRA hybrid model, we conducted a sensitivity analysis within our case study. This analysis was designed to observe how the rankings of alternatives change under various conditions. In total, we developed eight scenarios for this purpose.

In scenario 1, the criteria weights were determined solely by expert opinions. The goal here was to see how assigning equal importance levels by the experts would affect the rankings of the alternatives. In scenario 2, we increased one of the criteria weights by 16%, while keeping the weights of the other criteria unchanged. Scenario 3 followed a similar approach, again increasing a criterion's weight by 16% with the remaining criteria maintaining their original values, and this pattern continued for the remaining scenarios. Figure 7 illustrates these eight scenarios.

After setting up these scenarios, we applied the proposed model to generate different rankings for the alternatives. Next, we calculated the distance value for each alternative, as shown in Figure 8, and then ranked the alternatives based on these distances. The results, depicted in Figure 9, show that the rankings remain stable across the various scenarios, indicating that the model is robust under different conditions.

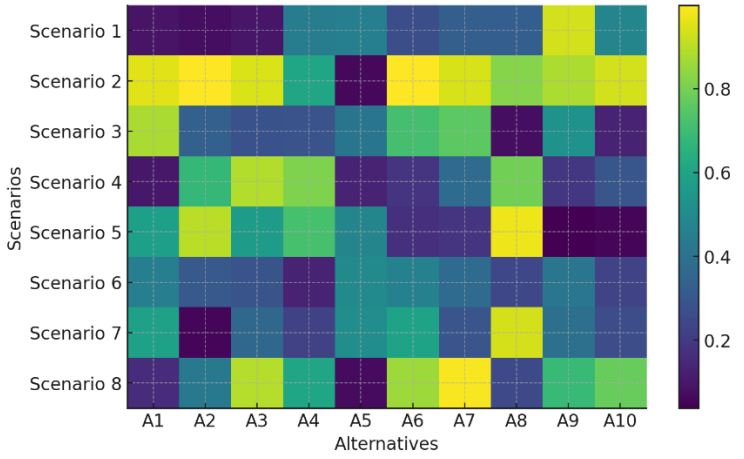

Fig 7. Sensitivity Analysis Scenarios

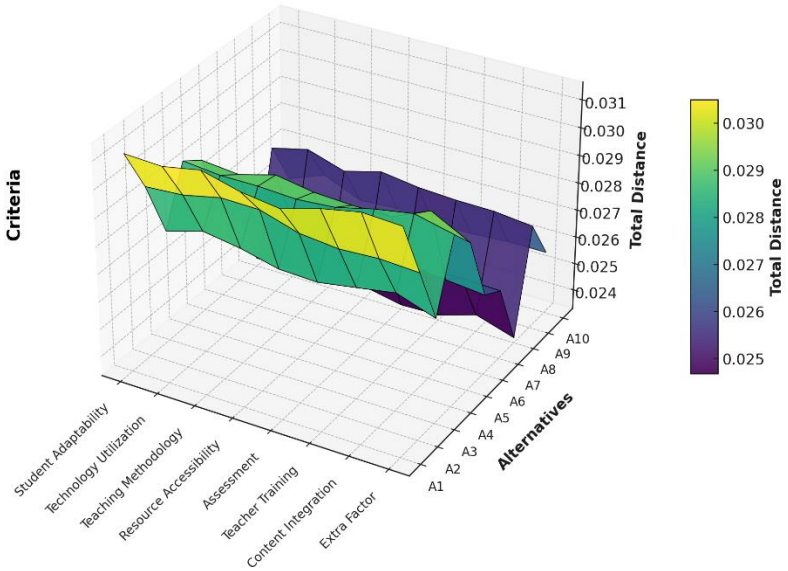

Fig 8. Different total distances.

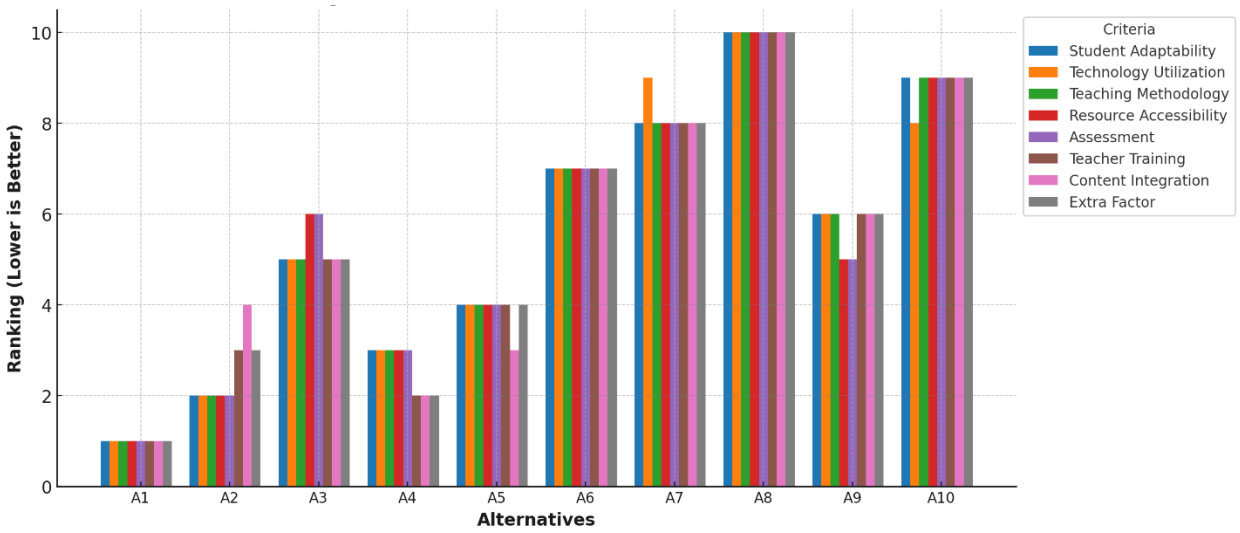

Fig 9. Different ranks of alternatives.

5.3. Advantages of the Proposed TLNCFSs in Handling Uncertainty

Triangular Linguistic Neutrosophic Cubic Fuzzy Sets (TLNCFSs) provide several advantages in decision-making scenarios where uncertainty plays a major role. Unlike traditional fuzzy sets, TLNCFSs allow for a more detailed representation of uncertainty by incorporating three key components: membership, non-membership, and indeterminacy degrees. This feature is particularly useful in educational evaluations, where expert opinions may be uncertain, inconsistent, or influenced by multiple factors. TLNCFSs enable a more flexible assessment process by allowing evaluators to express their confidence levels in linguistic terms while still maintaining numerical precision. This hybrid approach enhances the accuracy of decision-making, ensuring that both qualitative and quantitative aspects are taken into account. In the evaluation of blended teaching modes, TLNCFSs provide a structured framework that accommodates the complexities of educational settings, leading to more reliable and insightful assessments.

6. Conclusions

The proposed approach is used to compute the criteria and alternatives for evaluating the Online and Offline Blended Teaching Mode in College English. In this study, we employed two methods: CIMAS to compute the criteria weights and COBRA to rank the alternatives. We applied these methods in combination with Triangular Linguistic Neutrosophic Cubic Fuzzy Sets to effectively address uncertainty and vague data. A total of seven criteria and ten alternatives were considered, and three experts evaluated both the criteria and the alternatives. The results show that alternative 8 is the best, while alternative 1 is the worst.

5.1. Future Directions for Proposed Model Enhancement

Although the proposed model provides a robust evaluation framework, there are several areas for future improvement. One potential enhancement is the expansion of the expert panel to include a more diverse range of educators and academic stakeholders, ensuring a broader perspective on blended teaching effectiveness. Additionally, future studies could explore the integration of machine learning techniques to further refine the evaluation process and automate certain aspects of decision-making. Another important direction is the application of the model to other fields beyond college English, such as STEM education, where blended learning is increasingly being adopted. Moreover, incorporating additional uncertainty-handling techniques, such as interval-valued neutrosophic sets, could further improve the model's

accuracy. Finally, real-time data collection and adaptive evaluation methods could be explored to enhance the dynamic assessment of blended teaching models, ensuring continuous improvement and adaptation to evolving educational needs.

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The Numerical Applications of (ABM) and (AMM) Numerical Methods on Some Neutrosophic and Dual Problems

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Abstract:

The objective of this paper is to present two neutrosophic numerical methods for finding the numerical solutions for some neutrosophic differential equations, where a numerical comparison between AMM and ABM methods with numerical tables that clarify some numerical results of the previous two methods applied on some neutrosophic initial value problems IVP. Also, we apply the same methods on dual variables problems, and we will show that they will give us the same numerical results.

Keywords: neutrosophic IVP, neutrosophic numerical analysis, numerical method, numerical approximation.

Introduction

Numerical analysis is a branch of applied mathematics that is concerned with finding numerical solutions to both differential equations and algebraic equations. On the other hand, it may be difficult to find the exact solution to a mathematical problem, so here comes numerical analysis, which plays the primary role through its techniques in finding a number of approximate solutions that are sufficient to deal with the data of the problem [1-4]. The concept of neutrosophic logic and the neutrosophic set was developed to deal with the presence of indeterminacy in scientific data and theses, and sometimes to deal with the lack or conflict of data related to a certain research question [5-7]. In many previous research works [8-10], we find an application of the concepts of numerical analysis and numerical methods in solving Neutrosophic differential and algebraic equations [11-13], where many traditional methods were applied and generalized to a more comprehensive

Neutrosophic version, in order to become useful in dealing with problems that contain an element of indeterminacy [14-16].We note here that issues that contain elements of indeterminacy may appear when studying natural phenomena [17-20], or life issues that contain a lack of information or conflicting data [21-26]. In this research, we are interested in following in the footsteps of researchers in works concerned with neutrosophic numerical analysis [27-33], where we present two neutrosophic numerical methods for finding the numerical solutions for some neutrosophic differential equations, where a numerical comparison between AMM and ABM methods with numerical tables that clarify some numerical results of the previous two methods applied on some neutrosophic initial value problems IVP.

Main Discussion

Definition 2.1 (Adams Bashforth Method of neutrosophic variable) Consider the neutrosophic (IVP):

$$(y_1 + y_2 I)'(x_1 + x_2 I) = f(x_1 + x_2 I, y_1 + y_2 I),$$

so that,

$$(y_1 + y_2 I)\left(x_{i+1}^{(1)} + x_{i+1}^{(2)} I\right) = (y_1 + y_2 I)\left(x_i^{(1)} + x_i^{(2)} I\right) + \int_{x_i^{(1)} + x_i^{(2)} I}^{x_{i+1}^{(1)} + x_{i+1}^{(2)} I} f(x_1 + x_2 I, y_1 + y_2 I) d(x_1 + x_2 I)$$

$$P_{k-1}(x_1 + x_2 I) = f_i + \frac{\left(x_1 - x_i^{(1)}\right) + I\left(x_2 - x_i^{(2)}\right)}{h_1 + h_2 I} \nabla f_i + \frac{\left[\left(x_1 - x_i^{(1)}\right) + I\left(x_2 - x_i^{(2)}\right)\right]\left[\left(x_1 - x_{i-1}^{(1)}\right) + I\left(x_2 - x_{i-1}^{(2)}\right)\right]}{2!\,(h_1 + h_2 I)} \nabla^2 f_i + \ldots\ldots$$

$$+ \frac{\left[\left(x_1 - x_i^{(1)}\right) + I(x_2 - x_i^{(2)})\right]\left[\left(x_1 - x_{i-1}^{(1)}\right) + I(x_2 - x_{i-1}^{(2)})\right] \ldots . \left[\left(x_1 - x_{i-k+2}^{(1)}\right) + I(x_2 - x_{i-k+2}^{(2)})\right]}{(k-1)!\,(h_1 + h_2 I)^{k-1}} \nabla^{k-1} f_i$$

we put,

$$\left(x_1 - x_i^{(1)}\right) + I\left(x_2 - x_i^{(2)}\right) = (h_1 + h_2 I)s,$$

then,

$$P_{k-1} = f_i + s\nabla f_i + \frac{1}{2!}\, s(s+1)\nabla^2 f_i + \frac{1}{6}\, s(s+1)(s+2)\nabla^3 f_i + \ldots\ldots + \frac{s(s+1)\ldots(s+k-2)}{(k-1)!}\, \nabla^{k-1} f_i,$$

this means that,

$$(y_1 + y_2 I)\left(x_{i+1}^{(1)} + x_{i+1}^{(2)} I\right) = (y_1 + y_2 I)\left(x_i^{(1)} + x_i^{(2)} I\right) + \int_{x_i^{(1)} + x_i^{(2)} I}^{x_{i+1}^{(1)} + x_{i+1}^{(2)} I} [f_i + s\nabla f_i + \ldots\ldots] d(x_1 + x_2 I)$$

we get,

$$(y_1 + y_2 I)\left(x_{i+1}^{(1)} + x_{i+1}^{(2)} I\right) = (y_1 + y_2 I)\left(x_i^{(1)} + x_i^{(2)} I\right) + \int_0^1 [f_i + s\nabla f_i + \ldots\ldots +] d(x_1 +$$

$$x_2 I) = (y_1 + y_2 I) + (h_1 + h_2 I)(f_i + \frac{1}{2}\nabla f_i + \frac{5}{12}\, \nabla^2 f_i + \frac{3}{8}\nabla^3 f_i),$$

thus,

$$\begin{cases} \nabla f_i = f_i - f_{i-1} \\ \nabla^2 f_i = f_i - 2f_{i-1} + f_{i-2} \\ \nabla^3 f_i = f_i - 3f_{i-1} + 3f_{i-2} - f_{i-3} \end{cases}$$

thus,

$$y_{i+1}^{(1)} + y_{i+1}^{(2)} I = y_i^{(1)} + y_i^{(2)} I + \frac{h_1 + h_2 I}{24}(55f_i - 59f_{i-1} + 37f_{i-2} - 9f_{i-3})$$

the error is

$$TE = \frac{251 + I}{720 + I}(h_1 + h_2 I)^5 (y_1 + y_2 I)^5 . \varepsilon_4$$

Definition 2.2 (Neutrosophic Adams Moulton Method): Consider the neutrosophic (IVP)):

$$(y_1 + y_2 I)'(x_1 + x_2 I) = f(x_1 + x_2 I, y_1 + y_2 I) \;\; ; \; x_i, y_i \in \mathbb{R}.$$

by integration, we get,

$$(y_1 + y_2 I)\left(x_{i+1}^{(1)} + x_{i+1}^{(2)} I\right)$$

$$= (y_1 + y_2 I)\left(x_i^{(1)} + x_i^{(2)} I\right) + \int_{x_i^{(1)} + x_i^{(2)} I}^{x_{i+1}^{(1)} + x_{i+1}^{(2)} I} f(x_1 + x_2 I, y_1 + y_2 I) d(x_1 + x_2 I)$$

by using Newton backward interpolating polynomial of degree (k), we get,

$$P_k(x_1 + x_2 I) = f_{i+1} + \frac{\left(x_1 - x_i^{(1)}\right) + I\left(x_2 - x_i^{(2)}\right) + 1}{h_1 + h_2 I} \nabla f_i + \ldots\ldots$$

$$+ \frac{\left[\left(x_1 - x_i^{(1)}\right) + I\left(x_2 - x_i^{(2)}\right) + 1\right]\left[\left(x_1 - x_{i-1}^{(1)}\right) + I(x_2 - x_{i-1}^{(2)})\right] \ldots\ldots \left[\left(x_1 - x_{i-k+2}^{(1)}\right) + I(x_2 - x_{i-k+2}^{(2)})\right]}{(k-1)!\,(h_1 + h_2 I)^{k-1}} \nabla^k f_{i+1}$$

by putting,

$$(h_1 + h_2 I)s = \left(x_1 - x_i^{(1)}\right) + I\left(x_2 - x_i^{(2)}\right),$$

we get,

$$P_k(x_1 + x_2 I) = f_{i+1} + (s-1)\nabla f_{i+1} + \ldots\ldots + \frac{s(s-1)\ldots(s+k-2)}{k!}\, \nabla^k f_{i+1},$$

so that,

$$y_{i+1}^{(1)} + y_{i+1}^{(2)} I = y_i^{(1)} + y_i^{(2)} I$$

$$+ (h_1 + h_2 I) \int_{x_i^{(1)} + x_i^{(2)} I}^{x_{i+1}^{(1)} + x_{i+1}^{(2)} I} \left[f_{i-1} + (s-1)\nabla f_{i+1} + \frac{1}{2!}\,(s-1)s\nabla^2 f_{i+1} + \ldots\ldots\right] ds$$

this can be written as,

$$y_{i+1}^{(1)} + y_{i+1}^{(2)} I = y_i^{(1)} + y_i^{(2)} I + (h_1 + h_2 I) \int_{-1}^{0} [f_{i+1} + \ldots\ldots] ds = y_i^{(1)} + y_i^{(2)} I$$

$$+ (h_1 + h_2 I)\left[f_{i+1} - \frac{1}{2}\nabla f_{i+1} - \frac{1}{12}\,\nabla^2 f_{i+1} - \frac{1}{24}\nabla^3 f_{i+1}\right]$$

where,

$$\begin{cases} \nabla f_{i+1} = f_{i+1} - f_i \\ \nabla^2 f_{i+1} = f_{i+1} - 2f_i + f_{i-1} \\ \nabla^3 f_{i+1} = f_{i+1} - 3f_i + 3f_{i-1} - f_{i-2} \end{cases}$$

hence,

$$y_{i+1}^{(1)} + y_{i+1}^{(2)} I = y_i^{(1)} + y_i^{(2)} I + \frac{h_1 + h_2 I}{24}(9f_{i+1} + (19 + I)f_i - (5 + I)f_{i-1} + If_{i-2}).$$

Numerical Result:

Example 2.1 Consider the neutrosophic differential equation

$$(x_1 + x_2 I)(y_1 + y_2 I)" - (2 + I)(y_1 + y_2 I)' = (10 + I)(x_1 + x_2 I)^4$$

with initial conditions $(y_1 + y_2 I)(1 + I) = 2 + I$ and $(y_1 + y_2 I)'^{(1+I)} = 2 + I$

over the interval $0 \leq x_1 + x_2 I \leq 2$ with step size *h = 0.1+I*.

$$\begin{cases} (y_1 + y_2 I)' = Z_1 + Z_2 I \\ (Z_1 + Z_2 I)' = 10(x_1 + x_2 I)^3 + \dfrac{2}{x_1 + x_2 I}(Z_1 + Z_2 I) \end{cases}$$

with initial conditions $y(1) = 2$ and $z(2) = 2$. The exact solutions are:

$$y_1 + y_2 I = (x_1 + x_2 I)^5 - (x_1 + x_2 I)^3 + 2, \qquad Z_1 + Z_2 I = 5(x_1 + x_2 I)^5 - 3(x_1 + x_2 I)^2.$$

Example 2.2 Consider the neutrosophic differential equation $(y_1 + y_2 I)" - 2(y_1 + y_2 I)' = 4(x_1 + x_2 I)$ with initial conditions $(y_1 + y_2 I)(I) = 1 + I$ and $(y_1 + y_2 I)'(I) = 2 + I$ over the interval $0 \leq x_1 + x_2 I \leq 1$ with step size.

We first convert the second-order ODE into a system of first-order ODEs:

$$\begin{cases} (y_1 + y_2 I)' = Z_1 + Z_2 I \\ z' = 4(x_1 + x_2 I) + 2(Z_1 + Z_2 I) \end{cases}$$

with initial conditions $(y_1 + y_2 I)(I) = 1 + I$ and $(Z_1 + Z_2 I)(I) = 2 + I$.

The exact solutions are:

$$y_1 + y_2 I = -((x_1 + x_2 I)^2 + x_1 + x_2 I) + 1.5\exp(2(x_1 + x_2 I)) - 0.5,$$

$$Z_1 + Z_2 I = -(2(x_1 + x_2 I) + 1) + 3\exp(2(x_1 + x_2 I)).$$

Table 1. Approximation solution of 1 by using (Euler), ABM (RK) and (MS)

and with step size $h = 0.1+I$.

$x_1 + x_2 I$	Exact	ABM	ABM (RK)	AMM	Error (ABM)	Error (AMM)	Error MS
1+I	2+I	2+I	2+I	2+I	I	I	I
1.1+I	1.3998 +I	1.3228 +I	1.3568 +I	1.36918 +I	0.0445+I	0.0445+I	0.0445+I
1.2+I	2.5578+I	2.52278+I	2.5178+I	2.54568+I	0.121032+I	0.121032+I	0.121032+I
1.3+I	3.31006+I	3.224006+I	3.31176+I	3.328706+I	0.33101182+I	0.33101182+I	0.332102+I
1.4+I	4.23115+I	4.116115+I	4.23235+I	4.23385+I	0.76642161+I	0.76642161+I	0.64264216+I
1.5+I	6.3095+I	6.304595+I	6.35615+I	6.37915+I	0.948211621+I	0.948211621+I	0.948211621+I
1.6+I	8.441002+I	8.439002+I	8.49802+I	8.4299812+I	1.18813780+I	1.18813780+I	1.18813780+I
1.7+I	9.446+I	9.4216+I	9.455427+I	9.487646+I	1.36687140+I	1.36687140+I	1.366210140+I
1.8+I	16.88901+I	16.8789901+I	16.878701+I	16.8101901+I	1.671715365+I	1.671715365+I	1.6421715365+I
1.9+I	17.00123+I	17.001153+I	17.057913+I	17.0016413+I	2.096091486+I	2.096091486+I	2.092101486+I
2 +I	22.332102+I	22.321902+I	22.31102+I	22.3498102+I	2.443149+I	2.34419+I	2.88764+I

Table 2. Approximation solution of 1 using (Euler), ABM (RK) and

(MS) and the error estimation for step size $h = 0.1+I$.

x	Exact	ABM	ABM (RK)	MS	Error ABM	Error AMM	Error MS
1+I	2+I	2+I	2+I	2+I	I	I	I
1.1+I	3.4439+I	3.4+I	3.32836065+I	3.699355+I	0.22345+I	0.0000395+I	0.00003935+I
1.2+I	6.02214+I	5.34918182+I	6.64191587+I	6.04793587+I	0.371918+I	0.0000843+I	0.001348413+I
1.3+I	9.11231+I	7.88971212+I	9.2164194+I	9.01677594+I	1.24178788+I	0.0001346+I	0.008275136+I
1.4+I	13.2593+I	11.83823047+I	13.326413+I	11.327881+I	1.4482759+I	0.000153+I	0.08275011+I
1.5+I	18.18896+I	16+I.838271+I	18.32832+I	16.562332+I	1.6597442+I	0.000178+I	0.08275016+I
1.6+I	23.044380+I	23.88979212+I	25.08641+I	23.016772+I	1.8892078+I	0.000203+I	0.013416177+I
1.7+I	30.11345+I	30.95766292+I	33.0641005+I	30.09028981+I	2.13827508+I	0.0002295+I	0.013421019+I
1.8+I	41.445+I	40.37687609+I	42.32834220+I	41.7935046+I	2.382752391+I	0.00025780+I	0.0134020954+I
1.9+I	52.3987+I	51.88931560+I	54.3641126+I	52.3935172+I	2.68275440+I	0.00028724+I	0.082752588+I
2+I	66+I	65.04838248+I	67. 3283173+I	67.90167747+I	2.95827517+I	0.00031827+I	0.013426253+I

Table 3. Approximation solution of 1 using ABM (Euler), ABM (RK) and (MS)

and the error estimation for step size $h = 0.05+I$.

x	Exact	ABM	AMM	MS	Error ABM	Error AMM	Error MS
1+I	2+I	2+I	2+I	2+I	I	I	I
1.1+I	2.25541+I	2.28871+I	2.5560897+I	2.27950897+I	0.00003935+I	0.22345+I	0.00003935+I
1.2+I	2.76032+I	2.44914306+I	2.76031761+I	2.76031904+I	0.001348413+I	0.371918+I	0.00008413+I
1.3+I	3.767593+I	3.88181936+I	3.539352895+I	3.51592900+I	0.00827513406+I	1.24178788+I	0.00013406+I
1.4+I	4.332424+I	4.6688294+I	4. 393524018+I	4.110890+I	0.08275011894+I	1.448275953+I	0.00015623+I
1.5+I	6.115187+I	5.171701+I	6.21875130+I	6.213142+I	0.08275016779+I	1.65974429+I	0.00017896+I
1.6+I	8.48976+I	8.065518+I	8.38976230+I	8.3590146+I	0.013416177+I	1.88920788+I	0.00020369+I
1.7+I	11.44157+I	10.491675+I	11.2853935+I	11.281736811+I	0.013421019+I	2.13827508+I	0.00022995+I
1.8+I	15.069168+I	13.114943+I	15.082755389+I	15.05417767+I	0.0134020954+I	2.382752391+I	0.00025780+I
1.9+I	19.80199+I	17.377873+I	19.90199447+I	19.214713+I	0.0827525828+I	2.68275440+I	0.00028724+I
2+I	25+I	22.1991698+I	26.00082759+I	25.5579649+I	0.013426253+I	2.95827517+I	0.00031827+I

Table 4. Approximation solution of 1 of using ABM (Euler), ABM (RK) and

(MS) and the error estimation for step size $h = 0.05+I$.

x	Exact	ABM	AMM	MS	Error ABM	Error AMM	Error MS
1+I	2+I	2+I	2+I	2+I	I	I	I
1.1+I	3.67787+ I	3.5357787+I	3.6907787+I	3.69049724+I	0.1545446+I	0.000113+I	0.0000115+I
1.2+I	6. 169+I	5.761693+I	6.01699169+I	6.04799663+I	0.27828957+I	0.0000478+I	0.0000489+I
1.3+I	9. 4415+I	8. 4415+I	9. 4415+I	9.21049590+I	0.32617295+I	0.0000658+I	0.000066741+I
1.4+I	13.68275+I	12. 68275+I	13. 68275+I	13.32799513+I	0.37828757+I	0.0000513+I	0.00000487+I
1.5+I	16.04970+I	18.04972587+I	18.04979431+I	18.56249431+I	0.43425871+I	0.000722+I	0.0000688+I
1.6+I	22.13149+I	24.59391009+I	25.08799170+I	25.08799344+I	0.49408991+I	0.0008872+I	0.0000456+I
1.7+I	33.2286+I	32.5394089+I	33.0994089+I	33.09049251+I	0.55778118+I	0.0016937+I	0.0011749+I
1.8+I	42.6678+I	42.14266746+I	42.76798949+I	42.76799153+I	0.62533254+I	0.00241051+I	0.00021847+I
1.9+I	54.40918+I	53.63375603+I	54.33048829+I	54.33049050+I	0.69674397+I	0.00351171+I	0.0031950+I
2+I	68.33209+I	67.22798452+I	67.99998703+I	67.99998942+I	0.77201548+I	0.0041297+I	0.00421058+I

Definition 2.3 (Adams Bashforth Method of dual variable) Consider the dual (IVP):

$$(y_1 + y_2J)'(x_1 + x_2J) = f(x_1 + x_2J, y_1 + y_2J),$$

so that,

$$(y_1 + y_2J)\left(x_{i+1}^{(1)} + x_{i+1}^{(2)}J\right)$$

$$= (y_1 + y_2J)\left(x_i^{(1)} + x_i^{(2)}J\right) + \int_{x_i^{(1)}+x_i^{(2)}J}^{x_{i+1}^{(1)}+x_{i+1}^{(2)}J} f(x_1 + x_2J, y_1 + y_2J)d(x_1 + x_2J)$$

$$P_{k-1}(x_1 + x_2J) = f_i + \frac{\left(x_1 - x_i^{(1)}\right) + J\left(x_2 - x_i^{(2)}\right)}{h_1 + h_2J}\nabla f_i$$

$$+ \frac{\left[\left(x_1 - x_i^{(1)}\right) + J\left(x_2 - x_i^{(2)}\right)\right]\left[\left(x_1 - x_{i-1}^{(1)}\right) + J\left(x_2 - x_{i-1}^{(2)}\right)\right]}{2!\,(h_1 + h_2J)}\nabla^2 f_i + \ldots\ldots$$

$$+ \frac{\left[\left(x_1 - x_i^{(1)}\right) + J(x_2 - x_i^{(2)})\right]\left[\left(x_1 - x_{i-1}^{(1)}\right) + J(x_2 - x_{i-1}^{(2)})\right]\ldots\ldots\left[\left(x_1 - x_{i-k+2}^{(1)}\right) + J(x_2 - x_{i-k+2}^{(2)})\right]}{(k-1)!\,(h_1 + h_2J)^{k-1}}\nabla^{k-1} f_i$$

we put,

$$\left(x_1 - x_i^{(1)}\right) + I\left(x_2 - x_i^{(2)}\right) = (h_1 + h_2J)s,$$

then,

$$P_{k-1} = f_i + s\nabla f_i + \frac{1}{2!}\,s(s+1)\nabla^2 f_i + \frac{1}{6}\,s(s+1)(s+2)\nabla^3 f_i + \ldots\ldots + \frac{s(s+1)\ldots(s+k-2)}{(k-1)!}\,\nabla^{k-1} f_i,$$

this means that,

$$(y_1 + y_2J)\left(x_{i+1}^{(1)} + x_{i+1}^{(2)}J\right) = (y_1 + y_2J)\left(x_i^{(1)} + x_i^{(2)}J\right) + \int_{x_i^{(1)}+x_i^{(2)}J}^{x_{i+1}^{(1)}+x_{i+1}^{(2)}J} [f_i + s\nabla f_i + \ldots\ldots]d(x_1 + x_2J)$$

we get,

$$(y_1 + y_2J)\left(x_{i+1}^{(1)} + x_{i+1}^{(2)}J\right)$$

$$= (y_1 + y_2J)\left(x_i^{(1)} + x_i^{(2)}J\right) + \int_0^1 [f_i + s\nabla f_i + \ \ldots\ldots +]d(x_1 + x_2J) = (y_1 + y_2J)$$

$$+ (h_1 + h_2J)(f_i + \frac{1}{2}\nabla f_i + \frac{5}{12}\ \nabla^2 f_i + \frac{3}{8}\nabla^3 f_i)$$

thus,

$$\begin{cases} \nabla f_i = f_i - f_{i-1} \\ \nabla^2 f_i = f_i - 2f_{i-1} + f_{i-2} \\ \nabla^3 f_i = f_i - 3f_{i-1} + 3f_{i-2} - f_{i-3} \end{cases}$$

thus,

$$y_{i+1}^{(1)} + y_{i+1}^{(2)}J = y_i^{(1)} + y_i^{(2)}J + \frac{h_1 + h_2J}{24}(55f_i - 59f_{i-1} + 37f_{i-2} - 9f_{i-3}).$$

the error is:

$$TE = \frac{251 + J}{720 + J}(h_1 + h_2J)^5(y_1 + y_2J)^5.\varepsilon_4.$$

Definition2.4 (Adams Moulton Method for Dual variables): Consider the dual (IVP):

$$(y_1 + y_2J)'(x_1 + x_2J) = f(x_1 + x_2J, y_1 + y_2J)\ \ ;\ x_i, y_i \in \mathbb{R}$$

by integration, we get,

$$(y_1 + y_2J)\left(x_{i+1}^{(1)} + x_{i+1}^{(2)}J\right)$$

$$= (y_1 + y_2J)\left(x_i^{(1)} + x_i^{(2)}J\right) + \int_{x_i^{(1)} + x_i^{(2)}J}^{x_{i+1}^{(1)} + x_{i+1}^{(2)}J} f(x_1 + x_2J, y_1 + y_2J)d(x_1 + x_2J)$$

by using Newton backward interpolating polynomial of degree (*k*),

we get,

$$P_k(x_1 + x_2J) = f_{i+1} + \frac{\left(x_1 - x_i^{(1)}\right) + J\left(x_2 - x_i^{(2)}\right) + 1}{h_1 + h_2J}\nabla f_i + \ldots\ldots$$

$$+\frac{\left[\left(x_1-x_i^{(1)}\right)+J\left(x_2-x_i^{(2)}\right)+1\right]\left[\left(x_1-x_{i-1}^{(1)}\right)+J(x_2-x_{i-1}^{(2)})\right]\ldots..\left[\left(x_1-x_{i-k+2}^{(1)}\right)+I(x_2-x_{i-k+2}^{(2)})\right]}{(k-1)!\,(h_1+h_2I)^{k-1}}\nabla^k f_{i+1}$$

by putting,

$$(h_1+h_2J)s=\left(x_1-x_i^{(1)}\right)+J\left(x_2-x_i^{(2)}\right),$$

we get,

$$P_k(x_1+x_2J)=f_{i+1}+(s-1)\nabla f_{i+1}+.\ldots.+\frac{s(s-1)\ldots(s+k-2)}{k!}\,\nabla^k f_{i+1},$$

so that,

$$y_{i+1}^{(1)}+y_{i+1}^{(2)}J=y_i^{(1)}+y_i^{(2)}J$$

$$+(h_1+h_2J)\int_{x_i^{(1)}+x_i^{(2)}J}^{x_{i+1}^{(1)}+x_{i+1}^{(2)}J}\left[f_{i-1}+(s-1)\nabla f_{i+1}+\frac{1}{2!}\,(s-1)s\nabla^2 f_{i+1}+\ \ldots..\right]ds$$

this can be written as

$$y_{i+1}^{(1)}+y_{i+1}^{(2)}J=y_i^{(1)}+y_i^{(2)}J+(h_1+h_2J)\int_{-1}^{0}[f_{i+1}+\ \ldots..]ds=y_i^{(1)}+y_i^{(2)}J$$

$$+(h_1+h_2J)\left[f_{i+1}-\frac{1}{2}\nabla f_{i+1}-\frac{1}{12}\,\nabla^2 f_{i+1}-\frac{1}{24}\nabla^3 f_{i+1}\right]$$

where,

$$\begin{cases}\nabla f_{i+1}=f_{i+1}-f_i\\ \nabla^2 f_{i+1}=f_{i+1}-2f_i+f_{i-1}\\ \nabla^3 f_{i+1}=f_{i+1}-3f_i+3f_{i-1}-f_{i-2}\end{cases}$$

hence,

$$y_{i+1}^{(1)}+y_{i+1}^{(2)}J=y_i^{(1)}+y_i^{(2)}J+\frac{h_1+h_2J}{24}(9f_{i+1}+(19+J)f_i-(5+J)f_{i-1}+Jf_{i-2}).$$

Numerical Result:

Example 2.3 Consider the dual differential equation

$$(x_1 + x_2J)(y_1 + y_2J)'' - (2 + J)(y_1 + y_2J)' = (10 + J)(x_1 + x_2J)^4$$

with initial conditions $(y_1 + y_2J)(1 + J) = 2 + J$ and $(y_1 + y_2J)'^{(1+J)} = 2 + J$

over the interval $0 \leq x_1 + x_2J \leq 2$ with step size h = 0.1+J.

$$\begin{cases} (y_1 + y_2J)' = Z_1 + Z_2J \\ (Z_1 + Z_2J)' = 10(x_1 + x_2J)^3 + \dfrac{2}{x_1 + x_2J}(Z_1 + Z_2J) \end{cases}$$

with initial conditions $y(1) = 2$ and $z(2) = 2$.

The exact solutions are:

$$y_1 + y_2I = (x_1 + x_2I)^5 - (x_1 + x_2I)^3 + 2, \qquad Z_1 + Z_2I = 5(x_1 + x_2I)^5 - 3(x_1 + x_2I)^2.$$

Table 5. Approximation solution of 3 by using (dual Euler), dual ABM (RK) and dual (MS) and with step size *h = 0.1+J*.

$x_1 + x_2$**J**	**Exact**	**ABM**	**ABM (RK)**	**AMM**	**Error (ABM)**	**Error (AMM)**	**Error MS**
1+J	2+ J	2+ J	2+ J	2+ J	J	J	J
1.1+J	1.3998 + J	1.3228 + J	1.3568 + J	1.36918 + J	0.0445+ J	0.0445+ J	0.0445+ J
1.2+J	2.5578+ J	2.52278+ J	2.5178+ J	2.54568+ J	0.121032+ J	0.121032+ J	0.121032+ J
1.3+J	3.31006+ J	3.224006+ J	3.31176+ J	3.328706+ J	0.33101182+J	0.33101182+ J	0.332102+ J
1.4+J	4.23115+ J	4.116115+ J	4.23235+ J	4.23385+ J	0.76642161+J	0.76642161+J	0.642642161+J
1.5+J	6.3095+ J	6.304595+ J	6.35615+ J	6.37915+ J	0.948211621+J	0.948211621+J	0.948211621+J
1.6+J	8.441002+ J	8.439002+ J	8.49802+ J	8.4299812+ J	1.18813780+ J	1.18813780+ J	1.18813780+I
1.7+J	9.446+ J	9.4216+ J	9.455427+ J	9.487646+ J	1.36687140+ J	1.36687140+ J	1.366210140+J
1.8+J	16.88901+ J	16.8789901+J	16.878701+J	16.8101901+J	1.671715365+J	1.671715365+J	1.6421715365+J
1.9+J	17.00123+I	17.001153+J	17.057913+J	17.0016413+J	2.096091486+J	2.096091486+J	2.092101486+ J
2+J	22.332102+J	22.321902+J	22.31102+ J	22.3498102+J	2.443149+ J	2.34419+ J	2.88764+ J

Table 6. Approximation solution of 3 using (Dual Euler), dual ABM (RK) and Dual (MS) and the error estimation for step size $h = 0.1+J$.

x	Exact	ABM	ABM (RK)	MS	Error ABM	Error AMM	Error MS
1+ J	2+ J	2+ J	2+ J	2+ J	J	J	J
1.1+J	3.4439+ J	3.4+ J	3.32836065+J	3.699355+ J	0.22345+ J	0.00003935+J	0.00003935+ J
1.2+J	6.02214+I	5.34918182+J	6.64191587+J	6.04793587+J	0.371918+ J	0.00008413+J	0.001348413+ J
1.3+J	9.11231+ J	7. 88971212+ J	9.2164194+ J	9. 01677594+ J	1.24178788+J	0.00013406+J	0.00827513406+J
1.4+J	13.2593+ J	11. 83823047	13.32641377+J	11.32788106+J	1.448275953+J	0.00015623+J	0.08275011894+J
1.5+J	18.188916+J	16838275571+J	18.32832104+J	16.56233221+J	1.65974429 JJ	0.00017896+J	0.08275016779+J
1.6+J	23.044380+J	23. 88979212+J	25.0864131+ J	23. 0167723+ J	1.88920788+J	0.00020369+J	0.013416177+ J
1.7+J	30.113405+J	30.95766292+J	33.0641005+ J	30.09028981+J	2.13827508+J	0.00022995+J	0.013421019+ J
1.8+J	41.445+ J	40.37687609+ J	42.32834220+J	41.7935046+ J	2.382752391+J	0.00025780+J	0.0134020954+ J
1.9+J	52.3987+ J	51. 88931560+J	54.36411276+J	52.3935172+ J	2.68275440+ J	0.00028724+J	0.0827525828+ J
2+ J	66+ J	65.04838248+ J	67. 3283173+ J	67.90167747+J	2.95827517+ J	0.00031827+J	0.013426253+ J

Table 7. Approximation solution of 3 using dual ABM (Euler), dual ABM (RK) and dual (MS) and the error estimation for step size $h = 0.05+ J$.

x	Exact	ABM	AMM	MS	Error ABM	Error AMM	Error MS
1+J	2+ J	2+I	2+ J	2+ J	J	J	J
1.+ J	2.25541+ J	2.28871+ J	2.5560897+ J	2.27950897+ J	0.00003935+I	0.22345+ J	0.0000393+J
1.2+J	2.76032+ J	2.44914306+J	2.76031761+ J	2.76031904+ J	0.001348413+I	0.371918+ J	0.00008413+J
1.3+J	3.767593+J	3.88181936+J	3.539352895+J	3.51592900+ J	0.00827513406+J	1.24178788+J	0.00013406+J
.4+J	4.332424+J	4.6688294+ J	4.393524018+J	4.110890+ J	0.08275011894+J	1.448275953+J	0.00015623+J
1.5+J	6.115187+ J	5.171701+ J	6.21875130+ J	6.213142+ J	0.08275016779+J	1.65974429+ J	0.00017896+J
1.6+J	8.48976+ J	8.065518+ J	8.38976230+ J	8.3590146+ J	0.013416177+ J	1.88920788+ J	0.00020369+J
1.7+J	11.44157+ J	10.491675+ J	11.2853935+ J	11.281736811+J	0.013421019+ J	2.13827508+ J	0.00022995+J
1.8+J	15.069168+J	13.114943+ J	15.082755389+J	15.05417767+ J	0.0134020954+ J	2.382752391+J	0.00025780+J
1.9+J	19.80199+ J	17.377873+ J	19.90199447 J	19.214713+ J	0.0827525828+ J	2.68275440+ J	0.00028724+J
2+ J	25+ J	22.1991698+J	26.00082759+ J	25.5579649+ J	0.013426253+ J	2.95827517+ J	0.0031827+ J

Table 8. Approximation solution of 3 of using dual ABM (Euler), dual ABM (RK) and Dual (MS) and the error estimation for step size $h = 0.05+J$.

x	Exact	ABM	AMM	MS	Error ABM	Error AMM	Error MS
1+J	2+ J	2+ J	2+ J	2+ J	J	J	J
1+J	3.67787J	3.5357787+J	3.6907787+J	3.69049724**+**J	0.1545446**+**J	0.000113+J	0.0000115+J
1+J	6. 169+ J	5.761693+ J	6.01699169+ J	6.04799663+ J	0.27828957+J	0.0000478+J	0.0000489+ J
1+J	9. 4415+ J	8. 4415+ J	9. 4415+ J	9.21049590+ J	0.32617295+J	0.0000658+J	0.000066741+J
1.4+J	13.68275+J	12. 68275+ J	13. 68275+ J	13.32799513+J	0.37828757+J	0.0000513+J	0.00000487+ J
1.5+J	16.04970+J	18.04972587+J	18.04979431+J	18.56249431+J	0.43425871+J	0.000722+ J	0.0000688+ J
1.6+J	22.13149+J	24.59391009+J	25.08799170+J	25.08799344+J	0.49408991+J	0.0008872+J	0.0000456+ J
1.7+J	33.2286+ J	32.5394089+J	33.0994089+ J	33.09049251+J	0.55778118+J	0.0016937+J	0.0011749+ J
1.8+J	42.6678+ J	42.14266746+J	42.76798949+J	42.7699153+ J	0.62533254+J	0.00241051+J	0.00021847+ J
1.9+	54.40918+	53.63375603+	54.33048829+	54.33049050+	0.69674397+	0.00351171+	0.0031950+ J
2+J	68.3320+J	67.2279845+ J	67.99998703+J	67.99998942+J	0.77201548+J	0.0041297+ J	0.00421058+ J

Conclusion

In this paper, we have presented two neutrosophic numerical methods for finding the numerical solutions for some neutrosophic differential equations, where a numerical comparison between AMM and ABM methods with numerical tables that clarify some numerical results of the previous two methods applied on some neutrosophic initial value problems IVP. Also, we have applied the same methods on dual variables, and we got the same numerical solutions. We have noticed from the numerical study and application of the two mentioned methods that they provide good results compared to exact solutions, with an acceptable small error of approximation. We hope that in the future the possibility of applying some other numerical methods to neutrosophic problems will be discussed and the numerical results compared through numerical tables.

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Landscape Design Quality Evaluation of Abandoned Coal Mine Sites Using Single-Valued Triangular Neutrosophic Numbers

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Abstract: This research uses Single-Valued Triangular Neutrosophic Numbers (SVTNNs) to address uncertainty and vague problems in the Abandoned Coal Mine Sites evaluation. By adjusting the subjective assessments, Single-Valued Triangular Neutrosophic Numbers handle degrees of affiliation and disassociation as well as additional unclear degrees of neutral opinion. The WENSLO approach is used to compute criteria weights. To develop a hybrid model that handles ambiguity more adaptable, this study combines the WENSLO method with the Single-Valued Triangular Neutrosophic Numbers-based REGIME strategy. In conclusion, we used these methods to confirm the precision of the calculations made using the suggested approach. The fundamental idea behind the REGIME technique is to use a preference function to capture the degree of dominance and rank alternatives by evaluating their pairwise performance across several criteria. Combining qualitative and quantitative data eliminates the need for preset weights and preserves consistency in the way decisions are made. For simplicity of comprehension, a flowchart is provided to show the Single-Valued Triangular Neutrosophic Numbers WENSLO-REGIME approach. The Single-Valued Triangular Neutrosophic Numbers WENSLO-REGIME approach is used to solve a thorough case study and identify the optimal alternative to evaluate the effectiveness of the proposed method. Analysis of sensitivity is carried out.

Keywords: Single-Valued Triangular Neutrosophic Numbers; Abandoned Coal Mine Sites; WENSLO Method; REGIME Method.

1. Introduction

A country may benefit economically from coal extraction. However, if safeguards are not taken, it could cause significant environmental harm. Degradation of the land, topographical changes, pollution of the soil and water, and ecological decline can all result from mining operations. Another environmental issue associated with coal mining is acid mine drainage, which can cause soil and water bodies to become more acidic[1], [2]. Acidification would accelerate the breakdown of hazardous metals from open pits, waste rock heaps, and water-permeable tailings. Ecosystems

in the soil and water will be affected in exchange. Abandoned mining sites may also be linked to these issues[3], [4]. For years, these locations have kept contaminating the land and water.

For instance, Navarro et al. demonstrate that metals released downstream and downslope from tailings by surface runoff continue to contaminate the environment at an abandoned mining site in Cabezo Rajao, Spain[5], [6].

Contaminants from coal mining operations can travel great distances. The primary transport mechanisms are mechanical dispersion brought on by wind action and surface runoff. Transport is further improved by enhanced acidification-induced breakdown of hazardous metals[7], [8]. The significance of streams and surface runoff channels in the movement of pollutants from coal mining sites has been demonstrated in a small number of case studies. For example, the Handlova–Cigel brown coal district's freely flowing water carried a lot of harmful substances into surface streams, which subsequently poisoned a larger area. In 1980, the Ohio River in Pittsburgh received 1.2 million tons of sulfate from the Allegheny River and 1.35 million tons from the Monongahela River. Sams and Beer demonstrated that the trends in coal production rates in sub-basins that contribute to pollution transport are consistent with the trends in pollutant concentrations over time[9], [10]. This study evaluates the Abandoned Coal Mine Sites by using a set of criteria.

Traditional MCDM models need precise data, which might not always be available in practical settings. However, data are often complex, imprecise, and unstable, making it impossible to measure them precisely. The theory of fuzzy sets (FSs) was first put forth by Zadeh in opposition to specific logic. Numerous researchers investigated this subject after this study; specifics of various methods are visible. Additionally, several scholars put out certain MCDM models in a fuzzy context.

But Zadeh's fuzzy sets only consider the membership function; they are unable to handle other vague characteristics. Atanassov developed intuitionistic fuzzy sets (IFSs), an extension of FSs, to address this knowledge gap. Additionally, several MCDM models use intuitionistic fuzzy data[11], [12].

While incomplete information for a variety of real-world problems can be handled by the theory of IFSs, it is unable to manage all forms of uncertainty, including inconsistent and ambiguous data. Smarandache therefore developed the neutrosophic set (NS) as a strong generic framework that extends classical and various fuzzy sets (FSs and IFSs)[13], [14].

When the indeterminacy is quantified, NSs can handle contradictory, ambiguous, and indeterminate information. They can also separately create three types of membership functions. A variety of NSs have been introduced in recent years, including interval neutrosophic sets, bipolar neutrosophic sets, single-valued neutrosophic sets, and neutrosophic linguistic sets[15], [16]. Furthermore, logic, measure, probability, statistics, pre-calculus, calculus, and their applications in various fields have been expanded in the field of neutrosophic sets.

2. Basic Definitions

The neutrosophic set, as defined by Smarandache, is distinguished by its truth-membership function, indeterminacy-membership function, and falsity membership function. From a philosophical perspective, the idea of a neutrosophic set was developed to convey ambiguous and inconsistent data[17], [18]. The following is a definition of a neutrophilic set:

Definition 1

Let the $a = \left((a,b,c); T_a, I_a, F_a\right)$ and their membership functions can be computed such as:

$$T_a(x) = \begin{cases} \frac{(x-a)T_a}{b-a}, & a \le x \le b \\ \frac{(c-x)T_a}{c-b}, & b \le x \le c \\ 0 \quad otherwise \end{cases} \tag{1}$$

$$I_a(x) = \begin{cases} \frac{(b-x+I_a\,(x-a)\,)}{b-a}, & a \le x \le b \\ \frac{(x-b+I_a(c-x))}{c-b}, & b \le x \le c \\ 0 \quad otherwise \end{cases} \tag{2}$$

$$F_a(x) = \begin{cases} \frac{(b-x+F_a\,(x-a)\,)}{b-a}, & a \le x \le b \\ \frac{(x-b+F_a(c-x))}{c-b}, & b \le x \le c \\ 0 \quad otherwise \end{cases} \tag{3}$$

Let two single valued triangular neutrosophic numbers (SVTNNs) such as:

$$a_1 = \left((a_1,b_1,c_1); T_{a_1}(x), I_{a_1}(x), F_{a_1}(x)\right)$$

$$a_2 = \left((a_2,b_2,c_2); T_{a_2}(x), I_{a_2}(x), F_{a_2}(x)\right)$$

$$a_1a_2 = \begin{pmatrix} \left((a_1a_2, b_1b_2, c_1c_2); \min\{T_{a_1}(x), T_{a_2}(x)\}, \right. \\ \left. \max\{I_{a_1}(x), I_{a_2}(x)\}, \max\{F_{a_1}(x), F_{a_2}(x)\} \right. \\ \left. , \ (c_1 > 0, c_2 > 0) \right) \\ \left((a_1c_2, b_1b_2, c_1a_2); \min\{T_{a_1}(x), T_{a_2}(x)\} \right. \\ \left. , \max\{I_{a_1}(x), I_{a_2}(x)\}, \max\{F_{a_1}(x), F_{a_2}(x)\} \right. \\ \left. , \ (c_1 < 0, c_2 > 0) \right), \\ \left((c_1c_2, b_1b_2, a_1a_2); \min\{T_{a_1}(x), T_{a_2}(x)\}, \right. \\ \left. \max\{I_{a_1}(x), I_{a_2}(x)\}, \max\{F_{a_1}(x), F_{a_2}(x)\} \right. \\ \left. , \ (c_1 < 0, c_2 < 0) \right) \end{pmatrix} \tag{4}$$

$$a_1 + a_2 = \begin{pmatrix} (a_1 + a_2, b_1 + b_1.\, c_1 + c_1); \min\{T_{a_1}(x), T_{a_2}(x)\}; \\ \max\{i_{a_1}(x), I_{a_2}(x)\}; \\ \max\{F_{a_1}(x), F_{a_2}(x)\} \end{pmatrix} \tag{5}$$

$$\varphi a_1 = \left((\varphi a_1, \varphi b_1, \varphi c_1); T_{a_1}(x), I_{a_1}(x), F_{a_1}(x)\right) \tag{6}$$

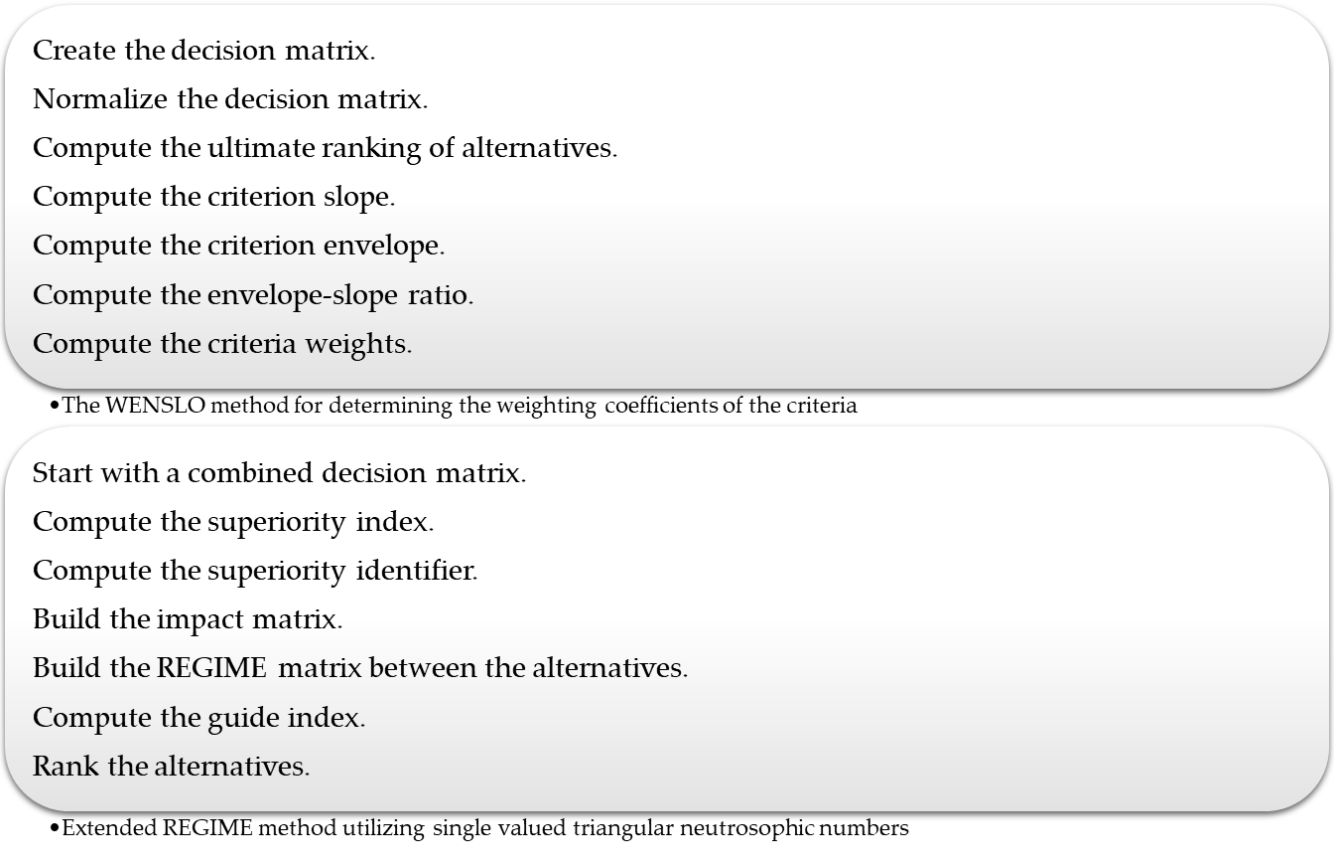
Create the decision matrix.
Normalize the decision matrix.
Compute the ultimate ranking of alternatives.
Compute the criterion slope.
Compute the criterion envelope.
Compute the envelope-slope ratio.
Compute the criteria weights.
•The WENSLO method for determining the weighting coefficients of the criteria
Start with a combined decision matrix.
Compute the superiority index.
Compute the superiority identifier.
Build the impact matrix.
Build the REGIME matrix between the alternatives.
Compute the guide index.
Rank the alternatives.
•Extended REGIME method utilizing single valued triangular neutrosophic numbers

Fig 1. The MCDM steps.

3. MCDM Approach

This section shows the steps of the proposed approach. Fig 1 shows the steps of the proposed approach.

The WENSLO method for determining the weighting coefficients of the criteria.

It is advised that criteria weights be determined using the methods suggested in this study. For this reason, Pamucar et al. have developed a novel technique that falls under the domain of objective weighing methods and is called WENSLO (Weights by ENvelope and SLOpe). The technique depends on each criterion's envelope-to-slope ratio[19]. The computation process is simple and easy to comprehend. When the slope value is low and the envelope value is large, a criterion is given a higher weight[19]. The basic phases of the WENSLO technique for determining criteria weight are described in the next section.

Step 1. Create the decision matrix

$$[x_{ij}]_{m\times n} = \begin{pmatrix} x_{11} & \cdots & x_{1n} \\ \vdots & \ddots & \vdots \\ x_{m1} & \cdots & x_{mn} \end{pmatrix} \tag{7}$$

Step 2. Normalize the decision matrix

$$r_{ij} = \frac{x_{ij}}{\sum_{i=1}^{m} x_{ij}} \tag{8}$$

Step 3. Compute the ultimate ranking of alternatives

$$z_j = \frac{\max\limits_{1<i\leq m} r_{ij} - \min\limits_{1<i\leq m} r_{ij}}{1+3.322\times\log(m)} \tag{9}$$

Step 4. Compute the criterion slope

$$q_j = \frac{\sum_{i=1}^{m} r_{ij}}{(m-1)\times z_j} \tag{10}$$

Step 5. Compute the criterion envelope.

$$N_j = \sum_{i=1}^{m-1} \sqrt{\left(r_{i+1j} - z_{ij}\right)^2 + z_j^2} \tag{11}$$

Step 6. Compute the envelope-slope ratio

$$s_j = \frac{N_j}{q_j} \tag{12}$$

Step 7. Compute the criteria weights.

$$w_j = \frac{s_j}{\sum_{i=1}^{n} s_j} \tag{13}$$

Extended REGIME method utilizing single valued triangular neutrosophic numbers[20]

Step 1. Start with a combined decision matrix.

Step 2. Compute the superiority index E_{fl}.

Step 3. Compute the superiority identifier.

$$E_{fl} = \sum_{j\in E_{fl}} w_j \tag{14}$$

Step 4. Build the impact matrix.

Step 5. Build the REGIME matrix between the alternatives.

$$E_{fl,j} = \begin{cases} -1 & if\ a_{fj} < a_{lj} \\ 0 & if\ a_{fj} = a_{lj} \\ +1 & if\ a_{fj} > a_{lj} \end{cases} \tag{15}$$

Step 6. Compute the guide index.

$$E_{fl} = \sum_{j=1}^{n} E_{fl}\, w_j \tag{16}$$

Step 7. Rank the alternatives.

4. An application

This section shows the outcomes of the proposed approach to computing the criteria weights and rank the alternatives. Six criteria and ten alternatives are used in this study as shown in Fig 2. These criteria and alternatives are evaluated using three experts and decision-makers who have experience in MCDM issues.

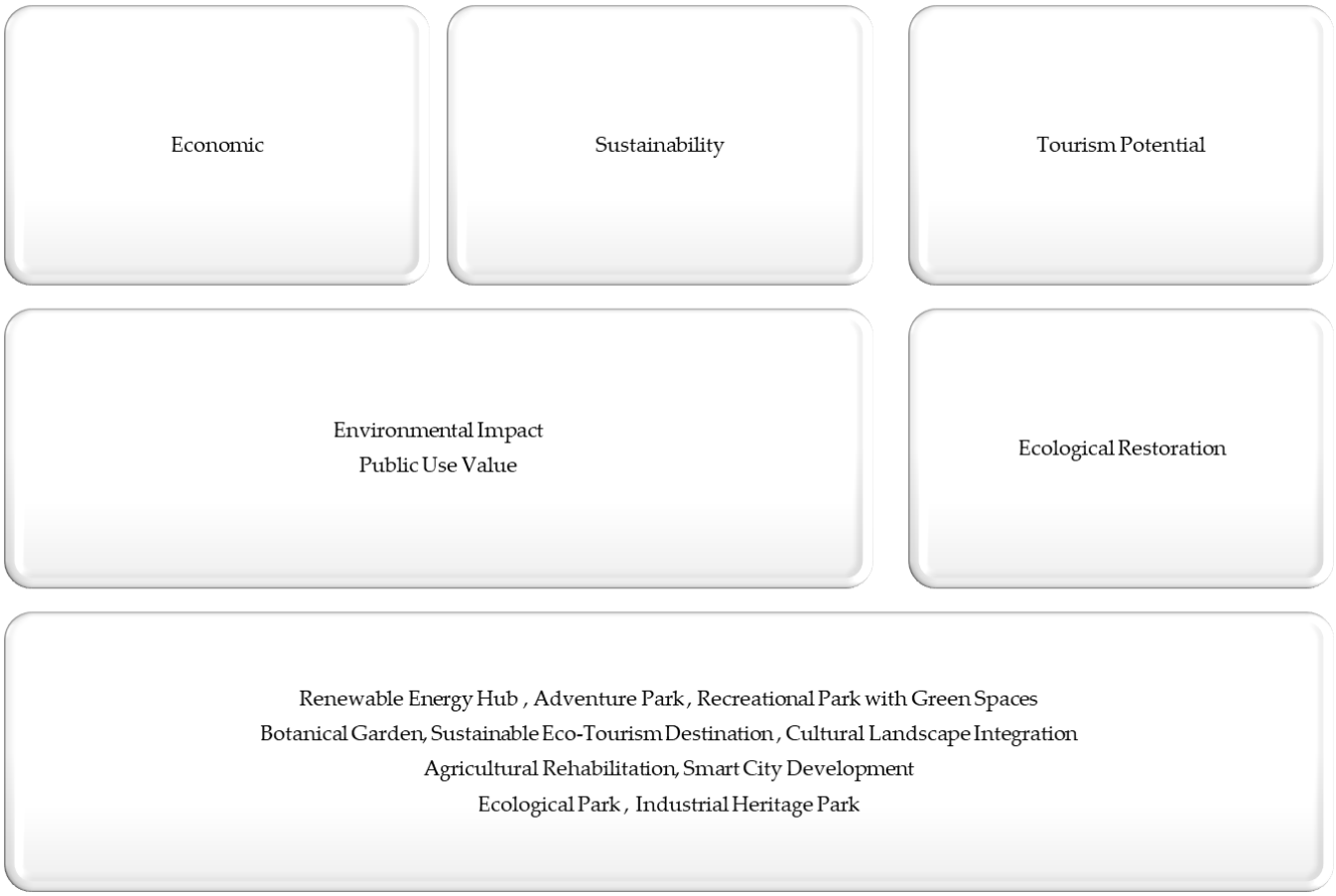
Economic
Sustainability
Tourism Potential
Environmental Impact
Public Use Value
Ecological Restoration
Renewable Energy Hub , Adventure Park , Recreational Park with Green Spaces
Botanical Garden, Sustainable Eco-Tourism Destination , Cultural Landscape Integration
Agricultural Rehabilitation, Smart City Development
Ecological Park , Industrial Heritage Park

Fig 2. The criteria of Abandoned Coal Mine Sites.

The results of the WENSLO method.

Step 1. Eq. (7) is used to create the decision matrix. We use the neutrosophic numbers to evaluate the decision matrix as shown in Table 1.

Step 2. Eq. (8) is used to normalize the decision matrix as shown in Table 2.

Step 3. Eq. (9) is used to Compute the ultimate ranking of alternatives.

Step 4. Eq. (10) is used to Compute the criterion slope.

Step 5. Eq. (11) is used to compute the criterion envelope.

Step 6. Eq. (12) is used to compute the envelope-slope ratio.

Step 7. Eq. (13) is used to compute the criteria weights as shown in fig 3.

Table 1. The SVTNNs matrix.

	C_1	C_2	C_3	C_4	C_5	C_6
A_1	((5.3, 7.3, 8.7); 0.7, 0.4, 0.8)	((6.2,8.9,9.1);0.6,0.3,0.5)	((7.1,7.7,8.3);0.5,0.2,0.4)	((4.4,5.9,7.2);0.7,0.2,0.3)	((4.4,5.9,7.2);0.7,0.2,0.3)	((5.3, 7.3, 8.7); 0.7, 0.4, 0.8)
A_2	((5.3, 7.3, 8.7); 0.7, 0.4, 0.8)	((4.4,5.9,7.2);0.7,0.2,0.3)	((5.3, 7.3, 8.7); 0.7, 0.4, 0.8)	((6.2,8.9,9.1);0.6,0.3,0.5)	((7.1,7.7,8.3);0.5,0.2,0.4)	((4.4,5.9,7.2);0.7,0.2,0.3)
A_3	((5.8,6.9,8.5);0.6,0.2,0.3)	((7.1,7.7,8.3);0.5,0.2,0.4)	((7.1,7.7,8.3);0.5,0.2,0.4)	((4.4,5.9,7.2);0.7,0.2,0.3)	((6.2,8.9,9.1);0.6,0.3,0.5)	((7.1,7.7,8.3);0.5,0.2,0.4)
A_4	((4.4,5.9,7.2);0.7,0.2,0.3)	((6.2,8.9,9.1);0.6,0.3,0.5)	((4.4,5.9,7.2);0.7,0.2,0.3)	((5.3, 7.3, 8.7); 0.7, 0.4, 0.8)	((5.3, 7.3, 8.7); 0.7, 0.4, 0.8)	((6.2,8.9,9.1);0.6,0.3,0.5)
A_5	((7.1,7.7,8.3);0.5,0.2,0.4)	((5.3, 7.3, 8.7); 0.7, 0.4, 0.8)	((7.1,7.7,8.3);0.5,0.2,0.4)	((4.4,5.9,7.2);0.7,0.2,0.3)	((4.4,5.9,7.2);0.7,0.2,0.3)	((5.3, 7.3, 8.7); 0.7, 0.4, 0.8)
A_6	((6.2,8.9,9.1);0.6,0.3,0.5)	((4.4,5.9,7.2);0.7,0.2,0.3)	((6.2,8.9,9.1);0.6,0.3,0.5)	((7.1,7.7,8.3);0.5,0.2,0.4)	((7.1,7.7,8.3);0.5,0.2,0.4)	((4.4,5.9,7.2);0.7,0.2,0.3)
A_7	((5.3, 7.3, 8.7); 0.7, 0.4, 0.8)	((7.1,7.7,8.3);0.5,0.2,0.4)	((5.3, 7.3, 8.7); 0.7, 0.4, 0.8)	((6.2,8.9,9.1);0.6,0.3,0.5)	((6.2,8.9,9.1);0.6,0.3,0.5)	((7.1,7.7,8.3);0.5,0.2,0.4)
A_8	((4.6,5.5,8.6);0.4,0.7,0.2)	((6.2,8.9,9.1);0.6,0.3,0.5)	((5.8,6.9,8.5);0.6,0.2,0.3)	((5.3, 7.3, 8.7); 0.7, 0.4, 0.8)	((5.3, 7.3, 8.7); 0.7, 0.4, 0.8)	((6.2,8.9,9.1);0.6,0.3,0.5)
A_9	((5.3, 7.3, 8.7); 0.7, 0.4, 0.8)	((5.3, 7.3, 8.7); 0.7, 0.4, 0.8)	((5.3, 7.3, 8.7); 0.7, 0.4, 0.8)	((5.3, 7.3, 8.7); 0.7, 0.4, 0.8)	((4.4,5.9,7.2);0.7,0.2,0.3)	((5.3, 7.3, 8.7); 0.7, 0.4, 0.8)
A_{10}	((5.8,6.9,8.5);0.6,0.2,0.3)	((4.4,5.9,7.2);0.7,0.2,0.3)	((7.1,7.7,8.3);0.5,0.2,0.4)	((6.2,8.9,9.1);0.6,0.3,0.5)	((5.3, 7.3, 8.7); 0.7, 0.4, 0.8)	((4.6,5.5,8.6);0.4,0.7,0.2)
	C_1	C_2	C_3	C_4	C_5	C_6
A_1	((4.6,5.5,8.6);0.4,0.7,0.2)	((6.2,8.9,9.1);0.6,0.3,0.5)	((7.1,7.7,8.3);0.5,0.2,0.4)	((4.4,5.9,7.2);0.7,0.2,0.3)	((4.6,5.5,8.6);0.4,0.7,0.2)	((5.3, 7.3, 8.7); 0.7, 0.4, 0.8)
A_2	((5.3, 7.3, 8.7); 0.7, 0.4, 0.8)	((4.6,5.5,8.6);0.4,0.7,0.2)	((5.3, 7.3, 8.7); 0.7, 0.4, 0.8)	((6.2,8.9,9.1);0.6,0.3,0.5)	((7.1,7.7,8.3);0.5,0.2,0.4)	((4.4,5.9,7.2);0.7,0.2,0.3)
A_3	((6.2,8.9,9.1);0.6,0.3,0.5)	((4.6,5.5,8.6);0.4,0.7,0.2)	((7.1,7.7,8.3);0.5,0.2,0.4)	((4.4,5.9,7.2);0.7,0.2,0.3)	((5.8,6.9,8.5);0.6,0.2,0.3)	((4.6,5.5,8.6);0.4,0.7,0.2)
A_4	((7.1,7.7,8.3);0.5,0.2,0.4)	((5.3, 7.3, 8.7); 0.7, 0.4, 0.8)	((6.2,8.9,9.1);0.6,0.3,0.5)	((5.3, 7.3, 8.7); 0.7, 0.4, 0.8)	((4.6,5.5,8.6);0.4,0.7,0.2)	((5.3, 7.3, 8.7); 0.7, 0.4, 0.8)
A_5	((4.4,5.9,7.2);0.7,0.2,0.3)	((4.6,5.5,8.6);0.4,0.7,0.2)	((4.4,5.9,7.2);0.7,0.2,0.3)	((4.6,5.5,8.6);0.4,0.7,0.2)	((5.3, 7.3, 8.7); 0.7, 0.4, 0.8)	((6.2,8.9,9.1);0.6,0.3,0.5)
A_6	((6.2,8.9,9.1);0.6,0.3,0.5)	((5.3, 7.3, 8.7); 0.7, 0.4, 0.8)	((4.6,5.5,8.6);0.4,0.7,0.2)	((5.3, 7.3, 8.7); 0.7, 0.4, 0.8)	((6.2,8.9,9.1);0.6,0.3,0.5)	((7.1,7.7,8.3);0.5,0.2,0.4)
A_7	((5.3, 7.3, 8.7); 0.7, 0.4, 0.8)	((6.2,8.9,9.1);0.6,0.3,0.5)	((5.3, 7.3, 8.7); 0.7, 0.4, 0.8)	((6.2,8.9,9.1);0.6,0.3,0.5)	((7.1,7.7,8.3);0.5,0.2,0.4)	((4.4,5.9,7.2);0.7,0.2,0.3)
A_8	((4.6,5.5,8.6);0.4,0.7,0.2)	((7.1,7.7,8.3);0.5,0.2,0.4)	((6.2,8.9,9.1);0.6,0.3,0.5)	((7.1,7.7,8.3);0.5,0.2,0.4)	((4.4,5.9,7.2);0.7,0.2,0.3)	((7.1,7.7,8.3);0.5,0.2,0.4)
A_9	((5.3, 7.3, 8.7); 0.7, 0.4, 0.8)	((4.4,5.9,7.2);0.7,0.2,0.3)	((7.1,7.7,8.3);0.5,0.2,0.4)	((4.4,5.9,7.2);0.7,0.2,0.3)	((4.4,5.9,7.2);0.7,0.2,0.3)	((4.4,5.9,7.2);0.7,0.2,0.3)
A_{10}	((5.8,6.9,8.5);0.6,0.2,0.3)	((4.4,5.9,7.2);0.7,0.2,0.3)	((4.4,5.9,7.2);0.7,0.2,0.3)	((6.2,8.9,9.1);0.6,0.3,0.5)	((5.3, 7.3, 8.7); 0.7, 0.4, 0.8)	((4.6,5.5,8.6);0.4,0.7,0.2)
	C_1	C_2	C_3	C_4	C_5	C_6
A_1	((5.3, 7.3, 8.7); 0.7, 0.4, 0.8)	((6.2,8.9,9.1);0.6,0.3,0.5)	((7.1,7.7,8.3);0.5,0.2,0.4)	((4.4,5.9,7.2);0.7,0.2,0.3)	((5.8,6.9,8.5);0.6,0.2,0.3)	((5.3, 7.3, 8.7); 0.7, 0.4, 0.8)
A_2	((5.3, 7.3, 8.7); 0.7, 0.4, 0.8)	((4.6,5.5,8.6);0.4,0.7,0.2)	((5.3, 7.3, 8.7); 0.7, 0.4, 0.8)	((6.2,8.9,9.1);0.6,0.3,0.5)	((7.1,7.7,8.3);0.5,0.2,0.4)	((4.4,5.9,7.2);0.7,0.2,0.3)
A_3	((5.8,6.9,8.5);0.6,0.2,0.3)	((4.6,5.5,8.6);0.4,0.7,0.2)	((7.1,7.7,8.3);0.5,0.2,0.4)	((4.4,5.9,7.2);0.7,0.2,0.3)	((5.8,6.9,8.5);0.6,0.2,0.3)	((5.8,6.9,8.5);0.6,0.2,0.3)
A_4	((4.4,5.9,7.2);0.7,0.2,0.3)	((5.3, 7.3, 8.7); 0.7, 0.4, 0.8)	((6.2,8.9,9.1);0.6,0.3,0.5)	((5.3, 7.3, 8.7); 0.7, 0.4, 0.8)	((4.6,5.5,8.6);0.4,0.7,0.2)	((5.3, 7.3, 8.7); 0.7, 0.4, 0.8)
A_5	((7.1,7.7,8.3);0.5,0.2,0.4)	((5.8,6.9,8.5);0.6,0.2,0.3)	((4.4,5.9,7.2);0.7,0.2,0.3)	((4.6,5.5,8.6);0.4,0.7,0.2)	((5.3, 7.3, 8.7); 0.7, 0.4, 0.8)	((4.6,5.5,8.6);0.4,0.7,0.2)
A_6	((6.2,8.9,9.1);0.6,0.3,0.5)	((4.4,5.9,7.2);0.7,0.2,0.3)	((7.1,7.7,8.3);0.5,0.2,0.4)	((5.3, 7.3, 8.7); 0.7, 0.4, 0.8)	((5.3, 7.3, 8.7); 0.7, 0.4, 0.8)	((5.3, 7.3, 8.7); 0.7, 0.4, 0.8)
A_7	((5.3, 7.3, 8.7); 0.7, 0.4, 0.8)	((7.1,7.7,8.3);0.5,0.2,0.4)	((4.4,5.9,7.2);0.7,0.2,0.3)	((5.8,6.9,8.5);0.6,0.2,0.3)	((6.2,8.9,9.1);0.6,0.3,0.5)	((6.2,8.9,9.1);0.6,0.3,0.5)
A_8	((4.6,5.5,8.6);0.4,0.7,0.2)	((6.2,8.9,9.1);0.6,0.3,0.5)	((5.8,6.9,8.5);0.6,0.2,0.3)	((4.4,5.9,7.2);0.7,0.2,0.3)	((7.1,7.7,8.3);0.5,0.2,0.4)	((7.1,7.7,8.3);0.5,0.2,0.4)
A_9	((5.3, 7.3, 8.7); 0.7, 0.4, 0.8)	((5.3, 7.3, 8.7); 0.7, 0.4, 0.8)	((5.3, 7.3, 8.7); 0.7, 0.4, 0.8)	((5.3, 7.3, 8.7); 0.7, 0.4, 0.8)	((4.4,5.9,7.2);0.7,0.2,0.3)	((4.4,5.9,7.2);0.7,0.2,0.3)
A_{10}	((5.8,6.9,8.5);0.6,0.2,0.3)	((4.4,5.9,7.2);0.7,0.2,0.3)	((7.1,7.7,8.3);0.5,0.2,0.4)	((6.2,8.9,9.1);0.6,0.3,0.5)	((5.3, 7.3, 8.7); 0.7, 0.4, 0.8)	((4.6,5.5,8.6);0.4,0.7,0.2)

Table 2. The normalized matrix for criteria weights.

	C_1	C_2	C_3	C_4	C_5	C_6
A_1	0.098355927	0.113736	0.105649	0.085976	0.090975	0.105572

A_2	0.100374964	0.087635	0.101423	0.113471	0.10796	0.086657
A_3	0.104701471	0.095363	0.105649	0.085976	0.104506	0.10044
A_4	0.092875685	0.104404	0.101	0.104743	0.092702	0.106745
A_5	0.100519181	0.098425	0.090717	0.087431	0.097452	0.102933
A_6	0.112489184	0.092447	0.100296	0.104452	0.10796	0.100733
A_7	0.10383617	0.110819	0.095366	0.109543	0.110839	0.103666
A_8	0.087395443	0.112278	0.102268	0.099942	0.098892	0.111437
A_9	0.098644361	0.098717	0.099451	0.094996	0.085073	0.092962
A_{10}	0.100807615	0.086177	0.098183	0.113471	0.103642	0.088856

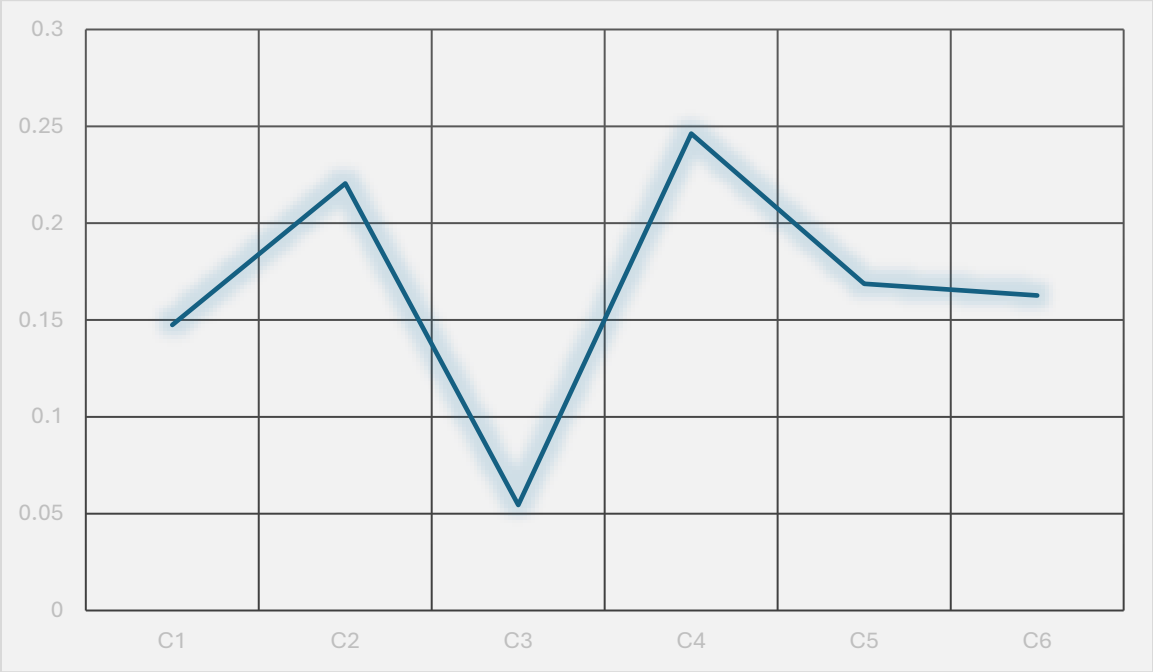

Fig 3. The criteria weights.

Results of REGIME method

Step 1. We used the combined decision matrix.

Step 2. We computed the superiority index E_{fl}.

Step 3. We computed the superiority identifier.

Step 4. We built the impact matrix.

Step 5. We built the REGIME matrix between the alternatives using Eq. (15) as shown in Fig 4.

Step 6. Eq. (16) is used to compute the guide index as shown in Fig 5.

Step 7. Rank the alternatives as shown in Fig 6.

	C1	C2	C3	C4	C5	C6		C1	C2	C3	C4	C5	C6
A1	0	0	0	0	0	0	A1	1	-1	-1	1	1	-1
A2	-1	1	1	-1	-1	1	A2	1	1	-1	-1	-1	1
A3	-1	1	-1	-1	-1	1	A3	1	-1	-1	1	1	1
A4	1	1	1	-1	-1	-1	A4	1	-1	-1	-1	1	-1
A5	-1	1	1	-1	-1	1	A5	1	-1	1	1	1	-1
A6	-1	1	1	-1	-1	1	A6	0	0	0	0	0	0
A7	-1	1	1	-1	-1	1	A7	1	-1	1	-1	-1	-1
A8	1	1	1	-1	-1	-1	A8	1	-1	-1	1	1	-1
A9	-1	1	1	-1	1	1	A9	1	-1	1	1	1	1
A10	-1	1	1	-1	-1	1	A10	1	1	1	-1	1	1
	C1	C2	C3	C4	C5	C6		C1	C2	C3	C4	C5	C6
A1	1	-1	-1	1	1	-1	A1	1	-1	-1	1	1	-1
A2	0	0	0	0	0	0	A2	1	1	-1	-1	1	1
A3	-1	-1	-1	1	1	-1	A3	-1	1	-1	1	1	1
A4	1	-1	1	1	1	-1	A4	1	1	-1	1	1	-1
A5	-1	-1	1	1	1	-1	A5	1	1	1	1	1	1
A6	-1	-1	1	1	1	-1	A6	-1	1	-1	1	1	1
A7	-1	-1	1	1	-1	-1	A7	0	0	0	0	0	0
A8	1	-1	-1	1	1	-1	A8	1	-1	-1	1	1	-1
A9	1	-1	1	1	1	-1	A9	1	1	-1	1	1	1
A10	-1	1	1	-1	1	-1	A10	1	1	-1	-1	1	1
	C1	C2	C3	C4	C5	C6		C1	C2	C3	C4	C5	C6
A1	1	-1	-1	-1	1	-1	A1	-1	-1	-1	1	1	1
A2	1	1	1	-1	-1	1	A2	-1	1	1	-1	-1	1
A3	0	0	0	0	0	0	A3	-1	1	-1	1	-1	1
A4	1	-1	1	-1	1	-1	A4	-1	1	1	-1	1	1
A5	1	-1	1	-1	1	-1	A5	-1	1	1	1	1	1
A6	-1	1	1	-1	-1	-1	A6	-1	1	1	-1	-1	1
A7	1	-1	1	-1	-1	-1	A7	-1	1	1	-1	-1	1
A8	1	-1	1	-1	1	-1	A8	0	0	0	0	0	0
A9	1	-1	1	-1	1	1	A9	-1	1	1	1	1	1
A10	1	1	1	-1	1	1	A10	-1	1	1	-1	-1	1
	C1	C2	C3	C4	C5	C6		C1	C2	C3	C4	C5	C6
A1	-1	-1	-1	1	1	1	A1	1	-1	-1	1	-1	-1
A2	-1	1	-1	-1	-1	1	A2	-1	1	-1	-1	-1	1
A3	-1	1	-1	1	-1	1	A3	-1	1	-1	1	-1	-1
A4	0	0	0	0	0	0	A4	1	-1	-1	-1	-1	-1
A5	-1	1	1	1	-1	1	A5	-1	1	1	1	-1	-1
A6	-1	1	1	1	-1	1	A6	-1	1	-1	-1	-1	-1
A7	-1	-1	1	-1	-1	1	A7	-1	-1	1	-1	-1	-1
A8	1	-1	-1	1	-1	-1	A8	1	-1	-1	-1	-1	-1
A9	-1	1	1	1	1	1	A9	0	0	0	0	0	0
A10	-1	1	1	-1	-1	1	A10	-1	1	1	-1	-1	1
	C1	C2	C3	C4	C5	C6		C1	C2	C3	C4	C5	C6
A1	1	-1	-1	1	1	-1	A1	1	-1	-1	1	1	-1
A2	1	1	-1	-1	-1	1	A2	1	-1	-1	-1	-1	1
A3	-1	1	-1	1	-1	1	A3	-1	-1	-1	1	-1	-1
A4	1	-1	-1	-1	1	-1	A4	1	-1	-1	1	1	-1
A5	0	0	0	0	0	0	A5	1	-1	1	1	1	-1
A6	-1	1	-1	-1	-1	1	A6	-1	-1	-1	1	-1	-1
A7	-1	-1	-1	-1	-1	-1	A7	-1	-1	1	1	-1	-1
A8	1	-1	-1	-1	-1	-1	A8	1	-1	-1	1	1	-1
A9	1	-1	-1	-1	1	1	A9	1	-1	-1	1	1	-1
A10	-1	1	-1	-1	-1	1	A10	0	0	0	0	0	0

Fig 4. The REGIME matrix.

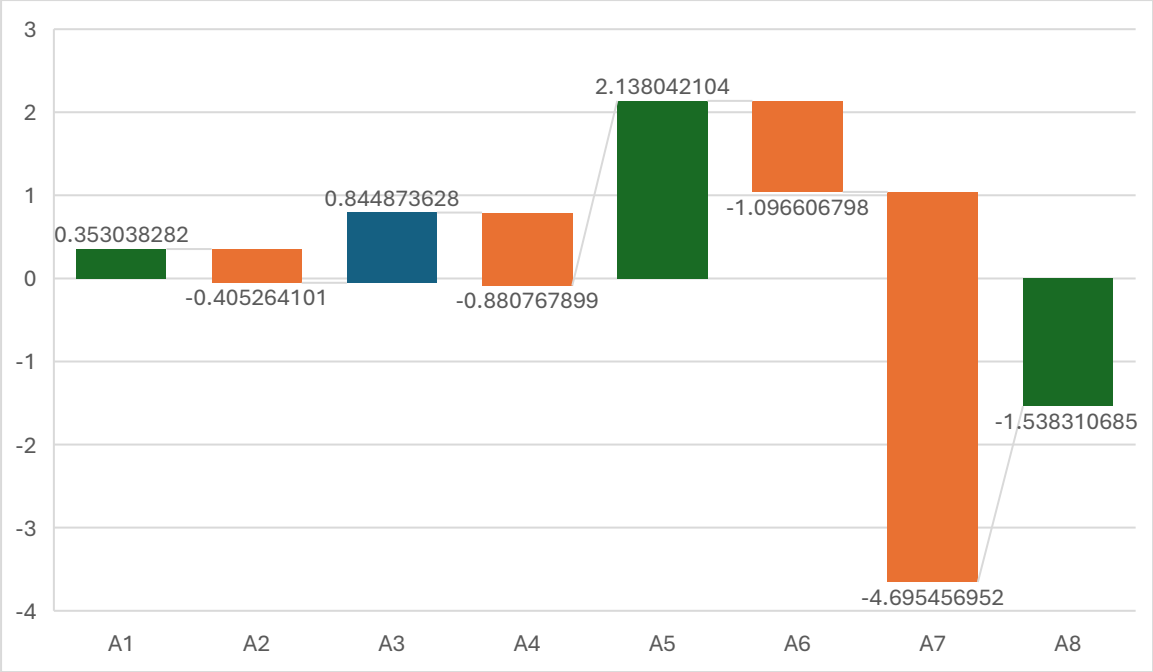

Fig 5. The guide index.

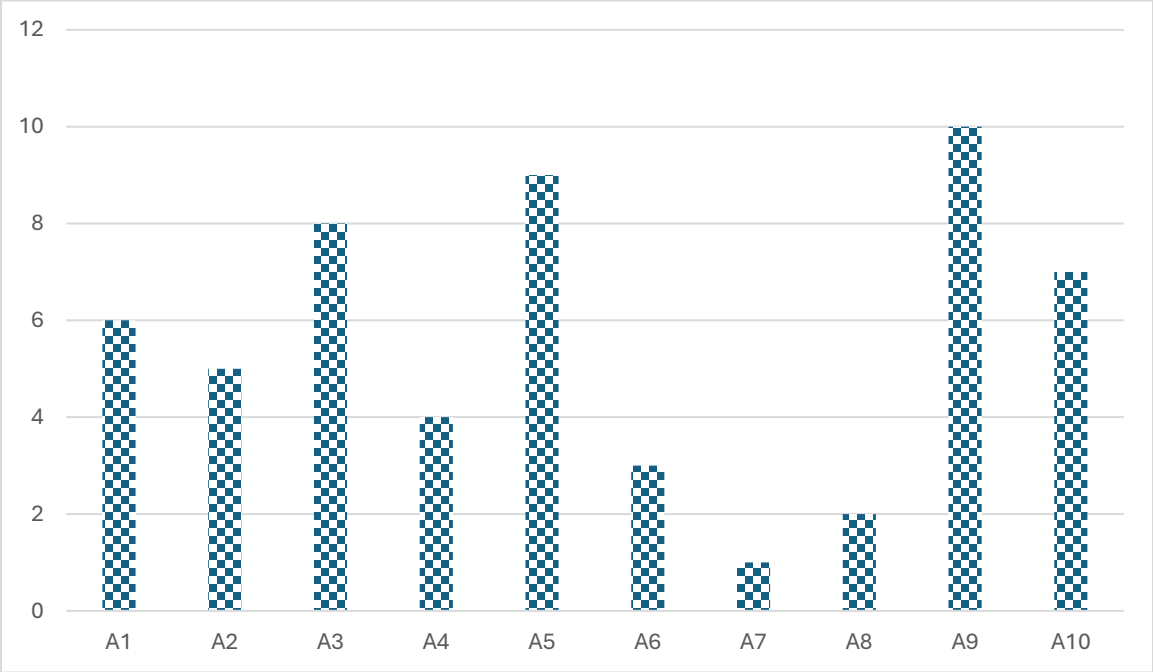

Fig 6. The rank of alternatives.

By conducting sensitivity analysis, we may learn more about how the framework behaves in various situations and use the results to inform our decision-making. This part does sensitivity analysis by adjusting the criteria weights that are established using the WENSLO technique to verify the criteria weights of our WENSLO-REGIME technique. The criteria weights are flipped

pairwise throughout each sensitivity analysis step, while the other weights remain unchanged. Then, using a new set of weights and the same REGIME approach, the options are ranked.

Based on the guide index, the REGIME model is then used to rate each round. For every round, the ranking order is almost the same. The finest option out of all of them is the alternative 9. We are confident in the WENSLO-REGIME model's dependability because our sensitivity analysis demonstrates that its results are consistent over a wide range of input values. Fig 7 shows the ranks of alternatives. Fig 8 shows the guide index values.

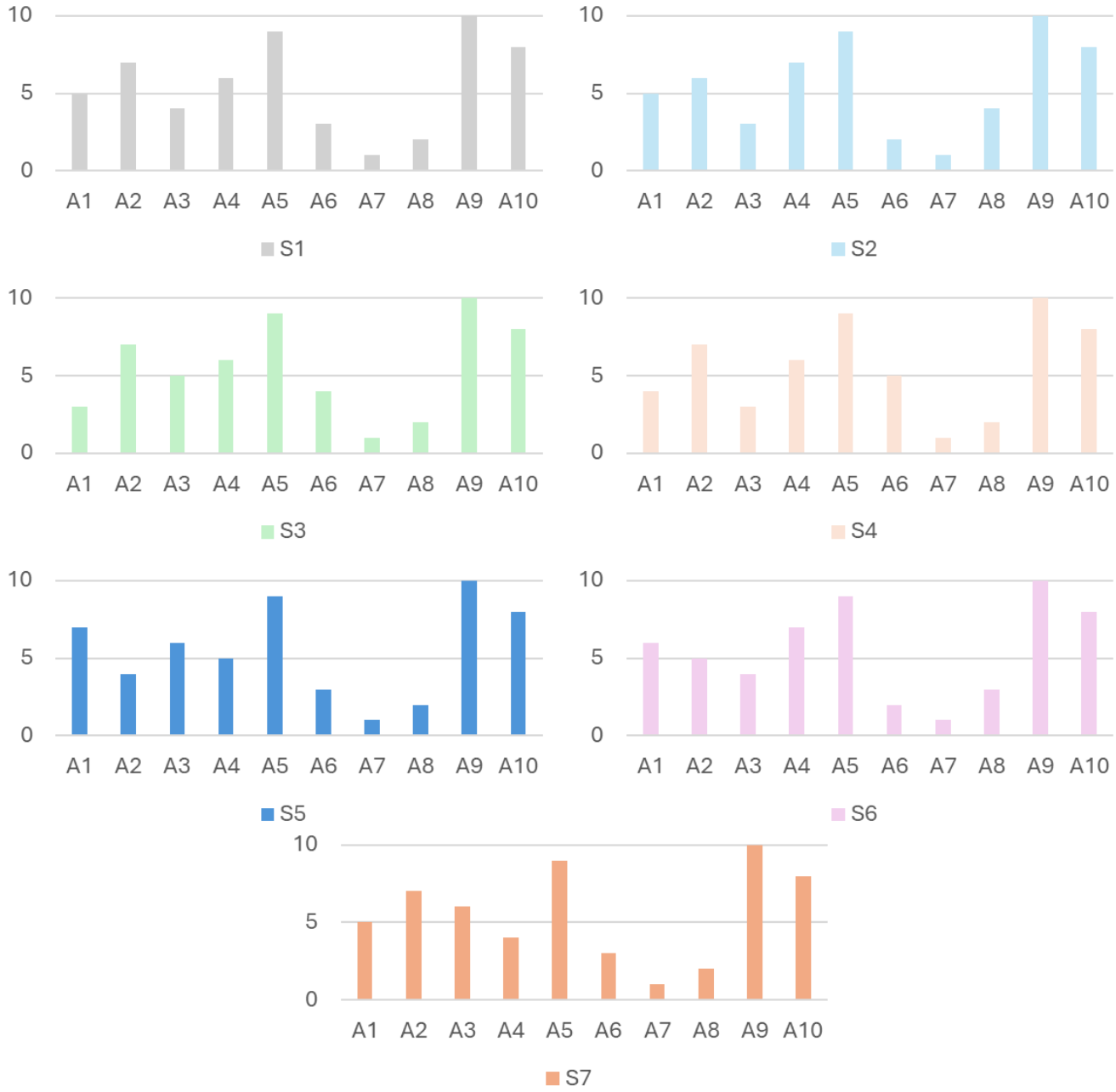

Fig 7. Results of sensitivity analysis.

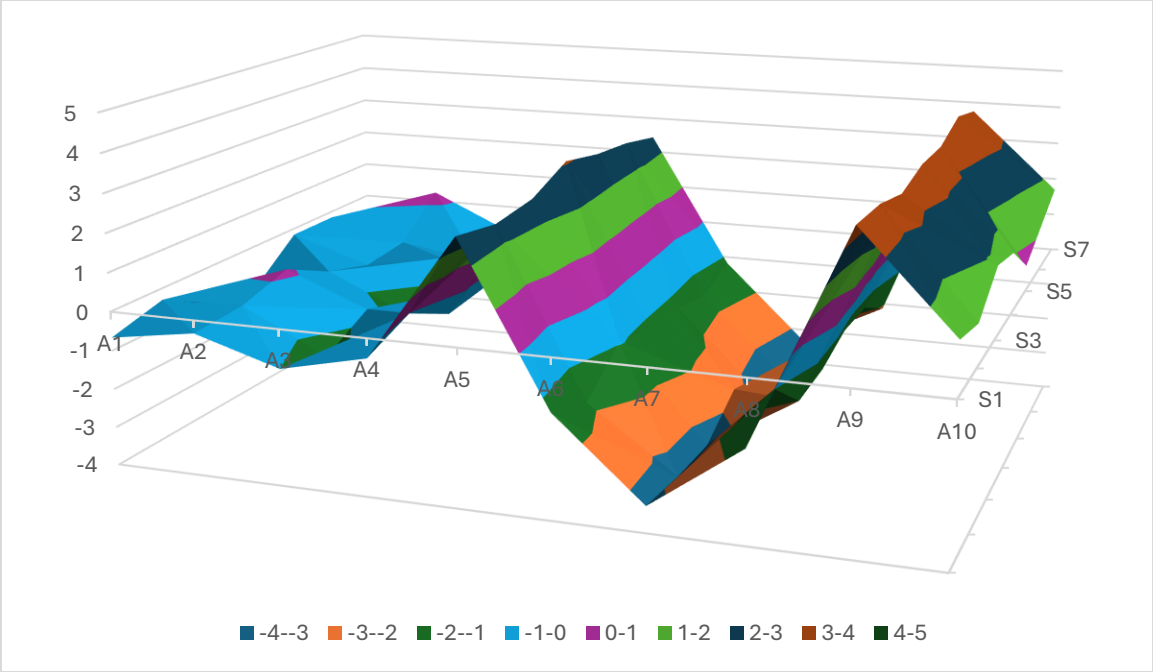

Fig 8. The guide index values.

Comprative analysis

This part comapres the proposed approach between other MCDM methods to show the effectiveness of the proposed approach. Fig shows the racnks of each method. Fig shows the coreelation between the proposed approach and other MCDM methods. The comparative aanlysis shows that the proposed is effective compred with other methods.

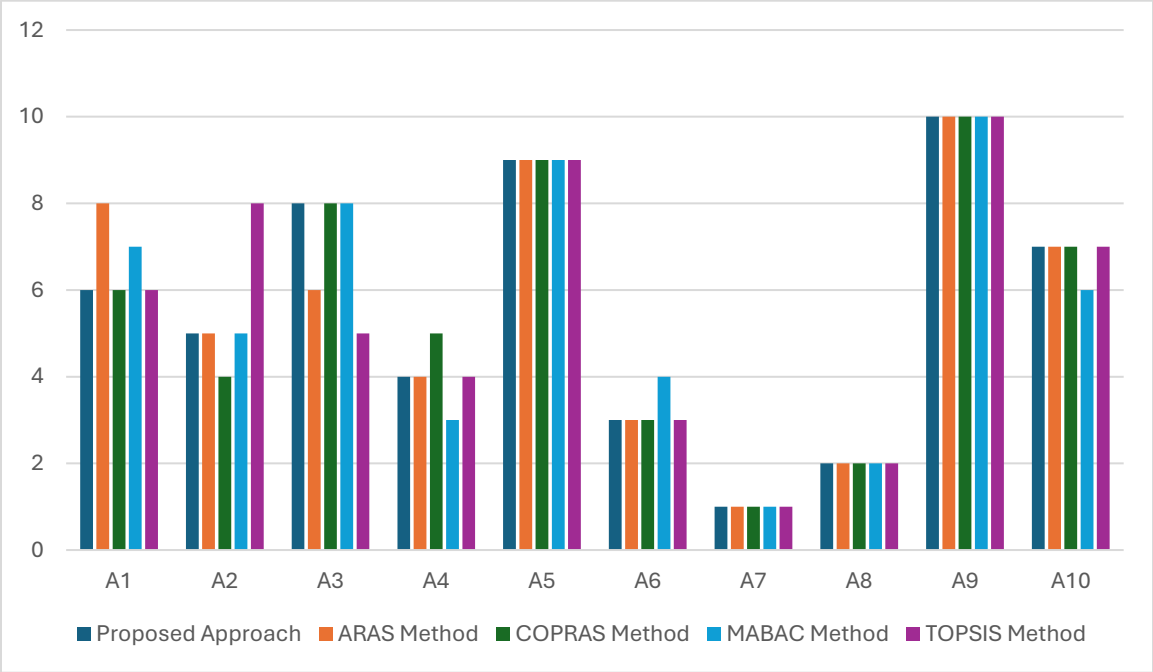

Fig . The ranks of each model.

	Proposed Approach	ARAS Method	COPRAS Method	MABAC Method	TOPSIS Method
Proposed Approach	1	0.951515152	0.987878788	0.975757576	0.890909091
ARAS Method	0.951515152	1	0.939393939	0.951515152	0.915151515
COPRAS Method	0.987878788	0.939393939	1	0.951515152	0.842424242
MABAC Method	0.975757576	0.951515152	0.951515152	1	0.866666667
TOPSIS Method	0.890909091	0.915151515	0.842424242	0.866666667	1

Fig . The correlation between our model and other models.

ARAS (Additive Ratio Assessment Method): ARAS evaluates alternatives by comparing their performance to an ideal alternative. It normalizes decision criteria and assigns weights to each factor before summing the weighted values. The alternative with the highest total utility value is considered the best. ARAS is simple, easy to implement, and does not require complex calculations. It is effective for straightforward decision-making problems where summation of benefits makes sense. It can be sensitive to the choice of normalization method, which may impact results. Additionally, it does not explicitly consider negative aspects of the alternatives. ARAS is often used in project evaluations, supplier selection, and economic analysis due to its straightforward approach.

COPRAS considers both beneficial and non-beneficial criteria while calculating a relative significance coefficient for each alternative. It determines positive and negative utility values to rank the alternatives proportionally. This method explicitly accounts for both advantages and disadvantages, offering a balanced evaluation. It is easy to use and provides a clear ranking system. COPRAS can be sensitive to subjective weight assignments and does not account for interdependencies between criteria. It is widely used in investment analysis, project selection, and risk assessment because of its ability to handle both costs and benefits effectively.

MABAC (Multi-Attributive Border Approximation Area Comparison Method): MABAC evaluates alternatives by measuring their relative distance from a border approximation area, which represents an ideal threshold between good and bad solutions. Unlike other methods that rely purely on distance-based ranking, MABAC also allows for more flexibility in handling nonlinear and complex decision environments. MABAC is useful in real-world scenarios where data is uncertain or when qualitative and quantitative criteria need to be combined. It is less sensitive to small changes in input data than other methods. The method is more computationally complex than ARAS and COPRAS, requiring advanced mathematical operations. Precise data inputs are necessary for accurate decision-making. It is commonly used in engineering design, environmental management, and policy evaluation, where decisions involve high levels of uncertainty.

TOPSIS (Technique for Order Preference by Similarity to Ideal Solution): TOPSIS ranks alternatives based on their closeness to an ideal solution and distance from the worst solution. It

uses Euclidean distance calculations to determine the best option. TOPSIS is widely recognized and applied in various industries because it effectively balances positive and negative aspects. It also provides a clear and logical ranking of alternatives. It does not consider interdependencies between criteria, meaning that each factor is evaluated independently. Additionally, small changes in data or weight assignments can significantly impact rankings. TOPSIS is commonly used in engineering, supply chain management, finance, and healthcare, where decisions need to be made by comparing multiple options in terms of their best and worst aspects.

5. Conclusions

By using the rough boundary approximation to capture the maximum unknown degrees, the neutrosophic- WENSLO -REGIME methodology. The study that is being presented is being used for the evaluation of abandoned coal mine sites. The significance of each criterion and its effect on the decision-making process have been assessed using the neutrosophic-WENSLO technique. Through pairwise comparisons of criteria, the neutrosophic-REGIME technique ranks alternatives, allocating scores according to their relative desirability. It makes decision-making flexible and objective by doing away with the requirement for preset weights. Alternative 9 was found to be the best option after we used the suggested approach to choosing the best alternative. The effectiveness and validity of the proposed approach have since been assessed through a comprehensive sensitivity analysis and comparative study.

Future directions

In the future directions, there are different MCDM methods can be applied to this MCDM problem to compute the criteria weights and rnak the alternatives such as AHP, TOPSIS, VIKOR, and EDAS methods. These methods can be applied to different crietria and alternatievs in the future directions. Extended number of crietria in the future durections can be considered to obtain accure and reliable results. Other extensions of neutrosophic sets can be used to dela with uncertainty and vgaue information in the future directions.

Acknowledgment

This work was supported by the Research on Sponge Design of Coal Mine Waste Land Landscape under the Dual Carbon Goal, Project No. 2022AH051550. The university-level key scientific research project "Research on the Application Design of the Fifth Facade Space of Public Buildings from the Perspective of Sponge - A Case Study of Huainan Xinhua Bookstore", Project No.: 2022XJZD008.

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Neutrosophic Sets and Systems: A Decade of Scientific Contribution and Growth

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Abstract: Neutrosophic Sets and Systems (NSS) has become an important Journal for neutrosophic theory and its applications in uncertainty modeling and decision sciences. In 2023, NSS celebrated its 10th anniversary, marking a decade of contributions to scientific literature. Motivated by this event, this paper provides a critical bibliometric analysis of NSS's evolution as a leading academic journal in neutrosophy. This study explores the publication trends, citation impact, most productive authors, institutions, and global collaboration networks of NSS. The study maps bibliographic coupling, keyword co-occurrence and co-citation analysis using data from the Scopus database, secondary search using NSS webpage, and visualization tools like the VOSviewer. The findings identify Neutrosophic Sets, Decision-Making, and Fuzzy Sets as the most frequently studied keywords that highlight the increasing integration of neutrosophic approaches into multi-criteria decision-making (MCDM) and soft computing. The results indicate the most significant areas of research that are topology, soft set theory, hypersoft sets, and uncertainty modeling. The findings highlight the NSS increasing international collaboration, with contributions from America, Middle East, and Asia. Additionally, the analysis explores NSS's alignment with sustainable development goals (SDGs), revealing notable contributions to good health and well-being (SDG 3), quality education (SDG 4), and peace, justice, and strong institutions (SDG 16). These contributions highlight how neutrosophic sets and systems is valuable across many researchers, from public health to governance to education. Overall, this paper serves as a fundamental study for researchers looking to explore emerging trends and novel applications of neutrosophic theories across diverse scientific and practical domains.

Keywords: Bibliometrics; Scopus database; Research Journal; VOS viewer software.

1. Introduction

Neutrosophic Sets and Systems (NSS) [1], is an international peer-reviewed, open access, free publishing research journal on neutrosophic sets [2], neutrosophic logic [3], and uncertainty

modelling [4]. Published by the University of New Mexico (UNM), United State and was founded by Florentin Smarandache [5], a pioneer in neutrosophic logic and set structure. Its current editor-in-chief includes Emeritus Professor Florentin Smarandache, Prof. Dr. Maikel Leyva-Vázquez [6], and Prof. Dr. Mohamed Abdel-Basset [7]. The journal covers both theoretical developments and practical applications, helping researchers tackle complex decision-making problems involving indeterminacy [8]. NSS was founded in 2013, and has been growing rapidly, it was indexed by Scopus [9] database in 2018, and 1,178 research papers has been published in NSS till the end 2023. The papers are also indexed by Scopus. Currently Q1 in the database [10] for Mathematics and Logic, while at 4th position globally in logic. Original research articles, reviews, case studies, and special issues on current topics are published.

Bibliometric analysis [11, 12] is a quantitative technique used to evaluate academic publications based on the trends, highly cited papers, influential authors and leading institutions or countries in the field. Since it has an ability to reveal trends in the evolution of research, present topics and structures along with developments, it can help the possible future directions [13]. Pioneering bibliometric work was initiated by early researchers in the field [14, 15] but advances in computing and visualization techniques have significantly supported the continuing growth of bibliometric studies. Bibliometric analysis has substantial meaning to the scientific literature in the perspectives of recognizing the most influential contributions, emerging topics and research [16, 17]. It is used to explore the development of scientific fields, pointing out those areas where more investigations are required [18]. It helps to find the highly cited papers, influential authors and areas in the field of study [19]. The findings provide a meaningful guidance for institutions and policymakers who need to know the trends of global research to design effective research policies and strategies [20]. Bibliometric indicators are also used to evaluate the quality of funding programs for research purposes, thus for the efficient use of these resources, making a recommendation about promotion, funding, and management of resources [21].

However, bibliometric analysis has its limitations. For example, Language of the data that can influence citation data, e.g. self-citation, discipline specific citation practices can change the results of some citation analyses [22]. Despite these difficulties, bibliometric methods are widely employed for the study of research trends and to evaluate research performance among a range of disciplines [23]. Bibliometric analyses have been widely applied to journals such as the European Journal of Operational Research [24], International Transactions in Operational Research [25], Operations Research [26], IEEE Transactions on Fuzzy Systems [27], Computers and Industrial Engineering [28], Communications and Applications [29], and International Journal of Information Technology & Decision Making [30].

In recent years, research on neutrosophic sets has gained significance, with *Neutrosophic Sets and Systems (NSS)* emerging as a key platform for publishing advancements in this field. Research papers published in NSS cover diverse applications, including decision-making [8], artificial intelligence

[31], uncertainty modeling, and engineering optimization. However, a comprehensive evaluation of the journal's impact, research trends, and scholarly contributions remains limited.

This study is carried out by integrating both statistical and graphical bibliometric techniques to discover the key contributors, emerging topics and global patterns of collaboration for neutrosophic research areas. The paper is structured as follows: Section 2 details the methodology, Section 3 presents key bibliometric indicators such as publication count, citation impact, influential authors, institutions, and collaborating countries. Section 4 visualizes bibliometric relationships using VOSviewer, and Section 5 discusses findings and provides conclusions.

2. Method

This study employs bibliometric analysis [32, 10] a quantitative method to assess research trends, impact, and collaboration networks in *Neutrosophic Sets and Systems*. The dataset was extracted from Scopus database [9], using the filter "source title" and searching with "Neutrosophic Sets and Systems" for the period 2018–2023 refining results to articles and reviews while excluding early access and non-relevant categories. From 2013 to 2017, data was extracted from the NSS webpage, and citation metrics were calculated using the Scopus filter for secondary documents. Notably, the journal was indexed in Scopus in 2018, covering volumes 21, 22, and 23, while two other volumes from 2018 remain unindexed.

Some of the bibliometric indicators such as total research papers (TRP), total citations (TC), h-index [33], citations per paper, and trends of publications were analyzed [34]. VOSviewer [35] was used to produce co-authorship networks, keyword clusters, institutional collaborations and research trends to visualize the pattern. Influential works are determined from co-citation analysis [36], major keywords are mapped through co-occurrence analysis, and bibliographic coupling [37].

We conducted this search in the Scopus database to identify documents related to the journal *Neutrosophic Sets and Systems*. To start, we filtered the source title, selecting *Neutrosophic Sets and Systems*, which initially yielded 1,353 documents. Next, we excluded publications from the years 2024 and 2025, narrowing the count to 1,178 documents. To refine the results further, we excluded items categorized as "Article" and "Review article" to ensure we focused solely on relevant research articles. However, this filter did not affect the total, which remained at 1,178 documents. Finally, we applied a filter to include only articles at the "Final" publication stage, ensuring that the search captured fully peer-reviewed and finalized works. The result included 1,178 documents from *Neutrosophic Sets and Systems* that met all these criteria.

Further, our search in secondary documents identified 188 research papers with citation metrics. We then cross-matched the publication numbers from the journal's webpage and found 55 articles that had not received any citations and were not indexed in the Scopus database in any category. This brings the total number of research papers analyzed in this bibliometric evaluation of NSS to 1,421.

This study was based on this refined dataset. A systematic and reproducible literature review process was ensured using the SPAR-4-SLR [38; 39] methodology. The paper gives some new indications on the path of evolution, the impact, and future directions of *Neutrosophic Sets and Systems* (NSS) journal.

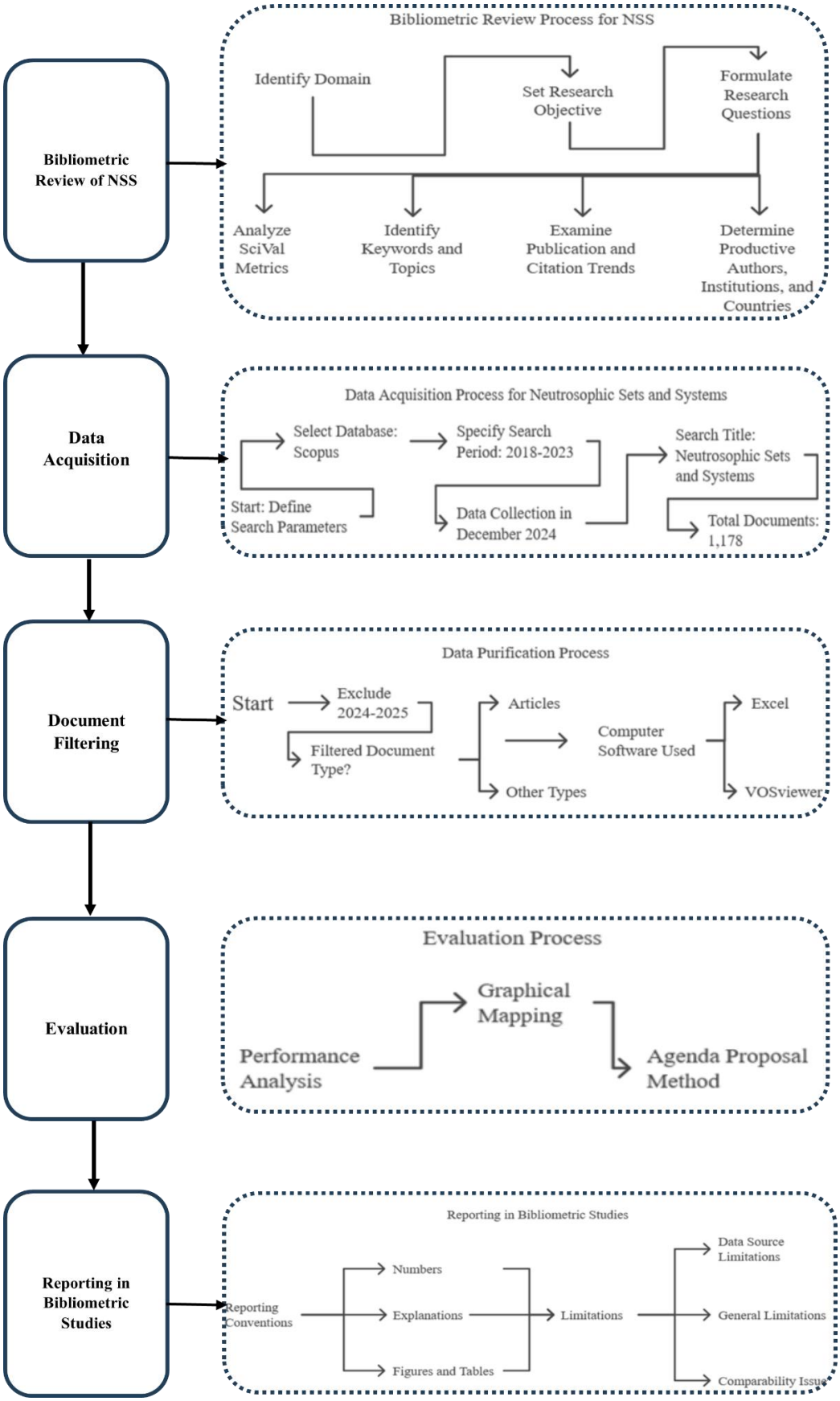

Figure 1: Framework of the proposed bibliometric study for the journal *Neutrosophic Sets and Systems (NSS)*

3. Metric Analysis

This section presents the bibliometric analysis of the journal *Neutrosophic Sets and Systems (NSS)*, divided into three subsections. The first subsection examines the number of publications, citation structure, and SciVal metrics highlighting key trends in NSS. The second part presents the most productive authors, leading institutions, and key international research contributing countries in NSS. The third subsection identifies participation of NSS to UN's Sustainable Development Goals (SDGs).

3.1 Publication and Citation Structure of NSS

The data presented in Table 1 shows the annual publication structure and citation of NSS. The data highlights remarkable growth in 2023, with the total research papers (TRP) published in NSS reaching a record high of 322 papers compared to just 10 papers in 2013. This significant growth reflects the journal's increasing contributions and rising popularity within the research community. Similarly, the total citations (TC) followed a fluctuating yet generally upward trend, peaking at 3,007 in 2020. The lower citation counts in 2022 (710) and 2023 (724) likely indicate the expected delay in citation accumulation for more recent publications, as they have had less time to gather citations. The citation distribution across thresholds (≥100, ≥75, ≥50, etc.) provides insight into the disciplinary impact of individual articles. Over 54 manuscripts received citations in the 50–75 range, followed closely by 197 articles cited between 25–50 times. However, most articles are clustered in the lower citation brackets, particularly in the ≥1 (82.34%) and ≥5 (40.49%) categories.

Table 1. Annual citation structure of NSS from 2013-2023

Year	TRP	TC	≥100	≥75	≥50	≥25	≥10	≥5	≥1
2013	10	420	2	2	3	4	6	8	9
2014	48	947	3	5	6	14	20	20	30
2015	62	677	0	1	5	11	15	17	28
2016	53	709	1	3	4	9	17	24	45
2017	46	469	0	1	1	7	11	19	40
2018	67	2,116	2	3	6	20	34	38	41
2019	116	1,513	0	0	3	16	63	87	109
2020	245	3,007	0	1	7	40	100	159	226
2021	241	1,843	0	0	0	19	61	116	197
2022	211	710	0	0	0	3	24	41	151
2023	322	724	0	0	1	4	10	36	199
13-17	219	3,222	6	12	19	45	69	88	152
18-20	428	6,636	2	4	16	76	197	284	376
21-23	774	3,277	0	0	1	26	95	193	547
Total	1,421	13,135	8	16	38	159	384	594	1170
%	100	100	0.56	1.13	2.67	11.19	27.02	41.80	82.34

Abbreviations: TRP = Total research papers, TC = total citations; ≥100, ≥75, ≥50, ≥25, ≥10, ≥5, ≥1 = Number of papers with equal or more than 100, 75, 50, 25, 10, 5 and 1 citations.

Analyzing the periods (D1: 2013–2017, D2: 2018–2020, D3: 2021–2023) reveals an evolution in publication, 219 papers were published during D1, 428 during D2, and 774 in D3. However, citation performance varied, with D2 seeing the highest TC (6,636), marking a peak in the journal's citation

impact. This surge might correlate with key publications or thematic areas that gained prominence during 2020–2021.

Figure 2 represents NSS publications and citations chart of scholarly output and citations from 2013 to 2023. The number of publications followed a steady rise, starting with 10 in 2013 and peaking at 322 in 2023. This consistent growth reflects increasing research activity of academic engagement over time. In contrast, the citation trend reveals a different story.

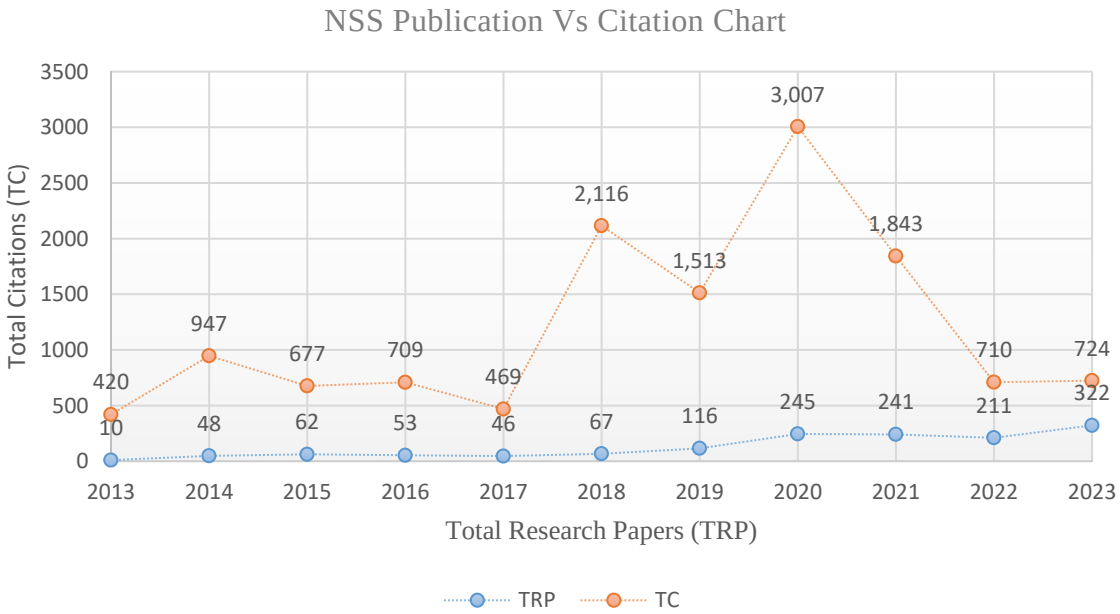

Figure 2. NSS annual publications vs citations chart

In 2020, citations reached 3,007 highlighting the impactful publications from prior years. However, the years followed showcase a steep decline, with citations dropping to 1,843 in 2021 and further tumbling to 710 in 2022. Despite the rise in publication numbers during this time, the decline in citations suggests that newer research has not achieved the same recognition or influence. This highlights a potential gap in the quality or visibility of the research being produced. The data showcase the need to go beyond quantity and focus on publishing impactful, high-quality research papers to provide a new direction to the young researchers of the academic community and to uplift the journal quality.

The data in Table 2, highlights real growth in the *Neutrosophic Sets and Systems (NSS)* journal's output and performance. The number of publications rose from 10 in 2013 to 322 in 2023, reflecting the journal's growing contributions and higher visibility. This progress can be further observed by its rise in quartile, from Quartile 3 (Q3) in 2018 to Quartile 1 (Q1) in 2023. However, this surge in publication volume has not increased the number of citations. Citations peaked at 3,007 in 2020 but declined to 710 in 2022 and 724 in 2023, likely due to the time needed for newer articles to gather citations.
The journal achieved a field-weighted citation impact (FWCI) of 0.86 in 2023, this shows a significant drop from its 2018 value of 3.08, suggesting a broader scope of articles with varied impacts is now being published. International collaboration has played a crucial role in the journal's success, in 2019

it was 43.9% and dipped to 25.1% in 2022, and now recovered to 41% in 2023. This trend demonstrates the journal's ability to attract global contributions to increase the journal's influence. Academic-corporate collaborations remained minimal, indicating the need for working with industry to broaden the journal's scope. In terms of visibility, the journal has received its highest number of views 2,775 in 2020.

Table 2. SciVal metrics of NSS

Journal Metrics	2018	2019	2020	2021	2022	2023
Scholarly Output	43	116	245	241	211	322
Citations	1,132	1,513	3,007	1,843	710	724
Views	554	1,465	2,775	2,676	2,183	2,670
Field-Weighted View Impact	0.57	0.63	0.67	0.92	0.9	0.84
International Collaboration (%)	37.2	43.9	34	34.2	25.1	41
Academic-Corporate Collaboration (%)	0	0	0.4	0.4	0	0.3
Field-Weighted Citation Impact	3.08	1.61	1.44	1.16	0.77	0.86
Field-Weighted Citation Impact (median)	2.47	1.25	0.82	0.54	0.23	0.39
Outputs in Top 10%, Citation Percentiles	55.8	20.7	20.4	17	9	5.3
Citations per Publication	26.3	13	12.2	7.6	3.3	2.2
Outputs in Top Views Percentiles (top 10%)	2.3	0	2	2.5	3.3	5
Views per Publication	12.9	12.6	11.3	11.1	10.3	8.3
Authors	89	286	503	554	443	641
Cite Score (Logic)	0.9	1.4	2.7	4.9	4.9	4.5
Best Percentile (Logic)	25	58	83	93	93	91
Journal Quartile	Q3	Q2	Q1	Q1	Q1	Q1

However, views per publication steadily declined from 12.9 in 2018 to 8.3 in 2023, possibly due to the increasing number of articles. The journal's field-weighted view impact peaked at 0.92 in 2021, signifying a period of highest visibility. Despite the increase in output, the percentage of articles in the top 10% citation percentiles has decreased from 55.8% in 2018 to 5.3% in 2023. The top view percentiles improved slightly in 2023. These trends point out the challenges in the sustainability of the journal's impact as its volume increases. Author participation has increased significantly, from 89 authors in 2018 to 641 in 2023, reflecting the journal's increasing reach within the research community. Increasing number of authors can be helpful in decreasing the number of research papers for individual researchers within the journal.

Table 3 consists of the top 40 most cited documents in NSS, that highlight the key publications leading in the influence of the journal *Neutrosophic Sets and Systems*. Leading the list is the paper " Extension of Soft Set to Hypersoft Set, and then to Plithogenic Hypersoft Set," [40] published in 2018, which has received more than 300 citations and an impressive 54 citation-per-year (C/Y) value. Followed by "Several similarity measures of neutrosophic sets" [41] which received 158 citations and 14 (C/Y) value, showcasing the journal's critical role in advancing decision-making frameworks. The papers published in 2018 and 2020 dominate this list, which reflects how quickly the journal began to grow, and how important it became during that time.

Table 3. Top 40 most cited documents in NSS

Rank	TC	Authors	Vol	Title	Year	C/Y
1	322	Smarandache F.	22	Extension of Soft Set to Hypersoft Set, and then to Plithogenic Hypersoft Set	2018	54
2	158	Broumi S.; Smarandache F.	1	Several similarity measures of neutrosophic sets	2013	14
3	138	Chi, P.; Peide L.	1	An extended topsis method for multiple attribute decision making problems based on interval neutrosophic set	2013	13
4	123	Biswas P.; Pramanik S.; Giri B.C.	2	Entropy based grey relational analysis method for multi-attribute decision making under single valued neutrosophic assessments	2014	12
5	106	Biswas P.; Pramanik S.; Giri B.C.	12	Value and ambiguity index based ranking method of single-valued trapezoidal neutrosophic numbers and its application to multi-attribute decision making	2016	13
6	105	Salama, A. A.; Smarandache F.; Kroumov V.	4	Neutrosophic closed set and neutrosophic continuous functions	2014	11
7	100	Alhabib R.; Ranna M.M.; Farah H.; Salama A.A.	22	Some Neutrosophic Probability Distributions	2018	17
8	100	Ye, Shan and Jun Ye.	6	Dice similarity measure between single valued neutrosophic multisets and its application in medical diagnosis	2014	10
9	98	Smarandache F.	21	Plithogenic set, an extension of crisp, fuzzy, intuitionistic fuzzy, and neutrosophic sets-revisited	2018	16
10	96	Saqlain M.; Jafar N.; Moin S.; Saeed M.; Broumi S.	32	Single and Multi-valued Neutrosophic Hypersoft set and Tangent Similarity Measure of Single valued Neutrosophic Hypersoft Sets	2020	24
11	95	Arokiarani, I.; R. Dhavaseelan; S. Jafari; and M. Parimala.	16	On some new notions and functions in neutrosophic topological spaces	2017	14
12	93	Salama, A. A.; Smarandache F.; Kroumov V.	2	Neutrosophic crisp sets and neutrosophic crisp topological spaces	2014	9
13	92	Biswas P.; Pramanik S.; Giri B.C.	3	A new methodology for neutrosophic multi-attribute decision-making with unknown weight information	2014	9
14	85	Biswas P.; Pramanik S.; Giri B.C.	12	Aggregation of triangular fuzzy neutrosophic set information and its application to multi-attribute decision making	2016	11
15	84	Sinha K.; Majumdar P.	11	Isolated single valued neutrosophic graphs	2016	11
16	74	Smarandache F.; Ricardo J.E.; Caballero E.G.; Vázquez M.Y.L.; Hernández N.B.	34	Delphi method for evaluating scientific research proposals in a neutrosophic environment	2020	19
17	76		9	Neutrosophic tangent similarity measure and its application to multiple attribute decision making	2015	8

18	69	Saqlain M.; Moin S.; Jafar M.N.; Saeed M.; Smarandache F.	32	Aggregate Operators of Neutrosophic Hypersoft Set	2020	17
19	69	A.A.A. Agboola	10	On Refined Neutrosophic Algebraic Structures	2015	8
20	69	Mondal, Kalyan and Surapati Pramanik.	7	Neutrosophic decision making model of school choice	2015	8
21	67	Al-Subhi S.H.S.; Pupo I.P.; Vacacela R.G.; Pérez P.Y.P.; Vázquez M.Y.L.	22	A New Neutrosophic Cognitive Map with Neutrosophic Sets on Connections, Application in Project Management	2018	11
22	66	Abbas M.; Murtaza G.; Smarandache F.	35	Basic operations on hypersoft sets and hypersoft point	2020	17
23	65		6	Multi-criteria group decision making approach for teacher recruitment in higher education under simplified neutrosophic environment	2014	7
24	64	Vasantha W.B.; Kandasamy I.; Smarandache F.	23	Algebraic structure of neutrosophic duplets in neutrosophic rings $\langle Z \cup I \rangle$, $\langle Q \cup I \rangle$ *and* $\langle R \cup I \rangle$	2018	11
25	63	Jansi R.; Mohana K.; Smarandache F.	30	Correlation Measure for Pythagorean Neutrosophic Sets with T and F as Dependent Neutrosophic Components	2019	13
26	63	Biswas P.; Pramanik S.; Giri B.C.	19	Distance measure based MADM strategy with interval trapezoidal neutrosophic numbers	2018	11
27	59	Patro, S. K. and Smarandache F.	12	The Neutrosophic Statistical Distribution - More Problems, More Solutions	2016	7
28	58	Rana S.; Qayyum M.; Saeed M.; Smarandache F.; Khan B.A.	28	Plithogenic Fuzzy Whole Hypersoft set, construction of operators and their application in frequency matrix multi attribute decision making technique	2019	12
29	57	Saqlain M.; Saeed M.; Ahmad M.R.; Smarandache F.	27	Generalization of TOPSIS for Neutrosophic Hypersoft set using Accuracy Function and its Application	2019	11
30	55	Rahman A.U.; Saeed M.; Smarandache F.; Ahmad M.R.	38	Development of Hybrids of Hypersoft Set with Complex Fuzzy Set, Complex Intuitionistic Fuzzy set and Complex Neutrosophic Set	2020	14
31	54	Ye, Shan; Jing Fu; and Jun Ye	7	Medical diagnosis using distance-based similarity measures of single valued neutrosophic multisets	2015	6
32	52	Chakraborty A.; Banik B.; Mondal S.P.; Alam S.	32	Arithmetic and Geometric Operators of Pentagonal Neutrosophic Number and its Application in Mobile Communication Service Based MCGDM Problem	2020	13
33	51	Mallick R.; Pramanik S.	36	Pentapartitioned neutrosophic set and its properties	2020	13
34	51	Mallick R.; Pramanik S.	22	VIKOR based MAGDM strategy with trapezoidal neutrosophic numbers	2018	9
35	51	Deli I.	22	Operators on Single Valued Trapezoidal Neutrosophic Numbers and SVTN-Group Decision Making	2018	9

36	51	Smarandache F.	9	Refined literal indeterminacy and the multiplication law of sub-indeterminacies	2015	6
37	50	Smarandache F.	53	Introduction to the Symbolic Plithogenic Algebraic Structures (revisited)	2023	50
38	49	Chakraborty A.; Mondal S.; Broumi S.	29	De-Neutrosophication Technique of Pentagonal Neutrosophic Number and Application in Minimal Spanning Tree	2019	10
39	49	Biswas; Pramanik S.; Giri B.C.	19	TOPSIS strategy for multi-attribute decision making with trapezoidal neutrosophic numbers	2018	8
40	48	Chakraborty A.; Broumi S.; Singh P.K.	28	Some properties of pentagonal neutrosophic numbers and its applications in transportation problem environment	2019	10

At number three "Entropy based grey relational analysis method for multi-attribute decision making under single valued neutrosophic assessments" [42] received 123 citations and C/Y = 12 proves the applicability of the work in its practical application. In term of C/Y some papers are worth mentioning "Introduction to the Symbolic Plithogenic Algebraic Structures (revisited) [43]" and "Single and Multi-valued Neutrosophic Hypersoft set and Tangent Similarity Measure of Single valued Neutrosophic Hypersoft Sets" [44] received 50 and 19 C/Y respectively. The difference in citation-per-year (C/Y) values of some documents shows that they received a decrease in interest. The papers with higher C/Y values (Rank = 1, 37, 7, 10, 16, 18 and 22) [40] [43-48] have a strong initial impact over time. Last, the appearance of multiple works in top cited documents by the leading authors Smarandache, Biswas, Pramanik, Giri, Saqlain, Chakraborty, Mondal, Broumi, Saeeed, and Bouzara demonstrate their core role in molding the journal's academic output and expanding the neutrosophic research area.

3.2 Most influential Authors, Institutes, and Countries in NSS

Table 4 showcases the top 20 most productive authors in NSS, highlighting key individuals who have significantly advanced the journal Neutrosophic Sets and Systems. Leading the list is Florentin Smarandache from the University of New Mexico, USA, with an impressive 143 total research publications (TRP), 2,034 total citations (TC), an h-index of 26, and 84 highly cited papers (over 5 citations). His foundational work has played a critical role in shaping neutrosophic research on a global scale.

Table 4. Top 20 most productive authors in NSS

R	Author Name	University	Country	TRP	TC	H	C/P	≥25	≥5
1	Smarandache, F.	U New Mexico	USA	143	2,034	26	14	28	84
2	Broumi, S.	U Hassan II Casablanca	MOR	108	881	16	8	11	46
3	Saeed, M.	U Management and Technology	PAK	32	681	14	21	10	23
4	Das, S.	Tripura U, Agartala	IND	30	392	12	13	7	21
5	Pramanik, S.	Nandalal Ghosh BT College, Kolkata	IND	28	596	15	21	13	22
6	Salama, A.A.	Port Said U	EGY	26	297	8	11	3	13
7	Abobala, M.	U Tichrine	SYR	19	363	13	19	10	16
8	Das, R.	ICFAI U Tripura	IND	19	256	9	13	5	12

9	Saqlain, M.	King Mongkut's U Technology	THA	18	458	11	25	8	12
10	Tripathy, B.C.	Department of Mathematics, Agartala	IND	16	219	9	14	4	9
11	Jafari, S.	Mathematical and Physical Sci Foundation	DEN	16	170	7	11	3	8
12	Jun, Y.B.	Gyeongsang National U	KOR	15	115	8	8	0	9
13	Talea, M.	U Hassan II Casablanca	MOR	15	132	6	9	1	8
14	Zulqarnain, R.M.	Zhejiang Normal U	CHN	12	178	6	15	4	6
15	Rahman, A.U.	U Management and Technology	PAK	11	216	7	20	4	8
16	Edalatpanah, S.A.	Ayandegan Institute of Higher Education	IRN	11	151	7	14	3	7
17	Kumar, R.	VIT-AP U, Amaravati	IND	9	119	4	13	3	4
18	Xin, X.L.	Northwest U Xian China	CHN	8	174	6	22	4	6
19	Chakraborty, A.	Academy of Technology	IND	6	181	5	30	3	5
20	Ihsan, M.	U Management and Technology	PAK	5	65	4	13	1	4

Abbreviations in previous tables.

Followed by Said Broumi from the University of Hassan II Casablanca, Morocco, with 108 publications, 881 citations, and an h-index of 16. At number three is Muhammad Saeed from the University of Management and Technology, Lahore, Pakistan, with 32 publications and 681 citations, showcase the significant contributions to the practical applications of neutrosophic methodologies. Among other prominent contributing authors, Das, from Tripura University, India, with 392 citations for 30 publications, Pramanik, Nandalal Ghosh B.T. College, India with 596 citations for 28 publications. Salama from Port Said University in Egypt, and Abobala from Al Tichrine University in Syria, have made substantial research contributions in the journal. Saqlain from King Mongkut's University of Technology Thonburi, Thailand is an emerging researcher with 16 publications and 219 citations. These authors showcase the expansion of the neutrosophic sets and systems journal within the research community and its global collaborators.

There is a tie between many authors based on total number of published papers i.e. 19, 16,15,11, so these can be ranked based on total number of citations they have received. This highlights the global collaboration that reflects increased recognition in decision making, mathematics, and technology. These top authors continue to increase the growth and visibility of *Neutrosophic Sets and Systems* throughout the world with their remarkable productivity and citation impact.

The top collaborating institutions performing research in NSS journals are shown in Table 5. The leading institute is the University Regional Autónoma de los Andes in Ecuador, which produced 211 total research publications (TRP) and received 1,075 TC, with an impressive h-index of 14. This illustrates the university’s leadership position in neutrosophic research worldwide.

Followed by the University of New Mexico in the USA, which has produced 136 publications that received 1,826 citations with an h-index of 25. The university has the largest number of papers that received (≥25 citations) and holds great influence in the field.

Table 5. Top 10 contributing Institutes in NSS

R	Institution	Country	TRP	TC	C/P	H	≥25	≥5	FWCI
1	U Regional Autónoma de los Andes	Ecuador	211	1,075	5	14	6	69	0.32
2	U New Mexico	USA	136	1,826	13	25	25	79	1.66
3	U Hassan II Casablanca	Morocco	116	935	8	17	11	49	1.20
4	U Management and Tech	Pakistan	45	695	15	14	10	27	2.64
5	Zagazig U	Egypt	39	186	5	8	2	11	1.34
6	Tripura U	India	31	514	17	14	10	24	0.96
7	Prince Sattam Bin Abdulaziz U	KSA	34	79	2	5	1	5	1.54
8	Nirmala College for Women	India	32	169	5	7	1	9	-
9	Port Said U	Egypt	27	298	11	8	3	13	1.26
10	U Telafer	Iraq	26	132	5	6	1	7	1.48

Abbreviations in previous tables. Except, Field-Weighted Citation Impact (FWCI).

University of Hassan II Casablanca in Morocco with 116 publications, 935 citations and an h index of 17 stands at the number three position in the table. University of Management and Technology, Lahore, Pakistan stands at number 4 with 45 publications and 695 citations, and Egypt's Zagazig University with 39 publications and 186 citations is at number 5.

NSS has been expanding globally across academic and applied disciplines due to the efforts of these institutions. The growing interest in publishing in NSS from the Asia region are be represented by Tripura University in India, the University of Management and Technology, in Pakistan, and Prince Sattam bin Abdulaziz University from Saudi Arabia. While Port Said University of Egypt and the University of Telafer of Iraq represent the active participation in the development of NSS from the Middle East region. Further insight and impact of research from these institutions has been analyzed by utilizing the Field-Weighted Citation Impact (FWCI) metric. With the highest FWCI of 2.64, the University of Management and Technology takes the lead in terms of the impact of its publications in the academic community. The collaboration from this global network has continued the growth of NSS research, developing these areas of decision-making, technology, and mathematics.

Table 6 highlights the most contributing countries in NSS. India stands first in the table with 496 publications, 3,334 citations, and an h-index of 29. Its 110 papers received at least 10 citations, and 6 papers are in highly cited papers with at least 50 citations. Ecuador is in second place with 270 publications, 1,592 citations, and an h-index of 20. The United States is in third position with 156 publications and 2,100 citations with an impressive h-index of 27 and received 13 cites per paper (C/P).

Table 6. Top 10 contributing Countries in NSS

R	Country	TRP	TC	H	C/P	≥50	≥10
1	India	496	3,334	29	7	6	110
2	Ecuador	270	1,592	20	6	2	49
3	United States	156	2,100	27	13	9	63
4	Morocco	119	951	17	8	2	24

5	Egypt	102	613	12	6	1	16
6	Pakistan	94	1,128	17	12	6	32
7	Saudi Arabia	88	468	12	5	0	16
8	Syrian Arab Republic	71	758	19	11	1	24
9	Iraq	70	603	15	9	0	22
10	China	54	304	9	6	0	9

Abbreviations in previous tables.

Morocco with 119 publications, 951 citations, and an h-indices of 17 stands at 4th position. While Egypt with 102, publications, 613 citations, and an h-index of 12 stand at number five in the Table 6. Contributions from Pakistan are also quite significant, with 94 publications, 1,128 citations, h-index of 17, and 6 highly cited papers. Saudi Arabia (88 publications), Syria (71 publications), Iraq (70 publications), and China (54 publications) have moderate outputs. While the h-index and cites per paper (C/P) are impressive that played an influential role in the development of the journal. This data highlights the international collaboration and participation in the NSS journal, with significant contributions from Asia, America, the Middle East, and Africa that played a role in the growth of NSS.

Table 7 represents the top 20 journals citing NSS. The contribution insights and impact of research journals were analyzed by utilizing the Field-Weighted Citation Impact (FWCI) metric. The *International Journal of Neutrosophic Science* leads the table with a total of 229 research publications (TRP), 1,260 citations, and a Field-Weighted Citation Impact (FWCI) of 1.33. Followed by the *Journal of Intelligent and Fuzzy Systems*, with 128 publications, 1,954 citations, and 1.5 FWCI. *Symmetry* and *Soft Computing* stand out for their significant citation impact, with FWCI values of 2.01 and 3.06, respectively, underscoring the high quality of research they publish that cited the work published in NSS. Meanwhile, *IEEE Access* with an impressive FWCI of 1.88, known for its multidisciplinary approach, contributed 41 publications that cited the journal NSS.

Table 7. Top 20 Journals citing NSS Papers

R	Journal	TRP	TC	FWCI
1	International Journal of Neutrosophic Science	229	1,260	1.33
2	Journal of Intelligent and Fuzzy Systems	128	1,954	1.5
3	Symmetry	61	1,086	2.01
4	Soft Computing	53	1,396	3.06
5	AIP Conference Proceedings	46	46	1.04
6	IEEE Access	41	1,050	1.88
7	Advances in Intelligent Systems and Computing	34	307	1.89
8	Mathematics	29	414	2.44
9	Studies in Fuzziness and Soft Computing	28	332	3.61
10	Lecture Notes in Networks and Systems	24	41	0.59
11	AIMS Mathematics	18	117	2.45
12	Advances in Mathematics: Scientific Journal	17	57	0.66
13	Applied Soft Computing	16	802	3.14

14	International Journal of Fuzzy Systems	16	229	1.13
15	Journal of Mathematics	16	118	1.8
16	Complex and Intelligent Systems	15	508	2.79
17	Journal of Ambient Intelligence and Humanized Computing	15	377	1.8
18	Journal of Fuzzy Extension and Applications	15	124	1.53
19	Mathematical Problems in Engineering	15	116	1.39
20	Neutrosophic Operational Research: Methods and Applications	15	41	1.61

Abbreviations in previous tables.

Similarly, *Mathematics and Studies in Fuzziness* and *Soft Computing* exhibit strong FWCI values of 2.44 and 3.61, highlighting their influence within the academic community that cited the work published in NSS. On the other hand, *AIP Conference Proceedings* and *Lecture Notes in Networks and Systems* have lower FWCI scores of 1.04 and 0.59, but they continue to play a role in disseminating research in the field. High-impact journals such as *Applied Soft Computing* and *Complex and Intelligent Systems* stand out with robust FWCI scores of 3.14 and 2.79, reflecting their commitment to publishing high-quality, impactful work. Other contributors, including *AIMS Mathematics*, *Advances in Intelligent Systems and Computing*, and *Journal of Fuzzy Extension and Applications*, maintained a balanced citation impact. These journals fostered progress in the growth of *Neutrosophic Sets and Systems* and showcased varying levels of reach and influence in the academic literature and research papers.

3.3 Influence of NSS to Sustainable Development Goals (SDGs)

The concentrating influences of NSS articles on the UN's Sustainable Development Goals (SDGs) are represented in Table 8. The most significant contribution was observed in good health and wellbeing (SDG 3) with 36 total research papers (TRP), 200 total citations (TC), and FWCI of 0.94. That's 17.0% of the total contributions, thus indicating NSS research's application to healthcare and medical decision making. Followed by quality education (SDG 4) with 30 papers, 159 citations, and 13.5% of the contributions. This means that NSS has a role in the enhancement of methodologies of education, in making decision on academic institutions and in intelligent learning systems. At number three, Peace, Justice and Strong Institutions (SDG 16) with 31 papers, 125 citations and 10.6% of the contribution.

Table 8. The contribution to SDG goals of the papers published in NSS

Name	TRP	FWCI	TC	Percentage
SDG 1: No Poverty (2023)	6	0.73	36	3.1
SDG 2: Zero Hunger (2023)	3	0	2	0.2
SDG 3: Good Health and Well-being (2023)	36	0.94	200	17.0
SDG 4: Quality Education (2023)	30	0.85	159	13.5
SDG 5: Gender Equality (2023)	9	0.5	30	2.5
SDG 6: Clean Water and Sanitation (2023)	7	2.34	111	9.4
SDG 7: Affordable and Clean Energy (2023)	10	1.13	67	5.7
SDG 8: Decent Work and Economic Growth (2023)	17	0.6	66	5.6
SDG 9: Industry, Innovation and Infrastructure (2023)	24	0.66	76	6.5
SDG 10: Reduced Inequality (2023)	15	0.84	76	6.5

SDG 11: Sustainable Cities and Communities (2023)	24	0.87	112	9.5
SDG 12: Responsible Consumption and Production (2023)	15	0.62	50	4.2
SDG 13: Climate Action (2023)	1	1.25	10	0.8
SDG 14: Life Below Water (2023)	0	0	0	0.0
SDG 15: Life on Land (2023)	1	0.92	4	0.3
SDG 16: Peace, Justice and Strong Institutions (2023)	31	0.67	125	10.6

Abbreviations in previous tables. Except Sustainable Development Goals (SDGs)

It demonstrates the application of research papers published in NSS are being used in areas of governance, policy making and institutional decision making.

Other key contributions include SDG 6: Clean Water and Sanitation (7 papers, 111 citations, 9.4%) and SDG 11: Sustainable Cities and Communities (24 papers, 112 citations, 9.5%). These contributions highlight the growing application of neutrosophic models in environmental sustainability, urban planning, and infrastructure development. Industry, Innovation, and Infrastructure (SDG 9) with 24 papers, 76 citations, and 6.5% of the contribution optimizing industrial process and fostering innovation and increased use of research and technology. Decent Work and Economic Growth (SDG 8) with 17 papers, 66 citations, and 5.6% contribution highlight the role in economic development.

However, some SDGs have contributed less. For example, life below water (SDG 14) has no paper, while life on land (SDG 15) has just 1 research paper with 0.3% contribution in NSS publications and growth. Overall, this data shows that NSS is a valuable journal that is contributing globally to sustainability across health, education, infrastructure and governance. Yet there is a need for expanding research coverage to other areas such as biodiversity and conservation of the environment to increase its participation in SGDs.

4. Mapping of NSS with VOSviewer Software

This section provides a visual analysis of different bibliometric indicators to identify the patterns and research trends that are emerging in the journal of *Neutrosophic Sets and Systems* (NSS) by using the VOS viewer software [35]. The visual bibliometric analysis is crucial for findings the intellectual landscape of a research domain. VOSviewer is widely used in bibliometric analysis as a graphing tool. It helps in analyzing the key indicators like most cited journals, the influential authors, the prominent collaborators, the trending keywords with the major research area and the important research papers. This method helps researchers to find out the structural trends and development of the topic under consideration.

For this study, the data was extracted from the Scopus database and was analyzed through VOSviewer software. By doing this, we were able to visualize and interpret important bibliometric trends, the most important contributors toward the development of the journal. With this analysis, we explored the insights about growth, impact, and collaboration in the NSS. For more information regarding the use of the VOSviewer software see the webpage: https://www.vosviewer.com/.

For a more structured and detailed exploration, this section is divided into three subsections:

- **Subsection 4.1** presents a co-citation analysis, which highlights the relationships between research journals, and authors that have a significant influence in NSS.
- **Subsection 4.2** provides the bibliographic coupling of documents, illustrating how different research papers of NSS are interconnected through shared references.
- **Subsection 4.3** focuses on keywords and topical analysis, identifying the most frequently used keywords, emerging topics, and case studies within NSS.

This structured approach provides a comprehensive visual representation of the key indicators within NSS and helps the scholars to identify influential works, collaboration networks, emerging research directions, and research gaps in the field.

4.1 Co-Citation Analysis

Co-citation analysis is a powerful bibliometric tool that helps identify the relationships between academic papers based on how frequently they are cited together [36]. It helps to map research trends and identify influential studies. By analyzing the co-cited works researchers can track how ideas have evolved, identify leading authors and journals, and uncover emerging topics. To find out the scientific structure and trend of the journal *Neutrosophic Sets and Systems (NSS),* co-citation analysis plays a key role and efficiently guides future research directions.

The co-citation network is presented in Fig. 3. It shows how most cited research journals are connected to each other in NSS. This network is centered around the *Neutrosophic Sets and Systems* journal, which represents the main resource in neutrosophic sets, logic, and research.

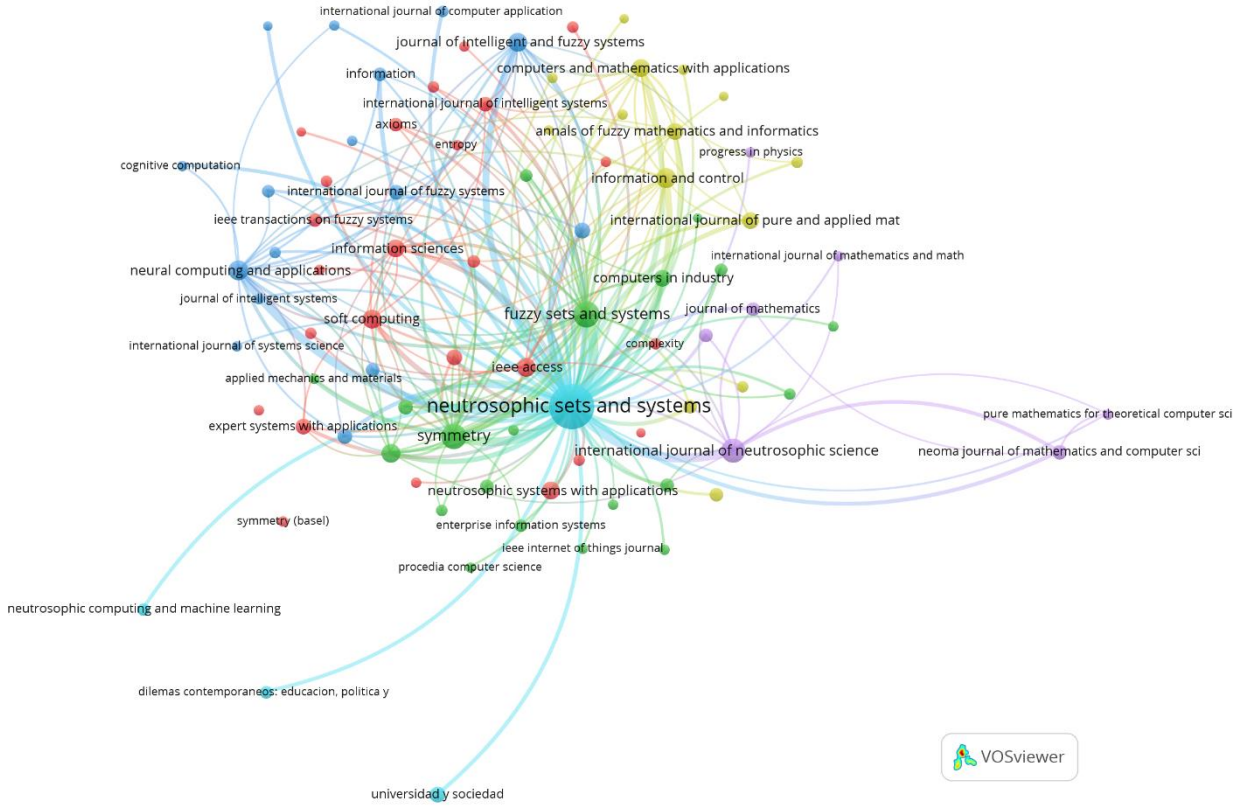

Fig. 3 Co-Citation of journals citing NSS: minimum 30 citations and 100 links

The visuals show that the most prominent node of the NSS journal is well-cited, indicating its leading role in the development of the theory and applications of neutrosophic sets, that provide a model to

confront uncertainty, imprecision, and indeterminacy across various fields. The node of the *Journal of Fuzzy Sets and Systems* holds an impressive connection with NSS, and their interlinks with other journals highlight the common research topics and trends. Among other prominent nodes reflecting *International Journal of Neutrosophic Science*, *Information Sciences*, and Soft Computing showcase the interlink with NSS. The research published in these journals has a strong relation with neutrosophic sets and their applications in applied areas, particularly information systems in decision-making.

Some other connected journals include *Computers and Industrial Engineering*, and *Neural Computing and Applications*, for their relevance to industrial applications applying the research published in NSS. The relationship between fuzzy sets and neutrosophic research continues to be reflected in the research mentioned above. Some less-connected journals with NSS highlight the interdisciplinary concepts connecting mathematics, fuzzy logic, AI, and systems theories.

Figure 4 showcases the co-citation network of authors for the Neutrosophic Sets and Systems (NSS) journal. Abdel-Basset and Smarandache are the most central authors in the network, indicating their leading role in neutrosophic sets and systems research. The interlinks of their works show that they are pioneers in the field. These authors are playing a one man show in the development of NSS.

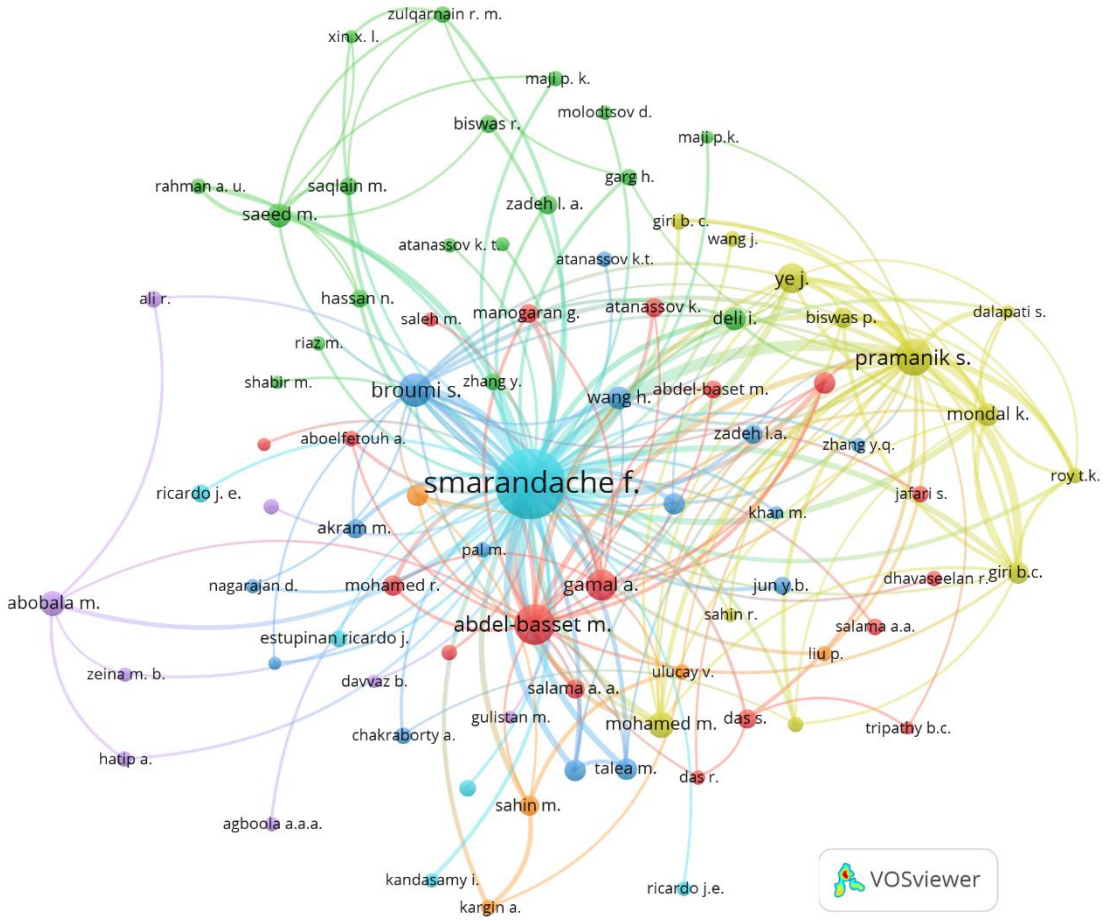

Fig. 4 Co-Citation of authors in NSS with minimum 100 citations and 100 links

The interlinks with other authors, e.g. Atanassov, Zadeh and Pramanik, highlight that their works have played a substantially significant role in the development of their subdomains from fuzzy

systems to neutrosophic sets, and their applications. The network clusters in different colours like, yellow, red, blue, green and orange showcases different research groups within the NSS and collaborate on some specific topics and trends. For example, Abdel-Basset, who has multiple collaborators of his own, including Gamal, Chang, and Manogran, on the one hand, Pramanik and Smarandache also have distinct research group. It highlights the applications and research topics they are working together. The focus of Sunderraman and Mondal is on the applications of research published in NSS towards optimization, and artificial intelligence.

The less connected clusters in the network demonstrate the diversity of scientific research that NSS has published on, ranging from theoretical work to application-oriented contributions.

4.2 Bibliographic coupling

Bibliographic coupling is a fundamental bibliometric analysis technique to detect the similarities between research works through shared references. Bibliographic coupling refers to the situation when two documents cite a common third source which is different from the two documents [37]. It helps to discover the extent of similarity between studies based on set of references and reflects the structure of the study. It highlights the relevant research topics, influential research papers and authors, and showcase the development of certain topics through time.

Figure 5 shows the bibliographic coupling network of documents in *Neutrosophic Sets and Systems* (NSS) journal. In this network, documents that cite the same sources were grouped, so that the relationships between them can be shown by the pattern of citations. The visualization highlights the foundational documents, and the emerging research areas, for understanding the structure of knowledge in the Journal of Neutrosophic Sets and systems.

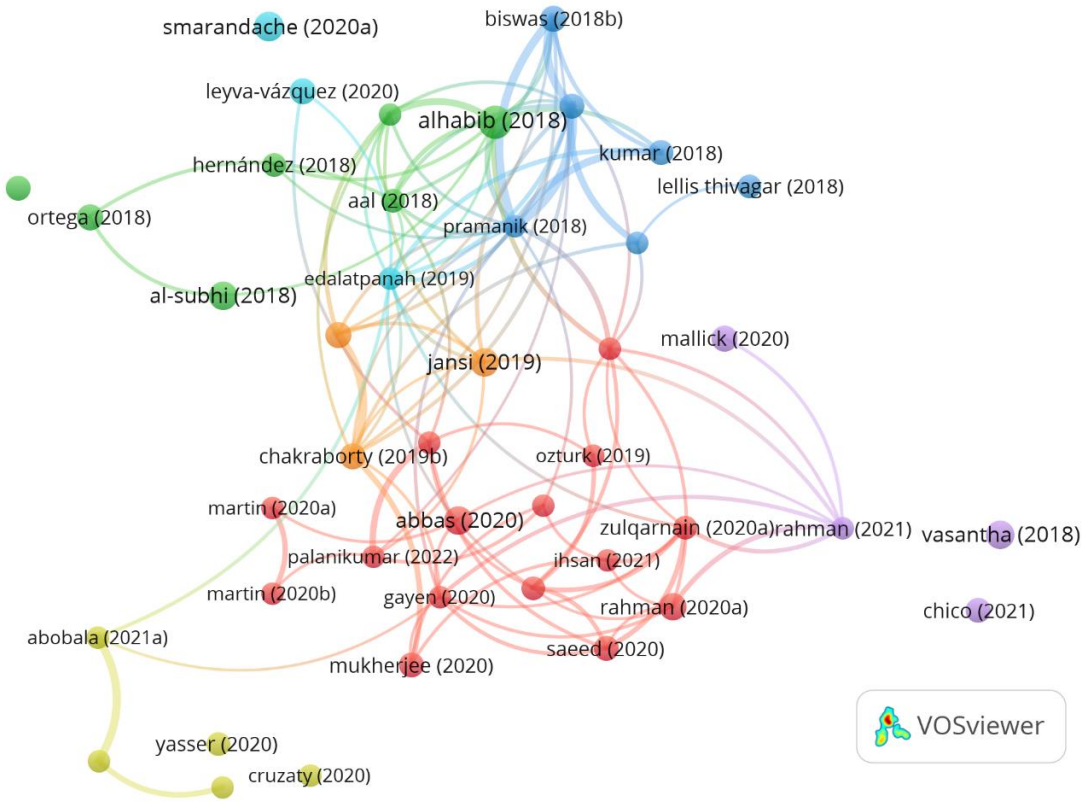

Fig. 5 Bibliographic coupling of documents in NSS with minimum 30 publications and 100 links

The prominent nodes, like Alhabib (2018), Smarandache (2020a, 2021, and 2023), Saqlain (2019, 2020b, 2020c), Biswas (2018a, 2018b), Chakraborty (2019b), and Rahman (2019) represent the influential papers in the journal. Clusters in different colors indicate document groups that are closely related to each other and share common references, i.e. different subtopics and evolving trends of subtopics within the field are reflected. They also show how the research of neutrosophic sets and systems was extended, with newer articles based on the concepts published earlier, and the large number of connections among the recent articles of both Smarandache and Saqlain are found. Some less connected nodes show the interdisciplinary diversity of scientific research that was published in NSS.

The bibliographic coupling of *Neutrosophic Sets and Systems (NSS)* journal is represented by Figure 6. The bibliographic coupling analysis of NSS journals provides insights into trends, research groups and the intellectual structure of the journal along with influential authors.

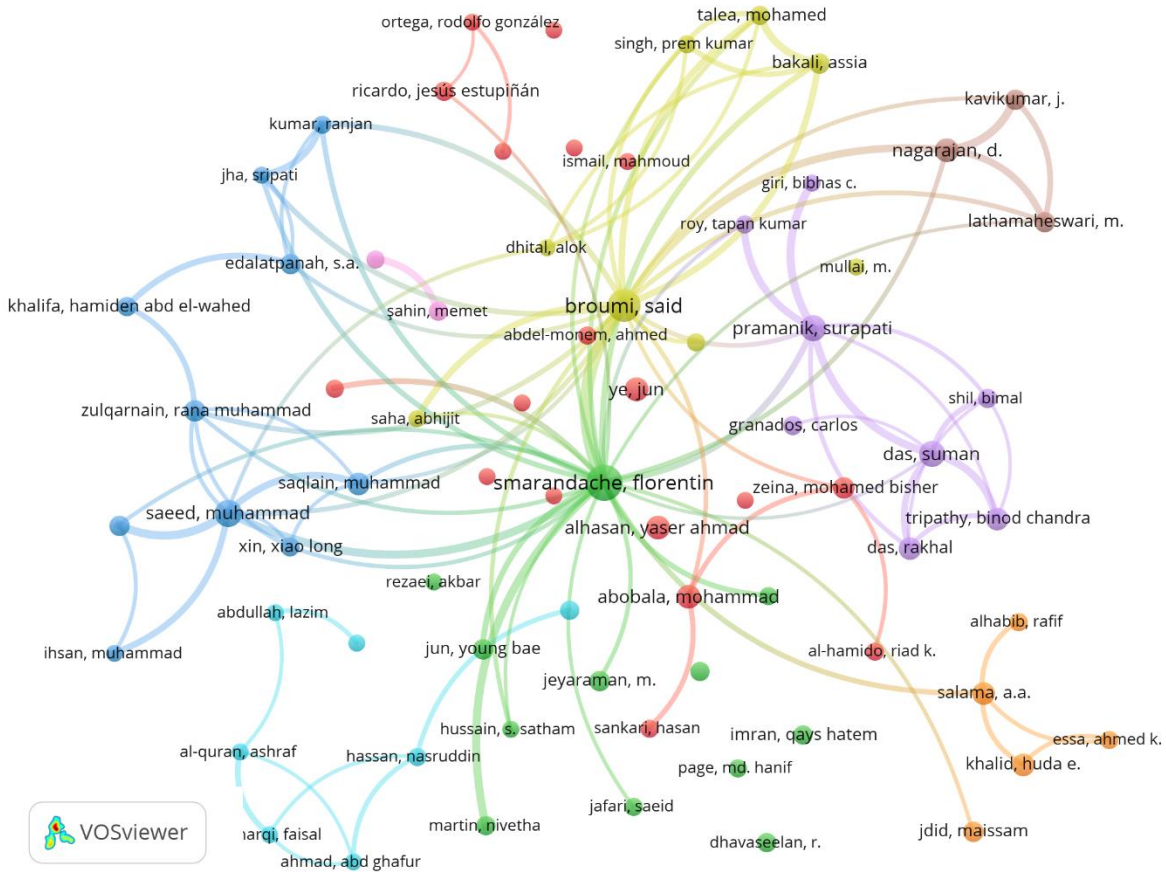

Fig. 6 Bibliographic coupling of authors in NSS: minimum 30 publications and 100 links

The prominent clusters of this network representing Smarandache, Broumi, Saqlain, and Saeed they cited the works together reflecting their immense impact on the development of neutrosophic sets and systems. The big red cluster represents the group of researchers that frequently co cite each other, which implies that their research is very interrelated. Smarandache and Broumi are coauthors whose joint publications are repeatedly cited together as they are key contributors to the theory.

The yellow cluster highlights the authors Pramanik, Surapati, and Das, the interlinks show that these authors are collaborators and working in a close research group. The green cluster of Saeed, Saqlain, Xiao Long Xin, Zulqarnain, Rahman, and Ihsan shows the connections and research groups within NSS. The focus of this research group is to provide the foundational theories and the solution to the optimization, decision-making and artificial intelligence problems. Smaller nodes of less interlinked authors show that their research revolves around the central topics and trends of the journal and the major contributors.

Figure 7 represents the bibliographic coupling network of countries contributing to the Neutrosophic Sets and Systems (NSS). The visual relationships between countries are based on shared citations showcasing the prominent contributors of the journal.
India as a central node is well connected to other countries, like the United States, Pakistan, Turkey, Egypt, and others, indicating India's important role in NSS and contributing to the research of other countries by providing the fundamentals and new directions. Some more prominent clusters of countries like Iran, Saudi Arabia, and Jordan illustrate the regional collaborations in NSS and participation of Middle East and North Africa areas. A separate cluster, including Ecuador, Peru, and Cuba, highlights the participation of Latin American countries that contribute to NSS. Among all the countries, Ecuador has the maximum number of connections, therefore, has a major role in NSS development in that region. Additionally, the network indicates relationships with Europe like Portugal, Spain, and Italy, and other continents like Australia. This network shows interest worldwide in neutrosophic sets and systems.

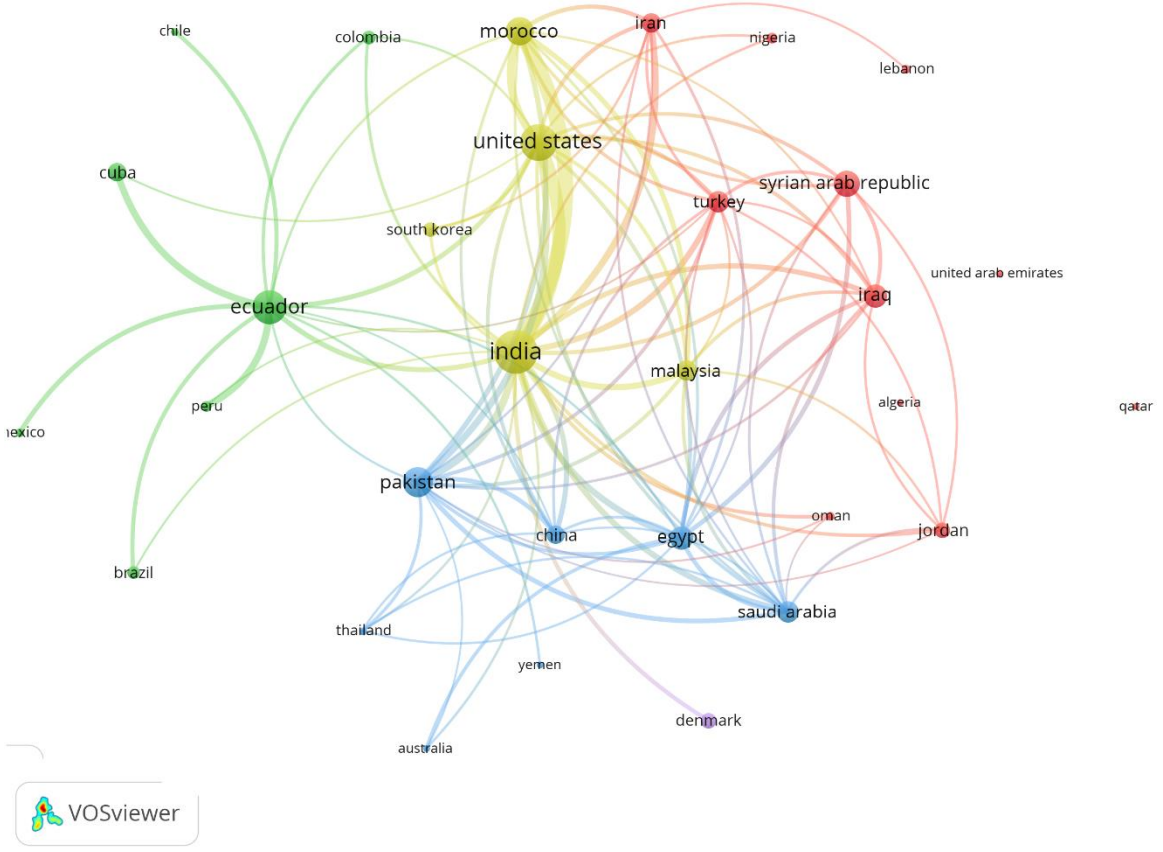

Fig. 7 Bibliographic coupling of Countries in NSS with minimum 5 citations and 100 links

The less connected nodes represent countries with less shared references in their published research. These nodes may indicate regions where neutrosophic research is emerging, with limited collaboration or integration into the broader academic network. Identifying these less connected nodes is valuable for understanding regional research gaps and fostering stronger global collaborations.

4.3 Co-occurrence

Bibliometric analysis of the co-occurrence of author keywords is a useful tool to investigate the relationships among research topics in various papers. Since authors use similar keywords in their publications, it indicates thematic similarities or focus areas. The clusters of co-occurrence keywords highlight the emerging trends, of related topics and the key areas of interest in each domain. The co-occurrence of author keywords helps to determine the research gaps in the literature, lead to future research directions, and promote collaboration among researchers who are working on similar or related research [49].

Figure 8 showcases the co-occurrence network of the author keywords of the Neutrosophic Sets and Systems (NSS). The most prominent clusters represent Neutrosophy, Neutrosophic Set, Neutrosophic Logic, Neutrosophic Statistics, Hypersoft Set, Soft Set, MCDM, and indeterminacy as the key trends in research published in NSS. The keywords relate to broad domains and applications where neutrosophic sets are applied, for example, healthcare, education, supply chain, and data analysis. These keywords are used together because neutrosophic sets are interdisciplinary in nature, where mathematical, logical, and stochastic tools are used in solving decision-making problems. For example, neutrosophic sets are applied in the areas of Human resources and Healthcare.

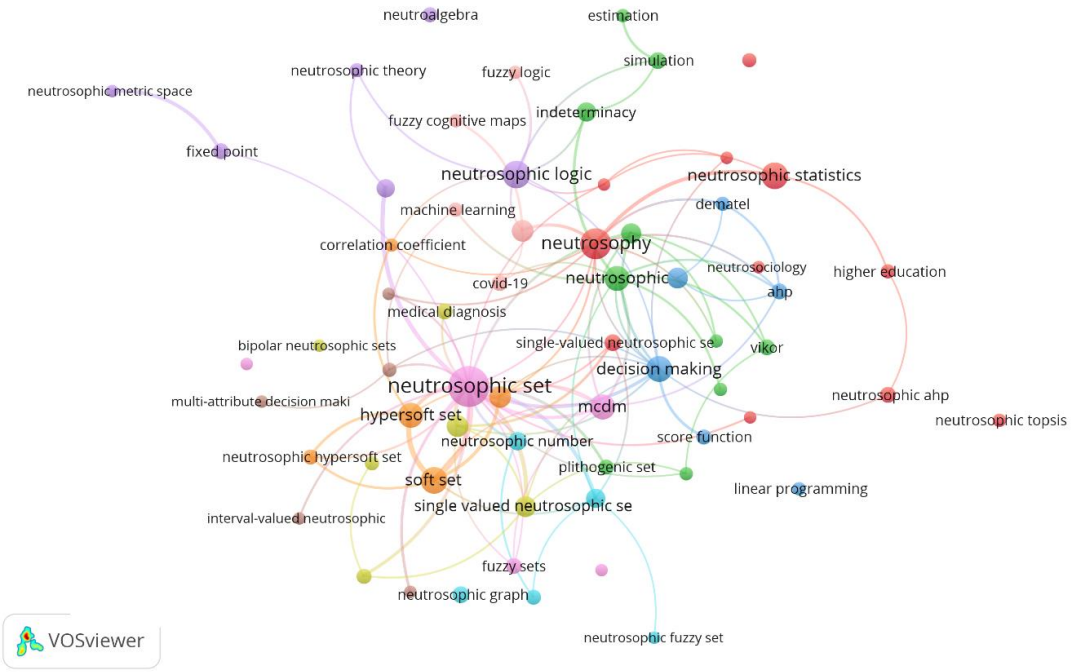

Fig. 8 Co-occurrence of author keywords in NSS with minimum 5 citations and 100 links

The clusters of keywords related to the optimization of Criteria Decision Making (MCDM) and Multi-Attribute Decision Making (MADM) problems are also prominent and have close interlinks with the theoretical development. The result of the co-occurrence analysis shows the major research areas and trends in the neutrosophic sets and systems that help identify the emerging trends in research. Some less-connected clusters of keywords represent the specific subfields or applications and indicate emerging topics with limited integration into the broader research landscape. Strengthening their connections can enhance research visibility and foster interdisciplinary studies. The integration of AI and ML with neutrosophic theory should be a key focus, as it enhances decision-making, pattern recognition, and uncertainty modelling. Publishing on this can drive innovation and enhance journal metrics and exposure.

Table 9 showcases the most productive and influential keywords in NSS. The keywords in *Neutrosophic Sets and Systems* (NSS) reflect the journal's broad research scope, spanning foundational theory, decision science, mathematical structures, and computational applications. These trends underscore NSS's role as a hub for both theoretical and applied research, fostering advancements in scientific and engineering disciplines. Neutrosophic Sets ranks first with 355 total research papers (TRP), 2,770 total citations (TC), and an impressive h-index (H) of 28, reflecting its central role in the field.

It highlights the importance of this keyword since neutrosophic sets plays a fundamental role in resolving the problem of uncertainty, indeterminacy, and inconsistency in decision-making problems. The second most impactful keyword is "Decision Making," which has been used in 293 research papers, and received 2,448 citations with an h-index of 8, indicating its importance in applying neutrosophic concepts to practical problems in multi-criteria decision making (MCDM). Fuzzy Set ranks third, with 135 TRP and 1,047 TC, highlighting the interdisciplinary connection between neutrosophic and fuzzy logic. Other notable keywords include Topology and Set Theory, which demonstrate the integration of neutrosophic sets into the mathematical and computational basis of NSS.

The table also reveals trends in recent research activity. For instance, Neutrosophic Sets and Decision-Making show sustained interest, with 150 and 128 papers published in the last two years, respectively. Keywords like Soft Set, Uncertainty, and MCDM (Multi-Criteria Decision Making) further emphasize the journal's focus on decision-making and uncertainty modeling. There is a tie between keywords "Soft set"and "Uncertainiity" based on total number of published papers i.e. 99, so these keywords can be ranked based on total number of citations they have received.

Table 9. The most productive and influential keywords in NSS.

R	Keyword	TRP	TC	H	C/P	≥100	≥10	D1	D2	D3
1	Neutrosophic Sets	355	2,770	28	8	0	87	46	150	159
2	Decision Making	293	2,448	27	8	0	73	20	128	145
3	Fuzzy Set	135	1,047	18	8	0	31	18	50	67
4	Topology	112	775	15	7	0	28	14	52	46
5	Set Theory	103	988	19	10	0	31	7	46	50

6	Soft Set	99	1,073	18	11	0	32	5	43	51
7	Uncertainty	99	605	13	6	0	19	5	37	57
8	MCDM	72	517	13	7	0	16	8	28	36
9	Neutrosophic logic	70	545	12	8	1	15	15	25	30
10	Algebra	65	433	12	7	0	14	6	27	32
11	Neutrosophy	62	322	10	5	0	10	10	18	34
12	Intuitionistic Fuzzy Set	56	587	14	10	0	19	11	24	21
13	Hypersoft Set	56	1017	19	18	0	30	2	26	28
14	Interval Valued	52	420	13	8	0	15	5	23	24
15	Computer Circuits	46	210	9	5	0	6	3	16	27
16	Similarity Measure	44	548	12	12	0	14	6	21	17
17	Mathematical Operators	44	311	9	7	0	9	3	16	25
18	Closed Set	42	280	12	7	0	13	11	17	14
19	Diagnosis	41	467	12	11	0	14	5	18	18
20	Neutrosophic statistics	40	208	8	5	0	6	4	19	17

Abbreviations are available in the previous tables.

Metrics like the h-index and citations per paper (C/P) provide further context. For example, Set Theory has a high C/P of 10, suggesting its influential role despite a lower TRP. The absence of papers with ≥100 citations for most keywords suggests that while the field is active, highly cited works are rare. Additionally, emerging keywords such as Hypersoft Set and Neutrosophic Logic indicate evolving research directions. Overall, the table underscores the interdisciplinary nature of NSS, bridging neutrosophic theory with applications in decision-making, topology, and fuzzy logic, while also highlighting emerging areas of interest.

5. Conclusion

The section is divided into three subsections. First subsection describes the contribution and main findings of this paper. Second subsection investigates the practical implications of the study. Lastly, the limitations and future directions are presented.

5.1 Contribution and Main Findings of the Study

The paper presents a bibliometric analysis of journal *Neutrosophic Sets and Systems (NSS)* focusing on research trends, collaborating countries, authors, institutes, and thematic developments. The study shows that the NSS journal has expanded significantly in terms of publication volume, but maintaining a high citation impact has been challenging. The peak in receiving citations per paper in 2020 followed by a decline could indicate saturation, a shift in research trends, or variability in article quality. Increasing submissions show that the journal remains an important publishing journal, but ensuring high-impact research contributions remains a key challenge. Strategies such as strict peer review, and special issues on trending topics could help improve citation impact.

An analysis of the field-weighted citation impact (FWCI) and international collaboration trends in the NSS journal reveals both achievements and areas for improvement. While the journal attained Q1 status in 2020, reflecting its growing recognition within the field, the declining proportion of highly cited papers suggests a need to prioritize modern and high-impact research. Additionally, although

NSS has successfully expanded its publications per author maintaining research quality is crucial for sustaining its influential.

The journal primarily focuses on key contributions to neutrosophic theory, particularly in decision-making algorithms and aggregation methodologies. Its development has been shaped by prominent researchers, including Smarandache, Broumi, Abdel-Baset, Saqlain, and Saeed, whose work has significantly influenced its research direction. However, citation patterns indicate that a relatively small number of papers received the high citations, while recent research struggles to gain visibility. This underscores the need for enhanced distribution strategies to increase the reach and impact of published work.

A comprehensive bibliometric analysis of NSS was done by analyzing co-citation, bibliographic coupling, and keyword co-occurrence, for a complete view of research trends. Co-citation analysis highlights the foundational contributions of key authors such as Smarandache, Abdel-Basset, Saqlain, Saeed, and Broumi, who have played a pivotal role in advancing neutrosophic logic. Bibliographic coupling reveals that neutrosophic research is gaining global attraction, with leading contributions from institutions such as the University of New Mexico (USA) and Universidad Regional Autónoma de los Andes (Ecuador). Additionally, international collaboration patterns show strong research ties between India, the United States, and Pakistan, along with emerging contributions from Latin America and the Middle East.

Keyword co-occurrence analysis identifies dominant research topics and trends, including neutrosophic logic, neutrosophic sets, neutrosophic statistics, and hypersoft set. Moreover, the growing application of neutrosophic sets in domains such as healthcare, multi-attribute decision-making (MADM), multi-criteria decision-making (MCDM), and optimization highlights their practical utility in addressing complex real-world problems. This study not only maps the intellectual structure and evolution of NSS research but also identifies emerging trends and potential areas for future study. Despite the significant growth of NSS journal in fostering a research community around neutrosophic theory, challenges remain in improving citation impact and visibility.

5.2 Practical Implementation

The increasing number of volumes of NSS publications highlights its significance in various research domains, particularly in decision support systems. Thus, NSS serves as a key journal to extend the decision-making algorithms along with aggregation operators of neutrosophic theories to provide practical solutions. The data highlight the need to prioritize high-quality, impactful research over quantity, particularly in AI and ML, as these fields enhance decision-making, pattern recognition, and uncertainty modeling, ultimately improving journal standards, visibility, and indexing. While leading authors have significantly contributed to the global visibility of *Neutrosophic Sets and Systems (NSS)*, diversifying contributions and encouraging new impactful publications can prevent over-reliance on a few individuals.

To strengthen NSS's role in Sustainable Development Goals (SDGs), increasing publications from Europe and Australia is essential to leverage their expertise and best practices. Addressing limited collaboration in these regions presents an opportunity to integrate emerging institutions into the broader academic network. Strengthening these collaborations and prioritizing interdisciplinary research will drive innovation, boost journal metrics, and expand NSS's global impact.

5.3 Limitations and Future Directions

This study provides a bibliometric analysis of the *Neutrosophic Sets and Systems* (NSS) journal up to 2023, highlighting key trends in publications, authorship, and collaborations. However, as research evolves, new topics, contributors, and institutions will likely emerge. Another limitation of this study is its reliance on the Scopus database, which may exclude valuable contributions from other sources like Web of Science (WoS) and Google Scholar. To address these limitations, future research should incorporate multiple bibliographic databases, including WoS and Google Scholar, to provide a more accurate representation of the journal development.

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Convolutional Interval-Valued Neutrosophic Network for Intelligent Evaluation of Smart Clothing Design Choices

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Abstract—Smart clothing design turned out to be a set of complex decision-making processes requiring complementary aesthetics, functionality, as well as customer preferences. This, in turn, makes the traditional evaluation methods struggle to deal with uncertainty and subjectivity in customer feedback about smart clothing design. In an attempt to address this challenge, this research article proposes a novel Neutrosophic approach that integrates Interval-Valued Neutrosophic (IVN) with Convolutional Network to build an intelligent tool for the evaluation of smart clothing design choices. The Neutrosophic representation enables modeling uncertainty, inconsistency, and hesitancy in decision-making by assigning interval-ed membership degrees for different views of smart clothes design. Using the interval-valued representations, we enable robust learning and interpretation of user partialities while handling vague feedback. Proof of concept experiments are conducted on a case study for a smart fashion dataset, and the quantitative results and analysis demonstrate that the proposed approach outperforms the standard techniques for smart clothes classification and ranking design choices. The findings from this analysis prove the ability of our approach to facilitate intelligent decision support in the fashion industry.

Keywords: Neutrosophic theory, Interval-Valued Neutrosophic (IVN), Smart Clothing, Neutrosophic Intelligence.

1. Introduction

The integration of artificial intelligence (AI) in fashion design has revolutionized the way designers and consumers evaluate clothing choices [1]. The design of smart clothing is a complicated process that entails advanced materials and embedded technologies and requires intelligent decision-making frameworks that balance aesthetics, functionality, and user preferences [2]. Nevertheless, the task of evaluating smart clothing designs and

making appropriate choices persists as characteristically subjective and uncertain, as customer feedbacks always differ according to different factors such as comfort, style, and usability [3]. This in turn makes the standard evaluation techniques hard to consider for this kind of uncertainty causing inconsistency or biased evaluations.

The neutrosophic sets (NS) were proposed by Smarandache [4] as a mathematical model extending the original concept of fuzzy logic to handle inconsistency, uncertainty, and impreciseness inherent in different life sectors. They designed NS based on a neutrality degree that was presented as a new and autonomous component not exist in fuzzy logic [5], [6]. Thereby, NS can be defined by three membership components which are truth, indeterminacy, and falsity. With the integration of indeterminacy components, we can allow a more detailed description of membership, facilitating the application of NS in decision-making. Accordingly, Neutrosophic theory can be very valuable for clothes designers to differentiate between complete truth and partial truth or between complete untruth and partial falsehood in their work. This can be achieved through systematic comparison between diverse kinds of membership. This qualifies the NS to have acted as an assistive tool for either human or AI decisions to handle the inherent levels of uncertainty within the smart clothes industry [7]. Recently, there is several categories of NS have been proposed in different publications. Trapezoidal NS (TNS) [8], interval-valued NS (IVNSs) [9], multi-valued NSs (MVNs) [10], single-valued NSs (SVNSs) [11], [12], and Intuitionistic NS [13] are some types of these NSs [14].

This research proposes a Convolutional Interval-Valued Neutrosophic Network for the intelligent evaluation and categorization of smart clothing design choices. Instead of scalar membership for truth (T), indeterminacy (I), and falsity (F) components, IVN theory extends original NS by offering intervals of values, which allows a more nuanced representation of uncertainty and disinclination in decision-making. Our framework develops an IVN encoding module to transform the clothes images into IVN-embedding. Then, we pass the IVN embedding to stacked convolutional kernels to learn discriminatory patterns from the IVN components. By the end of the model, we propose an improved IVN loss function to update the gradients of the model according to a decision made by three components.

The remainder of this paper is structured as follows: Section 2 presents the preliminaries and mathematical definitions of IVN operations, Section 3 outlines the case study and the proposed IVN framework, Section 4 discusses the results, and Section 5 concludes the findings.

2. Foundational Concepts

Definition 1. Neutrosophic set [4] to extend the fuzzy set for modeling uncertainty, and is defined as:

$$A = \{\langle x, T_A(x), I_A(x), F_A(x)\rangle : x \in U\} \tag{1}$$

$$0^- \leq \mathrm{T_A(x)} + \mathrm{I_A(x)} + \mathrm{F_A(x)} \leq 3^+ \tag{2}$$

$$\mathrm{T_A}: \mathrm{X} \rightarrow]^-0,1[^+, \quad \mathrm{I_A}: \mathrm{X} \rightarrow]^-0,1[^+, \quad \mathrm{F_A}: \mathrm{X} \rightarrow]^-0,1[^+ \tag{3}$$

Definition 2. An IVNS redefine the membership of each component to be interval instead of scalars, which can be expressed as:

$$A = \{\langle x, [T_L(x), T_U(x)], [I_L(x), I_U(x)], [F_L(x), F_U(x)]\rangle : x \in U\} \tag{4}$$

where the constituting intervals satisfy the following condition:

$$T_L(x) + I_L(x) + F_L(x) \leq 3,\ T_U(x) + I_U(x) + F_U(x) \leq 3 \tag{5}$$

In the above formula, $T_L(x)$ and $T_U(x)$ denote the lower and upper bounds of the truth membership, and similarly for I and F.

Definition 3. Given two IVNSs A and B, the union $A \cup B$ is defined as:

$$A \cup B = \left\{ \left\langle x, \begin{bmatrix} \max\left(T_L^A(x), T_L^B(x)\right), \max\left(T_U^A(x), T_U^B(x)\right) \\ \left[\min\left(I_L^A(x), I_L^B(x)\right), \min\left(I_U^A(x), I_U^B(x)\right)\right], \\ \left[\min\left(F_L^A(x), F_L^B(x)\right), \min\left(F_U^A(x), F_U^B(x)\right)\right] \end{bmatrix} \right\rangle : x \in U \right\} \tag{6}$$

Definition 4. Given two IVNSs A and B, the intersection $A \cap B$ is defined as:

$$A \cap B = \left\{ \left\langle x, \begin{bmatrix} \left[\min\left(T_L^A(x), T_L^B(x)\right), \min\left(T_U^A(x), T_U^B(x)\right)\right], \\ \left[\max\left(I_L^A(x), I_L^B(x)\right), \max\left(I_U^A(x), I_U^B(x)\right)\right], \\ \left[\max\left(F_L^A(x), F_L^B(x)\right), \max\left(F_U^A(x), F_U^B(x)\right)\right] \end{bmatrix} \right\rangle : x \in U \right\} \tag{7}$$

Definition 5. Given an IVNSs $A = \left\{\left\langle x, \left[T_A^L, T_A^U\right], \left[I_A^L, I_A^U\right], \left[F_A^L, F_A^U\right]\right\rangle : x \in U\right\}$, then the complement A^c can be expressed as:

$$A^c = \left\{\left\langle x, \left[F_A^L, F_A^U\right], \left[1 - I_A^L, 1 - I_A^U\right], \left[T_A^L, T_A^U\right]\right\rangle : x \in U\right\} \tag{8}$$

Definition 6. Given two IVNSs $A = \left\{\left\langle x, \left[T_A^L, T_A^U\right], \left[I_A^L, I_A^U\right], \left[F_A^L, F_A^U\right]\right\rangle : x \in U\right\}$, and $B = \left\{\left\langle x, \left[T_B^L, T_B^U\right], \left[I_B^L, I_B^U\right], \left[F_B^L, F_B^U\right]\right\rangle : x \in U\right\}$, the following relations apply as follows:

(9)

$$A \subseteq B \; iff \; T_A^L \le T_B^L, T_A^U \le T_B^U; I_A^L \ge I_B^L, I_A^U \ge I_B^U; \; F_A^L \ge F_B^L, F_A^U \ge F_B^U$$

$$A = B \; if \; and \; only \; if \; A \subseteq B \; and \; B \subseteq A. \tag{10}$$

Definition 7. Given an IVNS $A = \{\langle x, [T_A^L, T_A^U], [I_A^L, I_A^U], [F_A^L, F_A^U]\rangle : x \in U\}$,we can compute the score and accuracy functions as follows:

$$S_A(x) = [T_A^L + 1 - I_A^U + 1 - F_A^U, T_A^U + 1 - I_A^L + 1 - F_A^U]. \tag{11}$$

$$a_A(x) = [min\{T_A^L - F_A^L, T_A^U - F_A^U\}, max\{T_A^L - F_A^L, T_A^U - F_A^U\}] \tag{12}$$

Definition 8. Given two IVNSs $A = \{\langle x, [T_A^L, T_A^U], [I_A^L, I_A^U], [F_A^L, F_A^U]\rangle : x \in U\}$, and $B = \{\langle x, [T_B^L, T_B^U], [I_B^L, I_B^U], [F_B^L, F_B^U]\rangle : x \in U\}$,there are many elementary operations on to be applied:

- Addition

$$A \oplus B = \left\{ \left\langle \begin{matrix} x, [T_A^L + T_B^L - T_A^L \cdot T_B^L, T_A^U + T_B^U - T_A^U \cdot T_B^U], \\ [I_A^L \cdot I_B^L, I_A^U \cdot I_B^U], \\ [F_A^L \cdot F_B^L, F_A^U \cdot F_B^U] \end{matrix} \right\rangle : x \in U \right\} \tag{13}$$

- Subtraction

$$A \ominus B = \left\{ \left\langle \begin{matrix} x, [T_A^L - T_B^L, T_A^U - T_B^U], \\ [\max(I_A^L, I_B^L), \max(I_A^U, I_B^U)], \\ [F_A^L - F_B^L, F_A^U - F_B^U] \end{matrix} \right\rangle : x \in U \right\} \tag{14}$$

- Multiplication

$$A \otimes B = \left\{ \left\langle \begin{matrix} x, [T_A^L \cdot T_B^L, T_A^U \cdot T_B^U], \\ [I_A^L + I_B^L - I_A^L \cdot I_B^L, I_A^U + I_B^U - I_A^U \cdot I_B^U], \\ [F_A^L + F_B^L - F_A^L \cdot F_B^L, F_A^U + F_B^U - F_A^U \cdot F_B^U] \end{matrix} \right\rangle : x \in U \right\} \tag{15}$$

- Scalar multiplications

$$\lambda A = \left\{ \left\langle \begin{matrix} x, [1 - (1 - T_A^L)^\lambda, 1 - (1 - T_A^U)^\lambda], \\ [(I_A^U)^\lambda, (I_B^U)^\lambda], \\ [(F_A^U)^\lambda, (F_B^U)^\lambda] \end{matrix} \right\rangle : x \in U \right\} \tag{16}$$

- Power

$$A^\lambda = \left\{ \left\langle \begin{matrix} x, [(T_A^L)^\lambda, (T_A^U)^\lambda], \\ [1 - (1 - T_A^U)^\lambda, 1 - (1 - T_B^U)^\lambda], \\ [1 - (1 - F_A^U)^\lambda, 1 - (1 - F_B^U)^\lambda] \end{matrix} \right\rangle : x \in U \right\} \tag{17}$$

Definition 9. Given an IVN $A = [T_A^L, T_A^U], [I_A^L, I_A^U], [F_A^L, F_A^U]$, the deneutrosophication can be computed as follows:

$$\mathfrak{D}(A)=\left(\frac{\left(T_A^L+T_A^U\right)}{2}+\left(1-\frac{\left(I_A^L+I_A^U\right)}{2}\right)\left(I_A^U\right)-\left(\frac{\left(F_A^L+F_A^U\right)}{2}\right)\left(1-F_A^U\right)\right) \tag{18}$$

Definition 10. Given a set of IVNSs $A_j=\left\{\left\langle x,\left[T_{A_j}^L,T_{A_j}^U\right],\left[I_{A_j}^L,I_{A_j}^U\right],\left[F_{A_j}^L,F_{A_j}^U\right]\right\rangle:x\in U\right\}$, and corresponding weight vector $W=(w_1,w_2,\dots,w_n)^T$, where $w_j\geq 0$, and $\sum w_j=\mathbf{1}$. Their general weighted aggregation function is formulated as follows:

$$Z=IVNS\text{GWA}(A_1,A_2,\dots,A_n)=\left(\sum_{j=1}^{n}w_jA_j^{\lambda}\right)^{\frac{1}{\lambda}}$$

$$=\left(\sum_{j=1}^{n}w_j\left(\left\langle\left[T_j^{l\lambda},T_j^{r\lambda}\right],\left[1-\left(1-I_j^l\right)^{\lambda},1-\left(1-I_j^r\right)^{\lambda}\right],\left[1-\left(1-F_j^l\right)^{\lambda},1-\left(1-F_j^r\right)^{\lambda}\right]\right\rangle\right)\right)^{\frac{1}{\lambda}}$$

$$=\left(\sum_{j=1}^{n}w_j\left(\left\langle\left[1-\left(1-\left(T_j^L\right)^{\lambda}\right)^{w_j},1-\left(1-\left(T_j^U\right)^{\lambda}\right)^{w_j}\right],\left[1-\left(1-\left(I_j^L\right)^{\lambda}\right)^{w_j},1\right.\right.\right. \tag{19}$$

$$\left.\left.\left.-\left(1-\left(I_j^U\right)^{\lambda}\right)^{w_j}\right],\left[1-\left(1-\left(F_j^L\right)^{\lambda}\right)^{w_j},1-\left(1-\left(F_j^U\right)^{\lambda}\right)^{w_j}\right]\right\rangle\right)\right)^{\frac{1}{\lambda}}$$

3. Case Study and Method

This section provides a holistic explanation of the proposed approaches for evaluating and assessing smart clothes design in uncertain and complicated environments. Our discussion begins by explaining the case study we chose to prove the concept of our methodology explaining their relevance to the subject matter.

3.1 Case Study: Fashion-MNIST Dataset

To evaluate the effectiveness of the proposed method in smart clothing design assessment, we utilize the Fashion-MNIST dataset as a case study. Fashion-MNIST is a widely used benchmark dataset designed as an alternative to MNIST, containing 70,000 grayscale images (60,000 for training and 10,000 for testing) of 10 fashion categories. Each image has a resolution of 28×28 pixels and represents various clothing items (refer to Table 1).

Table 1. Class distribution of the Fashion-MNIST dataset

Label	Class Name	# samples
0	T-shirt/top	1000
1	Trouser	1000
2	Pullover	1000
3	Dress	1000

4	Coat	1000
5	Sandal	1000
6	Shirt	1000
7	Sneaker	1000
8	Bag	1000
9	Ankle boot	1000

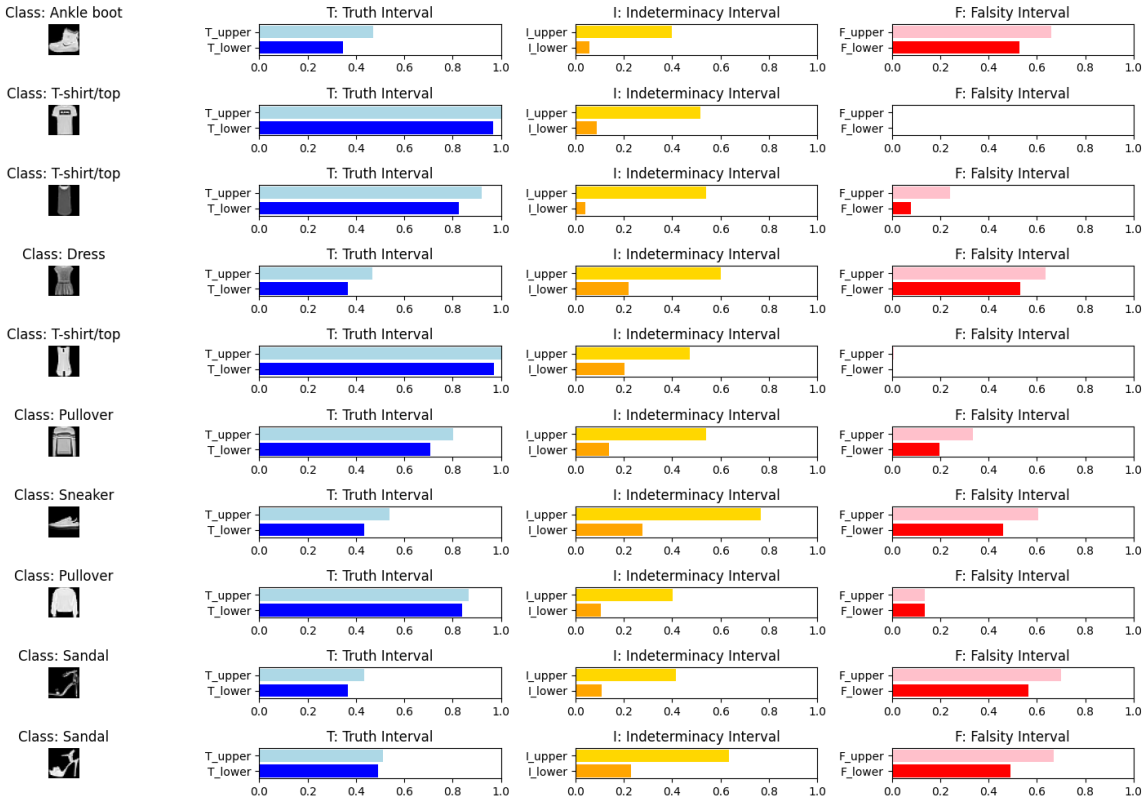

Figure 1. visualization of IVN representation of different fashion images.

The dataset contains diverse clothing items, enabling an initial validation of the model's capability in evaluating design choices. Fashion-MNIST is widely used in ML research, allowing a fair comparison between image classifiers and our IVN-enhanced classifier. While Fashion-MNIST contains simple grayscale images, the approach can be extended to more complex datasets containing high-resolution smart clothing images with additional design attributes.

3.2. Proposed Method

The first step in the proposed method emphasize integrating IVNS into our network to describe how fashion images are embedded into the three IVNS components, as mentioned earlier. Given an input image X, we can extract three interval-valued

components: $T = [T_{\mathrm{L}}, T_{\mathrm{U}}]$, $I = [I_L, I_U]$, and $F = [F_{\mathrm{L}}, F_{\mathrm{U}}]$, which are constrained within specific intervals as given below:

$$\begin{gathered} 0 \leq T_L \leq T_U \leq 1 \\ 0 \leq I_L \leq I_U \leq 1 \\ 0 \leq F_L \leq F_U \leq 1 \\ T_U + I_U + F_U \leq 3 \end{gathered} \tag{20}$$

To do so, we can extract expressive statistical features fusing stacked convolutional layers. These features are later distorted into IVNS components using mathematical encoding scheme (refer to Figure 1).

In our design, T should reflects powerful to which image belongs to a particular class. To do so, we apply a feature extraction function. $\phi_i(X)$, namely stack of convolution layers with activation layers, as shown in Figure 2.

$$\begin{aligned} T_L &= \sigma\left(\frac{1}{n}\sum_{i=1}^{n} \phi_i(X) - \alpha_T\right) \\ T_U &= \sigma\left(\frac{1}{n}\sum_{i=1}^{n} \phi_i(X) + \alpha_T\right) \end{aligned} \tag{21}$$

where $\phi(X)$ symbolizes feature vector extracted from the convolution layers, n is the number of feature values, $\sigma(x) = \frac{1}{1+e^{-z}}$ symbolizes the sigmoid function, and α_T symbolizes a small adjustment factor.

Next, in our design, the uncertainty in learning is often caused by noise, occlusion, or resemblance to manifold classes.

$$\begin{aligned} I_L &= \sigma\left(\frac{1}{n}\sum_{i=1}^{n} \left|\phi_i(X) - \mu_\phi(X)\right| - \alpha_I\right) \\ I_U &= \sigma\left(\frac{1}{n}\sum_{i=1}^{n} \left|\phi_i(X) - \mu_\phi(X)\right| + \alpha_I\right) \end{aligned} \tag{22}$$

Moreover, the falsity degree account for level to which an image does not belong to the target class. We compute this based on the distance from the prototype vector of the class:

$$\begin{aligned} F_{\mathrm{L}} &= \sigma(d(X, C) - \alpha_F) \\ F_{\mathrm{U}} &= \sigma(d(X, C) + \alpha_F) \end{aligned} \tag{23}$$

where $d(X, C)$ symbolizes Manhattan distance function. It is computed as:

$$d(X, C) = \frac{1}{n}\sum_{i=1}^{n} |\phi_i(X) - C_i| \tag{24}$$

where C_i symbolizes the class-centroid computed over all clothes images. α_F symbolize interval adjustment factor.

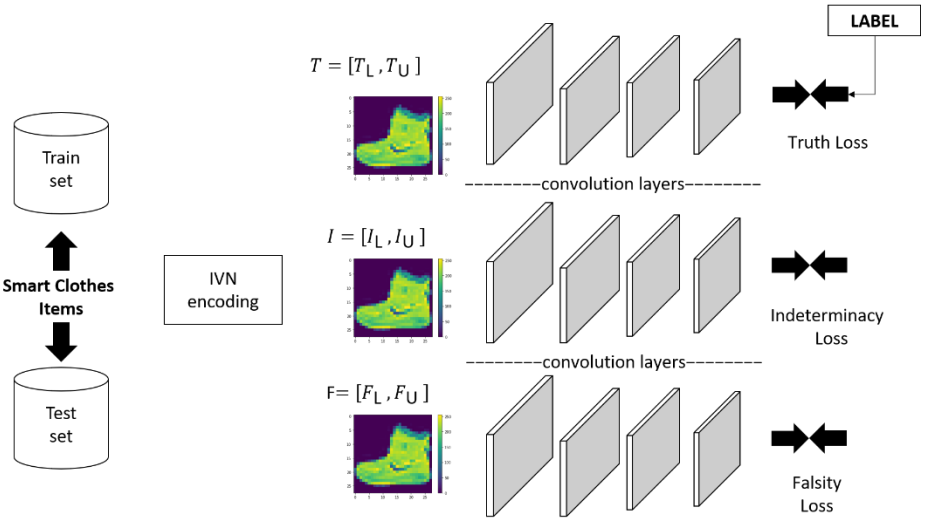

Figure 2. illustration of the architecture of the proposed approach

To train the proposed approach on smart clothing items, we propose and develop an objective function for minimizing the error for all three components. This function is inspired by the interval nature of IVNSs. Assume having an instance i with ground truth intervals are $\left[T_i^{\text{gt}}, I_i^{\text{gt}}, F_i^{\text{gt}}\right]$, while the prediction intervals are $\left[T_i^{\text{pred}}, I_i^{\text{pred}}, F_i^{\text{pred}}\right]$, then we can formulate the loss can as follows:

$$\mathcal{L}_{\text{IVNS}} = \sum_{i=1}^{N} \left(\underbrace{\mathcal{L}_{\text{T}}\left(T_i^{\text{gt}}, T_i^{\text{pred}}\right)}_{\text{Truth Loss}} + \underbrace{\mathcal{L}_{\text{I}}\left(I_i^{\text{gt}}, I_i^{\text{pred}}\right)}_{\text{Indeterminacy Loss}} + \underbrace{\mathcal{L}_{\text{F}}\left(F_i^{\text{gt}}, F_i^{\text{pred}}\right)}_{\text{Falsity Loss}} \right), \tag{25}$$

Where $\mathcal{L}_{\text{T}}, \mathcal{L}_{\text{I}}$ and $\mathcal{L}_{\text{F}}$ represent the interval mean squared error (IMSE) that is calculated as follows:

$$\mathcal{L}_T = \frac{1}{2}\left(\left(T_L^{pred} - T_L^{gt}\right)^2 + \left(T_U^{pred} - T_U^{gt}\right)^2\right), \tag{26}$$

We account for ensuring that the intervals remain valid such that $T_{\text{L}} \leq T_{\text{U}}$, $I_{\text{L}} \leq I_{\text{U}}$, and $F_{\text{L}} \leq F_{\text{U}}$.

$$\mathcal{L}_{penalty} = \sum_{i=1}^{N} \left(max(0, T_L - T_U) + max(0, I_L - I_U) + max(0, F_L - F_U)\right) \tag{27}$$

4. Results and discussions

Herein, we introduce and analyze the experimental results from the proposed method to be applied for evaluating smart clothing items. We emphases on the effectiveness of coupling the IVN representation to enhance the classification performance, handling uncertainty, and improve the decision-making in smart clothing design assessment.

Table 2. Quantitative results against competing methods.

Metric	Proposed	DT	RF
Accuracy	88.26	51.27	87.64
F1-score	88.28	40.37	87.49
Precision	88.53	35.70	87.53
Recall	88.26	51.27	87.64

In Table 2, we provide comparison between the results of our proposed method against the state of-the art approaches namely decision tree (DT), and random forest (RF). Our comparisons emphasize different performance metrics, namely accuracy, precision, recall, F1-score. These metrics are calculated as follows:

As shown, the tabulated results explain the ability of IVN-CNN model to outperforms traditional classifiers across all evaluation criteria, predominantly in modeling uncertainty in images of smart clothes. In addition, the proposed approach can generalize well to unseen smart clothes, which means reducing misclassification errors. This can over an improved interpretability in smart clothing design valuation.

$$Accuracy[\%] = \frac{TP_j+TN_j}{TP_j+TN_j+FP_j+FN_j} = \frac{\sum_{j=1}^{N_C} M_{j,j}}{\sum_{j=1}^{N_C}\sum_{i=1}^{N_C} M_{i,j}} \cdot 100\%. \quad (28)$$

$$Recall_i[\%] = \frac{TP_i}{TP_i+FN_i} = \frac{M_{i,i}}{\sum_{j=1}^{N_C} M_{i,j}} \cdot 100\% \quad (29)$$

$$Precision_j[\%] = \frac{TP_j}{TP_j+FP_j} = \frac{M_{j,j}}{\sum_{i=1}^{N_C} M_{i,j}} \cdot 100\%, \quad (30)$$

$$F1_{binary} = \frac{TP}{TP+\frac{1}{2}(FN+FP)} \quad (31)$$

To ensure the stability of learning performance on IVN-represented images of smart clothes, we display he progression of training and validation accuracy and loss curves of the models in Figure 3. It could be seen that the curves provide insights into the model's learning behavior, convergence rate, and generalization capability. Also, it is notable that the proposed model achieves stable training process, which means rapid convergence and lower validation loss. This can be attributed to the effect of IVN encoding in mitigating the effects of uncertainty, which means reducing overfitting and increase of robustness. Besides, there is small gap between training and validation, which indicate that model can

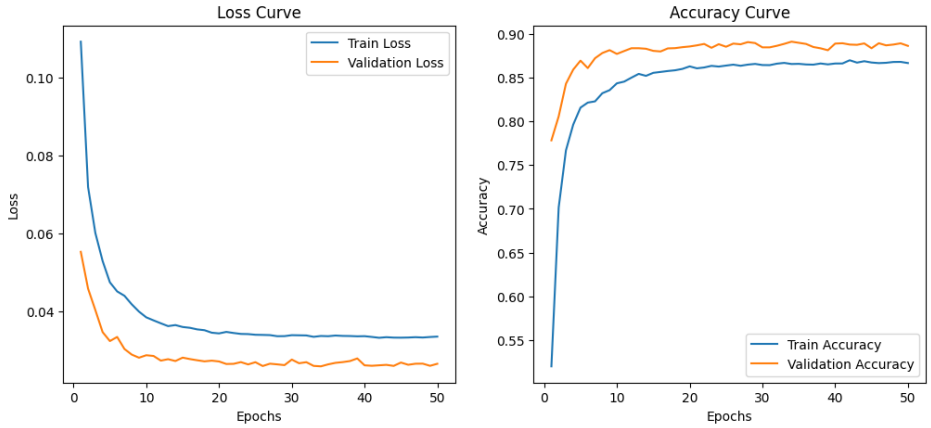

Figure 3. illustration of the learning history of the proposed approach

generalize well due its ability to handle the ambiguous patterns in the images of smart clothes.

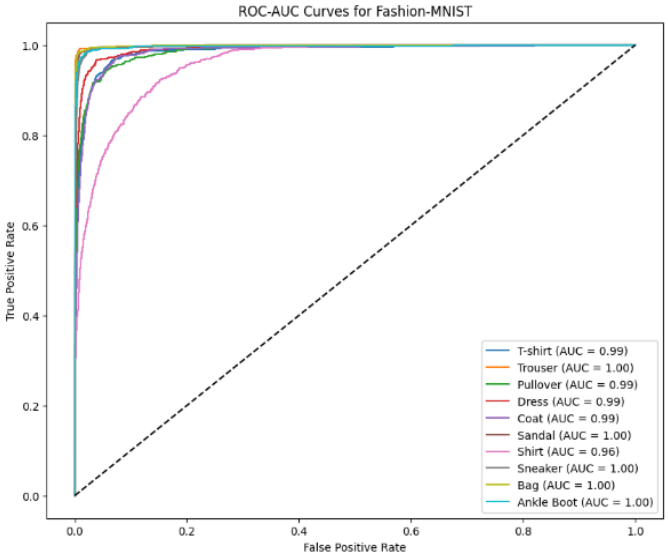

Figure 4. illustration of the ROCAUC curves for the proposed approach

Again, in in Figure 4, we display the trade-off between true positive rate (TPR) and false positive rate (FPR) across diverse clothes classification thresholds, which are encapsulated in form ROC- area under the curve (AUC) curves. This plot provides a fundamental indicator of the model's ability to differentiate between different fashion categories in the images of clothes. It is notable that the IVN-CNN dependably attains a higher AUC score, which demonstrate superior capability to discriminate between overlapping or visually alike clothing items.

In Figure 4, we plot the confusion matrices for the proposed method to provide a detailed breakdown of classification performance across different categories of smart clothes. Each

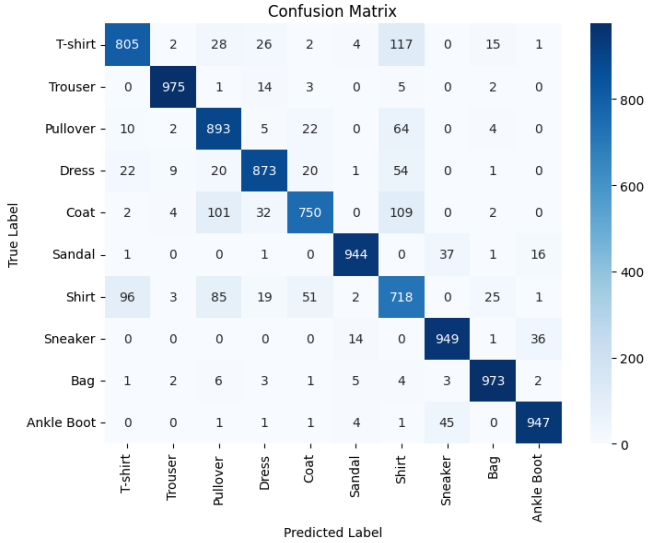

Figure 5. illustration of the confusion matrix of the proposed approach

matrix visualizes the number of correctly and incorrectly classified instances, offering insights into model accuracy and misclassification patterns. As observed, our method achieves a higher number of correct predictions and demonstrates a lower misclassification rate compared to the conventional CNN. Notably, the proposed method reduces confusion between overlapping categories, such as shirts and t-shirts or sneakers and ankle boots, leading to improved classification reliability.

5. Conclusion

In this work, we proposed a Neutrosophic approach that integrates the power of convolutional feature extractors with the uncertainty modeling power of IVN the neutrosophic to obtain improved evaluations of smart clothes design. Our approach extends the conventional convolution architecture to learn to estimate truth, indeterminacy, and falsity membership intervals as an output. We introduce neutrosophic loss function, our approach effectively captures uncertainty and improves decision reliability hence improving the overall gradient update of models. Experimental results demonstrate the potential of our approach in dealing with ambiguous data while making it a promising direction for applications requiring robust classification under uncertainty.

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Enhancing Missing Data Imputation for Migrants Data: A Neutrosophic Set-Based Machine Learning Approach

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Abstract: This study tackles the problem of missing data in migrant datasets by introducing a new framework that combines machine learning techniques with neutrosophic sets. These sets, which can represent uncertainty and ambiguity, are well-suited for managing the complex nature of missing information in sensitive fields like migration research. We test the effectiveness of KNN, SVM, decision tree, random forest, and Ada Boost algorithms on a migrant dataset, comparing their results using different imputation methods (mean/mode, model-based imputer (simple tree), and random values). Our research showed that our proposed approach, which used neutrosophic sets, improved imputation accuracy and strengthened model reliability. Our results underscored the potential of neutrosophic set-based machine learning for addressing missing data issues across various fields.

Keywords: Missing data imputation, neutrosophic sets, machine learning, migrant data, KNN, SVM, decision tree, random forest, Ada Boost, classification, accuracy, precision, recall, F1-score.

1. Introduction

In statistics, missing data, also known as missing values, occurs when no data value is saved for a variable in an observation. Incomplete data is a widespread issue in primary care studies, including clinical trials, observational research, and quality improvement projects. It refers to data points that are not available for analysis, such as migrants who drop out. Missing data is prevalent and can have a substantial impact on the conclusions formed from the data. Missing values in research can lead to bias, reduced validity, inaccurate conclusions and the loss of crucial information from study samples. When missing data is not correctly and handled improperly, it might result in statistical bias and hide the underlying relationship between variables. Missing data might lead to loss of knowledge, affecting study efficiency and interpretation. Also, missing data might reduce the validity and trustworthiness of research findings by increasing bias, causing information loss, and reducing statistical power (Suthar and Goswami, 2012; Hourarip et al., 2014; Ayilara et al., 2019).

The effect of missing values depends on how much data is missing, what type of data is missing, and why it's missing (Zhang, 2015). The most common methods which dealing with missing data is delectation missing values and using the mean to fill in missing values. This method resulted in biased estimation of parameters and uncertainty, and decreasing statistical power.

Machine learning methods can overcome the shortcomings of traditional approaches like mean, median, and regression imputation (Langkamp et al., 2010; Donders et al., 2006). Assessing these methods' performance requires careful planning and analysis. Factors like algorithm choice and sampling techniques are key. Therefore, selecting the right strategy for handling missing data is crucial, as improper treatment can lead to inaccurate results. Recently, advanced analysis models like neutrosophic models have gained popularity in studying the link between migration and missing persons. Neutrosophic statistics offers a robust framework for dealing with various uncertainties in statistical analysis, making it a powerful tool for predicting intervals in machine learning (Smarandache, 2022).

Neutrosophic machine learning combines neutrosophic theory, introduced by Florentin Smarandache, with machine learning algorithms to handle uncertain and incomplete data. While traditional machine learning assumes accurate and complete datasets, real-world data often contains ambiguities and gaps that can affect model performance. Neutrosophic methods offer a novel approach by directly incorporating uncertainty into the learning process, potentially improving model accuracy, flexibility, and interpretability. Neutrosophic sets provide a mathematical framework for representing information beyond simple true-false values, making them a powerful tool for managing uncertainty and indeterminacy (Salama et al., 2024). By integrating neutrosophic theory into machine learning algorithms, we can enhance model performance when dealing with missing data. Jdid et al. (2023, 2022) have explored the basics of neutrosophic simulation and its potential uses in handling randomness and uncertainty.

These models show promise in fields where data is often imperfect. However, implementing neutrosophic techniques can be complex, requiring significant customization of existing algorithms. Despite these challenges, neutrosophic machine learning has the potential to improve outcomes in various applications by better handling uncertainty and missing data. It represents a new direction in how models treat uncertainty and imprecision. While it currently faces obstacles and needs further validation, its ability to provide a richer and more flexible representation of data shows promise for increasing the accuracy and applicability of machine learning models across different fields (Maguina, et al., 2024).

Several studies have addressed missing data issues. For example, (Petersen et al., 2019) examined key implications of handling missing data using health indicator records in the UK. (Stiglic et al., 2019) highlighted challenges in electronic health records, demonstrating missed opportunities from incomplete data through simulations in predictive modeling. Their approach to missing data involved removing incomplete records. The rest of this study is designed as follows, in section 2, we proposed the related works, section 3 proposed the methodology of this study, in section 4, we presented missing data mechanisms, section 5, introduced machine learning algorithms, section 6, proposed neutrosophic machine learning and finally section 7, numerical analysis of this study.

2. Related Works

There are many studies related to this study like, Power and Freeman (2012) compared interpersonal psychotherapy (IPT) and cognitive-behavioral therapy (CBT) in primary care. Their study, which examined three treatments, encountered significant missing data. They suggested that complex imputation methods might offer the best estimates. Mirzaei et al. (2022) investigated handling missing survey data using various approaches, including deletion, calculation, and likelihood methods, providing guidance on their application. Nijman et al. (2022) applied machine learning to tackle missing data in model prediction, finding that deletion, complete-case analysis (CCA), and multiple imputation were most effective. Zhou et al. (2022) explored how missing data impacts comparative effectiveness research using electronic health records (EHRs), concluding that spline smoothing yielded results similar to having complete data. This study aims to employ neutrosophic set under machine learning to treat missing data issue.

3. Methodology

3.1. Data Collection and Processing

- **Dataset Acquisition:** A migrants dataset, either publicly available or proprietary, was obtained, containing relevant demographic, socioeconomic, and migration-related information.
- **Data Refinement:** The dataset underwent thorough cleaning to eliminate inconsistencies, errors, and outliers. Missing values were identified and classified based on their mechanisms like, missing completely at random (MCAR), missing at random (MAR), and missing not completely at random (MNAR).

3.2. Neutrosophic Set Application

Neutrosophic sets (NSs) introduced by (Smarandache,1998) which represented a generalization of fuzzy sets and intuitionistic fuzzy set, is a powerful tool to deal with incomplete, indeterminate and in consistent information, which exist in the real-world. Neutrosophic sets are determined by truth membership function (T), Indeterminacy membership function(I) and falsity membership function(F). This theory is very essential in various applications areas since indeterminacy is quantified explicitly and the determined memberships functions are independent. Wang, Smarandache, Zhang & Sunderraman (2010) proposed the concept of single – valued neutrosophic set. The single-valued neutrosophic set can convey truth-membership degree, indeterminacy-membership degree, and falsity-membership degree, addressing incomplete, indeterminate, and inconsistent data. The single-valued neutrosophic set accurately describes human thought due to the inadequacy of external knowledge.

- **Attribute Representation:** Each dataset attribute was represented as a neutrosophic set, assigning degrees of truth, falsity, and indeterminacy to possible values.
- **Distance Measures:** Appropriate measures (e.g., Hamming distance, Euclidean distance) were defined to compare neutrosophic sets, accounting for uncertainty and indeterminacy.

3.3. Machine Learning Algorithm Choice

- **Selected Algorithms:** The following algorithms were chosen based on their ability to handle missing data and incorporate neutrosophic sets:
  - K-Nearest Neighbors (KNN)
  - Support Vector Machines (SVM)
  - Decision Trees
  - Random Forest
  - AdaBoost
- **Algorithm Modification:** These algorithms were adapted to process neutrosophic data by incorporating neutrosophic distance measures or adjusting their decision-making processes.

3.4. Imputation Strategies

- **Conventional Imputation:** Standard imputation methods were employed as benchmarks, including:
  - Mean imputation
  - Median imputation
  - Multiple imputation
- **Neutrosophic Imputation:** An innovative imputation approach was developed, utilizing neutrosophic sets to fill in missing values based on the closest neighbors in the neutrosophic attribute space.

3.5. Model Training and Assessment

- **Model Training:** The chosen machine learning algorithms were trained on the imputed dataset using suitable training methods (e.g., cross-validation).
- **Model Assessment:** The models' performance was evaluated using common classification metrics, such as accuracy, precision, recall, and F1-score.
- **Comparison:** The effectiveness of the neutrosophic-based imputation method was compared to traditional imputation techniques to gauge its efficacy.

3.6. Sensitivity Analysis

- **Parameter Adjustment:** The sensitivity of the results to various algorithm parameters and imputation methods was examined.
- **Missing Data Patterns:** The effect of different missing data mechanisms (MCAR, MAR, MNAR) on model performance was explored.

3.7. Ethical Considerations

- **Data Protection:** Appropriate steps were taken to ensure data privacy and confidentiality throughout the study.
- **Fairness and Bias:** The potential for bias and unfairness in the dataset and models was addressed, with measures implemented to mitigate these issues.

4. Missing Data Mechanisms

The nature of missing data varies across studies based on the underlying mechanism of missingness. These mechanisms define the patterns of missing data. The three common types of missing data patterns are univariate, monotone, and non-monotone (Little and Rubin, 2019). Univariate patterns occur when missingness is related to a single variable (Demirtas, 2019), often seen in experimental studies (Lacerda et al., 2007). Monotone patterns are typically observed in ordered variables, such as those in longitudinal studies (Liu, 1995). Non-monotone patterns are the simplest to handle as missing values are easily identified through patterns (Dong and Peng, 2013). In this case, the missingness of one variable does not influence the missingness of others (Chen, 2020).

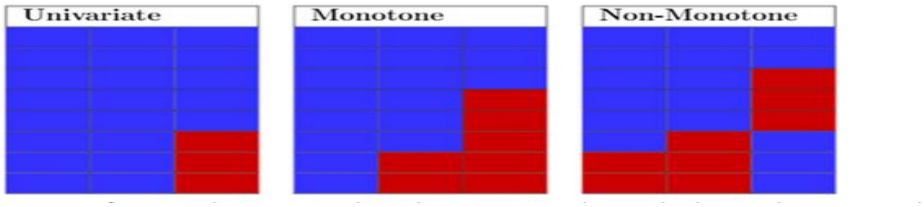


Figure 1: Missing data patterns.

The assumptions underlying missing data treatment methods are influenced by the mechanisms leading to data missingness. Therefore, it is crucial to understand these mechanisms. The theory of missing data, established by Rubin (1976), identifies three main mechanisms for missingness, which are determined based on the availability and absence of data.

To define the missing. Let Y be an entire data matrix, which separated into Y_0 observed data and Y_m missing data. Let R is a matrix of missing value known by,

$$R = \begin{cases} 0, & if\ Y\ is\ observed \\ 1, & if\ Y\ is\ missing \end{cases} \quad (1)$$

Let q a values vector, which determine the relation between the missing in R and the data set Y. The mechanisms of missing values defined by the probability of whether a value observed or missing as it displays below.

a) Missing at Random (MAR)

In MAR, the probability of missing data is solely linked to the observed data. This probability in MAR can be expressed as:

$$p(R|Y_0, q) \quad (2)$$

Missing at random (MAR) mostly appeared in medical studies data sets. The missing in this mechanism can be treated by observed predicator variables (Gomez et al., 2014).

b) Missing Completely at random (MCAR)

Occurs when the occurrence of missing data is unrelated to both observed and unobserved measurements. The probability of missing completely at random (MCAR) is represented as:

$$p(R|q) \quad (3)$$

c) Missing not at random (MNAR)

This situation occurs when the missing data is independent of the previous two mechanisms, and the missing values depend equally on both the missing and observed values. Dealing with missing data using this approach is challenging due to its connection with unobserved data.

The probability of MNAR is defined as: -

$$p(R|Y_0, Y_m, q) \quad (4)$$

The probability R depends on both Y_0 and Y_m and this mechanism is applied in various sectors like biomedicine in data sets of psychology and education (Grittner et al., 2011).

5. Machine Learning Algorithms.

The approaches of missing data vary from traditional to improved techniques of machine learning. As the imputation of traditional statistical methods has a bias at the final analysis of missing data. Therefore, this study introduces some machine learning methods for handling missing data problems. Such learning methods allow us to find challenging associations in data sets where exploratory analysis has failed to accurately determine the form of the underlying model. We do not want to provide an explicit formula for the distribution of our data; instead, we want the algorithm to figure out the pattern on its own, based on the facts. Learning methods allow us to challenge the assumptions underlying statistical methodology. Algorithms of machine learning imputation are used to develop a predictive technique to treat missing data. In addition, the estimation process of these methods relies on the availability of information from observed data using labeled or unlabeled data. The machine learning techniques are discussed below.

(5-1) K nearest Neighbor Classification

The algorithm of KNN conducts by classifying the nearest neighbors of missing values and using those neighbors for imputation using a distance measure between instances (Maillo et al., 2017). Different distance like, Euclidean distance, distance of cosine, Jaccard distance and distance of Minkowski can be used for imputation of KNN. However, the distance of Euclidean is the most distance measure used widely.

The imputation of KNN using distance of Euclidean distance can be written as follows:

$$Dist_{xy} = \sqrt{\sum_{k=1}^{m} \left(X_{ik} - X_{jk}\right)^2} \quad (5)$$

Where:-

- $\boldsymbol{Dist_{xy}}$: is the Euclidean distance.
- $\boldsymbol{k}$: data dimensions
- $\boldsymbol{X_{ik}}$: value for j .attribute involving missing data.
- $\boldsymbol{X_{jk}}$: is the value of j. Attribute including complete data.

The $\boldsymbol{k}$ points that have a minimum distance value are chosen, and then the estimation of weight mean is imputed as:

$$X_k = \frac{\sum_{j=1}^{J} W_j V_j}{\sum_{j=1}^{J} W_j} \quad (6)$$

Where:-

- $\boldsymbol{X_k}$: is the estimation.
- $\boldsymbol{J}$: is the parameters number with $j = 1,2,3, \ldots \ldots, k$.
- $\boldsymbol{V_j}$: are complete values on attributes including missing data.
- $\boldsymbol{W_j}$: is the observed nearest neighbors.

Hence, the weighted value given by the following equation:

$$W_j = \frac{1}{dis_{(x,y)^2}} \quad (7)$$

In both discrete and continuous data, The KNN calculation technique is used also, conducted as a handler of multiple missing data (Suthar et al., 2012 ; Mailo et al., 2017).

(5-2) Support Vector Machine (SVM)

This method represents the most used method for handling missing data (Honghai et al., 2005 ; Pelckmans et al., 2005). SVM discovers an optimal separating hyper- plane for a sample of a labeled training like the distance between the hyper- plane and the nearest data points is maximized (Stewart, 2018). The hyper- planes are known by:

$$w.\ x_1 + b \geq +1 \quad when \quad y_i = +1 \quad (8)$$

$$w.\ x_1 + b \leq -1 \quad when \quad y_i = -1 \quad (9)$$

Where:-

- $\boldsymbol{w}$: is a vector of weight.
- $\boldsymbol{x}$: is a vector of input.
- $\boldsymbol{b}$: is a bias.

Different researchers applied SVM in various studies. For instance, (Hong et al.,2005) handling missing data by using the SVM regression method. They concluded that this method gave precise results on the data set of SARS. In addition, (Smola et al., 2005) the authors treating missing data using SVM and Gaussian processes by using exponential families in feature peace. In another research (Ghazanfar and Prugel, 2013). The authors exchanged the missing values by using the findings assesses from applying the SVM classifier through the set of training and used SVM regression to treat the values.

(5-3) Decision tree

Describing a machine-learning algorithm that outlines all potential outcomes and the pathways to those outcomes in a tree structure format. This approach relies on constructing decision trees to analyze missing values for each variable, filling in these gaps by referencing the respective tree (Twala, 2009). Predicted missing values are displayed in the leaf nodes. Notably, this algorithm handles numerical and categorical variables with

equal effectiveness. Decision trees offer a lower bias in estimation compared to other methods, despite requiring a substantial amount of time for computation (Rockach, 2016).

(5-4) Random forest

This method enhances decision tree bagging by reducing correlation between trees. Similar to bagging, it involves training M decision trees on bootstrap samples. However, during tree construction, each split considers a random subset of predictors, allowing changes based only on this subset. In a random forest, each split restricts the use of most available predictors. The number of predictors used at each split, m, is typically the square root of the total number of predictors, p, rounded up:

$$m \approx \sqrt{p} \qquad (10)$$

It is necessary to utilize a small value of m when establishing a random forest in the presence of a large number of correlated predictors. The strategy of constructing a random forest, which containing$N$ trees, is as follows: -

1) Generate a bootstrap X_n for each tree.
2) Build each tree T_n on the sample X_n
3) At each tree split, the best predictor is selected from a random subset based on criteria like entropy or the Gini index. The tree continues to grow until its leaves contain no fewer than a specified minimum number of objects or the maximum tree depth is reached. This process results in a model (Boyko and Dypko, 2023):-

$$\widehat{f_{rand}}(x) = \frac{1}{M}\sum_{m=1}^{B} \hat{f}^m(x) \qquad (11)$$

(5-5) Ada Boost :-

This algorithm proposed by Frennd and Schapire in (1997) revolutionized ensemble modeling. This technique is extensively employed to tackle binary classification tasks. By converting numerous weak learners into formidable, strong learners, this potent algorithm significantly boosts prediction accuracy. The algorithm operates by initially constructing a model on the training dataset and subsequently creating a second model to rectify errors from the initial model. This iterative process continues until errors are minimized, ensuring accurate predictions on the dataset.

6. Neutrosophic Machine Learning:-

Neutrosophic statistics represents an extension of interval statistic, provides a robust framework for handling various indeterminacies in statistical analysis (Smarandache, 2022). The main concern of neutrosophic statistics is used for the analysis of the uncertainty observation data. Neutrosophic machine learning is an emerging branch that merges neutrosophic theory, introduced by Flor-entin Smarandache (Smarandache, 2022 & Smarandache, 1999) with algorithms of machine learning to handle uncertainty, imprecision and Missing data. Traditional machine learning supposes accurate and complete data, but data often includes ambiguities and gaps that can impact on model performance. Neutrosophic provides a novel approach by incorporating uncertainty .Directly into the process of learning, enhancing model accuracy, flexibility, and interpretability.

These models are useful in different fields where data is often imperfect. Although the complexity of implementing neutrosophic techniques because it requiring significant customizing of existing algorithms. Despite these challenges, neutrosophic machine learning has the potential to improve outcomes in various applications by better handling uncertainty and missing data. So, neytrosophic machine learning presents a new progression in the way models treat uncertainty and imprecision. However it currently faces hurdles and needs more validation, it is ability to give a richer and more flexible representation of data displays increasing of the accuracy and applying of machine learning models in different fields (Maguina, etal.,2024)**.**

To develop models that can learn from data to make prediction, machine learning uses mathematical formulations without being explicitly programmed to accomplish such tasks (Shinde and Shah, 2018).Interval prediction in machine learning refers to the approach of predicting a range of probable outcomes for a given input rather than a single point estimate. By offering intervals, these approaches give not just forecasts but also insight into the dependability and uncertainty of the predictions, which is vital for decision making in uncertain contexts.

For a data set with independent variables $X = [x_1, x_2, x_3, \dots., x_n]$ and a dependent variable, the main objective of regression analysis is to model the relationship between X and Y accurately.

This relationship is expressed mathematically as:

$$y \approx f(x; \theta) \qquad (12)$$

Where:-

- $\boldsymbol{y}$: is the dependent variable or the objective to be predicted.
- X: represents the independent variables that are used to predict
- f: Is the regression function, which can vary in shape depending on the type of regression model used (Linear, Polynomial, Logistic, etc).
- θ: are the parameters or coefficients of the model, adjusted during the training process to minimize a loss function, typically the mean square error(MSE) in regression (Anwar etal., 2024). For instance, the prediction interval for a new observation in a simple linear regression is given by:

$$\hat{y}_0 \pm t_{\frac{\alpha}{2}, n-2}.SE \qquad (13)$$

Where:-

- $\boldsymbol{\hat{y}_0}$: is the predicted value of y from the t distribution, for a specific confidence level $\propto$ and $n-2$ degrees of freedom.

Converting the traditional intervals into a neutrosophic number is enhanced to contain a component of indeterminacy. This addition captures the uncertainty and imprecision that is typically present in real- world data, offering a more nuanced understanding of data variability.

The interval neutrosophic treatment is as follows:-

$$\hat{y}_0 \pm t_{\frac{\alpha}{2}, n-2}.SE + \left(\hat{y}_0 \pm t_{\frac{\alpha}{2}, n-2}.SE\right) I_N \qquad (14)$$

I_N refers to the indeterminacy factor associated with the prediction. Where $I_N \in [I_l, I_u]$.

Where:-

- I_l: lower indeterminacy.
- I_u:upper indeterminacy.

Which define the range of possible deviations due to uncertain elements that affect the forecast (Delgado, etal., 2024). Neutrosophic techniques increasing the robustness and reliability of prediction models by presenting a more complete framework that takes into account many types of uncertainty. The neutrosopjic mean denoted as X_n, is calculated considering the neutrosophic inclusion I_N that connects to the interval $[I_l, I_u]$. This mean consists of two main elements X_l refers to the mean of the bottom part of the neutrosophic samples, and X_u is the mean of the top part.

The respective definitions are (Castro, etal., 2021):-

$$X_l = \frac{\sum_{i=1}^{n_l} X_{il}^1}{n_l} \quad (15)$$

$$X_u = \frac{\sum_{i=1}^{n_u} X_{iu}}{n_u} \quad (16)$$

Where n_l and n_u refers to the number of elements in the lower and upper parts of the neutrosophic samples respectively. Therefore, the neutrosophic mean X_n is defined as the sum of

X_l and X_u, handled by the interval of indeterminacy I_n(Sanchez etal., 2021).

$$X_N = X_l + X_u I_N; \quad I_N \in [I_l, I_u] \quad (17)$$

Where:-

- $I_l = 0$ and ,
- $I_u = \frac{X_u - X_l}{X_u}$

7. Numerical Analysis: -

This research utilized data obtained from the Missing Migrants Project, sourced from the International Organization for Migration. Specifically, the data originates from the Missing Migrants Project, a distinct initiative that monitors the deaths of migrants, including refugees, who have disappeared along complex migration paths globally, spanning from 2014 to June 2017. The dataset comprises 2420 individuals and was obtained from an open online repository available on the Missing Migrants and the dataset is publicly accessible at Global Missing Migrants Dataset (kaggle.com). The primary objective of this study revolves around handling missing data and determine the most appropriate machine learning algorithm for this dataset type, focusing on some accuracy metrics.

(7-1) Data set description: -

The Missing Migrants Project documents individuals who have perished or disappeared while attempting migration across international borders or en route to foreign destinations. This tally does not encompass fatalities within immigration detention centers, during deportation, or following the forced return of migrants to irregular statuses, such as those stemming from labor exploitation. Moreover, it excludes migrants who pass away or vanish after settling in a new residence, thus excluding deaths occurring in refugee camps or housing facilities.

The data available offers insights into evolving conditions and trends concerning migration paths and the individuals traversing them, providing valuable information for policy formulation and protective strategies. This dataset can aid in assessing the comparative risks associated with irregular migration routes. It comprises 10 predictor variables, including ID, cause of death, region of origin, nationality, missing persons, deceased individuals, incident region, date, latitude, and longitude. These variables play crucial roles in- depth analysis and discussion of numbers of missing and dead migrants around the world and the challenges involved in identification and tracking. But this analysis relied on eight variables that influence missing or death as displayed in table 1. Detailed description for the features of the missing and dead migrants are provided in table1.

Table1: Description of variables

Variable	Description
Id	Unique key documenting incident
Region of origin	Horn of Africa, null and other.
Missing persons	Counts (number of missing persons)
Dead	Counts (number of deaths)
Incident region (Source)	Region where incident was recorded.
Reliability	Verified, partially verified.
Latitude	Spatial coordinates
Longitude	Spatial coordinates

Table 1 introduced features description of eight variables for 2420 individuals. Firstly, the **id** variable represented numerical variable, and the key recorded for each incident. **The region** of individuals decomposed in three areas horn of Africa with percentage 20%, the percentage of persons with non-region was 18% and persons who belonged to other areas represented 62% of the total number of persons. The **incident – region** identified two places of missing or death north Africa contained 26% of instances, Mediterranean included 24% and the rest of percent (50%) represented other areas. **The source of** recording the states contained regional mixed, this source recorded 27% of migrants, 8% of people recorded by Pima county office of the medical examiner and the other sources recording 65% of people. Finally, **reliability,** the percent of verified individuals was 33% while partially verified involved 47% and the other percent 20%.

Table2: Feature statistics under missing data.

Variable	Mean	Mode	Median	Dispersion	Minimum	Maximum	Missing
Region- origin		Home of Africa(p)		2.19			443(18%)
Missing	39.66		10	2.12	0	750	2149(89%)
Dead	4.73	1	1	4.34	0	750	102(4%)
Incident – region		North Africa		1.91			10(0%)
Lat.	26.9014	12.5331	29.3489	0.428083	-26.2245	66.9672	4(0%)
Lon.	-13.9959	1.85869	14.4711	-4.40709	-117.071	116.225	4(0%)
Reliability		Partially verified		**0.908**			**324(13%)**
Id	95926.26	1	121178	0.65	1	184750	0(0%)

Table 2 presented the statistics of predicators using Orange software, under this study like mean, mode, median, dispersion, minimum, maximum value and missing number and percent of each variable. The highest variable of dispersion was missing. This proved that there were differences and variation between these variable values with a mean of 39.7 affected by missing observations. The less dispersion variable was dead variable this emphasized on symmetrical between its observation. This study depended on partially verified persons, which represented 13% from all instances with high dispersion equals 0.908. The number of missing migrants and the percentage of them through the covering period. The results showed that the highest percentage was for missing variable with 89%, then region origin with 18% and finally reliability with 13%. This variation in missing percentage relied on the number of missing persons to the total number of instances.

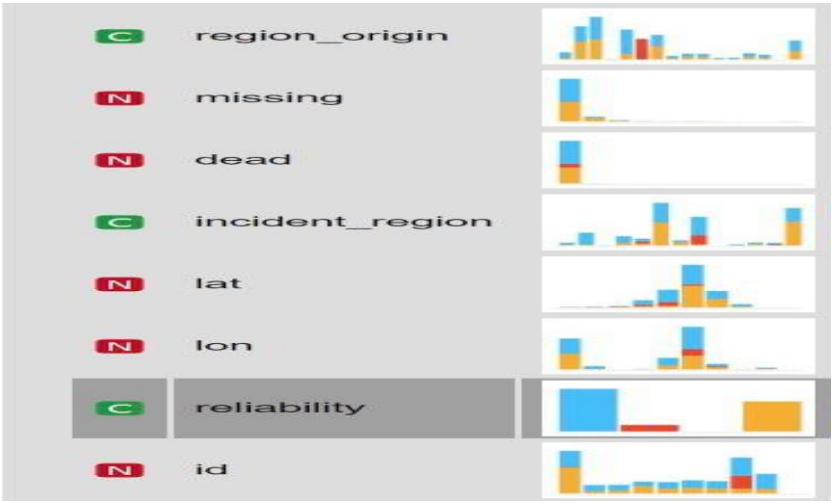


Figure 2: Distribution of study variables through missing data

Figure 2 proposed study predictors distributions. The figure displayed that the reliability of partially verified have a normal distribution, while missing is a right skewed variable, this emphasized on that the most extreme

values are on the right side and the mean or average is greater than the median because of the presence of missing values of some migrants.

The main objective of this study is to treat the missing values of people who are missing or partially verified in migration. This study depended on three methods to replace missing values they are , average/ most frequent method , a method of random values and model based imputer (simple tree). Also, using four precision measures to evaluate these methods under machine learning algorithms involving classification accuracy (CA), F_1 – score, precision and Recall. Classification accuracy represents the ratio of the number of correct predictions to the total number of input samples.

While F_1 is the weighted average of the accuracy and recall rates close to 1 indicating higher accuracy levels, F_1 – score equals: -

$$= \frac{2 \times precision \times Recall}{precision \times Recall} \qquad (18)$$

On the other hand, Recall or true positive rate (TPR) measures the percentage of relevant data points that were correctly identified by the model, represents a metric that measures how machine learning model correctly identifies positive instances. TPR can be written as follows: -

$$\frac{TP}{TP + FN} \qquad (19)$$

Finally, precision is one indicator of a machine learning model's performance- the quality of a positive prediction made by the model. Precision refers to the number of true positives divided by the total number of positive predictions (i.e. the number of true positives plus the number of false positives as follows: -

$$precision = \frac{Tp}{Tp + Fp} \qquad (20)$$

Table 3: Confusion matrix

		Predicted: Yes	**Predicted: No**
		1	0
Actual: Yes	1	TP	FN
Actual: No	0	FP	TN

Table 4: The effectiveness of replacing missing data under random values method

Model	**CA**	**F_1**	**Precision**	**Recall**
Tree	**0.924**	0.518	0.541	0.497
Logistic Regression	0.901	0.269	0.344	0.221
Naïve Bayes	0.869	0.539	0.379	**0.930**
SVM	0.898	0.283	0.333	0.246
KNN	0.905	0.346	0.396	0.307
Random forest	**0.919**	0.522	0.507	0.538

Ada boost	**0.919**	0.512	0.507	0.518

Table 4 proposed the imputation of random values, which is used to substitute missing values, there by constructing a data set with non-missing. Which this approach is straight forward, it doesn't utilize any information from the data set and may present randomness that affects subsequent analyses (Hui, et al., 2023). The precision measurements of machine learning under missing data issue introduced. The findings showed that the Recall measure for naïve Bayes has the highest precision with ratio 93%, then classified accuracy for tree algorithm, and after that random forest and ad boost have the same value under the same measure.

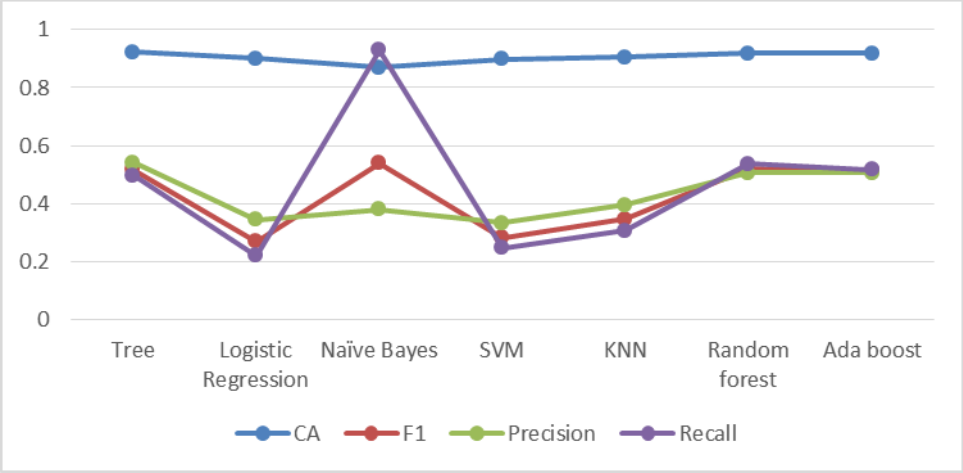


Figure 3 : Comparison of Machine Learning Algorithms for Missing Data Imputation and Classification (Random Values Method)

Figure 3 introduced (CA, F_1, precision and Recall) measures for proposed machine learning algorithms under random values method. The previous figure showed how these measures varies through different algorithms introduced. This figure displayed that the Naïve Bayes algorithm is the best for Recall measure with percent 93%, then the three algorithms of (tree, Ada boost and random forest) are the best for classified accuracy (CA) measure with percent (0.924, 0.919, 0.919) respectively.

Table 5: Handling missing data using average/ most frequent method

Model	CA	F_1	Precision	Recall
Tree	0.924	0.431	0.452	0.412
Logistic regression	**0.929**	0.023	0.333	0.012
Naïve Bayes	0.844	0.414	0.281	0.782
SVM	0.854	0.314	0.234	0.476
KNN	0.918	0.261	0.357	0.206
Random forest	**0.929**	0.387	0.495	0.318
Ada boost	0.924	0.454	0.461	0.447

Table 5 exhibited average/ most frequent method for fixing missing data. This method counted on replacing missing entries with the average (mean) or most frequent value(mode). This is quick and easy approach, but it

can introduce bias if the missing data is not randomly distributed (Tamboli, 2024). This study built on missing completely at random, hence the observations followed normal distribution, so, it reckoned on mean to substitute missing data. The findings clarified that logistic regression and random forest were the best methods under this technique according to CA measure with proportion 92.9.

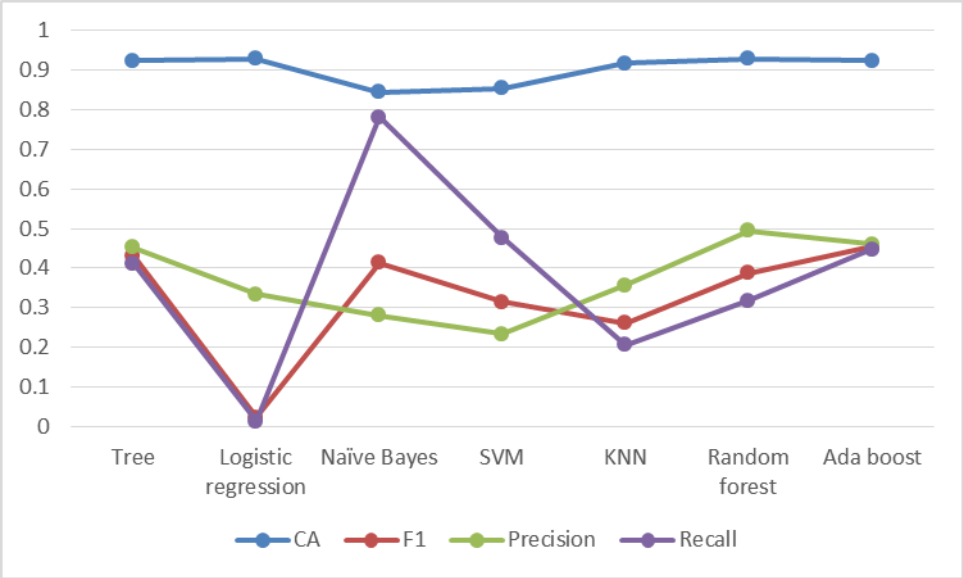


Figure 4: Comparison of Machine Learning Algorithms for Missing Data Imputation and Classification (Average/Most Frequent Method)

Figure 4 illustrated four different measures of accuracy to compare between (tree, logistic regression, Naïve Bayes, SVM, KNN, random forest and Ada boost) algorithms based on average/ most frequent enhancing missing data method. The figure concluded that according to classified accuracy measure, the logistic regression and random forest machine learning algorithms have the highest percent of accuracy with ratio(92.9).

Table 6: Treating missing data by model-based imputer (simple tree) through machine learning methods

Model	CA	F_1	Precision	Recall
Tree	0.922	0.420	0.442	0.400
Logistic regression	**0.928**	0.022	0.250	0.012
Naïve Bayes	0.845	0.420	0.285	0.800
SVM	0.874	0.278	0.232	0.341
KNN	0.918	0.250	0.351	0.194
Random Forest	**0.926**	0.382	0.466	0.324
Ada boost	**0.926**	0.464	0.470	0.459

Table 6 illustrated using model based imputer (simple tree) for replacing missing values. This methods strategr depended on constructs a model for predicting the missing values based on values of other attributes, a separate

model is constructed for each attribute. By another way involves creating astatistical model to predict the missing values based on other features in the data. This can be a powerful technique but it requires complex computations(Tamboli, 2024).This method displayed good results for logistics regression with percent 92.8.

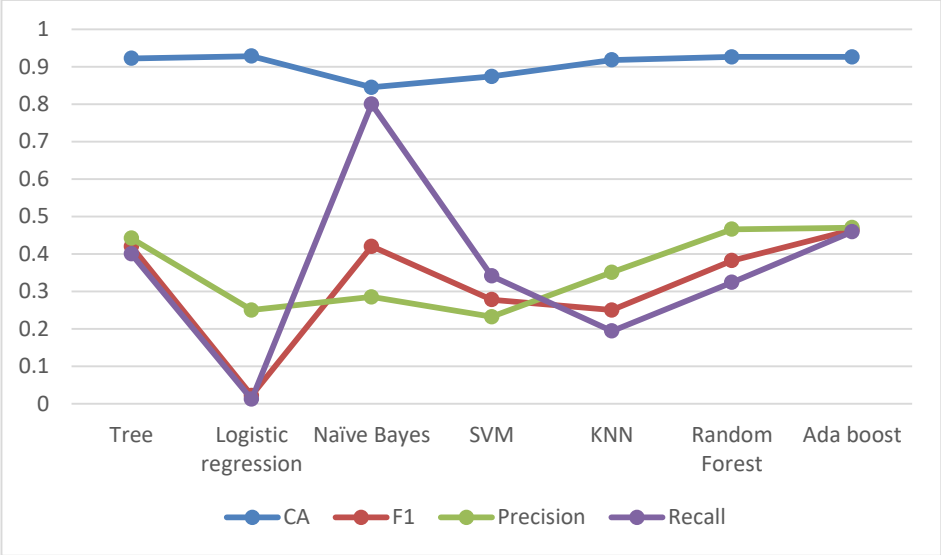
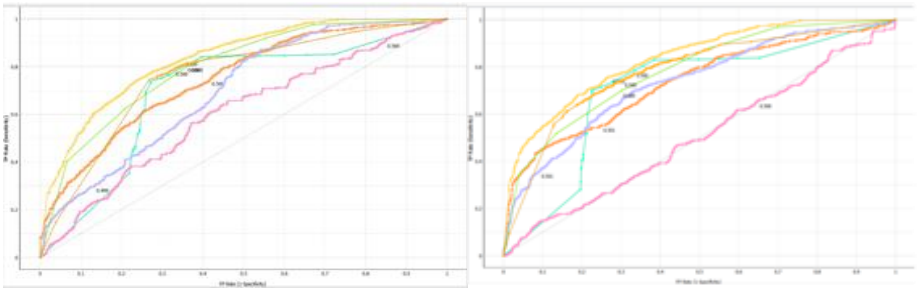
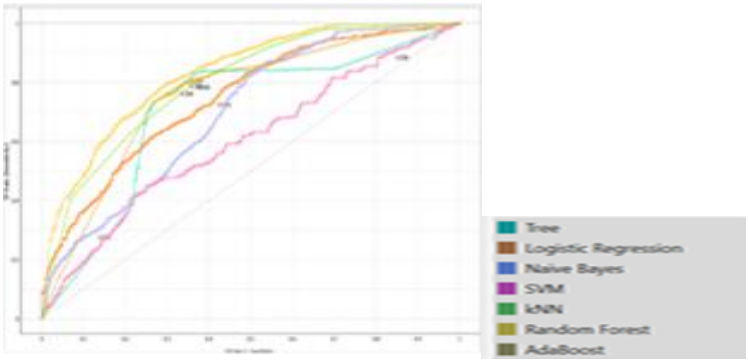


Figure 5: Comparison of Machine Learning Algorithms for Missing Data Imputation and Classification(simple tree)

Figure 5 conducted the seventh machine learning presented algorithms using simple tree treating missing data method and comparing between these algorithms relied on various measures of accuracy. The logistic regression , random forest and Ada boost algorithms displayed superiotity versus other methods poposed with (0.928, 0.926, 0.926) percents respectively



g.1 ROC curve of random values methods g.2 ROC curve of average method



g.3 Roc curve of simple tree method

Figure 6: ROC curve under missing data handling methods.

Figure 6 introduced a ROC curve of proposed methods, which illustrated the performance of a binary classifier model at varying threshold values, it also can be thought of as a plot of the statistical power as a function of type 1 error. The area under Roc curve (AUC) is widely recognized as the measure of a diagnostic test's discriminatory power (Fan et al., 2006). The maximum value for the AUC is 1.0, thereby identifying a (theoretically) perfect test (i.e., 100% sensitive and 100% specific). An AUC value of 0.5 indicates no discriminative value (i.e., 50% sensitive and 50% specific) and is represented by a straight, diagonal line extending from the lower left corner to the upper right. There are various scales for AUC value interpretation but, in general, ROC curves ≤0.75 are not useful and an AUC of 0.97 has a very high value, correlating with likelihood ratios of approximately 10 and 0.1 (Faraggi and Reiser, 2002). For all presented curves, the most dominant curve is the logistic regression curve and Ada Boost curve, introduced accurate tests under proposed methods in this study with 0.92% AUC.

Table 7: Merging between some machine learning algorithms.

Model	CA	F_1	Precision	Recall
Tree	0.922	0.420	0.442	0.400
Logistic Regression	**0.928**	0.022	0.250	0.012
Naïve Bayes	0.845	0.420	0.285	0.800
SVM	0.874	0.276	0.232	0.341
KNN	0.918	0.250	0.351	0.194
Random forest	**0.926**	0.382	0.466	0.324
Ada Boost	**0.926**	0.464	0.470	0.459
Merging	**0.926**	**0.151**	**0.381**	**0.094**

Table 7 presented merging between machine learning algorithms to get the best fit and the most effective methods of handling missing values of migration. Merging contained all proposed machine learning algorithms in this study. Also, merging logistic regression, random forest and Ada boost with values of classified accuracy 0.928, 0.926 and0.926 respectively. Merging results showed that the best merger was merging Ada boost and

logistic regression with accuracy measure equals 0.926. This emphasized that merging and using machine learning handling missing values with 92.6 percent.

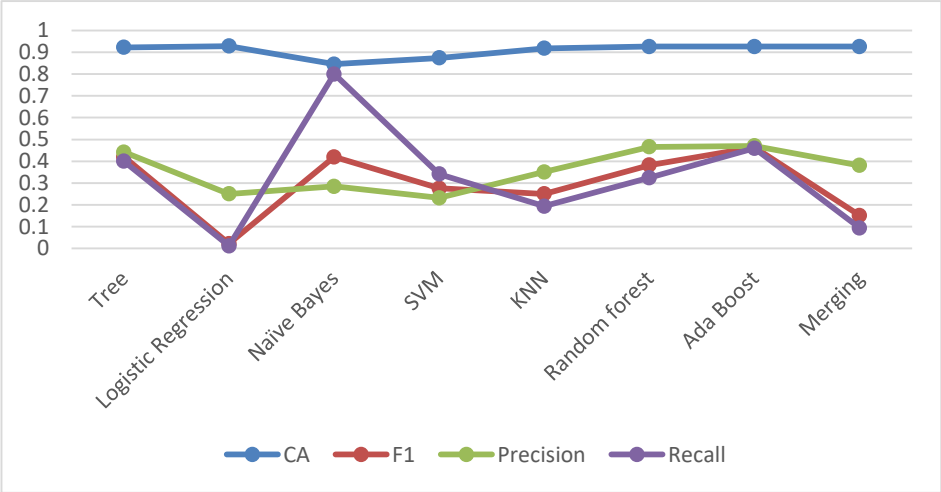


Figure7: Comparison of Machine Learning Algorithms for Missing Data Imputation and Classification (Merged Models)

Figure 7 revealed the accuracy of merging between proposed algorithms at this study. The merged models conducted that a best performance with accuracy was merging Ada boost and logistic regression algorithms with percent 0.926.

Table8: Feature statistics under methods of treating missing data.

Method	Variable	Mean	Mode	Median	Dispersion	Mini.	Max.	Missing
Simple tree	**Region-origin**		Home of Africa(p)		2.02			0(0%)
	Missing	40.66	1	10.21	2.22	0	750	0 (0%)
	Dead	5.02	1	1	4.07	0	750	0(0%)
	Incident – region		North Africa		1.9			0(0%)
	Lat.	26.9011	12.533	29.34	0.428083	-26.2245	66.9672	0(0%)
	Lon.	-13.97	1.85869	14.51	-4.41274	-117.071	116.225	0(0%)
	Reliability		Partially verified		0.862			0(0%)
	Id	95926.3	1	121178	0.65	1	184750	0(0%)
	Region- origin		**Home of Africa(p)**		**2**			**0(0%)**
	Missing	40.66	1	10.21	2.22	0	750	0 (0%)
	Dead	5.02	1	1	4.25	0	750	0(0%)
	Incident –		North Africa		1.9			0(0%)

Average/ mode value	**Region**							
	Lat.	26.9011	12.5331	29.34	0.428083	-26.2	66.9672	0(0%)
	Lon.	-13.9687	1.85869	14.4711	-4.41274	-117.1	116.225	0(0%)
	Reliability		Partially verified		0.862			0(0%)
	Id	95926.26	1	121178	0.65	1	184750	0(0%)
Random values	**Region-origin**		**Home of Africa(p)**		**2.19**			**0(0%)**
	Missing	40.66	1	10.21	2.22	0	750	0 (0%)
	Dead	4.69	1	1	4.29	0	750	0(0%)
	Incident Region				1.91			0(0%)
	Lat.	26.9011	12.5331	29.3489	0.42791	-26.2245	66.9672	0(0%)
	Lon.	-14.0435	1.85869	14.4711	-4.44	-117.071	116.225	0(0%)
	Reliability		Partially verified		0.912			0(0%)
	Id	95926.3	1	121178	0.65	1	184750	0(0%)

Table8 illustrated the statistics of variables after handling missing data using three methods of treatment, simple tree, average/ adequate value and random values. The percentage of missing after applying these techniques equals 0% with lower dispersion for both methods of simple tree and average/ adequate value. The less dispersion method for all variables is average/adequate method.

Figure8: Feature distributions after handling missing values

Figure 8 displayed the distributions of features after treatment of missing data. The figure showed that the distribution of some variables closed to normal distribution like, incident region, lat., and Lon. Also, reducing

the outliers value for both missing and reliability variables. This demonstrates on treating missing data improving the distribution of the data and hence getting more accurate results.

Table9: Accuracy Measures of Average Method for Handing Missing Data Under neutrosophic numbers

Model	CA_N	$F1_N$	$Prec_N$	$Recall_N$
Tree	0.996+0.996IN; IN ϵ[0,0.000]	0.97+0.97 IN ; IN ϵ[0,0.000]	0.976+0.976 IN ; IN ϵ[0,0.000]	0.965+0.965 IN ; IN ϵ[0,0.000]
Logistic Regression	0.917+0.918 IN ; IN ϵ[0,0.001]	0.082+0.092 IN ; IN ϵ[0,0.109]	0.184+0.208 IN ; IN ϵ[0,0.115]	0.053+0.059 IN ; IN ϵ[0,0.113]
Naive Bayes	0.843+0.844 IN ; IN ϵ[0,0.001]	0.449+0.449 IN ; IN ϵ[0,0.000]	0.298+0.298 IN ; IN ϵ[0,0.000]	0.906+0.912 IN ; IN ϵ[0,0.007]
SVM	0.856+0.858 IN ; IN ϵ[0,0.009]	0.335+0.336 IN ; IN ϵ[0,0.003]	0.243+0.248 IN ; IN ϵ[0,0.020]	0.518+0.547 IN ; IN ϵ[0,0.053]
kNN	0.995+0.995 IN ; IN ϵ[0,0.000]	0.965+0.965 IN ; IN ϵ[0,0.000]	0.959+0.959 IN ; IN ϵ[0,0.000]	0.971+0.971 IN ; IN ϵ[0,0.000]
Random Forest	0.996+0.997 IN ; IN ϵ[0,0.001]	0.97+0.976 IN ; IN ϵ[0,0.006]	0.988+1 IN ; IN ϵ[0,0.012]	0.953+0.953 IN ; IN ϵ[0,0.000]
AdaBoost	0.996+0.996 IN ; IN ϵ[0,0.000]	0.971+0.973 IN ; IN ϵ[0,0.002]	0.971+0.976 IN ; IN ϵ[0,0.005]	0.971+0.971 IN ; IN ϵ[0,0.000]
Merging	**0.997+0.998 IN ; IN ϵ[0,0.001]**	0.979+0.985 IN ; IN ϵ[0,0.006]	**0.994+1 IN ; IN ϵ[0,0.006]**	0.965+0.971 IN ; IN ϵ[0,0.006]

Table10: Accuracy Measures of Simple Tree Method for Handing Missing Data Under neutrosophic numbers

Model	CA_N	$F1_N$	$Prec_N$	$Recall_N$
Tree	0.996+0.996 IN ; IN ϵ[0,0.000]	0.97+0.97 IN ; IN ϵ[0,0.000]	0.976+0.976 IN ; IN ϵ[0,0.000]	0.965+0.965 IN ; IN ϵ[0,0.000]
Logistic Regression	0.915+0.920 IN ; IN ϵ[0,0.005]	0.072+0.085 IN ; IN ϵ[0,0.153]	0.154+0.22 IN ; IN ϵ[0,0.300]	0.047+0.053 IN ; IN ϵ[0,0.113]
Naive Bayes	0.841+0.843 IN ; IN ϵ[0,0.002]	0.443+0.445 IN ; IN ϵ[0,0.004]	0.29+0.296 IN ; IN ϵ[0,0.007]	0.9+0.9 IN ; IN ϵ[0,0.000]
SVM	0.858+0.871 IN ; IN ϵ[0,0.015]	0.292+0.317 IN ; IN ϵ[0,0.079]	0.238+0.24 IN ; IN ϵ[0,0.008]	0.376+0.471 IN ; IN ϵ[0,0.202]
kNN	0.995+0.995 IN ; IN ϵ[0,0.000]	0.965+0.965 IN ; IN ϵ[0,0.000]	0.959+0.959 IN ; IN ϵ[0,0.000]	0.971+0.971 IN ; IN ϵ[0,0.000]
Random Forest	0.996+0.996 IN ; IN ϵ[0,0.000]	0.97+0.973 IN ; IN ϵ[0,0.003]	0.994+1 IN ; IN ϵ[0,0.006]	0.947+0.947 IN ; IN ϵ99.[0,0.000]
AdaBoost	0.995+0.995 IN ; IN ϵ[0,0.000]	0.968+0.968 IN ; IN ϵ[0,0.000]	0.965+0.965 IN ; IN ϵ[0,0.000]	0.971+0.971 IN ; IN ϵ[0,0.000]

Merging	**0.998+0.998 IN ; IN ϵ[0,0.000]**	0.985+0.982 IN ; IN ϵ[0,0.003]	0.994+1 IN ; IN ϵ[0,0.006]	0.971+0.971 IN ; IN ϵ[0,0.000]

Table11: Accuracy Measures of Random Values Method for Handing Missing Data Under neutrosophic numbers

Model	CA_N	$F1_N$	$Prec_N$	$Recall_N$
Tree	0.969+0.978 IN ; IN ϵ[0,0.009]	0.817+0.863 IN ; IN ϵ[0,0.053]	0.797+0.861 IN ; IN ϵ[0,0.074]	0.838+0.865 IN ; IN ϵ[0,0.031]
Logistic Regression	0.902+0.903 IN ; IN ϵ[0,0.001]	0.1+0.125 IN ; IN ϵ[0,0.200]	0.194+0.227 IN ; IN ϵ[0,0.145]	0.067+0.086 IN ; IN ϵ[0,0.221]
Naive Bayes	0.866+0.868 IN ; IN ϵ[0,0.002]	0.53+0.54 IN ; IN ϵ[0,0.019]	0.367+0.377 IN ; IN ϵ[0,0.027]	0.948+0.949 IN ; IN ϵ[0,0.001]
SVM	0.882+0.9 IN ; IN ϵ[0,0.020]	0.219+0.297 IN ; IN ϵ[0,0.263]	0.238+0.338 IN ; IN ϵ[0,0.296]	0.203+0.264 IN ; IN ϵ[0,0.231]
KNN	0.981+0.984 IN ; IN ϵ[0,0.003]	0.88+0.896 IN ; IN ϵ[0,0.018]	0.923+0.927 IN ; IN ϵ[0,0.004]	0.838+0.87 IN ; IN ϵ[0,0.037]
Random Forest	0.98+0.986 IN ; IN ϵ[0,0.006]	0.869+0.906 IN ; IN ϵ[0,0.041]	0.916+0.97 IN ; IN ϵ[0,0.056]	0.827+0.85 IN ; IN ϵ[0,0.027]
Ada Boost	0.971+0.975 IN ; IN ϵ[0,0.004]	0.829+0.846 IN ; IN ϵ[0,0.020]	0.811+0.827 IN ; IN ϵ[0,0.019]	0.848+0.865 IN ; IN ϵ[0,0.020]
Merging	**0.982+0.986 IN ; IN ϵ[0,0.004]**	0.882+0.907 IN ; IN ϵ[0,0.028]	0.932+0.965 IN ; IN ϵ[0,0.034]	0.838+0.855 IN ; IN ϵ[0,0.020]

Tables 9, 10 &11 provided algorithms of machine learning and merging between these models under neutrosophic sets. These tables contained random values, average/adequate method and simple tree method to enhance missing data. For table 9, under average method and after entering neutrosophic sets on the data, we found that the classified accuracy increasing to 99.8 percent. While at table10 under simple tree method, neutrosophic sets increased accuracy to 99.8 percent. Finally at table 11, neutrosophic sets under random values method raised the accuracy to 98.2 percent. The findings from these tables proved on that using neutrosophic sets provided more accuracy rather than using algorithms only.

Conclusion: -

In this paper, we introduced seven machine learning algorithms namely, decision tree, logistic regression, naïve Bayes, support vector machine, K nearest neighbor, random forest and Ada Boost to handle missing data issue. These algorithms were applied on data of missing migrations to treat missing data. Handling missing data was conducted using three replacing missing data methods involving, random values, average value and simple tree method. Additionally accuracy criteria measurements were conducted for these algorithms and utilized in the analysis.

The analysis findings revealed that Naïve Bayes was superior performance in fitting missing data for random values method, logistic regression, while logistic regression was the best algorithm for both average method

and simple tree. Finally, merging between proposed algorithms emerged logistic regression as the best algorithm for CA criteria. Additionally applying neutrosophic sets gave more performance of accuracy, as it raised the ratio of accuracy for all proposed algorithms as shown in tables 9,10 and 10. The classified accuracy of merging increased from 92.6 percent to 99.8 percent, this emphasized on the good performance of proposed neutrosophic technique.

For future scope, we seek to apply type-2 Neutrosophic- logic to handle missing data. Also, we plan to develop a hybrid framework between a neutrosophic set and type-2 neutrosophic which incorporates machine learning. Furthermore, more applications will be conducted to test the efficiency of the proposed model with other membership functions or a higher number of membership functions.

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Practical Overview of Triangular Bipolar Neutrosophic Numbers for Design Effect Evaluation of Ethnic Minority Clothing: Comprehensive Guide

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Abstract: This paper offers a novel multi-criteria decision-making (MCDM) paradigm to handle the problem of assessing and prioritizing design effect evaluation of ethnic minority clothing. The framework incorporates the ARTASI (Alternative Ranking Technique based on Adaptive Standardized Intervals) technique for objective alternative evaluation and CRITIC method for robust weight determination. This novel method, CRITIC-ARTASI, makes use of a unique algorithm for logical reasoning and makes it easier to make decisions while designing effect evaluation of ethnic minority clothing. 7 Criteria and 13 are used in this study. Sensitivity studies were used to thoroughly assess the correctness and stability of the suggested CRITIC-ARTASI framework. After fine-tuning the utility function's parameters to evaluate their impact, the model's rank is tested under different cases. The results show the ranks of the alternatives are stable under different cases.

Keywords: Triangular Bipolar Neutrosophic Numbers (TBNNs); Uncertainty; Decision Making; Ethnic Minority Clothing.

1. Introduction and Literature Review

Zadeh introduced fuzzy set theory, which addresses the ideas of vagueness and uncertainty theory. To solve issues pertaining to statistical computation and engineering, vague theory is essential. It is frequently applied to social science, networking, decision-making, and other real-world issues. In 1986, Atanassov introduced the legerdemain concept of an intuitionistic fuzzy set in mathematics, based on Zadeh's paper[1], [2]. He took into consideration both the membership function and the non-membership function in the context of an intuitionistic fuzzy set. Later, in 2007, Liu and Yuan developed the idea of triangular intuitionistic fuzzy sets, which are a combination of intuitionistic fuzzy sets and triangle fuzzy sets.

Ye subsequently presented the basic concept of a trapezoidal intuitionistic fuzzy set, in which the truth function and falsity function are both trapezoidal numbers rather than triangular. The creation of intriguing models in a variety of domains with scientific and technical issues is influenced by uncertainty theory[3], [4]. But a simple question comes up: how can we apply or develop the concepts of uncertainty in our mathematical modeling in relation to everyday life? Scholars from all around the world have developed numerous strategies and techniques to define those ideas and have made various suggestions for applying

uncertainty philosophy. Various suggestions for categorizing some of the fundamental confusing criteria are presented in the literature[5], [6]. It should be mentioned that the vague parameter does not have an exclusive representation.

The option made by the decision maker to solve an issue can be given in a variety of ways depending on the application. In 1995, Smarandache proposed the concept of a neutrosophic set consisting of three distinct elements: truthiness, indeterminacies, and falseness. Our real-world systems are highly applicable to every facet of the neutrosophic set[7], [8]. The neutrosophic concept is a fascinating and successful idea. The idea of a single typed neutrosophic set was further developed by Wang et al. and is highly helpful in resolving any complicated issue[9], [10]. Triangular neutrosophic was first conceptualized and classified by Chakraborty et al. Using the removal area method, Chakraborty et al. demonstrated the perception of defuzzification.

Multi-criteria decision-making problems are currently of great interest to researchers worldwide. A finite number of alternatives and a finite number of attributes with various weight function types for varying numbers of decision makers were taken into consideration for that kind of problem. This method's objective was to compare the alternatives and attributes while keeping the decision makers' weight in place so that we could quickly identify the best and worst options. Although multi-criteria decision-making (MCDM) difficulties have been the subject of numerous research proposals, we focused on and examined an MCDM problem in this novel triangular bipolar neutrosophic arena using our de-polarization technique.

1.1 Literature Review

Chakraborty et al. [11] contributed to the generalized neutrosophic number theory from a unique perspective. Positive, indeterminacy, and non-belongingness membership functions are widely recognized to be closely related to and commonly associated with the idea of a neutrosophic number. Now, every membership function falls between 0 and 1. Nonetheless, they have developed a bipolar idea in which the membership has both positive and negative components within the ranges of 0 to 1 and -1 to 0. Regarding the membership functions that contain dependency or independency with one another, they describe many structures of generalized TBNNs.

The interaction and dependency between fuzzy and crisp numbers are always of interest to researchers from various domains. To enable the seamless conversion of any bipolar neutrosophic fuzzy number of any kind into a real number instantaneously, they also developed the perception of de-bipolarization for TBNNs using well-known techniques. Compared to other methods, creating a problem utilizing bipolar neutrosophic perception is more accurate, dependable, and trustworthy.

Graph theory is an essential part of mathematics that is used to solve problems in a variety of fields. One of the new options that is appearing in a neutrosophic chromatic number environment is graph coloring. Sudha et al. [10] investigated and illustrated the algebraic assumption and presented the concept of BTN chromatic graphs.

Abdel-Basset et al. [12] offered a new hybrid neutrosophic MCDM framework that used TOPSIS under bipolar neutrosophic numbers and a collection of neutrosophic ANP. Elsewedy Electric Group in Egypt is the subject of a case study that applies the MCDM framework to the selection of CEOs. They can accurately pick personnel by using the suggested method to compile individual assessments of the decision makers.

To demonstrate and validate the results, the suggested method's output is contrasted with those of comparable works.

Deli et al. [13] presented the idea of a bipolar neutrosophic set and some of its operations in this work. To compare the bipolar neutrosophic sets, they also suggested score, certainty, and accuracy functions. Additionally, they developed a bipolar neutrosophic MCDM approach based on the score, certainty, and accuracy functions. In their approach, the evaluation values of alternatives attributes are expressed as bipolar neutrosophic numbers, which are then used to choose the most desirable option or options. Lastly, a numerical example was provided to show how the suggested method may be applied and how effective it is.

ELECTRE and TOPSIS techniques are popular approaches for resolving MCDM issues. To address such issues, Akram et al. [14] introduced the bipolar neutrosophic TOPSIS and ELECTRE-I methods. In our bipolar neutrosophic TOPSIS technique, they ranked alternatives using the updated closeness degree. Using flow charts, they explained the bipolar neutrosophic TOPSIS and ELECTRE-I methods. They used the suggested techniques to solve numerical examples. They compare these approaches as well.

2. Methodology

Triangular Bipolar Neutrosophic Numbers (TBNNs)

This section shows some definitions of TBNNs such as[11], [15]:

Definition 1.

We can define the single types neutrosophic number (STNN) such as:

$$z = \begin{pmatrix} [(p^1, q^1, r^1, s^1); a], \\ [(p^2, q^2, r^2, s^2); b], \\ [(p^3, q^3, r^3, s^3); c] \end{pmatrix} \tag{1}$$

$$T_z(x) = \begin{cases} T_{zl}(x) & p^1 \le x \le q^1 \\ a & q^1 \le x \le r^1 \\ T_{zu}(x) & r^1 \le x \le s^1 \\ 0 & otherwise \end{cases} \tag{2}$$

$$I_z(x) = \begin{cases} I_{zl}(x) & p^2 \le x \le q^2 \\ b & q^2 \le x \le r^2 \\ I_{zu}(x) & r^2 \le x \le s^2 \\ 1 & otherwise \end{cases} \tag{3}$$

$$F_z(x) = \begin{cases} F_{zl}(x) & p^3 \le x \le q^3 \\ c & q^3 \le x \le r^3 \\ F_{zu}(x) & r^3 \le x \le s^3 \\ 1 & otherwise \end{cases} \tag{4}$$

The triangular single typed bipolar neutrosophic number (TSBNN) such as:

$A = \{i_1, i_2, i_3; j_1, j_2, j_3, k_1, k_2, k_3\}$

$$T_A^+(x) = \begin{cases} \frac{x-i_1}{i_2-i_1} & i_1 \le x \le i_2 \\ 1 & x = i_2 \\ \frac{i_3-x}{i_3-i_2} & i_2 \le x \le i_3 \\ 0 & otherwise \end{cases} \quad (5)$$

$$T_A^-(x) = \begin{cases} \frac{i_1-x}{i_2-i_1} & i_1 \le x \le i_2 \\ -1 & x = i_2 \\ \frac{x-i_3}{i_3-i_2} & i_2 \le x \le i_3 \\ 0 & otherwise \end{cases} \quad (6)$$

$$I_A^+(x) = \begin{cases} \frac{j_2-x}{j_2-j_1} & j_1 \le x \le j_2 \\ 0 & x = j_2 \\ \frac{x-j_3}{j_3-j_2} & j_2 \le x \le j_3 \\ 0 & otherwise \end{cases} \quad (7)$$

$$I_A^-(x) = \begin{cases} \frac{x-j_2}{j_2-j_1} & j_1 \le x \le j_2 \\ 0 & x = j_2 \\ \frac{j_2-x}{j_3-j_2} & j_2 \le x \le j_3 \\ -1 & otherwise \end{cases} \quad (8)$$

$$F_A^+(x) = \begin{cases} \frac{k_2-x}{k_2-k_1} & k_1 \le x \le k_2 \\ 0 & x = k_2 \\ \frac{x-k_3}{k_3-k_2} & k_2 \le x \le k_3 \\ 1 & otherwise \end{cases} \quad (9)$$

$$F_A^-(x) = \begin{cases} \frac{x-k_2}{k_2-k_1} & k_1 \le x \le k_2 \\ 0 & x = k_2 \\ \frac{k_3-x}{k_3-k_2} & k_2 \le x \le k_3 \\ -1 & otherwise \end{cases} \quad (10)$$

$$-3 \le T_A(x) + I_A(x) + F_A(x) \le 2 \quad (11)$$

CRITIC Model

This methodology is used to compute the criteria weights. The steps of the CRITIC methodology are summarized as follows[16], [17]:

Step 1. Build the initial decision matrix

Step 2. Normalize the decision matrix for positive and negative criteria

$$x_{ij} = \frac{r_{ij}-r_i^-}{r_i^+-r_i^-} \quad (12)$$

$$x_{ij} = \frac{r_{ij}-r_i^+}{r_i^--r_i^+} \quad (13)$$

Step 3. Compute the correlation coefficient

Step 4. Compute the standard deviation

Step 5. Compute the index H

$$H_j = stand \sum_{k=1}^{n}(1 - \text{correlation coefficient }) \tag{14}$$

Step 6. Compute the criteria weights.

$$w_j = \frac{H_j}{\sum_{j=1}^{n} H_j} \tag{15}$$

Evaluation Model: ARTASI.

The ARTASI methodology is used to rank the alternatives. The steps of the ARTASI methodology are summarized as follows[18]:

Step 1. Build the initial decision matrix

$$f = \begin{bmatrix} r_{11} & \cdots & r_{1n} \\ \vdots & \ddots & \vdots \\ r_{m1} & \cdots & r_{mn} \end{bmatrix} \tag{16}$$

Step 2. Compute the absolute max and min values.

$$p_j^{max} = \max_{1 \le i \le m}\left(r_{ij}\right) + \left\{\max_{1 \le i \le m}\left(r_{ij}\right)\right\}^{\frac{1}{m}} \tag{17}$$

$$p_j^{min} = \min_{1 \le i \le m}\left(r_{ij}\right) + \left\{\min_{1 \le i \le m}\left(r_{ij}\right)\right\}^{\frac{1}{m}} \tag{18}$$

Step 3. Normalize the initial decision matrix

The initial decision matrix is normalized such as:

$$n_{ij} = \frac{s^{(u)} - s^{(l)}}{p_j^{max} - p_j^{max}} r_{ij} + \frac{p_j^{max} s^{(u)} - p_j^{min} s^{(l)}}{p_j^{max} - p_j^{min}} \tag{19}$$

Where $s^{(u)}\ and\ s^{(l)}$ refer to the upper and lower limits of normalized interval.

Change the values, if the criterion is of the min type

$$c_{ij} = -n_{ij} + \max_{1 \le i \le m}\left(n_{ij}\right) + \min_{1 \le i \le m}\left(n_{ij}\right) \tag{20}$$

Step 4. Compute the degree of usefulness of alternatives about optimal and suboptimal values.

$$y_{ij}^{+} = \frac{c_{ij}}{\max_{1 \le i \le m}\left(c_{ij}\right)} w_j s^{(u)} \tag{21}$$

$$y_{ij} = \frac{\min_{1 \le i \le m}\left(c_{ij}\right)}{c_{ij}} w_j s^{(u)} \tag{22}$$

$$y_{ij}^{-} = -y_{ij} + \max_{1 \le i \le m}\left(y_{ij}\right) + \min_{1 \le i \le m}\left(y_{ij}\right) \tag{23}$$

Step 5. Compute the combined level of usefulness of alternatives for ideal value and anti-ideal value

$$t_i^{+} = \sum_{j=1}^{n} y_{ij}^{+} \tag{24}$$

$$t_i^{-} = \sum_{j=1}^{n} y_{ij}^{-} \tag{25}$$

Step 6. Compute the ultimate utility functions and rank the alternatives

$$t_i = (t_i^+ + t_i^-)\{af(t_i^+) + (1-a)f(t_i^-)\} \quad (26)$$

$$f(t_i^+) = \frac{t_i^+}{t_i^+ + t_i^-} \quad (27)$$

$$f(t_i^-) = \frac{t_i^-}{t_i^+ + t_i^-} \quad (28)$$

Where a between 0 and 1.

3. Results and Discussion

This section shows the results of the proposed approach. Three experts have evaluated the criteria and alternatives. This study evaluated seven criteria and 13 alternatives. Then These experts have experience in the related filed of the decision-making problem. These criteria and alternatives are used to aid experts to compute the criteria weights and rank the alternatives based on set of steps. The criteria of this study as: Functionality, cost-effective, Marketability, Production Techniques, Cultural Authenticity, Design Quality, Material Selection. The alternatives of this study are: High-Fashion Ethnic Couture Daily Wear Adaptation of Ethnic Styles, Authentic Traditional Replication, Functional Ethnic Wear, Wearable Technology Integration, Digitally Enhanced Ethnic Patterns, Souvenir Clothing Collections, Modernized Ethnic Fashion Fusion, Avant-Garde Ethnic Designs, Sustainable Ethnic Clothing, Minimalist Ethnic Fashion Approach, Cultural Festival Costumes, Urban Ethnic Fashion.

Results of CRITIC Model

We used the CRITIC method to compute the criteria weights.

- Step 1. We built the decision matrix based on the neutrosophic numbers as shown in Tables 1-3.
- Step 2. Then we normalize the decision matrix using Eq. (12) as shown in Table 4.
- Step 3. Then we compute the correlation coefficient as shown in Fig 1.
- Step 4. Then we compute the standard deviation for the criteria.
- Step 5. Eq. (14) is used to compute the H_j values as shown in fig 2.
- Step 6. Eq. (15) is used to compute the criteria weights as shown in fig 3.

Table 1. The first TBNNs.

	C_1	C_2	C_3	C_4	C_5	C_6	C_7
A_1	(1,2,3;0.5,1.5,2.5;1.3, 2.5,3.5)	(1,5,8;0.5,3,6;3.5,7,.5)	(2,4,6;1.5,2.5,3.5;3,5, 7)	(1,4,7;0.5,2,3;3.5,5.5, 7.5)	(1,3,5;0.5,2.5,4.5;2,4, 6)	(0.6,2,4;0.3,1,1.25;1.5 ,3,4.5)	(0.5,2.5,4.5;1,2,3;1.5, 3.5,5.5)
A_2	(0.5,2.5,4.5;1,2,3;1.5, 3.5,5.5)	(0.5,2.5,4.5;1,2,3;1.5, 3.5,5.5)	(0.6,2,4;0.3,1,1.25;1.5 ,3,4.5)	(1,3,5;0.5,2.5,4.5;2,4, 6)	(1,4,7;0.5,2,3;3.5,5.5, 7.5)	(2,4,6;1.5,2.5,3.5;3,5, 7)	(1,2,3;0.5,1.5,2.5;1.3, 2.5,3.5)
A_3	(0.6,2,4;0.3,1,1.25;1.5 ,3,4.5)	(1,2,3;0.5,1.5,2.5;1.3, 2.5,3.5)	(1,4,7;0.5,2,3;3.5,5.5, 7.5)	(1,3,5;0.5,2.5,4.5;2,4, 6)	(0.6,2,4;0.3,1,1.25;1.5 ,3,4.5)	(1,5,8;0.5,3,6;3.5,7,.5)	(0.5,2.5,4.5;1,2,3;1.5, 3.5,5.5)
A_4	(1,3,5;0.5,2.5,4.5;2,4, 6)	(0.5,2.5,4.5;1,2,3;1.5, 3.5,5.5)	(1,2,3;0.5,1.5,2.5;1.3, 2.5,3.5)	(1,2,3;0.5,1.5,2.5;1.3, 2.5,3.5)	(1,2,3;0.5,1.5,2.5;1.3, 2.5,3.5)	(1,2,3;0.5,1.5,2.5;1.3, 2.5,3.5)	(0.6,2,4;0.3,1,1.25;1.5 ,3,4.5)
A_5	(1,4,7;0.5,2,3;3.5,5.5, 7.5)	(0.6,2,4;0.3,1,1.25;1.5 ,3,4.5)	(0.5,2.5,4.5;1,2,3;1.5, 3.5,5.5)	(1,2,3;0.5,1.5,2.5;1.3, 2.5,3.5)	(0.5,2.5,4.5;1,2,3;1.5, 3.5,5.5)	(1,2,3;0.5,1.5,2.5;1.3, 2.5,3.5)	(1,3,5;0.5,2.5,4.5;2,4, 6)
A_6	(2,4,6;1.5,2.5,3.5;3,5, 7)	(1,3,5;0.5,2.5,4.5;2,4, 6)	(0.6,2,4;0.3,1,1.25;1.5 ,3,4.5)	(0.5,2.5,4.5;1,2,3;1.5, 3.5,5.5)	(0.6,2,4;0.3,1,1.25;1.5 ,3,4.5)	(0.5,2.5,4.5;1,2,3;1.5, 3.5,5.5)	(1,4,7;0.5,2,3;3.5,5.5, 7.5)
A_7	(1,5,8;0.5,3,6;3.5,7,.5)	(1,4,7;0.5,2,3;3.5,5.5, 7.5)	(1,3,5;0.5,2.5,4.5;2,4, 6)	(0.6,2,4;0.3,1,1.25;1.5 ,3,4.5)	(1,3,5;0.5,2.5,4.5;2,4, 6)	(0.6,2,4;0.3,1,1.25;1.5 ,3,4.5)	(2,4,6;1.5,2.5,3.5;3,5, 7)
A_8	(1,2,3;0.5,1.5,2.5;1.3, 2.5,3.5)	(2,4,6;1.5,2.5,3.5;3,5, 7)	(1,4,7;0.5,2,3;3.5,5.5, 7.5)	(1,3,5;0.5,2.5,4.5;2,4, 6)	(1,4,7;0.5,2,3;3.5,5.5, 7.5)	(1,3,5;0.5,2.5,4.5;2,4, 6)	(1,5,8;0.5,3,6;3.5,7,.5)
A_9	(0.5,2.5,4.5;1,2,3;1.5, 3.5,5.5)	(1,5,8;0.5,3,6;3.5,7,.5)	(2,4,6;1.5,2.5,3.5;3,5, 7)	(1,4,7;0.5,2,3;3.5,5.5, 7.5)	(2,4,6;1.5,2.5,3.5;3,5, 7)	(1,4,7;0.5,2,3;3.5,5.5, 7.5)	(1,2,3;0.5,1.5,2.5;1.3, 2.5,3.5)
A_{10}	(0.5,2.5,4.5;1,2,3;1.5, 3.5,5.5)	(1,2,3;0.5,1.5,2.5;1.3, 2.5,3.5)	(1,5,8;0.5,3,6;3.5,7,.5)	(2,4,6;1.5,2.5,3.5;3,5, 7)	(1,5,8;0.5,3,6;3.5,7,.5)	(2,4,6;1.5,2.5,3.5;3,5, 7)	(1,5,8;0.5,3,6;3.5,7,.5)

A_{11}	(1,2,3;0.5,1.5,2.5;1.3, 2.5,3.5)	(2,4,6;1.5,2.5,3.5;3,5, 7)	(1,2,3;0.5,1.5,2.5;1.3, 2.5,3.5)	(1,5,8;0.5,3,6;3.5,7,.5)	(1,2,3;0.5,1.5,2.5;1.3, 2.5,3.5)	(1,5,8;0.5,3,6;3.5,7,.5)	(2,4,6;1.5,2.5,3.5;3,5, 7)
A_{12}	(1,5,8;0.5,3,6;3.5,7,.5)	(1,4,7;0.5,2,3;3.5,5.5, 7.5)	(1,3,5;0.5,2.5,4.5;2,4, 6)	(1,2,3;0.5,1.5,2.5;1.3, 2.5,3.5)	(1,2,3;0.5,1.5,2.5;1.3, 2.5,3.5)	(1,2,3;0.5,1.5,2.5;1.3, 2.5,3.5)	(1,4,7;0.5,2,3;3.5,5.5, 7.5)
A_{13}	(2,4,6;1.5,2.5,3.5;3,5, 7)	(1,4,7;0.5,2,3;3.5,5.5, 7.5)	(1,3,5;0.5,2.5,4.5;2,4, 6)	(0.6,2,4;0.3,1,1.25;1.5 ,3,4.5)	(0.5,2.5,4.5;1,2,3;1.5, 3.5,5.5)	(0.6,2,4;0.3,1,1.25;1.5 ,3,4.5)	(1,3,5;0.5,2.5,4.5;2,4, 6)

Table 2. The second TBNNs.

	C_1	C_2	C_3	C_4	C_5	C_6	C_7
A_1	(1,4,7;0.5,2,3;3.5,5.5, 7.5)	(1,5,8;0.5,3,6;3.5,7,.5)	(2,4,6;1.5,2.5,3.5;3,5, 7)	(1,4,7;0.5,2,3;3.5,5.5, 7.5)	(1,3,5;0.5,2.5,4.5;2,4, 6)	(0.6,2,4;0.3,1,1.25;1.5 ,3,4.5)	(0.5,2.5,4.5;1,2,3;1.5, 3.5,5.5)
A_2	(2,4,6;1.5,2.5,3.5;3,5, 7)	(0.5,2.5,4.5;1,2,3;1.5, 3.5,5.5)	(0.6,2,4;0.3,1,1.25;1.5 ,3,4.5)	(1,4,7;0.5,2,3;3.5,5.5, 7.5)	(1,4,7;0.5,2,3;3.5,5.5, 7.5)	(2,4,6;1.5,2.5,3.5;3,5, 7)	(1,4,7;0.5,2,3;3.5,5.5, 7.5)
A_3	(1,5,8;0.5,3,6;3.5,7,.5)	(2,4,6;1.5,2.5,3.5;3,5, 7)	(1,4,7;0.5,2,3;3.5,5.5, 7.5)	(2,4,6;1.5,2.5,3.5;3,5, 7)	(1,4,7;0.5,2,3;3.5,5.5, 7.5)	(1,5,8;0.5,3,6;3.5,7,.5)	(2,4,6;1.5,2.5,3.5;3,5, 7)
A_4	(1,2,3;0.5,1.5,2.5;1.3, 2.5,3.5)	(2,4,6;1.5,2.5,3.5;3,5, 7)	(1,5,8;0.5,3,6;3.5,7,.5)	(1,5,8;0.5,3,6;3.5,7,.5)	(2,4,6;1.5,2.5,3.5;3,5, 7)	(1,2,3;0.5,1.5,2.5;1.3, 2.5,3.5)	(1,5,8;0.5,3,6;3.5,7,.5)
A_5	(0.5,2.5,4.5;1,2,3;1.5, 3.5,5.5)	(1,4,7;0.5,2,3;3.5,5.5, 7.5)	(2,4,6;1.5,2.5,3.5;3,5, 7)	(1,2,3;0.5,1.5,2.5;1.3, 2.5,3.5)	(1,5,8;0.5,3,6;3.5,7,.5)	(0.5,2.5,4.5;1,2,3;1.5, 3.5,5.5)	(1,2,3;0.5,1.5,2.5;1.3, 2.5,3.5)
A_6	(1,4,7;0.5,2,3;3.5,5.5, 7.5)	(1,4,7;0.5,2,3;3.5,5.5, 7.5)	(1,4,7;0.5,2,3;3.5,5.5, 7.5)	(0.5,2.5,4.5;1,2,3;1.5, 3.5,5.5)	(1,2,3;0.5,1.5,2.5;1.3, 2.5,3.5)	(1,4,7;0.5,2,3;3.5,5.5, 7.5)	(0.5,2.5,4.5;1,2,3;1.5, 3.5,5.5)
A_7	(2,4,6;1.5,2.5,3.5;3,5, 7)	(2,4,6;1.5,2.5,3.5;3,5, 7)	(2,4,6;1.5,2.5,3.5;3,5, 7)	(0.6,2,4;0.3,1,1.25;1.5 ,3,4.5)	(1,4,7;0.5,2,3;3.5,5.5, 7.5)	(2,4,6;1.5,2.5,3.5;3,5, 7)	(0.6,2,4;0.3,1,1.25;1.5 ,3,4.5)
A_8	(1,5,8;0.5,3,6;3.5,7,.5)	(1,5,8;0.5,3,6;3.5,7,.5)	(1,5,8;0.5,3,6;3.5,7,.5)	(1,3,5;0.5,2.5,4.5;2,4, 6)	(2,4,6;1.5,2.5,3.5;3,5, 7)	(1,5,8;0.5,3,6;3.5,7,.5)	(1,3,5;0.5,2.5,4.5;2,4, 6)
A_9	(1,2,3;0.5,1.5,2.5;1.3, 2.5,3.5)	(1,2,3;0.5,1.5,2.5;1.3, 2.5,3.5)	(1,2,3;0.5,1.5,2.5;1.3, 2.5,3.5)	(1,2,3;0.5,1.5,2.5;1.3, 2.5,3.5)	(1,5,8;0.5,3,6;3.5,7,.5)	(1,2,3;0.5,1.5,2.5;1.3, 2.5,3.5)	(1,5,8;0.5,3,6;3.5,7,.5)
A_{10}	(0.5,2.5,4.5;1,2,3;1.5, 3.5,5.5)	(0.5,2.5,4.5;1,2,3;1.5, 3.5,5.5)	(0.5,2.5,4.5;1,2,3;1.5, 3.5,5.5)	(1,5,8;0.5,3,6;3.5,7,.5)	(1,2,3;0.5,1.5,2.5;1.3, 2.5,3.5)	(0.5,2.5,4.5;1,2,3;1.5, 3.5,5.5)	(1,2,3;0.5,1.5,2.5;1.3, 2.5,3.5)
A_{11}	(0.6,2,4;0.3,1,1.25;1.5 ,3,4.5)	(0.6,2,4;0.3,1,1.25;1.5 ,3,4.5)	(0.6,2,4;0.3,1,1.25;1.5 ,3,4.5)	(2,4,6;1.5,2.5,3.5;3,5, 7)	(0.5,2.5,4.5;1,2,3;1.5, 3.5,5.5)	(0.6,2,4;0.3,1,1.25;1.5 ,3,4.5)	(0.5,2.5,4.5;1,2,3;1.5, 3.5,5.5)
A_{12}	(1,3,5;0.5,2.5,4.5;2,4, 6)	(1,3,5;0.5,2.5,4.5;2,4, 6)	(1,3,5;0.5,2.5,4.5;2,4, 6)	(1,4,7;0.5,2,3;3.5,5.5, 7.5)	(0.6,2,4;0.3,1,1.25;1.5 ,3,4.5)	(1,3,5;0.5,2.5,4.5;2,4, 6)	(0.6,2,4;0.3,1,1.25;1.5 ,3,4.5)
A_{13}	(2,4,6;1.5,2.5,3.5;3,5, 7)	(1,4,7;0.5,2,3;3.5,5.5, 7.5)	(1,3,5;0.5,2.5,4.5;2,4, 6)	(0.6,2,4;0.3,1,1.25;1.5 ,3,4.5)	(1,3,5;0.5,2.5,4.5;2,4, 6)	(0.6,2,4;0.3,1,1.25;1.5 ,3,4.5)	(1,3,5;0.5,2.5,4.5;2,4, 6)

Table 3. The third TBNNs.

	C_1	C_2	C_3	C_4	C_5	C_6	C_7
A_1	(1,2,3;0.5,1.5,2.5;1.3, 2.5,3.5)	(1,5,8;0.5,3,6;3.5,7,.5)	(2,4,6;1.5,2.5,3.5;3,5, 7)	(1,4,7;0.5,2,3;3.5,5.5, 7.5)	(1,3,5;0.5,2.5,4.5;2,4, 6)	(0.6,2,4;0.3,1,1.25;1.5 ,3,4.5)	(0.5,2.5,4.5;1,2,3;1.5, 3.5,5.5)
A_2	(1,2,3;0.5,1.5,2.5;1.3, 2.5,3.5)	(0.5,2.5,4.5;1,2,3;1.5, 3.5,5.5)	(0.6,2,4;0.3,1,1.25;1.5 ,3,4.5)	(1,3,5;0.5,2.5,4.5;2,4, 6)	(1,4,7;0.5,2,3;3.5,5.5, 7.5)	(2,4,6;1.5,2.5,3.5;3,5, 7)	(1,2,3;0.5,1.5,2.5;1.3, 2.5,3.5)
A_3	(1,5,8;0.5,3,6;3.5,7,.5)	(2,4,6;1.5,2.5,3.5;3,5, 7)	(1,4,7;0.5,2,3;3.5,5.5, 7.5)	(1,3,5;0.5,2.5,4.5;2,4, 6)	(0.6,2,4;0.3,1,1.25;1.5 ,3,4.5)	(1,5,8;0.5,3,6;3.5,7,.5)	(1,5,8;0.5,3,6;3.5,7,.5)
A_4	(1,4,7;0.5,2,3;3.5,5.5, 7.5)	(2,4,6;1.5,2.5,3.5;3,5, 7)	(1,5,8;0.5,3,6;3.5,7,.5)	(1,2,3;0.5,1.5,2.5;1.3, 2.5,3.5)	(0.5,2.5,4.5;1,2,3;1.5, 3.5,5.5)	(1,2,3;0.5,1.5,2.5;1.3, 2.5,3.5)	(2,4,6;1.5,2.5,3.5;3,5, 7)
A_5	(1,3,5;0.5,2.5,4.5;2,4, 6)	(1,4,7;0.5,2,3;3.5,5.5, 7.5)	(2,4,6;1.5,2.5,3.5;3,5, 7)	(1,5,8;0.5,3,6;3.5,7,.5)	(1,2,3;0.5,1.5,2.5;1.3, 2.5,3.5)	(0.5,2.5,4.5;1,2,3;1.5, 3.5,5.5)	(1,4,7;0.5,2,3;3.5,5.5, 7.5)
A_6	(1,3,5;0.5,2.5,4.5;2,4, 6)	(1,3,5;0.5,2.5,4.5;2,4, 6)	(1,4,7;0.5,2,3;3.5,5.5, 7.5)	(1,3,5;0.5,2.5,4.5;2,4, 6)	(0.5,2.5,4.5;1,2,3;1.5, 3.5,5.5)	(0.6,2,4;0.3,1,1.25;1.5 ,3,4.5)	(1,3,5;0.5,2.5,4.5;2,4, 6)
A_7	(0.6,2,4;0.3,1,1.25;1.5 ,3,4.5)	(0.6,2,4;0.3,1,1.25;1.5 ,3,4.5)	(2,4,6;1.5,2.5,3.5;3,5, 7)	(0.6,2,4;0.3,1,1.25;1.5 ,3,4.5)	(0.6,2,4;0.3,1,1.25;1.5 ,3,4.5)	(1,3,5;0.5,2.5,4.5;2,4, 6)	(0.6,2,4;0.3,1,1.25;1.5 ,3,4.5)
A_8	(0.6,2,4;0.3,1,1.25;1.5 ,3,4.5)	(0.5,2.5,4.5;1,2,3;1.5, 3.5,5.5)	(1,5,8;0.5,3,6;3.5,7,.5)	(0.5,2.5,4.5;1,2,3;1.5, 3.5,5.5)	(1,4,7;0.5,2,3;3.5,5.5, 7.5)	(1,4,7;0.5,2,3;3.5,5.5, 7.5)	(0.5,2.5,4.5;1,2,3;1.5, 3.5,5.5)
A_9	(0.5,2.5,4.5;1,2,3;1.5, 3.5,5.5)	(1,2,3;0.5,1.5,2.5;1.3, 2.5,3.5)	(1,2,3;0.5,1.5,2.5;1.3, 2.5,3.5)	(1,2,3;0.5,1.5,2.5;1.3, 2.5,3.5)	(1,4,7;0.5,2,3;3.5,5.5, 7.5)	(2,4,6;1.5,2.5,3.5;3,5, 7)	(1,2,3;0.5,1.5,2.5;1.3, 2.5,3.5)
A_{10}	(0.5,2.5,4.5;1,2,3;1.5, 3.5,5.5)	(1,5,8;0.5,3,6;3.5,7,.5)	(0.5,2.5,4.5;1,2,3;1.5, 3.5,5.5)	(1,5,8;0.5,3,6;3.5,7,.5)	(2,4,6;1.5,2.5,3.5;3,5, 7)	(1,5,8;0.5,3,6;3.5,7,.5)	(1,5,8;0.5,3,6;3.5,7,.5)
A_{11}	(1,2,3;0.5,1.5,2.5;1.3, 2.5,3.5)	(2,4,6;1.5,2.5,3.5;3,5, 7)	(0.6,2,4;0.3,1,1.25;1.5 ,3,4.5)	(2,4,6;1.5,2.5,3.5;3,5, 7)	(1,5,8;0.5,3,6;3.5,7,.5)	(1,2,3;0.5,1.5,2.5;1.3, 2.5,3.5)	(2,4,6;1.5,2.5,3.5;3,5, 7)
A_{12}	(1,5,8;0.5,3,6;3.5,7,.5)	(1,4,7;0.5,2,3;3.5,5.5, 7.5)	(1,3,5;0.5,2.5,4.5;2,4, 6)	(1,4,7;0.5,2,3;3.5,5.5, 7.5)	(1,2,3;0.5,1.5,2.5;1.3, 2.5,3.5)	(0.5,2.5,4.5;1,2,3;1.5, 3.5,5.5)	(1,4,7;0.5,2,3;3.5,5.5, 7.5)
A_{13}	(2,4,6;1.5,2.5,3.5;3,5, 7)	(1,4,7;0.5,2,3;3.5,5.5, 7.5)	(1,3,5;0.5,2.5,4.5;2,4, 6)	(0.6,2,4;0.3,1,1.25;1.5 ,3,4.5)	(0.5,2.5,4.5;1,2,3;1.5, 3.5,5.5)	(0.6,2,4;0.3,1,1.25;1.5 ,3,4.5)	(1,3,5;0.5,2.5,4.5;2,4, 6)

Table 4. The normalization of TBNNs by the CRITIC method.

	C_1	C_2	C_3	C_4	C_5	C_6	C_7
A_1	0.322264	1	0.910221	0.882839	0.610108	0.013619	0.084302
A_2	0.453585	0	0.004834	0.676085	0.935018	0.863813	0
A_3	0.775849	0.377143	0.889503	0.682977	0.318412	1	1
A_4	0.536604	0.533333	0.700276	0.339766	0.433935	0	0.771802

A_5	0.660377	0.364762	0.716851	0.339766	0.484477	0.212711	0.412791
A_6	0.871698	0.485714	0.594613	0.407305	0.118412	0.392348	0.651163
A_7	0.723019	0.374286	0.799724	0	0.518412	0.476654	0.034884
A_8	0.387925	0.6	1	0.490007	0.833935	0.798962	0.767442
A_9	0.242264	0.020952	0.296961	0.284631	1	0.56939	0.116279
A_{10}	0.366038	0.177143	0.571823	1	0.686643	0.727626	0.863372
A_{11}	0	0.38381	0	0.951757	0.484477	0.337873	0.898256
A_{12}	0.984906	0.628571	0.578729	0.583735	0	0.290532	0.640988
A_{13}	1	0.771429	0.578729	0	0.436823	0.013619	0.607558

Fig 1. The correlation between criteria.

Fig 2. The H_j values.

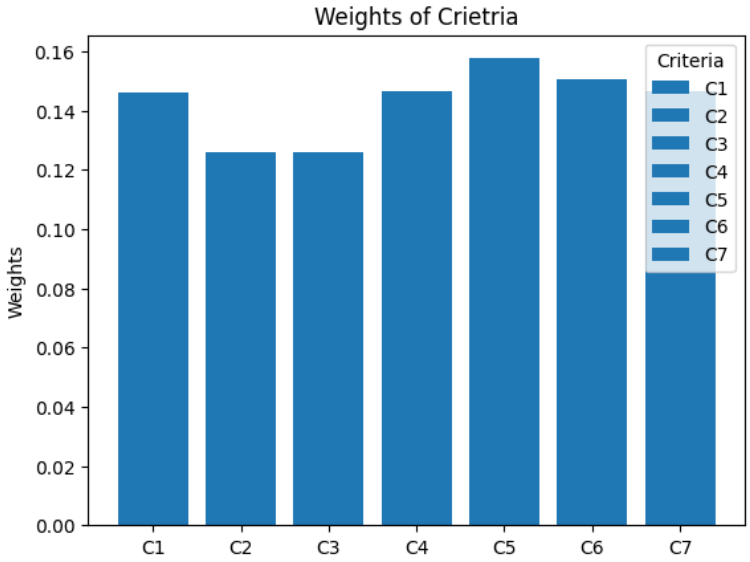

Fig 3. The criteria weights.

Evaluation Model: ARTASI.

We used the ARTASI to rank the alternatives.

- Step 1. Eq. (16) is used to build the initial decision matrix
- Step 2. Eq. (17) and (18) are used to compute the absolute max and min values.
- Step 3. Eq. (19) is used to normalize the initial decision matrix as shown in Table 5.
- Eq. (20) is used to change the values, if the criterion is of the min type as shown in Table 6.
- Step 4. Eqs. (21,22,23) are used to Compute the degree of usefulness of alternatives about optimal and suboptimal values.
- Step 5. Eqs. (24 and 15) are used to compute the combined level of usefulness of alternatives for ideal value and anti-ideal value as shown in Tables 7 and 8.
- Step 6. Eqs. (27,28,29) are used to compute the ultimate utility functions and rank the alternatives as shown in fig 4. we put the a with 0.5 as shown in Table 9.

Table 5. The normalization of TBNNs.

	C_1	C_2	C_3	C_4	C_5	C_6	C_7
A_1	3.472884	5.482748	3.817882	3.798251	3.633656	2.88445	6.403223
A_2	3.589259	4.594568	3.015231	3.614948	3.921634	3.638286	6.328438
A_3	3.874843	4.929539	3.799515	3.621058	3.375116	3.759038	7.215542
A_4	3.662828	5.068264	3.63176	3.316774	3.477508	2.872375	7.013107
A_5	3.772514	4.918542	3.646454	3.316774	3.522305	3.060978	6.694627
A_6	3.959783	5.02597	3.538087	3.376653	3.19785	3.220255	6.906087
A_7	3.828026	4.927001	3.719923	3.015546	3.552383	3.295006	6.359384
A_8	3.531071	5.127476	3.897474	3.449975	3.832041	3.580786	7.009239
A_9	3.40199	4.613178	3.27421	3.267893	3.97923	3.377232	6.43159
A_{10}	3.511676	4.751903	3.517883	3.902123	3.701491	3.517535	7.094339
A_{11}	3.1873	4.93546	3.010946	3.859353	3.522305	3.171954	7.125285
A_{12}	4.060105	5.152853	3.524005	3.533072	3.092898	3.129979	6.897062

A_{13}	4.073482	5.279735	3.524005	3.015546	3.480068	2.88445	6.867406

Table 6. The change values.

	C_1	C_2	C_3	C_4	C_5	C_6	C_7
A_1	10.73367	15.56006	10.7263	10.71592	10.70578	9.515863	19.9472
A_2	10.85004	14.67188	9.92365	10.53262	10.99376	10.2697	19.87242
A_3	11.13562	15.00685	10.70793	10.53873	10.44724	10.39045	20.75952
A_4	10.92361	15.14558	10.54018	10.23444	10.54964	9.503788	20.55709
A_5	11.0333	14.99586	10.55487	10.23444	10.59443	9.69239	20.23861
A_6	11.22056	15.10329	10.44651	10.29432	10.26998	9.851668	20.45007
A_7	11.08881	15.00432	10.62834	9.933215	10.62451	9.926419	19.90336
A_8	10.79185	15.20479	10.80589	10.36764	10.90417	10.2122	20.55322
A_9	10.66277	14.69049	10.18263	10.18556	11.05136	10.00865	19.97557
A_{10}	10.77246	14.82922	10.4263	10.81979	10.77362	10.14895	20.63832
A_{11}	10.44808	15.01278	9.919365	10.77702	10.59443	9.803367	20.66927
A_{12}	11.32089	15.23017	10.43242	10.45074	10.16503	9.761392	20.44104
A_{13}	11.33426	15.35705	10.43242	9.933215	10.5522	9.515863	20.41139

Table 7. The ideal values.

	C_1	C_2	C_3	C_4	C_5	C_6	C_7
A_1	0.138579	0.125865	0.125009	0.14537	0.152708	0.137977	0.141046
A_2	0.140082	0.118681	0.115654	0.142883	0.156816	0.148907	0.140517
A_3	0.143769	0.121391	0.124795	0.142966	0.149021	0.150658	0.14679
A_4	0.141032	0.122513	0.122839	0.138839	0.150481	0.137802	0.145358
A_5	0.142448	0.121302	0.123011	0.138839	0.15112	0.140537	0.143106
A_6	0.144866	0.122171	0.121748	0.139651	0.146492	0.142846	0.144601
A_7	0.143164	0.12137	0.123867	0.134752	0.151549	0.14393	0.140736
A_8	0.139331	0.122992	0.125936	0.140645	0.155538	0.148074	0.145331
A_9	0.137664	0.118832	0.118672	0.138175	0.157638	0.145122	0.141246
A_{10}	0.13908	0.119954	0.121512	0.146779	0.153676	0.147157	0.145933
A_{11}	0.134892	0.121438	0.115604	0.146199	0.15112	0.142146	0.146151
A_{12}	0.146161	0.123197	0.121584	0.141773	0.144995	0.141537	0.144538
A_{13}	0.146333	0.124223	0.121584	0.134752	0.150518	0.137977	0.144328

Table 8. The anti-ideal values.

	C_1	C_2	C_3	C_4	C_5	C_6	C_7
A_1	0.138945	0.125865	0.125078	0.145473	0.152957	0.137993	0.141067
A_2	0.140473	0.118681	0.115659	0.143105	0.156878	0.149038	0.140517
A_3	0.144087	0.12149	0.124879	0.143185	0.149253	0.150658	0.14679
A_4	0.141422	0.122618	0.123022	0.139072	0.150742	0.137802	0.145406
A_5	0.142813	0.1214	0.123187	0.139072	0.151384	0.140734	0.143173
A_6	0.145126	0.122276	0.121959	0.139901	0.146606	0.143122	0.144663
A_7	0.143507	0.12147	0.124005	0.134752	0.151813	0.144216	0.140745
A_8	0.139713	0.123092	0.125936	0.140903	0.155681	0.148253	0.145379
A_9	0.137998	0.11884	0.11886	0.138389	0.157638	0.145402	0.141275
A_{10}	0.139458	0.120016	0.121727	0.146779	0.1539	0.147379	0.145964
A_{11}	0.135052	0.121539	0.115604	0.146244	0.151384	0.142406	0.146176
A_{12}	0.146333	0.123295	0.121798	0.142021	0.144995	0.141778	0.1446
A_{13}	0.146493	0.124297	0.121798	0.134752	0.150779	0.137993	0.144393

Table 9. The alternative method results.

	t_i^+	t_i^-	$f(t_i^+)$	$f(t_i^-)$	t_i
A_1	0.966555	0.96738	0.499787	0.500213	1.450245
A_2	0.963541	0.964351	0.49979	0.50021	1.445716
A_3	0.979389	0.980342	0.499757	0.500243	1.46956
A_4	0.958864	0.960083	0.499682	0.500318	1.438905
A_5	0.960362	0.961763	0.499635	0.500365	1.441243
A_6	0.962374	0.963653	0.499668	0.500332	1.444201
A_7	0.959368	0.960508	0.499703	0.500297	1.439622
A_8	0.977847	0.978957	0.499716	0.500284	1.467325
A_9	0.95735	0.958401	0.499726	0.500274	1.43655
A_{10}	0.974091	0.975224	0.499709	0.500291	1.461703
A_{11}	0.957551	0.958405	0.499777	0.500223	1.436754
A_{12}	0.963784	0.96482	0.499731	0.500269	1.446194
A_{13}	0.959715	0.960504	0.499795	0.500205	1.439967

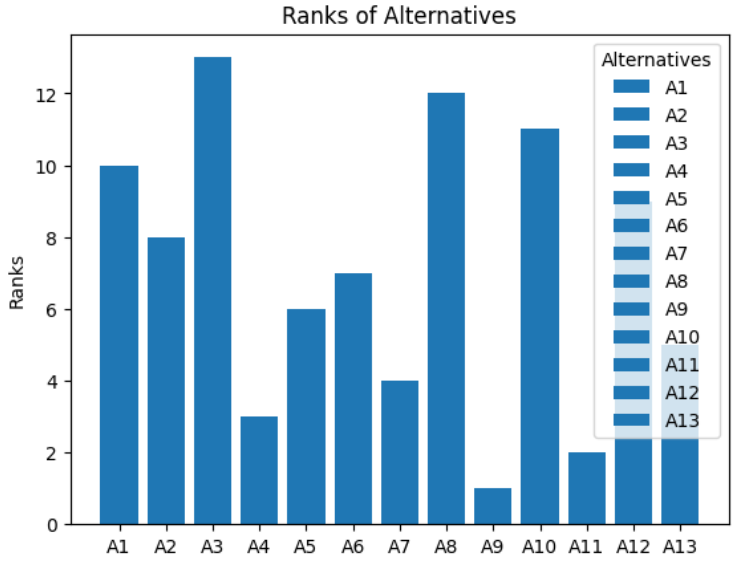

Fig 4. The rank of alternatives.

4. Analysis

The sensitivity analysis and validation methods used for the results obtained using the ARTASI model are described in this section. There are two successive stages to the sensitivity analysis. First, attention is focused on how the utility function's parameters (α) change. The discussion then moves on to changing threshold values within the criterion period. Additionally, the results generated by the MCDM models are under consideration compared with a statistical correlation to the preliminary results.

In the following part, the sensitivity of the model was analyzed in 11 scenarios in which the utility function's parameter varied systematically. In the first scenario, a = 0 was adopted, and for each following scenario, parameter a was successively increased by 0.1. These modifications affected the utility function's

transformation and, as a result, changed the alternatives' final values. Fig 5 shows the utility function values. Fig 6 shows the ranks of the alternatives.

Alternative 3 is the best and alternative 9 is the worst. We show the ranks of the alternatives under different cases are stable.

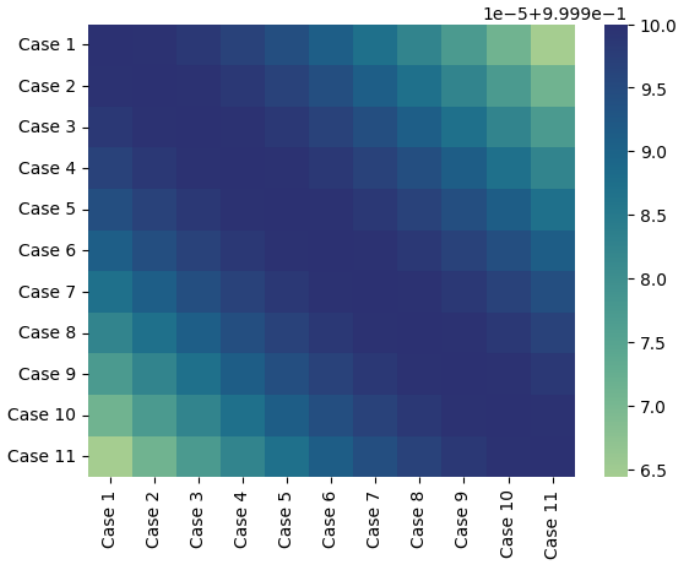

Fig 5. The utility functions values.

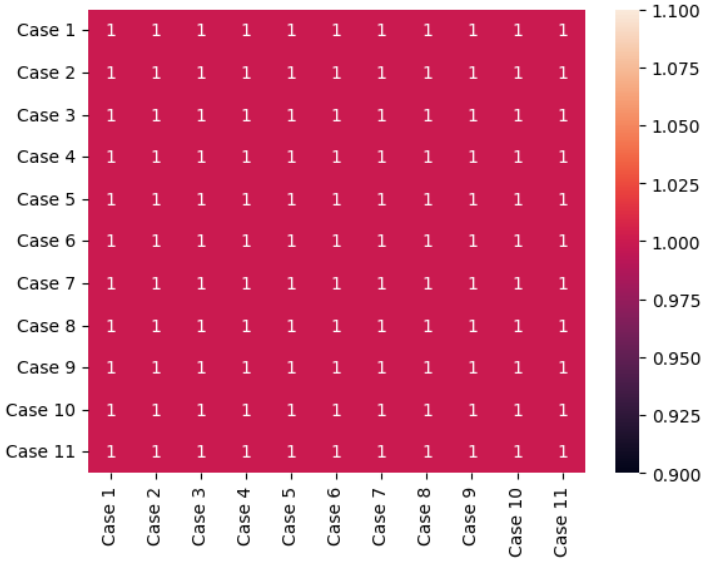

Fig 6. The rank of the alternatives under different cases.

Advantages of Proposed Method

The CRITIC method is a powerful objective weight determination technique used in MCDM. It assigns weights to criteria based on the contrast intensity (variance) and the degree of conflict (correlation) between criteria. Here are its key advantages:

1. Objective Weight Calculation

Unlike subjective weighting methods (e.g., AHP, SWARA), CRITIC assigns weights based on mathematical calculations rather than expert judgment. This eliminates human bias and makes the process more data-driven and reliable.

2. Considers Both Variability and Interdependencies

CRITIC evaluates two essential factors:

- Contrast intensity (higher variance means a criterion is more important).
- Correlation between criteria
- This ensures that less redundant and more informative criteria receive higher weights.

3. Effective Large-Scale Decision Problems

Since it is purely statistical, CRITIC works well in large and complex decision-making problems, where many criteria must be evaluated simultaneously.

4. Eliminates Subjectivity and Bias

Because the weights are computed using standard deviation and correlation coefficients, the method does not require human input, reducing subjective inconsistencies.

5. Compatible with Other MCDM Methods

CRITIC is frequently combined with methods like TOPSIS, ARAS, COPRAS, and MABAC to enhance decision-making accuracy by providing more reliable weight values.

6. Ensures Data-Driven Decision Making

By relying on statistical measures rather than personal judgment, CRITIC enhances the credibility and transparency of the decision-making process.

7. Suitable for Various Fields

5. Conclusions

This study used the new CRITIC-ARTASI multi-criteria decision-making model to evaluate the effectiveness of 13 alternatives. We have identified 7 criteria in this study. A unique CRITIC approach is used to compute the criteria weights for the seven criteria. According to the findings, the CRITIC-ARTASI multi-criteria decision-making model, when combined with rankings, offers a solid and reliable framework for assessing design effect evaluation of ethnic minority clothing. The results show alternative 3 is the best and alternative 9 is the worst. The sensitivity analysis is conducted to show the stability of the ranks.

Acknowledgment

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Human Capital and Innovation in Developed and Developing Countries: A Systematic Literature Review Using Methodi Ordinatio and HyperSoft Sets

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Abstract. This article aims to analyze the influence of human capital on innovation. To achieve this, the Methodi Ordinatio methodology was used. The search for scientific articles was done using the Scopus and Web of Science databases. Only documents published between 2013 and 2023 were selected. The systematic review of the literature shows that human capital is relevant in promoting innovation in any type of economy; however, the impact may vary depending on the economic and regional context being studied. The data obtained are analyzed with the help of the hybridization of the Methodi Ordinatio with the HyperSoft Sets theory. In this way, the search is more flexible, because it allows to automatically represent complex combinations of keywords to find patterns of knowledge from the scientific literature on the subject.

Keywords: Human capital, innovation, Methodi Ordinatio, Soft Set, HyperSoft Set.

1 Introduction

In today's business environment, the ability to innovate has become a relevant factor for the success of any organization. Likewise, innovation, which is defined as the incorporation of new products, services, and processes, also allows organizations to endure over time.

During the industrial era, land, buildings, and property were of greater importance to companies, as their fixed assets and natural resources were considered their main source of wealth. However, in recent years a knowledge-based economy known as the post-capitalist society has emerged. In this emerging society, the value of knowledge has surpassed that of traditional elements such as physical capital, land, or labor.

These changes in the economy are more common in developed countries because they have the tools necessary to move from an industrial economy to a knowledge-based one. Thus, human capital, which includes knowledge, skills, creativity, and health, has become one of the most valuable intangible assets in organizations.

This systematic literature review seeks to understand how human capital influences innovation, therefore the following research question is established: How does human capital influence innovation processes in different economic and regional contexts?

To answer this question, a systematic literature review is carried out that will address the relationship between human capital and innovation in various countries and sectors. The main characteristics, trends, limitations, and challenges in this field of study will also be identified. Therefore, the analysis of this relationship will provide valuable information for companies, governments, and academics interested in understanding the behavior of this phenomenon.

The methodology known as Methodi Ordinatio is based on an equation to rank papers according to their scientific relevance. Some bibliometric indicators are used for this purpose. We believe that this methodology can be reinforced by using the theory of Soft Set extensions defined by Professor F. Smarandache [1-4].

The HyperSoft Sets, SuperHyperSoft Sets, and Fuzzy_Extension SuperHyperSoft Sets extend Soft Sets to a parametric function with domain in more than one set of parameters, their power sets, and even associated fuzzy truth values or some of their extensions [1-6].

Specifically, we adapted some components of HyperSoft Sets to model the bibliometric study linked to the Methodi Ordinatio [7-16]. This connection reaffirms the usefulness of these tools that emerged in the field of Neutrosophy, which allows for the modeling of complex real-world situations, where multiple dimensions are not always structured. By studying the relationship between words, knowledge is obtained, especially the relationship that exists between human capital and innovation.

To achieve this objective, the article is divided into a section called Preliminaries where the basic notions of the Methodi Ordinatio methodology are presented, as well as Soft Sets, HyperSoft Sets, SuperHyperSoft Sets, and Fuzzy_Extension SuperHyperSoft Sets. The article continues with a section showing the results of the study, based on the theories and methodologies presented in the previous section. The last section is the Conclusion.

2 Preliminaries

This section contains the basic notions of the theories and concepts that we use in this study.

2.1 Methodi Ordinatio

This systematic literature review article uses the methodology called Methodi Ordinatio, which takes into account factors such as number of citations, year of publication of the articles, and impact factor (IF) to classify the documents according to the scores obtained by the equation. In addition, a descriptive-qualitative research approach is adopted. Thus, a systematic review is a synthesis of the available evidence on a particular topic, taking into consideration quantitative and qualitative aspects of the selected studies to summarize the existing information. On the other hand, descriptive-qualitative studies comprise an important methodological approach, especially when we wish to directly describe the phenomena to be studied.

The first figure summarizes the phases of this methodology, which are described below, in Figure 1.

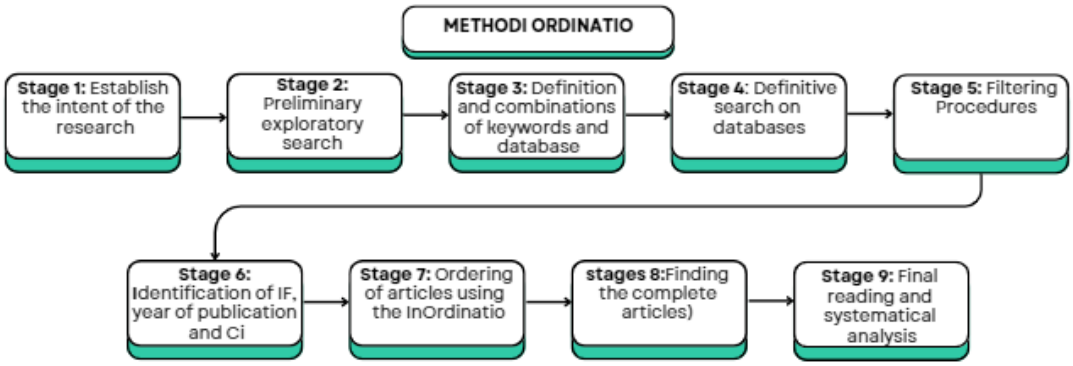
METHODI ORDINATIO
Stage 1: Establish the intent of the research
Stage 2: Preliminary exploratory search
Stage 3: Definition and combinations of keywords and database
Stage 4: Definitive search on databases
Stage 5: Filtering Procedures
Stage 6: Identification of IF, year of publication and Ci
Stage 7: Ordering of articles using the InOrdinatio
stages 8:Finding the complete articles)
Stage 9: Final reading and systematical analysis

Figure 1: Phases of Methodi Ordinatio.

Phase 1. Establish the intention to investigate.

Phase 2. Preliminary exploratory research with keywords in databases.

Phase 3. Definition and combination of keywords and databases.

Phase 4. Final search of the databases: The final search is carried out in important databases such as Web of Science (WoS) and Scopus. For this step, descriptors and Boolean operators such as "AND", and "OR", among others, are included.

Phase 5. Information filtering procedure.

Phase 6. Identification of the impact factor, year, and number of citations.

Step 7. Classifying documents using the *InOrdinatio* equation: In this step, the *InOrdinatio* value is calculated for the found items, in Equation 1.

$$InOrdinatio = \left(\frac{IF}{1000}\right) + \propto * [10 - (ResearchYear - PublishYear)] + (\sum C_i) \qquad (1)$$

Where:

IF = Impact Factor (JCR, SJR).

$\propto$ = coefficient (1 to 10) Importance value of the year of the article.

Research Year = Year in which the research was conducted.

PublishYear = Year of publication of the article.

$\sum C_i$ =Total number of citations.

Phase 8. Search for complete documents, reading, and systematic analysis of the documents.

2.2 Soft Sets and their Extensions

This subsection is dedicated to explaining the basic notions of Soft Sets, HyperSoft Sets, SuperHyperSoft Sets, and Fuzzy_Extension SuperHyperSoft Sets.

Definition 1 ([1, 17]). Given U is the initial universe set and E is the set of parameters. A pair (F, E) is called a *Soft Set* (over U) if and only if F is a mapping of E into the set of all subsets of U.

That is to say, having a set E of parameters and fixing a parameter $\varepsilon \in E$, then $F(\varepsilon) \in \mathcal{P}(U)$, where $\mathcal{P}(U)$

denotes the power set of U and $F(\varepsilon)$ is considered the set of ε-elements of the Soft Set (F, E) or the set of ε-approximate elements of the Soft Set.

It is not difficult to realize that fuzzy sets are soft sets, this is a consequence of the α-levels definition of a membership function μ_A where we have the following:

$F(\alpha) = \{x \in U \mid \mu_A(x) \geq \alpha\}$, $\alpha \in [0, 1]$. Thus, if we know the family F, then we can reconstruct the function μ_A by using the following formula:

$$\mu_A(x) = \sup_{\substack{\alpha \in [0,1] \\ x \in F(\alpha)}} \alpha$$

Thus, a fuzzy set is a $(F, [0, 1])$ soft set.

Given a binary operation * for subsets of the set U, where (F, A) and (G, B) are soft sets over U. Then, the operation * for soft sets is defined as follows:

$(F, A) * (G, B) = (J, A \times B)$, where $J(\alpha, \beta) = F(\alpha) * G(\beta)$; $\alpha \in A$, $\beta \in B$, and $A \times B$ is the Cartesian product of the sets A and B.

Definition 2 ([5, 6, 17]). Let U be a universe set, $\mathcal{P}(U)$ the power set of U. Let $a_1, a_2, \ldots, a_n$, for $n \geq 1$, be n distinct attributes, whose corresponding attribute values are respectively the sets $A_1, A_2, \ldots, A_n$, with $A_i \cap A_j = \emptyset$, for $i \neq j$, and $i, j \in \{1, 2, \ldots, n\}$. Then the pair $(F, A_1 \times A_2 \times \ldots \times A_n)$, where: $F: A_1 \times A_2 \times \ldots \times A_n \to \mathcal{P}(U)$ is called a *HyperSoft Set* over U.

Definition 3 ([5, 6, 17]). Let U be a universe set, $\mathcal{P}(U)$ the power set of U. Let $a_1, a_2, \ldots, a_n$, for $n \geq 1$, be n distinct attributes, whose corresponding attribute values are respectively the sets $A_1, A_2, \ldots, A_n$, with $A_i \cap A_j = \emptyset$, for $i \neq j$, and $i, j \in \{1, 2, \ldots, n\}$. Then the pair $(F, \mathcal{P}(A_1) \times \mathcal{P}(A_2) \times \ldots \times \mathcal{P}(A_n))$, where:

$F: \mathcal{P}(A_1) \times \mathcal{P}(A_2) \times \ldots \times \mathcal{P}(A_n) \to \mathcal{P}(U)$ is called a *SuperHyperSoft Set* over U.

Definition 4 ([5, 6, 17]). Let U be a universe set, $\mathcal{P}(U)$ the power set of U. Let $a_1, a_2, \ldots, a_n$, for $n \geq 1$, be n distinct attributes, whose corresponding attribute values are respectively the sets $A_1, A_2, \ldots, A_n$, with $A_i \cap A_j = \emptyset$, for $i \neq j$, and $i, j \in \{1, 2, \ldots, n\}$. Then the pair $(F, \mathcal{P}(A_1) \times \mathcal{P}(A_2) \times \ldots \times \mathcal{P}(A_n))$, where:

$F: \mathcal{P}(A_1) \times \mathcal{P}(A_2) \times \ldots \times \mathcal{P}(A_n) \to \mathcal{P}\left(U\left(x(d^0)\right)\right)$ is called a *Fuzzy_Extension SuperHyperSoft Set* over U.

Where $x(d^0)$ is the fuzzy or any fuzzy extension degree of appurtenance of the element x to the set U. Fuzzy extension means Fuzzy Set or Intuitionistic Fuzzy Set, Pythagorean Fuzzy Set, Fermatean Fuzzy Set, Neutrosophic Fuzzy Set, Plithogenic Fuzzy Set, etc.

3 Results

The model used in the study contains combinations of elements with different linguistic meanings. An automatic search for words and combinations of words is required to find patterns that allow evaluating the relationship between the concepts studied, specifically "human capital" and " innovation" in both developed and developing countries.

Here the universe U is the set of papers published in the most important databases; these are the Web of Science (WoS) and Scopus. Of these articles, we are interested in the journal's impact factor and the number of citations of the article.

On the other hand, the set of parameters is given by the keywords to be searched. That is, in general, we are going to determine a set of parametric functions:

$F = \{F_1, F_2, \cdots, F_n\}$, $i = 1, 2, \ldots, n$ such that F_i: $A_{i1} \times \ldots \times A_{im_i} \to \mathcal{P}(U)$, where $m_i \in \mathbb{Z}_*^+$ or the set of positive integers. A_{i1} is a set of specific keywords, for example, $F_1(\text{"}Human\text{"}, \text{"}Capital\text{"}) \in \mathcal{P}(U)$ is the set of articles found that have as keywords the combination "Human Capital". Of these articles, we are interested in their relevance, which is measured by the *InOrdinatio* that appears in Equation 1. In this way, the Methodi Ordinatio is hybridized with Smarandache's theory of HyperSoft Sets.

Within each function, F_i: $A_{i1} \times \ldots \times A_{im_i} \to \mathcal{P}(U)$ the articles that have the words $w_{i1} \in A_{i1}, \ldots, w_{im_i} \in A_{im_i}$ for $F_i(w_{i1}, \ldots, w_{im_i}) = \{p_{i1}, \ldots, p_{il_i}\}$ as keywords are obtained.

On the other hand, among the parametric functions that appear in F, other relationships are defined between them, for example, the "OR", such that if $F_i(w_{i1}, \ldots, w_{im_i}) = \{p_{i1}, \ldots, p_{il_i}\}$ and $F_j(w_{j1}, \ldots, w_{jm_j}) = \{p_{j1}, \ldots, p_{jl_j}\}$ then, $F_i(w_{i1}, \ldots, w_{im_i})\ OR\ F_j(w_{j1}, \ldots, w_{jm_j}) = \{p_{i1}, \ldots, p_{il_i}, p_{j1}, \ldots, p_{jl_j}\}$, that is, the articles that have as keywords $w_{i1}\ AND \ldots AND\ w_{im_i}$ are included with those that have as keywords $w_{j1}\ AND \ldots AND\ w_{jm_j}$.

The results obtained after following each of the phases were as follows:

Phase 1. The main objective of this research is to analyze the impact of human capital on innovation processes in different economic and regional contexts.

Phase 2. In this stage, an initial search was carried out in databases such as Web of Science and Scopus, using the terms "innovation" and "human capital". This first sweep was done to find synonyms and keywords in titles and abstracts, contributing to the creation of the search equations; in total, 17,253 scientific articles were found.

Phase 3. We include specifications regarding the way to innovate using the following terms: "technological innovation", "organizational innovation", "product innovation", and "innovative performance". New words related to socioeconomic aspects of the countries studied in the articles were also included, such as: "advanced countries", "industrialized countries", "wealthy countries", and "developed countries", among others. The list of complete words can be seen in Table 1.

Database	**Keywords, HyperSoft Sets and Constraints**	**Results**
WoS	(ALL=(human capital)) AND ALL=("innovation" OR "technological innovation" OR "organizational innovation" OR "product innovation" OR "Innovative Performance") AND ALL=("advanced countries" OR "industrialized countries" OR "wealthy countries" OR "developed countries" OR "more developed countries" OR "prosperous countries" OR "advanced economies" OR "developing countries" OR "underdeveloped countries" OR "less developed countries" OR "emerging countries" OR "developing economies" OR "industrial nations" OR "industrial economies" OR "industrialized nations" OR "advanced nations" OR "developed nations" OR "industrial states" OR "industrial societies" OR "rich countries" OR "wealthier countries" OR "wealthy nations" OR "industrially advanced countries" OR "civil countries" OR "civilized nations" OR "civilized powers")	360
Scopus	TITLE-ABS (("human capital" AND ("innovation" OR "technological innovation" OR "organizational innovation" OR "product innovation") AND ("advanced countries" OR "industrialized countries" OR "wealthy countries" OR "developed countries " OR "more developed countries" OR "prosperous countries" OR "advanced economies" OR "developing countries" OR "underdeveloped countries" OR "less developed countries" OR "emerging countries" OR "developing economies" OR "industrial nations" OR "industrial economies" OR "industrialized nations" OR "advanced nations" OR "developed nations" OR "industrial states" OR "industrial societies" OR "rich countries" OR "wealthier countries" OR "wealthy nations" OR "industrially advanced countries" OR "civil countries" OR "civilized nations" OR "civilized powers")))	378
Total		738

Table 1. Final search equations.

Tools such as VOSviewer enable the implementation of a node cluster to locate and categorize the keywords represented graphically. A bibliometric analysis of the articles is carried out through VOSviewer. Words such as "impact", "growth", "productivity", "economic growth" and "education" have a larger circle, which indicates the importance of these terms for the research found. It also shows how these words are interrelated with others, forming clusters differentiated by colors. Figure 2 shows the nodes related to the terms "innovation" and "human capital".

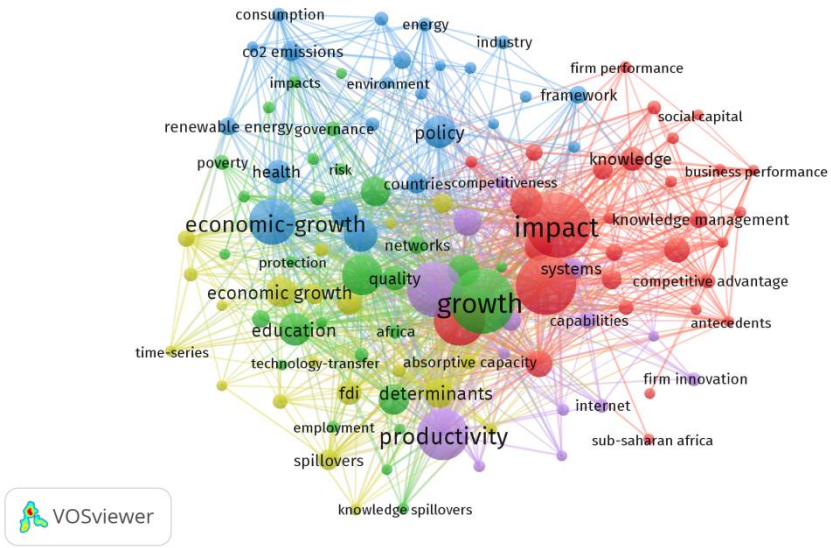

Figure 2. Bibliometric network of the terms innovation and human capital.

Phase 4. The final search was performed in major databases such as Web of Science (WoS) and Scopus. For this step, descriptors and Boolean operators were included, managing to find 360 articles on the WoS page and 378 on Scopus, for a total of 738 articles. The search equations used are shown in Table 1.

Phase 5. Using the Mendeley and Jabref tools, duplicate files were eliminated. The search was restricted to documents published between 2013 and 2023, excluding all types of documents other than articles, such as books or book chapters because these do not fit the Inordinatio equation by not having an impact factor. Documents not directly related to the research objective were also excluded, resulting in a portfolio of 27 documents.

Phase 6. Each document was assigned its corresponding impact factor calculated by Journal Citation Reports (JCR), as well as the number of citations, the year of publication, and the year of search for the articles. For journals without a JCR impact factor record, the SJR impact factor was chosen as a second option. Both indicators have demonstrated their capacity to measure the scientific influence of academic journals.

The number of citations and the year of publication were obtained through Google Scholar. We took into account that several authors recommend the use of this tool for its wide coverage and its high capacity to track citations; on the other hand, the search year was the same (2024) for all calculations because this bibliographic review was carried out on the same date.

Phase 7. The importance value of the year of publication of the articles ($\propto$) is assigned the lowest rating (equal to 1), because a filter had previously been applied to select only articles written during 10 years.

Phase 8. Finally, the Mendeley reference manager was used to collect the articles and perform the analysis of the documents obtained through the Methodi Ordinatio methodology.

Figure 3 represents a summary of the selection process carried out to build the final portfolio.

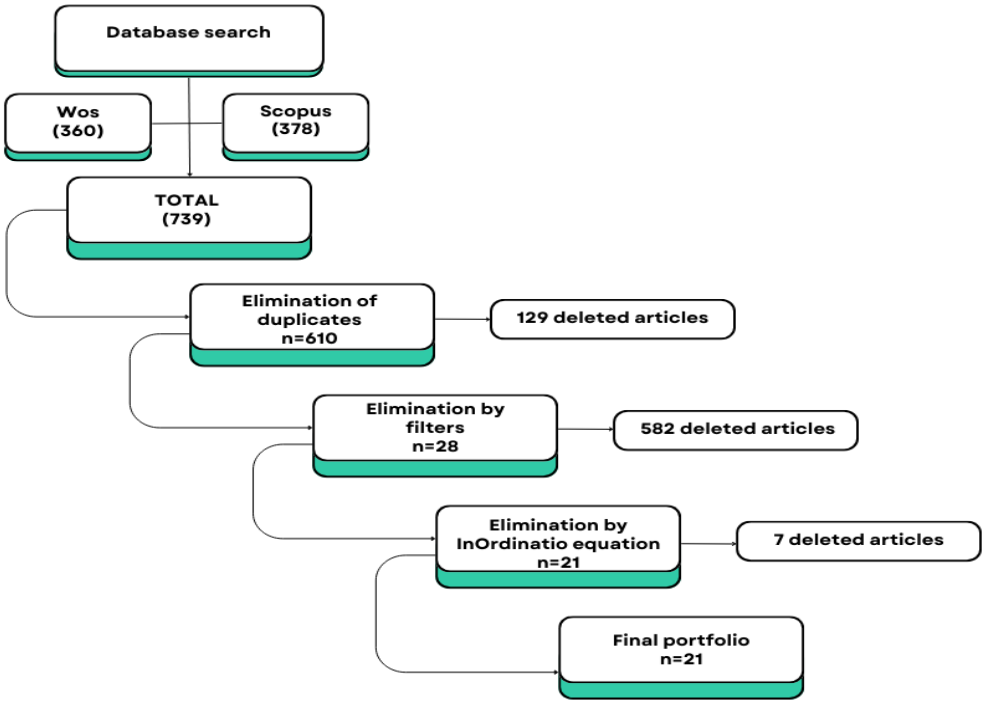

Figure 3. Summary of the bibliographic portfolio selection process

On the other hand, Figure 4 visually represents the countries with the highest number of publications related to the research topic, using a color scale ranging from light green for countries with fewer publications to dark green for countries with more publications. The United States and China are the countries with the most publications on the topic, followed by Canada, Australia, and Russia.

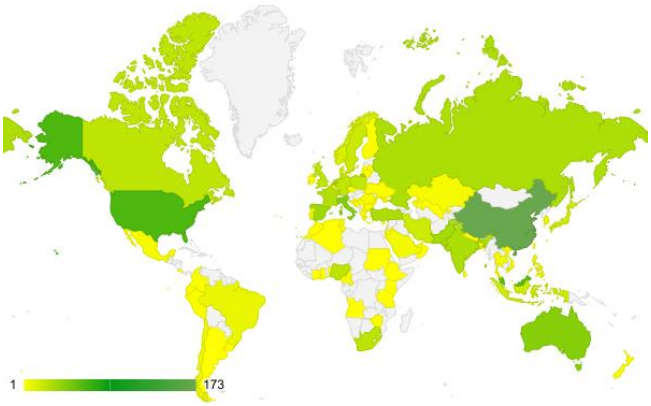

Figure 4. Countries with the highest number of publications.

Figure 5 shows the Sankey diagram of the selected articles, this is a graphical representation that shows the flow and relationships between different concepts or variables. In this case, the diagram focuses on the relationship between the terms "innovative", "capital", "innovation", and "human" with a series of subtopics and categories. Thus:

- **Innovative**: It is mainly connected with "knowledge" and "countries", suggesting that innovation is strongly influenced by knowledge and national context.
- **Capital**: Related to terms such as "relationship", "manufacture", and "competitive", indicating that capital has implications for competitiveness and relationships within the manufacturing process.
- **Innovation**: It is linked to "firms" and "development", suggesting that innovation is crucial both for firms and for development in general. In addition, it is connected to "technology" and "policy", showing the importance

of these factors in innovation processes.

- **Human**: It is associated with "impact", "institutions", and "process", indicating that human capital has a significant impact through institutions and processes within companies.

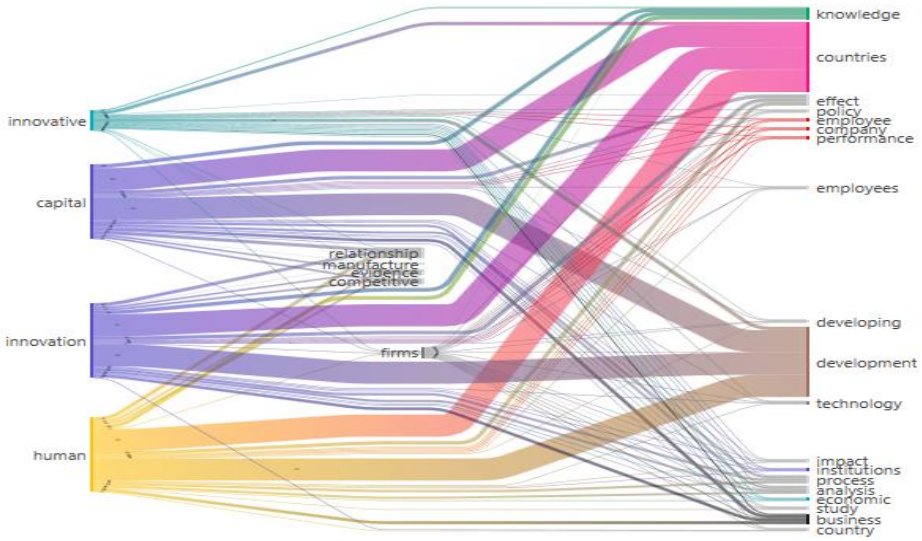

Figure 5. Sankey diagram of this study.

Finally, Figure 6 shows two trend lines that show the evolution of the number of publications in the Web of Science (WoS) and Scopus databases during the period between 2014 and 2022. These lines allow us to analyze the behavior of scientific production in the two sources of information over time, providing a longitudinal perspective on the development of research in the specific field addressed.

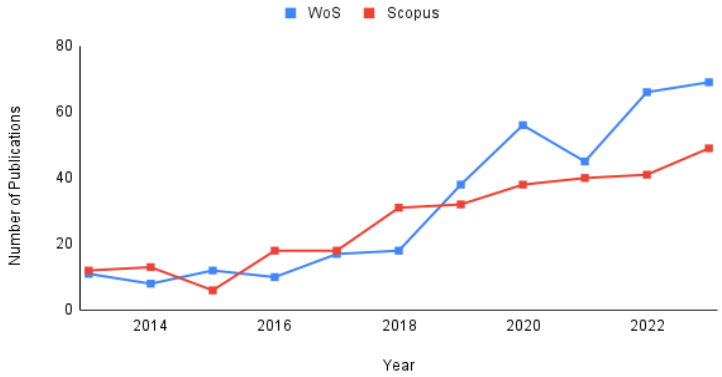

Figure 6. Evolution of the number of publications in the Web of Science (WoS) and Scopus databases.

Conclusion

Research consistently shows that human capital, characterized by education, skills, creativity, and health, is a critical factor for the generation and success of innovation. This finding is consistent with existing knowledge and expands the understanding of how human capital interacts with other factors, such as technological infrastructure and government policies.

In developed countries, there is a direct relationship between robust and well-developed human capital and the ability to generate relevant innovations. In contrast, in developing countries, human capital faces additional challenges, such as a lack of access to adequate educational and technological infrastructure, which limits its innovation potential.

The research also highlights the importance of public policies and cross-sector collaboration in promoting human capital for innovation. An important conclusion is the implication that governments and organizations must work together to develop and implement effective human capital strategies.

In terms of future research, this review suggests several directions. First, it would be valuable to conduct more

detailed studies in developing countries to better understand the specific challenges they face in terms of developing human capital for innovation. Furthermore, future research could further explore how rapidly evolving technology and changes in the labor market affect human capital needs in different industries.

It would also be beneficial to conduct longitudinal studies to better understand how investments in human capital influence innovation over time, providing a deeper understanding of the causal relationship between human capital and innovation.

This review provides a solid foundation for understanding the importance of human capital in innovation and highlights key areas for practice and future research. Although there are limitations that need to be addressed in future studies, the findings have important implications for those interested in fostering innovation through human capital development.

The findings of this systematic review have profound implications for human capital management strategies in organizations and governments. The research shows that investment in human capital development is crucial to fostering an enabling environment for innovation. Therefore, organizations should consider not only technical training but also the development of soft and creative skills that enhance the innovative capacity of their staff.

Collaboration between the educational and business sectors is also emerging as a key factor. Universities and other educational institutions must align their programs with the needs of the labor market, while companies can offer internship and training opportunities that complement formal education. This interaction could contribute to the creation of human capital that is better suited to the challenges of contemporary innovation.

This article once again shows the usefulness of using HyperSoft Sets to represent complex relationships between data automatically. In this research, these complex combinations led to obtaining useful knowledge about organizations, which is an example of text mining.

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Clear Approach to Education Quality Evaluation of Multilingual Higher Education Internationalization under Probabilistic Simplified Neutrosophic Sets for Academic Improvement

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Abstract: An integrated multi-criteria decision-making (MCDM) approach is used in this study to evaluate multilingual higher education internationalization. This study is the first to employ a novel combination technique that has not yet been used for corporate performance evaluation. It is based on root assessment method (RAM) and indifference threshold-based attribute ratio analysis (ITARA) methodologies. The weights of the criteria are determined using the ITARA approach, and the RAM method is used to rank the alternatives. these methods are used under the neutrosophic sets to deal with vague and uncertainty data. This study uses the Probabilistic Simplified Neutrosophic sets to overcome uncertainty issues. Three experts have evaluated the criteria and alternatives. We used seven criteria and ten alternatives to be evaluated and selected the best one.

Keywords: Probabilistic Simplified Neutrosophic sets; RAM Method; ITARA Method; Decision Making.

1. Introduction and Related Work

To assess multilingual higher education internationalization based on their criteria statements, several methodologies have been developed in the literature, including regression models, discriminant analysis, factor analysis, and multi-criteria decision-making (MCDM) techniques. To create a successful model for multilingual higher education internationalization analysis, several factors need to be considered. As a result, MCDM techniques, which are popular in multilingual higher education internationalization, can be utilized to handle such issues. Researchers favor MCDM approaches due to the intricate, multifaceted, and contradictory character of multilingual higher education internationalization performance. Furthermore, by combining data from multiple criteria into a single score, MCDM approaches simplify and make performance evaluation more straightforward[1], [2]. On the other hand, MCDM techniques use several criteria to rank options based on a mathematical foundation to aid in decision-making.

Many scholars use MCDMs for many domains. However, decision makers find it challenging to model issues when faced with ambiguous or insufficient information. Zadeh created fuzzy sets (FSs) to deal with this issue and model unclear information. The information about its members defines an FS. FSs are used as models for decision problems, pattern recognition, and approximation reasoning in several publications[3], [4]. However, the absence of information about uncertainty makes FSs inadequate for expressing some circumstances. Atanassov created the idea of intuition fuzzy sets (IFS) with this in mind. We have information about non-membership, and the total membership and non-membership is less than or equal to one, unlike FSs.

However, uncertainty in our lives can take many forms. When asked to consider a circumstance, for instance, an expert may state that the degree of truth membership is 0.5, the degree of indeterminacy membership is 0.3, and the degree of falsity membership is 0.7. The sets mentioned above are insufficient to explain this circumstance. Smarandache created neutrosophic logic and neutrosophic set (NS) based on this concept. Truth membership, indeterminacy membership, and falsity membership are the three memberships that define the NSs. None of these memberships add up to more than three[5], [6]. This is an extension of the standard unit interval where the membership degrees accept values in non-standard unit intervals.

Decision-makers may be reluctant to express their preferences during the decision-making process and may want information regarding the potential of each neutrosophic aspect. This issue is partially resolved by the possibility-induced simplified neutrosophic number (PISNN), which was recently presented by Şahin and Liu. The neutrosophic number is given a probability value by a PISNN. However, when decision makers voice their opinions about neutrosophic aspects, they could be hesitant to discuss each one[7], [8]. A new set theory is needed for this concept. In light of this, we provide some theoretical set attributes and define a new set known as the probabilistic simplified neutrosophic set. It is more efficient and trustworthy to define issues that are emerging in numerous fields when components and their potential are considered simultaneously.

For the first time, this study combines the ITARA and RAM methodologies, which is an innovative approach and contribution. The integration of the ITARA and RAM methodologies is further justified by the fact that they are recently introduced approaches with limited literature usage. Moreover, assessments of multilingual higher education internationalization performance have not used these two approaches[9], [10]. This study will add to the body of knowledge and assist scholars in assessing the multilingual higher education internationalization performance.

1.1 Related Work

PSNS is a useful tool for characterizing the ambiguity that exists in everyday life. Altun et al. [6] define a PSNS and go over some theoretical set operations in this study. They suggested the ideas of an inner product, a projection operator between two, and a module on. They present a PSNS in connection with this new collection. Truth membership degree, indeterminacy membership degree, and falsity membership degree are the three pieces that make up a PSNS. They provided a score function, an accurate function, and a few algebraic operational principles. They designed a way based on preference function to calculate the weight of each decision maker and use a fuzzy measure-based method to establish the weights of the criterion. For group decision problems, they offered an expanded PROMETHEE approach. To guarantee

the stability of the suggested approach, they concluded by providing an example of a MCDM problem based on the extended PROMETHEE method.

The widespread use of computer networks has greatly facilitated people's lives and careers, but it has also resulted in information security issues. Performing well in the assessment of computer network security is crucial. In the face of complicated computer network applications, a security assessment approach with good operability and a larger application range is required. Conventional computer network security evaluation can be achieved with the aid of firewalls, antivirus software, etc. It is believed that the assessment of computer network security is a MCDM problem. An extended PSNN grey relational analysis approach is developed by Wang et al. [11] for assessing the security of computer networks. The PSNN-GRA technique combined with the significance of criteria. A numerical example for computer network security evaluation was used to demonstrate the feasibility of the recently proposed method and to compare it with other approaches, even though the CRITIC method was used to rank the optional alternatives in PSNS scenario. Their findings show that the method is straightforward, reliable, and easy to calculate.

Neutosophic sets are a kind of intuitionistic fuzzy set and fuzzy sets that are used to represent ambiguous, imprecise, inconsistent, and incomplete information found in the actual world. The primary goal of SNS is to solve problems involving a given set of numbers. However, there are certain issues with the way SNS now operate, as well as with their comparison techniques and aggregation operators. Thus, Peng et al. [5] established a comparison approach based on the related research of intuitionistic fuzzy numbers and specifies the innovative operations of SNNs.

There are five sections to this study. Section 2 shows some definitions of PSNS. A summary of the MCDM framework is provided in the third section. The steps of the ITARA approach, a semi-objective weighting procedure suggested to establish the criteria weights, are first described. The steps of the new MCDM technique, the RAM method, are then presented and described. By using the integrated MCDM approach to evaluate criteria, the fourth section Additionally, the conclusion is shown in Section 5.

2. Probabilistic simplified neutrosophic set (PSNS)

We introduce some basic definitions of the PSNSs in this paper[11], [12].

Definition 1

We can define the PSNS such as:

$$K = \left\{\left(z; T_k(z)\left(P_{Tk}(z)\right), I_k(x)\left(P_{Ik}(z)\right), F_k(x)\left(P_{Fk}(z)\right)\right); z \in Z\right\} \quad (1)$$

$$0 < P_{Tk}(z), P_{Ik}(z), P_{Fk}(z) \leq 1 \quad (2)$$

$$0 < T_k(x) + I_k(x) + F_k(x) \leq 3 \quad (3)$$

Definition 2.

Let two PSNS such as:

$$A_1 = \left\{\left(z; T_{k_{A_1}}(z)\left(P_{Tk_{A_1}}(z)\right), I_{k_{A_1}}(x)\left(P_{Ik_{A_1}}(z)\right), F_{k_{A_1}}(x)\left(P_{Fk_{A_1}}(z)\right)\right); z \in Z\right\}$$

$$A_2 = \left\{ \left(z; T_{k_{A_2}}(z)\left(P_{Tk_{A_2}}(z)\right), I_{k_{A_2}}(x)\left(P_{Ik_{A_2}}(z)\right), F_{k_{A_2}}(x)\left(P_{Fk_{A_2}}(z)\right) \right); z \in Z \right\}$$

$$A_1 \sqsubseteq A_2 \leftrightarrow \begin{pmatrix} T_{k_{A_1}}(z) \le T_{k_{A_2}}(z), P_{Tk_{A_1}}(z) \le P_{Tk_{A_2}}(z); \\ I_{k_{A_1}}(x) \ge I_{k_{A_2}}(x), P_{Ik_{A_1}}(z) \ge P_{Ik_{A_2}}(z); \\ F_{k_{A_1}}(x) \ge F_{k_{A_2}}(x), P_{Fk_{A_1}}(z) \ge P_{Fk_{A_2}}(z) \end{pmatrix} \tag{4}$$

$$A_1^c = \begin{Bmatrix} F_{k_{A_1}}(x)\left(P_{Fk_{A_1}}(z)\right), \\ \left(1 - I_{k_{A_1}}(x)\right)\left(1 - P_{Ik_{A_1}}(z)\right), \\ T_{k_{A_1}}(z)\left(P_{Tk_{A_1}}(z)\right) \end{Bmatrix} \tag{5}$$

$$A_1 \cap A_2 = \begin{Bmatrix} \min\left\{T_{k_{A_1}}(z), T_{k_{A_2}}(z)\right\}\left(\min\left\{P_{Tk_{A_1}}(z), P_{Tk_{A_2}}(z)\right\}\right), \\ \max\left\{I_{k_{A_1}}(x), I_{k_{A_2}}(x)\right\}, \left(\max\left\{P_{Ik_{A_1}}(z), P_{Ik_{A_2}}(z)\right\}\right), \\ \max\left\{F_{k_{A_1}}(x), F_{k_{A_2}}(x)\right\}, \left(\max\left\{P_{Fk_{A_1}}(z), P_{Fk_{A_2}}(z)\right\}\right) \end{Bmatrix} \tag{6}$$

$$A_1 \cup A_2 = \begin{Bmatrix} \max\left\{T_{k_{A_1}}(z), T_{k_{A_2}}(z)\right\}\left(\max\left\{P_{Tk_{A_1}}(z), P_{Tk_{A_2}}(z)\right\}\right), \\ \min\left\{I_{k_{A_1}}(x), I_{k_{A_2}}(x)\right\}, \left(\min\left\{P_{Ik_{A_1}}(z), P_{Ik_{A_2}}(z)\right\}\right), \\ \min\left\{F_{k_{A_1}}(x), F_{k_{A_2}}(x)\right\}, \left(\min\left\{P_{Fk_{A_1}}(z), P_{Fk_{A_2}}(z)\right\}\right) \end{Bmatrix} \tag{7}$$

$$A_1 \oplus A_2 = \begin{pmatrix} T_{k_{A_1}}(z) + T_{k_{A_2}}(z) - \\ T_{k_{A_1}}(z)T_{k_{A_2}}(z)\left(2!\frac{P_{Tk_{A_1}}(z)P_{Tk_{A_2}}(z)}{P_{Tk_{A_1}}(z)+P_{Tk_{A_2}}(z)}\right), \\ I_{k_{A_1}}(x)I_{k_{A_2}}(x)\left(2!\frac{P_{Ik_{A_1}}(z)P_{Ik_{A_2}}(z)}{P_{Ik_{A_1}}(z)+P_{Ik_{A_2}}(z)}\right), \\ F_{k_{A_1}}(x)F_{k_{A_2}}(x)\left(2!\frac{P_{Fk_{A_1}}(z)P_{Fk_{A_2}}(z)}{P_{Fk_{A_1}}(z)+P_{Fk_{A_2}}(z)}\right) \end{pmatrix} \tag{8}$$

$$\gamma A_1 = \begin{pmatrix} 1 - \left(1 - T_{k_{A_1}}(z)\right)^{\gamma} P_{Tk_{A_1}}(z), \\ \left(I_{k_{A_1}}(z)\right)^{\gamma} P_{Ik_{A_1}}(z), \\ \left(F_{k_{A_1}}(z)\right)^{\gamma} P_{Fk_{A_1}}(z) \end{pmatrix} \tag{9}$$

$$A_1^{\gamma} = \begin{pmatrix} \left(T_{k_{A_1}}(z)\right)^{\gamma} P_{Tk_{A_1}}(z), \\ 1 - \left(1 - I_{k_{A_1}}(z)\right)^{\gamma} P_{Ik_{A_1}}(z), \\ 1 - \left(1 - F_{k_{A_1}}(z)\right)^{\gamma} P_{Fk_{A_1}}(z) \end{pmatrix} \tag{10}$$

Definition 3

We can compute the score function such as:

$$S(A) = \frac{TP_T + 1 - FP_I + 1 - FP_F}{3} \tag{11}$$

3. MCDM framework

The framework known as MCDM was created by combining several disciplines to help decision-makers by evaluating the issue they are trying to address from several angles. Another definition of MCDM is a procedure designed to manage decision-making in a transparent and consistent manner. For this procedure, more than 50 MCDM approaches have been proposed by researchers in literature, and the number of these methods is growing every day. It is clear from evaluating the suggested MCDM approaches for different choice issues that each approach has advantages and disadvantages; yet the literature analysis has shown that the MCDM approaches are not intrinsically better.

Determining the decision criteria, taking expert viewpoints into account, and performing a literature study are all part of the integrated method's initial step. The seven factors considered when evaluating the Multilingual Higher Education Internationalization are shown in Table 1. The criteria fall into one of two categories: cost or benefit. The benefit criterion is the one that is chosen to have greater values, while the cost criterion is the one that is favored to have lower values. Stated differently, the benefit criteria describe the circumstance in which greater values are more effective in reaching the goal.

Table 1. Criteria and alternatives considered in the evaluation of the MCDM issue.

Alternatives	Criteria
AI-Powered Translation Joint Degree Programs Cultural Integration Programs International Branch Campuses Open Education Resources Multilingual Degree Programs Faculty Exchange Programs Content and Language Integrated Learning Global Research Collaborations Hybrid Learning Platforms	Faculty Language Proficiency Technology-Enhanced Multilingual Learning International Student Support Multilingual Curriculum Design Global Research Collaboration Graduate Employability Cross-Cultural Communication

ITARA method

Hatefi devised the ITARA approach, a semi-objective weighting technique, to determine the criteria's weights[13], [14]. To establish criterion weights based on alternatives, it is predicated on the ideas of dispersion logic and indifference threshold (IT). Compared to the entropy approach, the ITARA method is more stable and simpler to apply. When there are more options, the ITARA approach has been used in literature.

The ITARA technique has been applied in the literature to establish the weights of criteria in several different domains.

Step 1. The decision matrix is built

$$X_{ij} = \begin{pmatrix} x_{11} & \cdots & x_{1j} \\ \vdots & \ddots & \vdots \\ x_{i1} & \cdots & x_{ij} \end{pmatrix}_{m \times n} \tag{12}$$

Where n refers to the number of criteria and m refers to the number of alternatives. The experts can determine the IT.

Step 2. Normalize the decision matrix and normalize IT (NIT).

$$y_{ij} = \frac{x_{ij}}{\sum_{i=1}^{m} x_{ij}} \tag{13}$$

$$NIT_j = \frac{IT_j}{\sum_{i=1}^{m} x_{ij}} \tag{14}$$

Step 3. Rank the normalized values and named them such as: $\delta_{ij}\ so\ that\ \delta_{ij} \leq \delta_{i+1j}$

Step 4. Compute the distance

$$d_{ij} = \delta_{i+1j} + \delta_{ij} \tag{15}$$

Step 5. Compute the distance between the ranked scores

$$y_{ij} = \begin{cases} d_{ij} - NIT_j \quad for\ d_{ij} > NIT_j \\ 0\ for\ d_{ij} \leq NIT_j \end{cases} \tag{16}$$

Step 6. Compute the criteria weights.

$$w_j = \frac{v_j}{\sum_{i=1}^{n} v_j} \tag{17}$$

$$v_j = \left(\sum_{i=1}^{m-1} y_{ij}^{p}\right)^{\frac{1}{p}} \tag{18}$$

Where p refers to the parameter model.

RAM Method

This part shows the steps of the RAM method to rank the alternatives[15], [16].

Step 1. Normalize the decision matrix

$$q_{ij} = \frac{x_{ij}}{\sum_{i=1}^{m} x_{ij}} \tag{19}$$

Step 2. Compute the weighted decision matrix.

$$f_{ij} = w_j q_{ij} \tag{20}$$

Step 3. Compute the sum of weighted decision matrix for positive and negative criteria

$$T_{+i} = \sum_{j=1}^{n} f_{+ij} \tag{21}$$

$$T_{-i} = \sum_{j=1}^{n} f_{-ij} \tag{22}$$

Step 4. Compute the total score

$$L_i = \sqrt[2+T_{-i}]{2 + T_{+i}} \tag{23}$$

Step 5. Rank the alternatives.

4. Case Study

This section shows the results of the proposed approach to compute the criteria weights and rank the alternatives. We invited three experts to evaluate the criteria and alternatives. We gathered seven criteria and ten alternatives to be evaluated. We proposed ten solutions to Multilingual Higher Education Internationalization to select the best one.

Determination of criterion weights by the ITARA method

Eq. (12) is used to build the decision matrix. We used the neutrosophic numbers to evaluate the criteria and alternatives as shown in Tables 2-4. Then we applied the score function to obtain the crisp values. Then we used the average method to combine these values.

Table 2. Expert 1 values.

	C_1	C_2	C_3	C_4	C_5	C_6	C_7
A_1	(0.8(1),0.3(0.8),0.5(0.7))	(0.8(0.9),0.2(0.7),0.3(1))	(0.7(1),0.2(1),0.2(0.7))	(0.5(0.9),0.3(0.9),0.5(0.8))	(0.5(0.9),0.3(0.9),0.5(0.8))	(0.8(1),0.3(0.8),0.5(0.7))	(0.5(0.9),0.3(0.9),0.5(0.8))
A_2	(0.4(0.9),0.2(0.9),0.1(1))	(0.5(0.9),0.3(0.9),0.5(0.8))	(0.8(1),0.3(0.8),0.5(0.7))	(0.8(0.9),0.2(0.7),0.3(1))	(0.7(1),0.2(1),0.2(0.7))	(0.5(0.9),0.3(0.9),0.5(0.8))	(0.7(1),0.2(1),0.2(0.7))
A_3	(0.3(0.7),0.2(0.9),0.3(0.9))	(0.7(1),0.2(1),0.2(0.7))	(0.7(1),0.2(1),0.2(0.7))	(0.5(0.9),0.3(0.9),0.5(0.8))	(0.8(0.9),0.2(0.7),0.3(1))	(0.7(1),0.2(1),0.2(0.7))	(0.8(0.9),0.2(0.7),0.3(1))
A_4	(0.5(0.9),0.3(0.9),0.5(0.8))	(0.8(0.9),0.2(0.7),0.3(1))	(0.5(0.9),0.3(0.9),0.5(0.8))	(0.8(1),0.3(0.8),0.5(0.7))	(0.8(1),0.3(0.8),0.5(0.7))	(0.8(0.9),0.2(0.7),0.3(1))	(0.8(1),0.3(0.8),0.5(0.7))
A_5	(0.7(1),0.2(1),0.2(0.7))	(0.8(1),0.3(0.8),0.5(0.7))	(0.7(1),0.2(1),0.2(0.7))	(0.5(0.9),0.3(0.9),0.5(0.8))	(0.5(0.9),0.3(0.9),0.5(0.8))	(0.8(1),0.3(0.8),0.5(0.7))	(0.4(0.9),0.2(0.9),0.1(1))
A_6	(0.8(0.9),0.2(0.7),0.3(1))	(0.5(0.9),0.3(0.9),0.5(0.8))	(0.8(0.9),0.2(0.7),0.3(1))	(0.7(1),0.2(1),0.2(0.7))	(0.7(1),0.2(1),0.2(0.7))	(0.5(0.9),0.3(0.9),0.5(0.8))	(0.5(0.9),0.3(0.9),0.5(0.8))
A_7	(0.8(1),0.3(0.8),0.5(0.7))	(0.7(1),0.2(1),0.2(0.7))	(0.8(1),0.3(0.8),0.5(0.7))	(0.8(0.9),0.2(0.7),0.3(1))	(0.8(0.9),0.2(0.7),0.3(1))	(0.7(1),0.2(1),0.2(0.7))	(0.7(1),0.2(1),0.2(0.7))
A_8	(0.6,(0.8),0.3(0.9),0.2(0.9))	(0.8(0.9),0.2(0.7),0.3(1))	(0.3(0.7),0.2(0.9),0.3(0.9))	(0.8(1),0.3(0.8),0.5(0.7))	(0.8(1),0.3(0.8),0.5(0.7))	(0.8(0.9),0.2(0.7),0.3(1))	(0.8(0.9),0.2(0.7),0.3(1))
A_9	(0.4(0.9),0.2(0.9),0.1(1))	(0.8(1),0.3(0.8),0.5(0.7))	(0.4(0.9),0.2(0.9),0.1(1))	(0.4(0.9),0.2(0.9),0.1(1))	(0.5(0.9),0.3(0.9),0.5(0.8))	(0.8(1),0.3(0.8),0.5(0.7))	(0.8(1),0.3(0.8),0.5(0.7))
A_{10}	(0.3(0.7),0.2(0.9),0.3(0.9))	(0.5(0.9),0.3(0.9),0.5(0.8))	(0.7(1),0.2(1),0.2(0.7))	(0.8(0.9),0.2(0.7),0.3(1))	(0.8(1),0.3(0.8),0.5(0.7))	(0.6,(0.8),0.3(0.9),0.2(0.9))	(0.5(0.9),0.3(0.9),0.5(0.8))

Table 3. Expert 2 values.

	C_1	C_2	C_3	C_4	C_5	C_6	C_7
A_1	(0.6,(0.8),0.3(0.9),0.2(0.9))	(0.8(0.9),0.2(0.7),0.3(1))	(0.7(1),0.2(1),0.2(0.7))	(0.5(0.9),0.3(0.9),0.5(0.8))	(0.6,(0.8),0.3(0.9),0.2(0.9))	(0.8(1),0.3(0.8),0.5(0.7))	(0.8(0.9),0.2(0.7),0.3(1))
A_2	(0.8(1),0.3(0.8),0.5(0.7))	(0.6,(0.8),0.3(0.9),0.2(0.9))	(0.8(1),0.3(0.8),0.5(0.7))	(0.8(0.9),0.2(0.7),0.3(1))	(0.7(1),0.2(1),0.2(0.7))	(0.5(0.9),0.3(0.9),0.5(0.8))	(0.6,(0.8),0.3(0.9),0.2(0.9))
A_3	(0.8(0.9),0.2(0.7),0.3(1))	(0.6,(0.8),0.3(0.9),0.2(0.9))	(0.7(1),0.2(1),0.2(0.7))	(0.5(0.9),0.3(0.9),0.5(0.8))	(0.3(0.7),0.2(0.9),0.3(0.9))	(0.6,(0.8),0.3(0.9),0.2(0.9))	(0.8(1),0.3(0.8),0.5(0.7))
A_4	(0.7(1),0.2(1),0.2(0.7))	(0.4(0.9),0.2(0.9),0.1(1))	(0.8(0.9),0.2(0.7),0.3(1))	(0.8(1),0.3(0.8),0.5(0.7))	(0.6,(0.8),0.3(0.9),0.2(0.9))	(0.8(1),0.3(0.8),0.5(0.7))	(0.8(0.9),0.2(0.7),0.3(1))
A_5	(0.5(0.9),0.3(0.9),0.5(0.8))	(0.6,(0.8),0.3(0.9),0.2(0.9))	(0.5(0.9),0.3(0.9),0.5(0.8))	(0.6,(0.8),0.3(0.9),0.2(0.9))	(0.8(1),0.3(0.8),0.5(0.7))	(0.8(0.9),0.2(0.7),0.3(1))	(0.7(1),0.2(1),0.2(0.7))
A_6	(0.8(0.9),0.2(0.7),0.3(1))	(0.8(1),0.3(0.8),0.5(0.7))	(0.6,(0.8),0.3(0.9),0.2(0.9))	(0.8(1),0.3(0.8),0.5(0.7))	(0.8(0.9),0.2(0.7),0.3(1))	(0.7(1),0.2(1),0.2(0.7))	(0.5(0.9),0.3(0.9),0.5(0.8))
A_7	(0.8(1),0.3(0.8),0.5(0.7))	(0.8(0.9),0.2(0.7),0.3(1))	(0.8(1),0.3(0.8),0.5(0.7))	(0.8(0.9),0.2(0.7),0.3(1))	(0.7(1),0.2(1),0.2(0.7))	(0.5(0.9),0.3(0.9),0.5(0.8))	(0.8(1),0.3(0.8),0.5(0.7))
A_8	(0.6,(0.8),0.3(0.9),0.2(0.9))	(0.7(1),0.2(1),0.2(0.7))	(0.8(0.9),0.2(0.7),0.3(1))	(0.7(1),0.2(1),0.2(0.7))	(0.5(0.9),0.3(0.9),0.5(0.8))	(0.7(1),0.2(1),0.2(0.7))	(0.8(0.9),0.2(0.7),0.3(1))
A_9	(0.4(0.9),0.2(0.9),0.1(1))	(0.5(0.9),0.3(0.9),0.5(0.8))	(0.7(1),0.2(1),0.2(0.7))	(0.5(0.9),0.3(0.9),0.5(0.8))	(0.5(0.9),0.3(0.9),0.5(0.8))	(0.5(0.9),0.3(0.9),0.5(0.8))	(0.7(1),0.2(1),0.2(0.7))

A_{10}	(0.3(0.7),0.2(0.9),0.3(0.9))	(0.5(0.9),0.3(0.9),0.5(0.8))	(0.5(0.9),0.3(0.9),0.5(0.8))	(0.8(0.9),0.2(0.7),0.3(1))	(0.8(1),0.3(0.8),0.5(0.7))	(0.6,(0.8),0.3(0.9),0.2(0.9))	(0.5(0.9),0.3(0.9),0.5(0.8))

Table 4. Expert 3 values.

	C_1	C_2	C_3	C_4	C_5	C_6	C_7
A_1	(0.8(1),0.3(0.8),0.5(0.7))	(0.8(0.9),0.2(0.7),0.3(1))	(0.7(1),0.2(1),0.2(0.7))	(0.5(0.9),0.3(0.9),0.5(0.8))	(0.3(0.7),0.2(0.9),0.3(0.9))	(0.8(1),0.3(0.8),0.5(0.7))	(0.8(0.9),0.2(0.7),0.3(1))
A_2	(0.4(0.9),0.2(0.9),0.1(1))	(0.6,(0.8),0.3(0.9),0.2(0.9))	(0.8(1),0.3(0.8),0.5(0.7))	(0.8(0.9),0.2(0.7),0.3(1))	(0.7(1),0.2(1),0.2(0.7))	(0.5(0.9),0.3(0.9),0.5(0.8))	(0.5(0.9),0.3(0.9),0.5(0.8))
A_3	(0.3(0.7),0.2(0.9),0.3(0.9))	(0.6,(0.8),0.3(0.9),0.2(0.9))	(0.7(1),0.2(1),0.2(0.7))	(0.5(0.9),0.3(0.9),0.5(0.8))	(0.3(0.7),0.2(0.9),0.3(0.9))	(0.3(0.7),0.2(0.9),0.3(0.9))	(0.3(0.7),0.2(0.9),0.3(0.9))
A_4	(0.5(0.9),0.3(0.9),0.5(0.8))	(0.4(0.9),0.2(0.9),0.1(1))	(0.8(0.9),0.2(0.7),0.3(1))	(0.8(1),0.3(0.8),0.5(0.7))	(0.6,(0.8),0.3(0.9),0.2(0.9))	(0.4(0.9),0.2(0.9),0.1(1))	(0.5(0.9),0.3(0.9),0.5(0.8))
A_5	(0.7(1),0.2(1),0.2(0.7))	(0.3(0.7),0.2(0.9),0.3(0.9))	(0.5(0.9),0.3(0.9),0.5(0.8))	(0.6,(0.8),0.3(0.9),0.2(0.9))	(0.8(1),0.3(0.8),0.5(0.7))	(0.6,(0.8),0.3(0.9),0.2(0.9))	(0.4(0.9),0.2(0.9),0.1(1))
A_6	(0.8(0.9),0.2(0.7),0.3(1))	(0.5(0.9),0.3(0.9),0.5(0.8))	(0.7(1),0.2(1),0.2(0.7))	(0.4(0.9),0.2(0.9),0.1(1))	(0.8(1),0.3(0.8),0.5(0.7))	(0.8(1),0.3(0.8),0.5(0.7))	(0.6,(0.8),0.3(0.9),0.2(0.9))
A_7	(0.8(1),0.3(0.8),0.5(0.7))	(0.7(1),0.2(1),0.2(0.7))	(0.5(0.9),0.3(0.9),0.5(0.8))	(0.3(0.7),0.2(0.9),0.3(0.9))	(0.8(0.9),0.2(0.7),0.3(1))	(0.8(0.9),0.2(0.7),0.3(1))	(0.8(1),0.3(0.8),0.5(0.7))
A_8	(0.6,(0.8),0.3(0.9),0.2(0.9))	(0.8(0.9),0.2(0.7),0.3(1))	(0.3(0.7),0.2(0.9),0.3(0.9))	(0.5(0.9),0.3(0.9),0.5(0.8))	(0.7(1),0.2(1),0.2(0.7))	(0.7(1),0.2(1),0.2(0.7))	(0.8(0.9),0.2(0.7),0.3(1))
A_9	(0.4(0.9),0.2(0.9),0.1(1))	(0.8(1),0.3(0.8),0.5(0.7))	(0.4(0.9),0.2(0.9),0.1(1))	(0.4(0.9),0.2(0.9),0.1(1))	(0.5(0.9),0.3(0.9),0.5(0.8))	(0.5(0.9),0.3(0.9),0.5(0.8))	(0.7(1),0.2(1),0.2(0.7))
A_{10}	(0.3(0.7),0.2(0.9),0.3(0.9))	(0.5(0.9),0.3(0.9),0.5(0.8))	(0.7(1),0.2(1),0.2(0.7))	(0.8(0.9),0.2(0.7),0.3(1))	(0.8(1),0.3(0.8),0.5(0.7))	(0.6,(0.8),0.3(0.9),0.2(0.9))	(0.5(0.9),0.3(0.9),0.5(0.8))

Eq. (13) is used to normalize the decision matrix between the criteria and alternatives as shown in Table 5. Then we ranked the normalization values. Then we compute the distance using the Eq. (15). Then we compute the distance between the ranked scores. Then we compute the criteria weights using Eq. (17). Then we ranked these criteria as shown in fig 1.

Table 5. The normalization matrix by the ITARA method.

	C_1	C_2	C_3	C_4	C_5	C_6	C_7
A_1	0.103831294	0.108969	0.109361	0.086004	0.088399	0.105506	0.100587
A_2	0.102543464	0.093038	0.10241	0.110163	0.112363	0.084978	0.09789
A_3	0.093367675	0.102278	0.109361	0.086004	0.092049	0.097868	0.099159
A_4	0.09529942	0.102597	0.09793	0.10678	0.099508	0.104551	0.099476
A_5	0.104636188	0.095587	0.091443	0.094057	0.098397	0.103756	0.103443
A_6	0.110109466	0.091923	0.103027	0.107103	0.108713	0.10105	0.088688
A_7	0.106728912	0.111518	0.095768	0.101788	0.109824	0.102164	0.107568
A_8	0.098036059	0.110244	0.089589	0.102271	0.100778	0.111394	0.10852
A_9	0.10045074	0.098773	0.100711	0.095668	0.084748	0.09182	0.109948
A_{10}	0.08499678	0.085072	0.100402	0.110163	0.105221	0.096913	0.084722

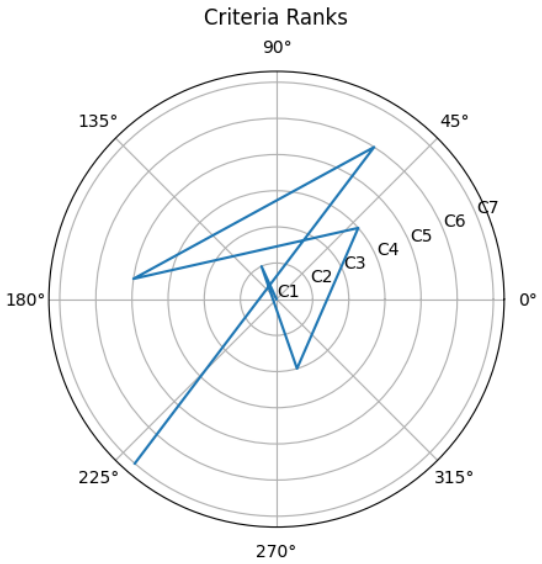

Fig 1. The rank of criteria.

Evaluation of alternatives with the RAM method

We used the criteria weights to rank the alternatives. Eq. (19) is used to normalize the decision matrix. Then we used Eq. (20) to compute the weighted decision matrix as shown in Table 6.

Then we compute the sum of weighted decision matrix for positive and negative criteria using Eqs. (21 and 22). Then we compute the total score using Eq. (23) as shown in Fig 2. Then we ranked the alternatives as shown in fig 3.

Table 6. The weighted normalization matrix.

	C_1	C_2	C_3	C_4	C_5	C_6	C_7
A_1	0.029233	0.003468	0.022763	0.030575	0.002964	0.00253	0.006585
A_2	0.02887	0.002961	0.021316	0.039164	0.003767	0.002037	0.006409
A_3	0.026287	0.003255	0.022763	0.030575	0.003086	0.002347	0.006492
A_4	0.026831	0.003265	0.020384	0.037961	0.003336	0.002507	0.006513
A_5	0.029459	0.003042	0.019034	0.033438	0.003299	0.002488	0.006772
A_6	0.031	0.002926	0.021445	0.038076	0.003645	0.002423	0.005806
A_7	0.030048	0.003549	0.019934	0.036186	0.003682	0.00245	0.007042
A_8	0.027601	0.003509	0.018648	0.036358	0.003379	0.002671	0.007105
A_9	0.028281	0.003144	0.020963	0.034011	0.002842	0.002202	0.007198
A_{10}	0.02393	0.002708	0.020898	0.039164	0.003528	0.002324	0.005547

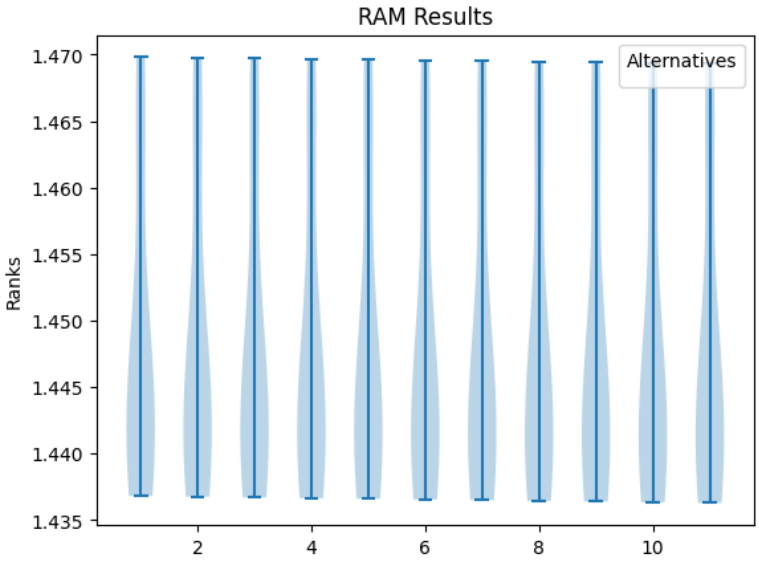

Fig 2. The total score by the RAM method.

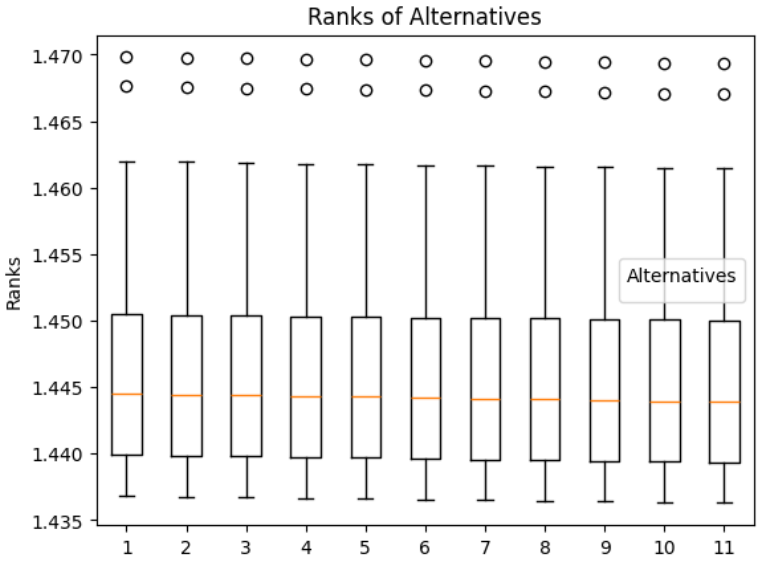

Fig 3. The rank of the alternatives.

Validation of the results

We validated the results of this proposed approach by conducting sensitivity analysis to show the different ranks of alternatives by different criteria weights. We proposed eight cases in the criteria weights. Fig 4. Show the different criteria weights. In each case we increase the criteria weights by the 28% weights and all other criteria have the same weights.

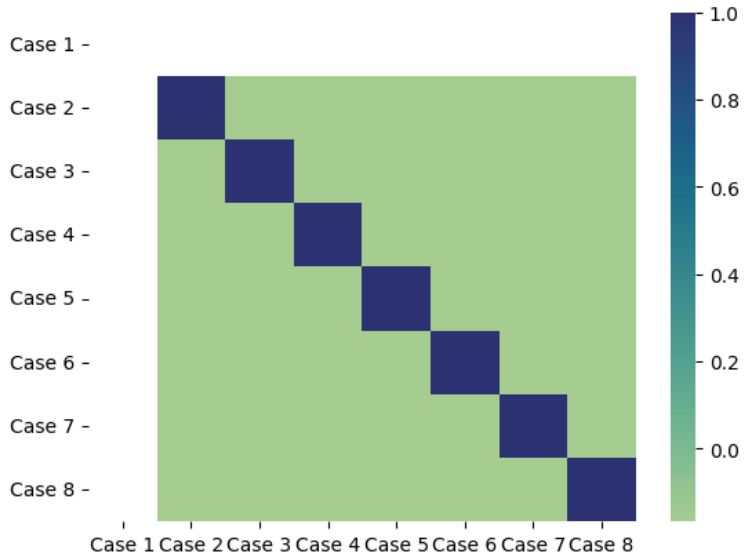

Fig 4. Different criteria weights.

Then we applied the RAM method under different criteria weights. In all cases we compute the rank of alternatives and compute the total score. The total scores of each alternatives with different criteria weights are shown in Fig 5. Then we ranked the alternatives under these criteria weights as shown in Fig 6. The results show the rank of alternatives are stable under different cases.

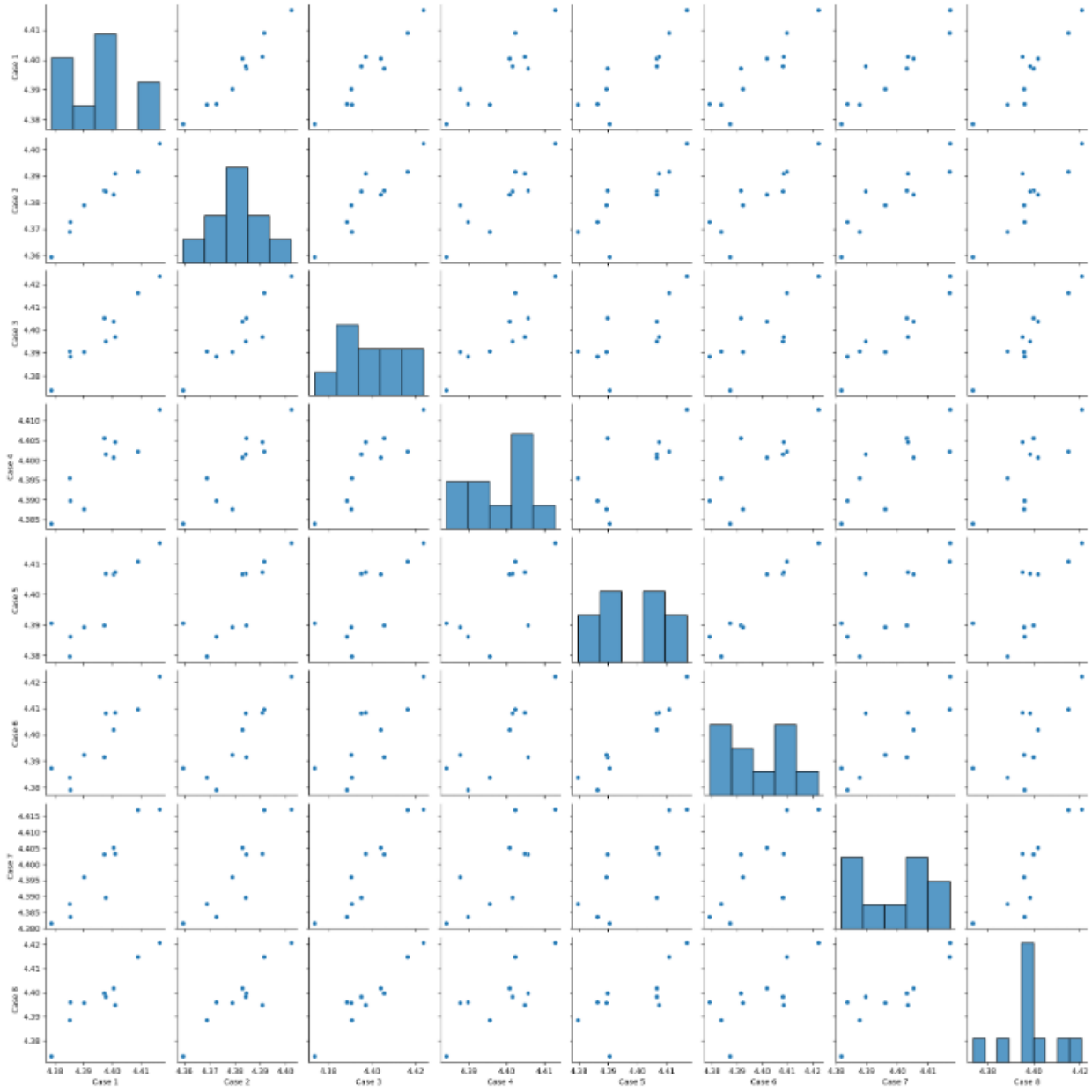

Fig 5. Different total scores of each alternative.

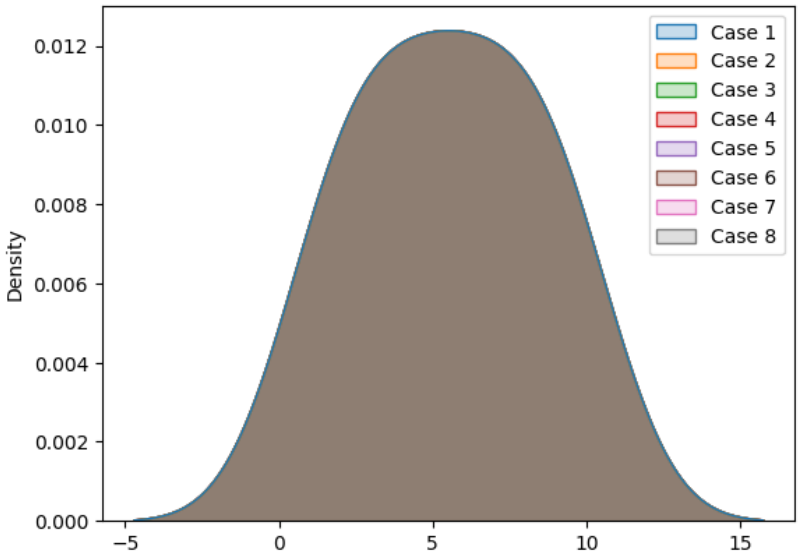

Fig 6. Different ranks of each alternative.

5. Conclusions

This study proposed the MCDM approach to compute the criteria weights and rank the alternatives of multilingual higher education internationalization. We used two methods, ITARA to compute the criteria weights and the RAM methodology to rank the alternatives. These two methods are used under the Probabilistic Simplified Neutrosophic sets to deal with uncertainty and vague information. We used seven criteria and ten alternatives to be evaluated in this study. Three experts evaluate these criteria and alternatives. The sensitivity analysis was conducted with eight cases to show the rank of alternatives. The results show the rank of alternatives is stable in different cases.

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A Comparison between Euler's Method and 4-th Order Runge-Kutta Method For Numerical Solutions of Neutrosophic and Dual Differential Problems

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Abstract:

This paper is dedicated to study some numerical solutions of neutrosophic and dual differential problems through a numerical comparison between neutrosophic Euler's method and 4-th order Runge-Kutta neutrosophic and dual method, where we apply both methods on the same problems and compare the numerical results of solutions and numerical estimations of errors by using numerical tables.

Keywords: Numerical solution, neutrosophic problem, neutrosophic error, numerical estimation

1. Introduction

Numerical analysis is a basic branch of mathematics and is primarily concerned with finding approximate values and approximate solutions to algebraic or differential equations, for which solutions are difficult to find using traditional methods [1-6]. Numerical analysis is closely related to algorithms and computer science [7-10]. The concept of neutrosophic logic and the neutrosophic set was developed to deal with the presence of indeterminacy in scientific data and theses, and sometimes to deal with the lack or conflict of data related to a certain research question [11-15]. In many research papers published recently [16-20], numerical solutions have been studied and found for many equations that contain an element of indeterminacy (neutrosophic), where the effectiveness of many methods in studying these solutions has been compared and classified into independent tables called numerical tables

[21-25]. In this research, we are interested in following in the footsteps of researchers in works concerned with neutrosophic numerical analysis [26-33], where we study some numerical solutions of neutrosophic and dual differential problems through a numerical comparison between neutrosophic Euler's method and 4-th order Runge-Kutta neutrosophic and dual method, where we apply both methods on the same problems and compare the numerical results of solutions and numerical estimations of errors by using numerical tables.

2. Main Discussion

Definition 2.1 Consider the following (IVP) of neutrosophic variables:

$$\begin{cases} (y_1 + y_2 I)' = \dfrac{d(y_1 + y_2 I)^{\emptyset}}{d(x_1 + x_2 I)} = f(x_1 + x_2 I, y_1 + y_2 I) \\ (y_1 + y_2 I)(x_0 + x_0' I) = y_0 + y_0' I \quad ; x_0, x_0', y_0, y_0' \in \mathbb{R}. \end{cases},$$

The method can be explained by the following equations:

$$(y_1 + y_2 I)(x_1 + x_2 I) = (y_n + y_n' I) + f(x_0 + x_0' I, y_0 + y_0' I)(x_0 + x_0' I,$$

when $x_1 + x_2 I = x_1' + x_2' I$, we have:

$$y_1 + y_1' I = y_0 + y_0' I + (h_1 + h_2 I) f(x_0 + x_0' I, y_0 + y_0' I),$$

The next approximations are:

$y_2 + y_2' I = y_1 + y_1' I + (h_1 + h_2 I) f(x_1 + x_1' I, y_1 + y_1' I)$, at the point $x_1 + x_2 I = x_2 + x_2' I$,

$y_3 + y_3' I = y_2 + y_2' I + (h_1 + h_2 I) f(x_2 + x_2' I, y_2 + y_2' I)$, and $y_{n+1} + y_{n+1}' I = y_n + y_n' I + (h_1 + h_2 I) f(x_n + x_n' I, y_n + y_n' I)$ for all $n = 0,1,2, \ldots..$

Definition 2.2 Consider the following (IVP) of neutrosophic variables:

$$(y_1 + y_2 I)' = f(x_1 + x_2 I, y_1 + y_2 I), (y_1 + y_2 I)(x_0 + x_0' I) = y_0 + y_0' I,$$

Let

$$h = (x_1 + x_1'I) - (x_0 + x_0'I) = (x_1 - x_0) + I(x_1' - x_0'),$$

then,

$$(y_1 + y_2I)(x + x'I + h + h'I)$$
$$= (y_1 + y_2I)(x + x'I) + (h + h'I)(y_1 + y_2I)'(x + x'I)$$
$$+ \frac{(h + h'I)^2}{2!}(y_1 + y_2I)^{"}(x + x'I) + \ldots.$$

Hence,

$$(y_1 + y_2I)(x + x'I + h + h'I) - (y_1 + y_2I)(x + x'I)$$
$$= (h + h'I)(y_1 + y_2I)'(x + x'I) + \frac{(h + h'I)^2}{2!}(y_1 + y_2I)^{"}(x + x'I) + \ldots.$$

By partial differenation, we get:

$$y' = (y_1 + y_2I)' = f(x_1 + x_2^*I, y_1 + y_2I) = f,$$

$$y^{"} = f_{x_1 + x_2I} + f.f_{y_1 + y_2I},$$

$$y''' = f_{(x_1 + x_2I)(x_1 + x_2I)} + 2ff_{(x_1 + x_2I)(y_1 + y_2I)} + f^2 f_{(y_1 + y_2I)(y_1 + y_2I)} + f_{(x_1 + x_2I)(y_1 + y_2I)} + ff_{y_1 + y_2I}^2,$$

Take:

$$F_1 + F_1'I = f_{x_1 + x_2I} + f.f_{y_1 + y_2I},$$

$$F_2 + F_2'I = f_{(x_1 + x_2I)(x_1 + x_2I)} + 2ff_{(x_1 + x_2I)(y_1 + y_2I)} + f^2 f_{(x_1 + x_2I)(y_1 + y_2I)},$$

$$F_3 + F_3'I = f_{(x_1 + x_2I)(x_1 + x_2I)(x_1 + x_2I)} + 3ff_{(x_1 + x_2I)(x_1 + x_2I)(y_1 + y_2I)} + f^2 f_{(y_1 + y_2I)(y_1 + y_2I)(y_1 + y_2I)} +$$
$$f^2 f_{(y_1 + y_2I)(y_1 + y_2I)(y_1 + y_2I)},$$

hence,

$$\begin{cases} F_1 + F_1'I = y^{"} \\ F_2 + F_2I + f_{(y_1 + y_2I)}(F_1 + F_1I) = y''' \\ (F_3 + F_3I) + f_{(y_1 + y_2I)}(F_2 + F_2I) + 3(F_1 + F_1I)(f_{(x_1 + x_2I)(y_1 + y_2I)} + f.f_{(y_1 + y_2I)(y_1 + y_2I)}) \end{cases},$$

$$(y_1 + y_2I)(x + x'I + h + h'I) - (y_1 + y_2I)(x + x'I) = (h_1 + h_2I)(f) + \frac{(h_1 + h_2I)^2}{2!}(F_1 +$$

$$F_1I) + \ldots.$$

$$K_1 + K_1'I = (h + h'I)f,$$

$$K_2 + K_2'I = (h + h'I)f\big(x + x'I + (m + m'I)(h + h'I), y + y'I + (m + m'I)(K_1 + K_1'I)\big),$$

$$K_3 + K_3'I = (h + h'I)f\big(x + x'I + (n + n'I)(h + h'I), y + y'I + (n + n'I)(K_2 + K_2'I)\big),$$

$$K_4 + K_4'I = (h + h'I)f(x + x'I + (p + p'I)(h + h'I), y + y'I + (p + p'I)(K_3 + K_3'I)),$$

$\Delta_{y_1+y_2I}= (y_1 + y_2I)(x + x'I, y_1 + y_2I) - (y_1 + y_2I)(x_1 + x_2I) = a(K_1 + K_1'I) + b(K_2 + K_2'I) + c(K_3 + K_3'I) + d(K_4 + K_4'I),$

Now, we can use Taylor's expansion for several variables:

$$K_1 + K_1'I = (h + h'I)f,$$

$K_2 + K_2'I = (h + h'I) + \Big[f + (m + m'I)(h + h'I)(F_1 + F_1'I) + \frac{1}{2}(m + m'I)^2(h + h'I)^2(F_2 + F_2'I) + \ldots..\Big],$

$K_3 + K_3'I = (h + h'I) + \Big[f + (n + n'I)(h + h'I)(F_1 + F_1'I) + \frac{1}{2}(h + h'I)^2\Big((n + n'I)^2(F_2 + F_2'I) + 2(m + m'I)(n + n'I)f_y(F_1 + F_1I)\Big) + \frac{1}{6}(h + h'I)^3\big((n + n'I)^3(F_3 + F_3'I) + 3(m + m'I)^2(n + n'I)f_yF_2I\big) + 6\,(m + m'I)(n + n'I)^2(F_1 + F_1I)\big(F_{xIyI} + ff_{yy}\big) + \ldots..\Big],$

$K_4 + K_4'I = (h + h'I) + \Big[f + (p + p'I)(h + h'I)(F_1 + F_1'I) + \frac{1}{2}(h + h'I)^2\big[(p + p'I)^2(F_2 + F_2'I) + 2(n + n'I)(p + p'I)f_y(F_1 + F_1'I)\big] + \frac{1}{6}(h + h'I)^3\big[(p + p'I)^3(F_3 + F_3'I) + 3(n + n'I)^2(p + p'I)f_y(F_2 + F_2'I)\big] + 6\,(n + n'I)(p + p'I)(F_1 + F_1I)\big(F_{xIyI} + ff_{y^2I} + \ldots..\big)\ldots.\Big],$

hence,

$-(y_1 + y_2I)(x + x'I) + (y_1 + y_2I)\big(x + h + I(x' + h')\big) = a(h + h'I)f + b(h + h'I)[f + (m + m'I)(h + h'I)(F_1 + F_1I) + \ldots..] + c(h + h'I)\Big[f + (n + n'I)(h + h'I)(F_1 + F_1'I) + \frac{1}{2}(h + h'I)^2(n + n'I)^2F_2I\Big] + \frac{1}{6}(h + h'I)^3[(p + p'I)^3(F_3 + F_3'I) + 3(n + n'I)^2(p + p'I)F_2I + \ldots..],$

thus,

$$\begin{aligned}(y_1 + y_2 I)\big(x + h + I(x' + h')\big) - (y_1 + y_2 I)(x + x' I) \\ = (a + b + c + d)(h + h' I) f \\ + [b(m + m' I) + c(n + n' I) + d(p + p' I)](h + h' I)^2 (F_2 + F_2' I) \\ + [b(m + m' I)^2 + c(n + n' I)^2 + d(p + p' I)^2] \frac{(h + h' I)^3 (F_2 I)}{2} \\ + [b(m + m' I)^3 + c(n + n' I)^3 + d(p + p' I)^3] \frac{(h + h' I)^4 (F_2 I)}{6} + \ldots\ldots,\end{aligned}$$

hence,

$$\begin{cases} a + b + c + d = 1 \\ b(m + m' I) + c(n + n' I) + d(p + p' I) = \frac{1}{2} + I \\ b(m + m' I)^2 + c(n + n' I)^2 + d(p + p' I)^2 = \frac{1}{3} + I \\ b(m + m' I)^3 + c(n + n' I)^3 + d(p + p' I)^3 = \frac{1}{4} + I \end{cases},$$

and

$$\begin{cases} c(m + m' I)(n + n' I) + d(n + n' I)(p + p' I) = \frac{1}{6} + I \\ c(m + m' I)^2 (n + n' I) + d(n + n' I)^2 (p + p' I) = \frac{1}{12} + I \\ c(m + m' I)(n + n' I)^2 + d(n + n' I)(p + p' I)^2 = \frac{1}{8} + I \\ d(m + m' I)(n + n' I)(p + p' I) = \frac{1}{24} \end{cases},$$

so that:

$$\begin{cases} m = \frac{1}{2} \\ n = \frac{1}{2} \\ m' = n' = 1 \end{cases}, \begin{cases} p = 1 \\ p' = 1 \end{cases}, \begin{cases} a = d = \frac{1}{6} \\ b = c = \frac{1}{3} \end{cases}.$$

thus,

$$K_1 + K_1' I = (h + h' I) f(x_0 + x_0' I, y_0 + y_0' I) = df.$$

$$K_2 + K_2'I = (h + h'I)f\left(x_0 + x_0'I + \frac{h+h'I}{2}, y_0 + y_0'I + \frac{K_1+K_1'I}{2}\right),$$

$$K_3 + K_3'I = (h + h'I)f\left(x_0 + x_0'I + \frac{h+h'I}{2}, y_0 + y_0'I + \frac{K_2+K_2'I}{2}\right),$$

$$K_4 + K_4'I = (h + h'I)f(x_0 + x_0'I + h + h'I, y_0 + y_0'I + K_3 + K_3'I)$$

$$\Delta_{y_1+y_2I} = \frac{1}{6}[(K_1 + 2K_2 + 2K_3 + K_4) + I(K_1' + 2K_2' + 2K_3' + K_4')],$$

$$y_1 + y_1'I = y_0 + y_0'I + \frac{1}{6}[(K_1 + 2K_2 + 2K_3 + K_4) + I(K_1' + 2K_2' + 2K_3' + K_4')],$$

this implies that the neutrosophic 4-th order Runge- Kutta formula is:

$$K_1 + K_1'I = (h + h'I)f(x_n + x_n'I, y_n + y_n'I) = df,$$

$$K_2 + K_2'I = (h + h'I)f\left(x_n + x_n'I + \frac{h+h'I}{2}, y_n + y_n'I + \frac{K_1+K_1'I}{2}\right),$$

$$K_3 + K_3'I = (h + h'I)f\left(x_n + x_n'I + \frac{h+h'I}{2}, y_n + y_n'I + \frac{K_2+K_2'I}{2}\right),$$

$$K_4 + K_4'I = (h + h'I)f(x_n + x_n'I + h + h'I, y_n + y_n'I + K_3 + K_3'I),$$

$$y_{n+1} + y_{n+1}'I = y_n + y_n'I + \frac{1}{6}(K_1 + 2K_2 + 2K_3 + K_4) + \frac{1}{6}I(K_1' + 2K_2' + 2K_3' + K_4'),$$

Numerical comparison:

Consider the following neutrosophic problem:

$$(x_1 + x_2I)(y_1 + y_2I)'' - 2(y_1 + y_2I)' = (10 + 10I)(x_1 + x_2I)^4,$$

$$\begin{cases} (y_1 + y_2I)(1 + I) = 2 + I \\ (y_1 + y_2I)'(1 + I) = 2 + I \\ h = 0.1 + 0.1I \\ 0 \le x_1 \le 2, 0 \le x_2 \le 1 \end{cases},$$

hence,

$$\begin{cases} y_1' + y_2'I = z_1 + z_2I \\ z_1' + z_2'I = (10 + I)(x_1 + x_2I)^3 + \dfrac{2}{x_1 + x_2I}(z_1 + z_2I), \end{cases}$$

Now, consider the following problem:

$$(y_1 + y_2 I)'' - 2(y_1 + y_2 I)' = 4(x_1 + x_2 I),$$

with

$$\begin{cases} (y_1 + y_2 I)(0) = 1 + I \\ (y_1 + y_2 I)'(0) = 2 + I \\ 0 \le x_1 \le 1 \\ 0 \le x_2 \le 1 \end{cases}.$$

hence,

$$\begin{cases} (y_1 + y_2 I)'' = z_1 + z_2 I \\ (z_1' + z_2' I) = (4 + I)(x_1 + x_2 I) + (2 + I)(z_1 + z_2 I), \end{cases}$$

with

$$\begin{cases} (y_1 + y_2 I)(0) = 1 + I \\ (z_1 + z_2 I)(0) = 2 + I \end{cases}.$$

Numerical Result:

Example 2.1 Consider the neutrosophic differential equation

$$(x_1 + x_2 I)(y_1 + y_2 I)'' - (2 + I)(y_1 + y_2 I)' = (10 + I)(x_1 + x_2 I)^4,$$

with initial conditions

$$(y_1 + y_2 I)(1 + I) = 2 + I,$$

and

$$(y_1 + y_2 I)'^{(1+I)} = 2 + I,$$

over the interval $0 \le x_1 + x_2 I \le 2$ with step size $h = 0.1 + I$.

$$\begin{cases} (y_1 + y_2 I)' = Z_1 + Z_2 I \\ (Z_1 + Z_2 I)' = 10(x_1 + x_2 I)^3 + \dfrac{2}{x_1 + x_2 I}(Z_1 + Z_2 I)' \end{cases}$$

with initial conditions $y(1) = 2$ and $z(2) = 2$.

The exact solutions are:

$$y_1 + y_2 I = (x_1 + x_2 I)^5 - (x_1 + x_2 I)^3 + 2, \qquad Z_1 + Z_2 I = 5(x_1 + x_2 I)^5 - 3(x_1 + x_2 I)^2.$$

Example 2.2 Consider the neutrosophic differential equation

$$(y_1 + y_2 I)'' - 2(y_1 + y_2 I)' = 4(x_1 + x_2 I),$$

with initial conditions

$$(y_1 + y_2 I)(I) = 1 + I,$$

and

$$(y_1 + y_2 I)'(I) = 2 + I,$$

over the interval $0 \leq x_1 + x_2 I \leq 1$ with step size.

We first convert the second-order ODE into a system of first-order ODEs:

$$\begin{cases} (y_1 + y_2 I)' = Z_1 + Z_2 I \\ z' = 4(x_1 + x_2 I) + 2(Z_1 + Z_2 I)' \end{cases}$$

with initial conditions

$$(y_1 + y_2 I)(I) = 1 + I,$$

and

$$(Z_1 + Z_2 I)(I) = 2 + I.$$

The exact solutions are:

$$y_1 + y_2 I = -((x_1 + x_2 I)^2 + x_1 + x_2 I) + 1.5 \exp(2(x_1 + x_2 I)) - 0.5,$$

$$Z_1 + Z_2 I = -(2(x_1 + x_2 I) + 1) + 3 \exp(2(x_1 + x_2 I)).$$

Table 1. Approximation solution of 1 by using (Euler), ABM (RK) and with step size $h = 0.1 + I$.

$x_1 + x_2 I$	Exact	Euler method	Runge-Kutta	Error (Euler)	Error(Runge-Kutta)
1+I	2+I	2+I	2+I	I	I
1.1+I	1.3998 +I	1.36918 +I	1.3568 +I	0.0445+I	0.0445+I
1.2+I	2.5578+I	2.544168+I	2.54418+I	0.12441032+I	0.1244132+I
1.3+I	3.31006+I	3.324416+I	3.3144176+I	0.331044182+I	0.344101182+I
1.4+I	4.23115+I	4.23385+I	4.23235+I	0.76642161+I	0.76642161+I
1.5+I	6.3095+I	6.378890115+I	6.3889015+I	0.9482188901+I	0.9482889011+I
1.6+I	8.441002+I	8.429889012+I	8.488901802+I	1.18813788901+I	1.18890113780+I
1.7+I	9.446+I	9.48890146+I	9.455427+I	1.36687140+I	1.36687140+I
1.8+I	16.88901+I	16.81889011901+I	16.888901701+I	1.88901715365+I	1.68890115365+I
1.9+I	17.00123+I	17.0016413+I	17.057913+I	2.096091486+I	2.096091486+I
2 +I	22.332102+I	22.3498102+I	22.31102+I	2.443149+I	2.34419+I

Table 2. Approximation solution of 1 using (Euler), ABM (RK) and the error estimation for step size $h = 0.1 + I$.

x	Exact	Euler method	Runge-Kutta	Error Euler	ErrorRunge-Kutta
1+I	2+I	2+I	2+I	I	I
1.1+I	3.4439+I	3.4+I	3.699355+I	0.22345+I	0.00003935+I
1.2+I	6.02214+I	5.34911123182+I	6.0471123187+I	0.31123118+I	0.00011231413+I
1.3+I	9.11231+I	7. 88971212+I	9.016711231594+I	1.241123178788+I	0.0011231406+I
1.4+I	13.2593+I	11. 83823047	11.32788106+I	1.448275953+I	0.00015623+I
1.5+I	18.188916+I	16+I.838438571+I	16.5643821+I	1.6543829+I	0.0043896+I
1.6+I	23.044380+I	23. 88979212+I	23. 0167723+I	1.88920788+I	0.00020369+I
1.7+I	30.113405+I	30.95743892+I	30.0943881+I	2.13438508+I	0.0043895+I

1.8+I	41.445+I	40.37687609+I	41.7935046+I	2.382752391+I	0.00025780+I
1.9+I	52.3987+I	51. 8893481960+I	52.3948192+I	2.684819440+I	0.0481924+I
2+I	66+I	65.04838248+I	67.90167747+I	2.95827517+I	0.00031827+I

Table 3. Approximation solution of 2 using ABM (Euler), ABM (RK) and the error estimation for step size $h = 0.05 + I$.

x	**Exact**	**Euler**	**Runge-Kutta**	**Error Euler**	**ErrorRunge-Kutta**
1+I	2+I	2+I	2+I	I	I
1.1+I	2.25541+I	2.2481997+I	2.5691697+I	0.00003935+I	0.22481945+I
1.2+I	2.76032+I	2.7220804+I	2.7669162208061+I	0.00008413+I	0.34819918+I
1.3+I	3.767593+I	3.5154819+I	3.53691652895+I	0.00013406+I	1.248198788+I
1.4+I	4.332424+I	4.11084819+I	4. 393220804018+I	0.00015623+I	1.448275953+I
1.5+I	6.115187+I	6.22208042+I	6.2122080130+I	0.0001691696+I	1.65974429+I
1.6+I	8.48976+I	8.3481946+I	8.3691676230+I	0.0002691669+I	1.88481920788+I
1.7+I	11.44157+I	11.281736811+I	11.2853935+I	0.00022995+I	2.13827508+I
1.8+I	15.069168+I	15.05481967+I	15.0220805389+I	0.00026916780+I	2.3848192391+I
1.9+I	19.80199+I	19.22208013+I	19.9022080447+I	0.000269164+I	2.648195440+I
2+I	25+I	25.5481949+I	26.00082759+I	0.0003691627+I	2.9481927517+I

Table 4. Approximation solution of 2 of using ABM (Euler), ABM (RK) and the error estimation for step size $h = 0.05 + I$.

x	**Exact**	**Euler**	**Runge-Kutta**	**Error ABM**	**Error AMM**
1+I	2+I	2+I	2+I	I	I
1.1+I	3.67787+I	3.69049724+I	3.6907787+I	0.0000115+I	0.000113+I
1.2+I	6. 169+I	6.04799663+I	6.01699169+I	0.0000489+I	0.0000478+I
1.3+I	9. 4415+I	9.21049590+I	9. 4415+I	0.000066741+I	0.0000658+I
1.4+I	13. 68275+I	13.32799513+I	13. 68275+I	0.00000487+I	0.0000513+I
1.5+I	16. 04970+I	18.56249431+I	18. 04979431+I	0.0000688+I	0.000722+I
1.6+I	22.13149+I	25.08799344+I	25.08799170+I	0.0000456+I	0.0008872+I

1.7+I	33.2286+I	33.09049251+I	33.0994089+I	0.0011749+I	0.0016937+I
1.8+I	42.6678+I	42.76799153+I	42.76798949+I	0.00021847+I	0.00241051+I
1.9+I	54.40918+I	54.33049050+I	54.33048829+I	0.0031950+I	0.00351171+I
2+I	68.33209+I	67.99998942+I	67.99998703+I	0.00421058+I	0.0041297+I

Definition 2.3 Consider the following (IVP) of dual variables:

$$\begin{cases} (y_1 + y_2J)' = \dfrac{d(y_1 + y_2J)^{\emptyset}}{d(x_1 + x_2J)} = f(x_1 + x_2J, y_1 + y_2J) \\ (y_1 + y_2J)(x_0 + x_0'J) = y_0 + y_0'J \quad ; x_0, x_0', y_0, y_0' \in \mathbb{R} \end{cases}.$$

The method can be explained by the following equations:

$$(y_1 + y_2J)(x_1 + x_2J) = (y_n + y_n'J) + f(x_0 + x_0'J, y_0 + y_0'J)(x_0 + x_0'J),$$

when,

$$x_1 + x_2J = x_1' + x_2'J,$$

we have:

$$y_1 + y_1'J = y_0 + y_0'J + (h_1 + h_2J)f(x_0 + x_0'J, y_0 + y_0'J),$$

The next approximations are:

$$y_2 + y_2'J = y_1 + y_1'J + (h_1 + h_2J)f(x_1 + x_1'J, y_1 + y_1'J),$$

at the point

$$x_1 + x_2J = x_2 + x_2'J,$$

$$y_3 + y_3'J = y_2 + y_2'J + (h_1 + h_2J)f(x_2 + x_2'J, y_2 + y_2'J),$$

and

$$y_{n+1} + y_{n+1}'J = y_n + y_n'J + (h_1 + h_2J)f(x_n + x_n'J, y_n + y_n'J) \text{ for all } n = 0,1,2, \ldots..$$

Definition 2.4 Consider the following (IVP) of dual variables:

$$(y_1 + y_2J)' = f(x_1 + x_2J, y_1 + y_2J), (y_1 + y_2J)(x_0 + x_0'J) = y_0 + y_0'J.$$

let

$$h = (x_1 + x_1'J) - (x_0 + x_0'J) = (x_1 - x_0) + J(x_1' - x_0'),$$

then,

$$(y_1 + y_2J)(x + x'J + h + h'J) = (y_1 + y_2J)(x + x'J) + (h + h'J)(y_1 + y_2J)'(x + x'J) +$$

$$\frac{(h+h'J)^2}{2!}(y_1 + y_2J)^{"}(x + x'I) + \ldots.,$$

hence,

$$(y_1 + y_2J)(x + x'J + h + h'J) - (y_1 + y_2J)(x + x'J)$$

$$= (h + h'J)(y_1 + y_2J)'(x + x'J) + \frac{(h + h'J)^2}{2!}(y_1 + y_2J)^{"}(x + x'J) + \ldots.$$

by partial differenation, we get:

$$y' = (y_1 + y_2J)' = f(x_1 + x_2^*J, y_1 + y_2J) = f,$$

$$y^{"} = f_{x_1+x_2J} + f.f_{y_1+y_2J},$$

$$y''' = f_{(x_1+x_2J)(x_1+x_2J)} + 2ff_{(x_1+x_2J)(y_1+y_2J)} + f^2f_{(y_1+y_2J)(y_1+y_2J)} + f_{(x_1+x_2J)(y_1+y_2J)} + ff_{y_1+y_2J}^2,$$

take,

$$F_1 + F_1'J = f_{x_1+x_2J} + f.f_{y_1+y_2J},$$

$$F_2 + F_2'J = f_{(x_1+x_2J)(x_1+x_2J)} + 2ff_{(x_1+x_2J)(y_1+y_2J)} + f^2f_{(x_1+x_2J)(y_1+y_2J)},$$

$$F_3 + F_3'J = f_{(x_1+x_2J)(x_1+x_2J)(x_1+x_2J)} + 3ff_{(x_1+x_2J)(x_1+x_2J)(y_1+y_2J)} + f^2f_{(y_1+y_2J)(y_1+y_2J)(y_1+y_2J)} +$$

$$f^2f_{(y_1+y_2J)(y_1+y_2J)(y_1+y_2J)},$$

hence,

$$\begin{cases} F_1 + F_1'J = y^{"} \\ F_2 + F_2J + f_{(y_1+y_2I)}(F_1 + F_1J) = y''' \\ (F_3 + J) + f_{(y_1+y_2J)}(F_2 + F_2J) + 3(F_1 + J)(f_{(x_1+x_2J)(y_1+y_2J)} + f.f_{(y_1+y_2J)(y_1+y_2J)}) \end{cases},$$

$(y_1 + y_2J)(x + x'J + h + h'J) - (y_1 + y_2J)(x + x'J) = (h_1 + h_2J)(f) + \frac{(h_1+h_2J)^2}{2!}(F_1 + F_1J) + \ldots.$

$$K_1 + K_1'J = (h + h'J)f,$$

$$K_2 + K_2'J = (h + h'J)f\big(x + x'J + (m + m'J)(h + h'J), y + y'J + (m + m'J)(K_1 + K_1'I)\big),$$

$$K_3 + K_3'J = (h + h'J)f\big(x + x'J + (n + n'J)(h + h'J), y + y'J + (n + n'J)(K_2 + K_2'I)\big),$$

$$K_4 + K_4'J = (h + h'J)f(x + x'J + (p + p'J)(h + h'J), y + y'J + (p + p'J)(K_3 + K_3'I)),$$

$$\Delta_{y_1+y_2I} = (y_1 + y_2J)(x + x'J, y_1 + y_2J) - (y_1 + y_2J)(x_1 + x_2J) = a(K_1 + K_1'J) + b(K_2 + K_2'J) + c(K_3 + K_3'J) + d(K_4 + K_4'J),$$

now, we can use Taylor's expansion for several variables:

$$K_1 + K_1'J = (h + h'J)f,$$

$$K_2 + K_2'J = (h + h'J) + \Big[f + (m + m'J)(h + h'J)(F_1 + F_1'J) + \frac{1}{2}(m + m'J)^2(h + h'J)^2(F_2 + F_2'J) + \ldots\Big],$$

$$K_3 + K_3'J = (h + h'J) + \Big[f + (n + n'J)(h + h'J)(F_1 + F_1'J) + \tfrac{1}{2}(h + h'J)^2\Big((n + n'J)^2(F_2 + J) + 2(m + m'J)(n + n'J)f_y(F_1 + F_1J)\Big) + \tfrac{1}{6}(h + h'J)^3\big((n + n'J)^3(F_3 + F_3'J) + 3(m + m'J)^2(n + n'J)f_yF_2J\big) + 6\,(m + m'J)(n + n'J)^2(F_1 + F_1J)\big(F_{xIyI} + ff_{yy}\big) + \ldots.\Big],$$

$$K_4 + K_4'J = (h + h'J) + \Big[f + (p + p'J)(h + h'J)(F_1 + F_1'J) + \tfrac{1}{2}(h + h'J)^2\big[(p + p'J)^2(F_2 + F_2'J) + 2(n + n'J)(p + p'J)f_y(F_1 + F_1'J)\big] + \tfrac{1}{6}(h + h'J)^3\big[(p + p'J)^3(F_3 + F_3'J) + 3(n + n'J)^2(p + p'J)f_y(F_2 + F_2'J)\big] + 6\,(n + n'J)(p + p'J)(F_1 + F_1J)\big(F_{xIyI} + ff_{y^2I} + \ldots.\big) \ldots.\Big],$$

hence,

$$-(y_1+y_2J)(x+x'J)+(y_1+y_2J)\big(x+h+J(x'+h')\big)=a(h+h'J)f+b(h+h'J)[f+$$

$$(m+m'J)(h+h'J)(F_1+F_1J)+\ldots.]+c(h+h'J)\Big[f+(n+n'J)(h+h'J)(F_1+F_1'J)+$$

$$\frac{1}{2}(h+h'J)^2(n+n'J)^2F_2J\Big]+\frac{1}{6}(h+h'J)^3[(p+p'J)^3(F_3+F_3'J)+3(n+n'J)^2(p+p'J)F_2J+$$

$$\ldots\ldots],$$

thus,

$$(y_1+y_2J)\big(x+h+J(x'+h')\big)-(y_1+y_2J)(x+x'J)=(a+b+c+d)(h+h'J)f+$$

$$[b(m+m'J)+c(n+n'J)+d(p+p'J)](h+h'J)^2(F_2+F_2'I)+[b(m+m'J)^2+c(n+n'J)^2+$$

$$d(p+p'J)^2]\frac{(h+h'J)^3(F_2J)}{2}+[b(m+m'J)^3+c(n+n'J)^3+d(p+p'J)^3]\frac{(h+h'J)^4(F_2I)}{6}+\ldots\ldots,$$

so that,

$$\begin{cases} m=\frac{1}{2} \\ n=\frac{1}{2} \\ m'=n'=1 \end{cases},\begin{cases} p=1 \\ p'=1 \end{cases},\begin{cases} a=d=\frac{1}{6} \\ b=c=\frac{1}{3} \end{cases},$$

thus,

$$K_1+K_1'J=(h+h'J)f(x_0+x_0'J,y_0+J)=df.$$

$$K_2+K_2'J=(h+h'J)f\left(x_0+x_0'J+\frac{h+h'J}{2},y_0+y_0'J+\frac{K_1+K_1'J}{2}\right),$$

$$K_3+K_3'J=(h+h'J)f\left(x_0+x_0'J+\frac{h+h'J}{2},y_0+y_0'J+\frac{K_2+K_2'J}{2}\right),$$

$$K_4+K_4'J=(h+h'J)f(x_0+x_0'J+h+h'J,y_0+y_0'J+K_3+K_3'J),$$

Numerical comparison:

Consider the following dual problem:

$$(x_1+x_2J)(y_1+y_2J)"-2(y_1+y_2J)'=(10+10J)(x_1+x_2J)^4,$$

$$\begin{cases}(y_1+y_2I)(1+J)=2+J\\(y_1+y_2J)'(1+I)=2+J\\h=0.1+0.1J\\0\le x_1\le 2,0\le x_2\le 1\end{cases},$$

hence,

$$\begin{cases}y_1'+y_2'J=z_1+z_2J\\z_1'+z_2'J=(10+J)(x_1+x_2J)^3+\dfrac{2}{x_1+x_2J}(z_1+z_2J).\end{cases}$$

now, consider the following problem:

$$(y_1+y_2J)"-2(y_1+y_2J)'=4(x_1+x_2J),$$

with

$$\begin{cases}(y_1+y_2J)(0)=1+J\\(y_1+y_2J)'(0)=2+J\\0\le x_1\le 1\\0\le x_2\le 1\end{cases}.$$

hence,

$$\begin{cases}(y_1+y_2J)"=z_1+z_2J\\(z_1'+z_2'J)=(4+J)(x_1+x_2J)+(2+J)(z_1+z_2J)'\end{cases}$$

with

$$\begin{cases}(y_1+y_2J)(0)=1+J\\(z_1+z_2J)(0)=2+J\end{cases}.$$

Table 1. Approximation solution of the example by using (Euler), ABM (RK) and with step size $h=0.1+J$.

x	Exact	Euler	Runge-Kutta	Error Euler	Error RK
1+ J	2+ J	2+ J	2+ J	J	J
1.1+J	3.67787+J	3.5357787+ J	3.6907787+ J	0.15454464+J	0.000113+ J
1.2+J	6. 169+ J	5.761693+ J	6.01699169+J	0.27828957+J	0.0000478+J
1.3+J	9. 4415+ J	8. 4415+ J	9. 4415+ J	0.32617295+J	0.0000658+J
1.4+J	13.68275+J	12. 68275+ J	13. 68275+ J	0.37828757+J	0.0000513+JJ
1.5+J	16.04970+J	18.04972587+J	18.04979431+J	0.43425871+J	0.000722+ J
1.6+J	22.13149+J	24.59391009+J	25.08799170+J	0.49408991+J	0.0008872+ J

1.7+J	33.2286+ J	32.5394089+ J	33.0994089+ J	0.55778118+J	0.0016937+ J
1.8+J	42.6678+ J	42.14266746+J	42.76798949+J	0.62533254+J	0.00241051+J
1.9+	54.40918+	53.63375603+	54.33048829+	0.69674397+	0.00351171+
2+J	68.33209+J	67.22798452+J	67.99998703+J	0.77201548+J	0.0041297+J

Table 2. Approximation solution of the example using (Euler), ABM (RK) and the error estimation for step size $h = 0.9 + J$.

x	Exact	Euler	Runge-Kutta	Error Euler	Error RK
1+ J	2+ J	2+I	2+ J	J	J
1.1+J	2.25541+ J	2.28871+ J	2.5560897+ J	0.00003935+I	0.22345+ J
1.2+J	2.76032+ J	2.44914306+J	2.76031761+ J	0.001348413+I	0.371918+ J
1.3+J	3.767593+ J	3.88181936+J	3.539352895+ J	0.00827513406+J	1.24178788+ J
1.4+J	4.332424+ J	4.6688294+ J	4. 393524018+ J	0.08275011894+J	1.448275953+J
1.5+J	6.115187+ J	5.171701+ J	6.21875130+ J	0.08275016779+J	1.65974429+ J
1.6+J	8.48976+ J	8.065518+ J	8.38976230+ J	0.013416177+ J	1.88920788+ J
1.7+J	11.44157+ J	10.491675+ J	11.2853935+ J	0.013421019+ J	2.13827508+ J
1.8+J	15.069168+J	13.114943+ J	15.082755389+J	0.0134020954+J	2.382752391+J
1.9+J	19.80199+ J	17.377873+ J	19.90199447 J	0.0827525828+J	2.68275440+J
2+ J	25+ J	22.1991698+J	26.00082759+J	0.013426253+ J	2.95827517+ J

Table 3. Approximation solution of the example using ABM (Euler), ABM (RK) and the error estimation for step size $h = 0.08 + J$.

x	Exact	Euler	Runge-Kutta	Error Euler	Error RK
1+ J	2+ J	2+ J	2+ J	J	J
1.1+J	3.67787+ J	3.5357787+ J	3.6907787+ J	0.15454464+J	0.000113+ J
1.2+J	6. 169+ J	5.761693+ J	6.01699169+J	0.27828957+J	0.0000478+J
1.3+J	9. 4415+ J	8. 4415+ J	9. 4415+ J	0.32617295+J	0.0000658+J
1.4+J	13.68275+J	12. 68275+ J	13. 68275+ J	0.37828757+J	0.0000513+J
1.5+J	16.04970+J	18.04972587+J	18.04979431+J	0.43425871+J	0.000722+ J
1.6+J	22.13149+J	24.59391009+J	25.08799170+J	0.49408991+J	0.0008872+J
1.7+J	33.2286+J	32.5394089+ J	33.0994089+ J	0.55778118+J	0.0016937+J
1.8+J	42.6678+J	42.14266746+J	42.76798949+J	0.62533254+J	0.00241051+J

1.9+	54.40918+	53.63375603+	54.33048829+	0.69674397+	0.00351171+
2+J	68.33209+J	67.22798452+J	67.99998703+J	0.77201548+J	0.0041297+ J

Conclusion

In this paper we have studied some numerical solutions of neutrosophic differential problems through a numerical comparison between neutrosophic Euler's method and 4-th order Runge-Kutta neutrosophic method, where we applied both methods on the same problems and compare the numerical results of solutions and numerical estimations of errors by using numerical tables. We have noticed from the numerical study and application of the two mentioned methods that they provide good results compared to exact solutions, with an acceptable small error of approximation. We hope that in the future the possibility of applying some other numerical methods to neutrosophic problems will be discussed and the numerical results compared through numerical tables.

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Effective Approach to Shock Absorption Performance Evaluation of Mechanical Component Springs Using the Single Valued Trapezoidal Neutrosophic Numbers for Structural Stability

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Abstract: We propose a decision-making approach for shock absorption performance evaluation of mechanical component springs. The main goal of this research is to provide a decision support system that will help decision makers to select the best alternative to alleviate this ambiguity. The selection of the best alternative is done using the multi-attribute group decision-making approach. The hybrid model is created utilizing two-step logarithmic normalization (ARLON) and Single Valued Trapezoidal Neutrosophic Numbers (SVTNNs) for MEREC and alternative ranking. The criteria's relevance levels are ascertained using the innovative SVTNN-MEREC approach. The alternatives are ranked using the recently developed SVTNN-ARLON technique. The suggested hybrid model's application procedures are illustrated by an algorithm that is created. This algorithm is used as the basis for real-world research.

Keywords: Single Valued Trapezoidal Neutrosophic Numbers; Uncertainty; Shock Absorption Performance; Mechanical Component Springs.

1. Introduction

The neutrophilic set, which emphasizes the nature and cause of neutralities in various sectors, is a generalization of the classical set, fuzzy set, intuitionistic fuzzy set, and so forth. Smarandache invented this multidimensional logic, which contains more information than fuzzy logic and imports the term indeterminacy. Compared to fuzzy logic, this results in greater performance. A proposition in neutrosophic logic has three degrees: truth (T), indeterminacy (I), and falsity (F). The single-valued neutrosophic set, which accepts values in the standard interval [0, 1], was introduced because it is challenging to manage data with non-standard intervals[1], [2]. Smarandache introduced the concepts of neutrosophic measure, neutrosophic integral, and neutrosophic probability. Since there are many kinds of indeterminacies depending on the problem, the neutrosophic integral and neutrosophic probability are also defined in a variety of ways. This is because neutrosophic measure presents numerous real-world examples. The neutrosophic logic has been used by numerous scholars in a variety of domains[3], [4].

The use of single-valued, triangular, and trapezoidal neutrosophic numbers in decision-making is examined regarding their applications. In addition to introducing the neutrosophic number from various perspectives, the various linear and non-linear generalized triangular neutrosophic numbers—which are crucial for uncertainty theory—and the de-neutrosophication concept for neutrosophic numbers—which aids in turning a neutrosophic number into a crisp number—are also covered. This has been used in the problems of MCDM selection assessment review techniques[5], [6].

Type 2 neutrosophic numbers (T2NN) are an enhanced sort of neutrosophic approach that was established. To demonstrate the effectiveness and usefulness of the type 2 neutrosophic number, a real-world scenario involving a decision-making based on T2NN was presented. According to the judgments of experts and decision-makers, the linguistic factors were presented using the triangular neutrosophic numbers (TriNs). One of the most important tools for resolving challenging and complex choice problems is the multicriteria decision-making (MCDM) methodology[7], [8]. Group decision-making and the definition of the bipolar neutrosophic number were established.

1.1. Aims of the study

The main goal is to create a MAGDM-based decision assistance system for Shock Absorption Performance Evaluation of Mechanical Component Springs. According to the main goal, the following are the predetermined goals:

Selecting the best alternative: Using the MAGDM framework, a new decision support system is developed to identify the optimal alternative.

Creation of hybrid methodology centers on the creation of the SVTN-MEREC-ARLON hybrid technique, which combines the phases of alternative selection, criteria weighing, and expertise appraisal into a unified approach to decision-making.

1.2. Contributions of the study

This study mainly adds a new hybrid model for choosing the best alternative to the body of literature. The following is a breakdown of the contributions made to the body of current knowledge:

- To identify the best alternative, a new decision support system is created within the MAGDM framework.
- To illustrate the usefulness of the established approach, a case study is created that contrasts the findings of assessments conducted by information system experts and users.
- We used the single valued triangular neutrosophic sets to deal with uncertainty and vague information.
- The MEREC method is used to compute the criteria weights and the ARLON method is used to rank the alternatives.

1.3. Organization of the study

Each of the five components that make up this study has a specific function within the research framework. The function of authentication in the performance evaluation system is explained in Section 2. The research approach is outlined in Section 3, which also provides a detailed explanation of the newly created SVTNS-MEREC-ARLON hybrid model. The hybrid model created using two case studies is put into practice and

put through robustness testing in Section 4, which is the application phase. The study's conclusions and ramifications are presented in Section 5.

2. Preliminaries

We can define some operations of neutrosophic numbers[9], [10].

Definition 2.

We can define the single valued trapezoidal neutrosophic sets (SVTNSs) such as:

$a = \left((a_1, a_2, a_3, a_4); \alpha_a(x)\right), \beta_a(x), \gamma_a(x)$. Then we can compute the three-membership functions such as:

$$T_a(x) = \begin{cases} \alpha_a\left(\frac{x-a_1}{a_2-a\_1}\right) & (a_1 \le x \le a_2) \\ \alpha_a & (a_2 \le x \le a_3) \\ \alpha_a\left(\frac{a_4-x}{a_4-a_3}\right) & (a_3 \le x \le a_4) \\ 0 \quad otherwise \end{cases} \tag{1}$$

$$I_a(x) = \begin{cases} \frac{(a_2-x+\beta_a(x-a_1))}{(a_2-a_1)} & (a_1 \le x \le a_2) \\ \beta_a & (a_2 \le x \le a_3) \\ \left(\frac{x-a_3+\beta_a(a_4-x)}{a_4-a_3}\right) & (a_3 \le x \le a_4) \\ 0 \quad otherwise \end{cases} \tag{2}$$

$$F_a(x) = \begin{cases} \frac{(a_2-x+\gamma_a(x-a_1))}{(a_2-a_1)} & (a_1 \le x \le a_2) \\ \gamma_a & (a_2 \le x \le a_3) \\ \left(\frac{x-a_3+\gamma_a(a_4-x)}{a_4-a_3}\right) & (a_3 \le x \le a_4) \\ 0 \quad otherwise \end{cases} \tag{3}$$

Where $\alpha_a, \beta_a, \gamma_a$ refer to the maximum truth-membership function, minimum indeterminacy-membership function, and minimum falsity-membership function.

Definition 2.

Let two single valued trapezoidal neutrosophic numbers (SVTNNs) such as:

$a = \left(\left((a_1, a_2, a_3, a_4)\right); \alpha_a(x), \beta_a(x), \gamma_a(x)\right)$

$b = \left(\left((b_1, b_2, b_3, b_4); \alpha_b(x)\right), \beta_b(x), \gamma_b(x)\right)$

Addition

$$a + b = \begin{pmatrix} (a_1 + b_1, a_2 + b_2, a_3 + b_3, a_4 + b_4); \\ \alpha_a(x) \wedge \alpha_b(x), \beta_a(x) \vee \beta_b(x), \gamma_a(x) \vee \gamma_b(x) \end{pmatrix} \tag{4}$$

Subtraction

$$a - b = \begin{pmatrix} (a_1 - b_4, a_2 - b_3, a_3 - b_2, a_4 - b_1); \\ \alpha_a(x) \wedge \alpha_b(x), \beta_a(x) \vee \beta_b(x), \gamma_a(x) \vee \gamma_b(x) \end{pmatrix} \tag{5}$$

Inverse

$$a^{-1} = \left(\left(\frac{1}{a_4}, \frac{1}{a_3}, \frac{1}{a_2}, \frac{1}{a_1}\right); \alpha_a(x), \beta_a(x), \gamma_a(x)\right) \; where \; a \neq 0 \tag{6}$$

Multiplication by constant

$$\mu a = \begin{cases} \begin{pmatrix} (\mu a_1, \mu a_2, \mu a_3, \mu a_4); \\ \alpha_a(x), \beta_a(x), \gamma_a(x) \end{pmatrix} & if \; \mu > 0 \\ \begin{pmatrix} (\mu a_1, \mu a_2, \mu a_3, \mu a_4); \\ \alpha_a(x), \beta_a(x), \gamma_a(x) \end{pmatrix} & if \; \mu > 0 \end{cases} \tag{7}$$

Division

$$\frac{a}{b} = \begin{cases} \begin{pmatrix} \left(\frac{a_1}{b_4}, \frac{a_2}{b_3}, \frac{a_3}{b_2}, \frac{a_4}{b_1}\right); \\ \alpha_a(x) \wedge \alpha_b(x), \beta_a(x) \vee \beta_b(x), \gamma_a(x) \vee \gamma_b(x) \end{pmatrix} \\ if \; a_4 > 0, b_4 > 0 \\ \begin{pmatrix} \left(\frac{a_4}{b_4}, \frac{a_3}{b_3}, \frac{a_2}{b_2}, \frac{a_1}{b_1}\right); \\ \alpha_a(x) \wedge \alpha_b(x), \beta_a(x) \vee \beta_b(x), \gamma_a(x) \vee \gamma_b(x) \end{pmatrix} \\ if \; a_4 < 0, b_4 > 0 \\ \begin{pmatrix} \left(\frac{a_4}{b_1}, \frac{a_3}{b_2}, \frac{a_2}{b_3}, \frac{a_1}{b_4}\right); \\ \alpha_a(x) \wedge \alpha_b(x), \beta_a(x) \vee \beta_b(x), \gamma_a(x) \vee \gamma_b(x) \end{pmatrix} \\ if \; a_4 < 0, b_4 < 0 \end{cases} \tag{8}$$

3. The SVTNS-MEREC-ARLON method

It is necessary to provide a summary of the use of the SVTNS-MEREC-ARLON hybrid approach before going into great depth about each stage. SVTNS sets are used in Stage 1 to assess the expertise levels of experts. Experts' effects on the decision-making process are then determined. In Stage 2, criteria are evaluated by generating an initial matrix using the SVTNS-MEREC approach. After that, SVTN numbers are converted into distinct values via a score function. After that, the criteria evaluation decision matrix is weighted and normalized. Weights are computed based on the identification of maximum and minimum values. The final criteria weights are then calculated, and reliability indices are used to verify the criteria weights.

An initial decision matrix is produced in Stage 3 by using the SVTNS-ARLON approach for expert evaluation of alternatives based on criteria. After combining expert opinions, values are subjected to a two-step logarithmic normalization procedure to further normalize them. The final ranking scores for authentication systems are then calculated by weighing the normalized data. The following describes the steps in the SVTNS-MEREC-ARLON hybrid methodology's procedural sequence:

Stage 1- Construction of the decision matrix by the opinions of experts

A1. Experts and decision makers are using the linguistic terms of the SVTNs to evaluate the criteria and alternatives.

A2. The linguistic terms are replaced by the SVTNNs. These numbers are used to evaluate the criteria and alternatives.

Stage 2 – Establishing criteria weighting utilizing the SVTN-MEREC method.

B1. Compute the crisp criterion assessment matrix

B2. Combine the assessment matrix

We use the average method to compute the one matrix.

B3. Normalize the assessment matrix.

$$n_{ij} = \begin{cases} \frac{\min x_{ij}}{x_{ij}} \; for\ positive\ criteria \\ \frac{x_{ij}}{\max x_{ij}} \; for\ negative\ criteria \end{cases} \tag{9}$$

B4. Compute the overall performance of the alternatives such as:

$$S_i = \ln\left(1 + \left(\frac{1}{m}\sum_j \left|\ln\left(n_{ij}\right)\right|\right)\right) \tag{10}$$

B5. Compute the performance of the alternatives by removing each criterion.

$$S_{ij}^{'} = \ln\left(1 + \left(\frac{1}{m}\sum_{k,k\neq j} \left|\ln(n_{ik})\right|\right)\right) \tag{11}$$

B6. Compute the summation of absolute deviation.

$$E_j = \sum_i \left|S_{ij}^{'} - S_i^{'}\right| \tag{12}$$

B7. Compute the final weights of criteria

$$w_j = \frac{E_j}{\sum_k E_k} \tag{13}$$

Stage 3 – The process of ranking alternatives employing the SVTNSs-ARLON method[11], [12].

D1. Generate the decision matrix

D2. Compute two logarithmic normalization methods such as

The first normalization

$$Y_{ij}^{1st} = \begin{cases} Y_{ij}^{1st(+)} = \frac{\ln(x_{ij})}{\ln\left(\prod_{i=1}^{m} x_{ij}\right)} \; positive\ criteria \\ Y_{ij}^{1st(-)} = \left(\frac{1 - \frac{\ln(x_{ij})}{\ln\left(\prod_{i=1}^{m} x_{ij}\right)}}{m-1}\right) negative\ criteria \end{cases} \tag{14}$$

The second normalization

$$Y_{ij}^{2st} = \begin{cases} Y_{ij}^{2st(+)} = \frac{\log_2(x_{ij})}{\sum_{i=1}^{m}(\log_2(x_{ij}))} \; positive\ criteria \\ Y_{ij}^{2st(-)} = \left(1 - \frac{\log_2(x_{ij})}{\sum_{i=1}^{m}(\log_2(x_{ij}))}\right) negative\ criteria \end{cases} \tag{15}$$

D3. Combine the normalization matrix.

$$Y_{ij}^{com} = \left((1-\alpha)\sqrt{\left(Y_{ij}^{1st} + Y_{ij}^{2st}\right)} + (\alpha)\left(\frac{Y_{ij}^{1st} + Y_{ij}^{2st}}{2}\right)\right) \tag{16}$$

D4. Compute the weighted normalized decision matrix

$$q_{ij} = w_j Y_{ij}^{com} \tag{17}$$

D5. Compute the summations of the weighted combined normalized values for positive and negative criteria

$$Z_i^- = sum\left(q_{ij}\right) for\ positive\ criteria \tag{18}$$

$$Z_i^+ = sum\left(q_{ij}\right) for\ negative\ criteria \tag{19}$$

D6. Obtain final rank of alternatives.

$$Z_i = (Z_i^+)^{\beta} + (Z_i^-)^{(1-\beta)} \tag{20}$$

The model steps are shown in Table 1.

Table 1. The algorithm steps.

Algorithm	**Computing the criteria weights and ranking the alternatives.**
Input	In the given model, we used a set of criteria ($C = \left(c_1, \dots, c_j\right)$), alternatives ($A = (A_1, \dots, A_i)$), criteria weights ($W = \left(w_1, \dots, w_j\right)$), and experts.
Output	Select the best alternative.
Begin	
First stage	A1. Experts and decision makers are using the linguistic terms of the SVTNs to evaluate the criteria and alternatives. A2. The linguistic terms are replaced by the SVTNNs. These numbers are used to evaluate the criteria and alternatives.
Second stage	B1. Compute the crisp criterion assessment matrix B2. Combine the assessment matrix We use the average method to compute the one matrix. B3. Normalize the assessment matrix. n_{ij} B4. Compute the overall performance of the alternatives such as: S_i B5. Compute the performance of the alternatives by removing each criterion. $S_{ij}^{'}$ B6. Compute the summation of absolute deviation. E_j B7. Compute the final weights of criteria w_j
Third stage	D1. Generate the decision matrix D2. Compute two logarithmic normalization methods such as The first normalization Y_{ij}^{1st} The second normalization Y_{ij}^{2st} D3. Combine the normalization matrix. Y_{ij}^{com} D4. Compute the weighted normalized decision matrix

q_{ij}

D5. Compute the summations of the weighted combined normalized values for positive and negative criteria

Z_i^-

Z_i^+

D6. Obtain final rank of alternatives.

Z_i

End

4. Application

This section shows the results of the Shock Absorption Performance Evaluation of Mechanical Component Springs under neutrosophic numbers. Three experts have evaluated the criteria and alternatives. This study has 8 criteria and 12 alternatives to be evaluated as Fatigue Resistance & Durability, Environmental Adaptability, Manufacturing Precision, Damping Efficiency, Material Quality and Composition, Energy Absorption Capacity, Load-Bearing Capacity, Cost-Effectiveness and Maintenance. The alternatives can be defined such as: Metal Mesh Springs, Progressive Rate Coil Springs, Rubber Springs, Compression Springs, Hydraulic-Damped Coil Springs, Wave Springs, Helical Compression Springs, Air Springs, Gas Springs, Torsion Bar Springs, Leaf Springs with Composite Materials, Nested Belleville Springs.

Stage 1- Construction of the decision matrix by the opinions of experts

We build the three-decision matrix between the criteria and the alternatives as shown in Tables 2-4. These assessment matrices are evaluated using the SVTNNs to deal with uncertainty information in the evaluation

Table 2. The first SVTNNs.

	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8
A_1	(0.2,0.3,0.4,0.5;0.7,0.2,0.5)	(0.3,0.5,0.65,0.9;0.5,0.2,0.1)	(0.1,0.4,0.65,1;0.5,0.2,0.1)	(0.5,0.6,0.7,0.8;0.7,0.2,0.5)	(0.1,0.35,0.5,0.7;0.8,0.2,0.1)	(0.6,0.7,0.75,0.9;0.5,0.2,0.1)	(0.3,0.4,0.5,0.8;0.7,0.2,0.5)	(0.1,0.4,0.65,1;0.5,0.2,0.1)
A_2	(0.3,0.4,0.5,0.8;0.7,0.2,0.5)	(0.3,0.4,0.5,0.8;0.7,0.2,0.5)	(0.6,0.7,0.75,0.9;0.5,0.2,0.1)	(0.1,0.35,0.5,0.7;0.8,0.2,0.1)	(0.5,0.6,0.7,0.8;0.7,0.2,0.5)	(0.1,0.4,0.65,1;0.5,0.2,0.1)	(0.2,0.3,0.4,0.5;0.7,0.2,0.5)	(0.5,0.6,0.7,0.8;0.7,0.2,0.5)
A_3	(0.6,0.7,0.75,0.9;0.5,0.2,0.1)	(0.2,0.3,0.4,0.5;0.7,0.2,0.5)	(0.5,0.6,0.7,0.8;0.7,0.2,0.5)	(0.1,0.35,0.5,0.7;0.8,0.2,0.1)	(0.6,0.7,0.75,0.9;0.5,0.2,0.1)	(0.3,0.5,0.65,0.9;0.5,0.2,0.1)	(0.3,0.4,0.5,0.8;0.7,0.2,0.5)	(0.1,0.35,0.5,0.7;0.8,0.2,0.1)
A_4	(0.1,0.35,0.5,0.7;0.8,0.2,0.1)	(0.3,0.4,0.5,0.8;0.7,0.2,0.5)	(0.2,0.3,0.4,0.5;0.7,0.2,0.5)	(0.2,0.3,0.4,0.5;0.7,0.2,0.5)	(0.2,0.3,0.4,0.5;0.7,0.2,0.5)	(0.2,0.3,0.4,0.5;0.7,0.2,0.5)	(0.6,0.7,0.75,0.9;0.5,0.2,0.1)	(0.6,0.7,0.75,0.9;0.5,0.2,0.1)
A_5	(0.5,0.6,0.7,0.8;0.7,0.2,0.5)	(0.6,0.7,0.75,0.9;0.5,0.2,0.1)	(0.3,0.4,0.5,0.8;0.7,0.2,0.5)	(0.2,0.3,0.4,0.5;0.7,0.2,0.5)	(0.3,0.4,0.5,0.8;0.7,0.2,0.5)	(0.2,0.3,0.4,0.5;0.7,0.2,0.5)	(0.1,0.35,0.5,0.7;0.8,0.2,0.1)	(0.3,0.4,0.5,0.8;0.7,0.2,0.5)
A_6	(0.1,0.4,0.65,1;0.5,0.2,0.1)	(0.1,0.35,0.5,0.7;0.8,0.2,0.1)	(0.6,0.7,0.75,0.9;0.5,0.2,0.1)	(0.3,0.4,0.5,0.8;0.7,0.2,0.5)	(0.6,0.7,0.75,0.9;0.5,0.2,0.1)	(0.3,0.4,0.5,0.8;0.7,0.2,0.5)	(0.5,0.6,0.7,0.8;0.7,0.2,0.5)	(0.6,0.7,0.75,0.9;0.5,0.2,0.1)
A_7	(0.3,0.5,0.65,0.9;0.5,0.2,0.1)	(0.5,0.6,0.7,0.8;0.7,0.2,0.5)	(0.1,0.35,0.5,0.7;0.8,0.2,0.1)	(0.6,0.7,0.75,0.9;0.5,0.2,0.1)	(0.1,0.35,0.5,0.7;0.8,0.2,0.1)	(0.6,0.7,0.75,0.9;0.5,0.2,0.1)	(0.1,0.4,0.65,1;0.5,0.2,0.1)	(0.1,0.35,0.5,0.7;0.8,0.2,0.1)

A8	(0.2,0.3,0.4,0.5;0.7,0.2,0.5)	(0.1,0.4,0.65,1;0.5,0.2,0.1)	(0.5,0.6,0.7,0.8;0.7,0.2,0.5)	(0.1,0.35,0.5,0.7;0.8,0.2,0.1)	(0.5,0.6,0.7,0.8;0.7,0.2,0.5)	(0.1,0.35,0.5,0.7;0.8,0.2,0.1)	(0.3,0.5,0.65,0.9;0.5,0.2,0.1)	(0.3,0.4,0.5,0.8;0.7,0.2,0.5)
A9	(0.3,0.4,0.5,0.8;0.7,0.2,0.5)	(0.3,0.5,0.65,0.9;0.5,0.2,0.1)	(0.1,0.4,0.65,1;0.5,0.2,0.1)	(0.5,0.6,0.7,0.8;0.7,0.2,0.5)	(0.1,0.4,0.65,1;0.5,0.2,0.1)	(0.5,0.6,0.7,0.8;0.7,0.2,0.5)	(0.2,0.3,0.4,0.5;0.7,0.2,0.5)	(0.6,0.7,0.75,0.9;0.5,0.2,0.1)
A10	(0.3,0.4,0.5,0.8;0.7,0.2,0.5)	(0.2,0.3,0.4,0.5;0.7,0.2,0.5)	(0.3,0.5,0.65,0.9;0.5,0.2,0.1)	(0.1,0.4,0.65,1;0.5,0.2,0.1)	(0.3,0.5,0.65,0.9;0.5,0.2,0.1)	(0.1,0.4,0.65,1;0.5,0.2,0.1)	(0.3,0.5,0.65,0.9;0.5,0.2,0.1)	(0.1,0.35,0.5,0.7;0.8,0.2,0.1)
A11	(0.2,0.3,0.4,0.5;0.7,0.2,0.5)	(0.1,0.4,0.65,1;0.5,0.2,0.1)	(0.2,0.3,0.4,0.5;0.7,0.2,0.5)	(0.3,0.5,0.65,0.9;0.5,0.2,0.1)	(0.2,0.3,0.4,0.5;0.7,0.2,0.5)	(0.3,0.5,0.65,0.9;0.5,0.2,0.1)	(0.1,0.4,0.65,1;0.5,0.2,0.1)	(0.5,0.6,0.7,0.8;0.7,0.2,0.5)
A12	(0.3,0.5,0.65,0.9;0.5,0.2,0.1)	(0.5,0.6,0.7,0.8;0.7,0.2,0.5)	(0.1,0.35,0.5,0.7;0.8,0.2,0.1)	(0.2,0.3,0.4,0.5;0.7,0.2,0.5)	(0.2,0.3,0.4,0.5;0.7,0.2,0.5)	(0.2,0.3,0.4,0.5;0.7,0.2,0.5)	(0.5,0.6,0.7,0.8;0.7,0.2,0.5)	(0.1,0.4,0.65,1;0.5,0.2,0.1)

Table 3. The second SVTNNs.

	C1	**C2**	**C3**	**C4**	**C5**	**C6**	**C7**	**C8**
A1	(0.5,0.6,0.7,0.8;0.7,0.2,0.5)	(0.3,0.5,0.65,0.9;0.5,0.2,0.1)	(0.1,0.4,0.65,1;0.5,0.2,0.1)	(0.5,0.6,0.7,0.8;0.7,0.2,0.5)	(0.1,0.35,0.5,0.7;0.8,0.2,0.1)	(0.6,0.7,0.75,0.9;0.5,0.2,0.1)	(0.3,0.4,0.5,0.8;0.7,0.2,0.5)	(0.1,0.4,0.65,1;0.5,0.2,0.1)
A2	(0.1,0.4,0.65,1;0.5,0.2,0.1)	(0.3,0.4,0.5,0.8;0.7,0.2,0.5)	(0.6,0.7,0.75,0.9;0.5,0.2,0.1)	(0.5,0.6,0.7,0.8;0.7,0.2,0.5)	(0.5,0.6,0.7,0.8;0.7,0.2,0.5)	(0.1,0.4,0.65,1;0.5,0.2,0.1)	(0.5,0.6,0.7,0.8;0.7,0.2,0.5)	(0.5,0.6,0.7,0.8;0.7,0.2,0.5)
A3	(0.3,0.5,0.65,0.9;0.5,0.2,0.1)	(0.1,0.4,0.65,1;0.5,0.2,0.1)	(0.5,0.6,0.7,0.8;0.7,0.2,0.5)	(0.1,0.4,0.65,1;0.5,0.2,0.1)	(0.5,0.6,0.7,0.8;0.7,0.2,0.5)	(0.3,0.5,0.65,0.9;0.5,0.2,0.1)	(0.1,0.4,0.65,1;0.5,0.2,0.1)	(0.1,0.35,0.5,0.7;0.8,0.2,0.1)
A4	(0.2,0.3,0.4,0.5;0.7,0.2,0.5)	(0.1,0.4,0.65,1;0.5,0.2,0.1)	(0.3,0.5,0.65,0.9;0.5,0.2,0.1)	(0.3,0.5,0.65,0.9;0.5,0.2,0.1)	(0.1,0.4,0.65,1;0.5,0.2,0.1)	(0.2,0.3,0.4,0.5;0.7,0.2,0.5)	(0.3,0.5,0.65,0.9;0.5,0.2,0.1)	(0.6,0.7,0.75,0.9;0.5,0.2,0.1)
A5	(0.3,0.4,0.5,0.8;0.7,0.2,0.5)	(0.5,0.6,0.7,0.8;0.7,0.2,0.5)	(0.1,0.4,0.65,1;0.5,0.2,0.1)	(0.2,0.3,0.4,0.5;0.7,0.2,0.5)	(0.3,0.5,0.65,0.9;0.5,0.2,0.1)	(0.3,0.4,0.5,0.8;0.7,0.2,0.5)	(0.2,0.3,0.4,0.5;0.7,0.2,0.5)	(0.1,0.35,0.5,0.7;0.8,0.2,0.1)
A6	(0.5,0.6,0.7,0.8;0.7,0.2,0.5)	(0.5,0.6,0.7,0.8;0.7,0.2,0.5)	(0.5,0.6,0.7,0.8;0.7,0.2,0.5)	(0.3,0.4,0.5,0.8;0.7,0.2,0.5)	(0.2,0.3,0.4,0.5;0.7,0.2,0.5)	(0.5,0.6,0.7,0.8;0.7,0.2,0.5)	(0.3,0.4,0.5,0.8;0.7,0.2,0.5)	(0.6,0.7,0.75,0.9;0.5,0.2,0.1)
A7	(0.1,0.4,0.65,1;0.5,0.2,0.1)	(0.1,0.4,0.65,1;0.5,0.2,0.1)	(0.1,0.4,0.65,1;0.5,0.2,0.1)	(0.6,0.7,0.75,0.9;0.5,0.2,0.1)	(0.5,0.6,0.7,0.8;0.7,0.2,0.5)	(0.1,0.4,0.65,1;0.5,0.2,0.1)	(0.6,0.7,0.75,0.9;0.5,0.2,0.1)	(0.1,0.35,0.5,0.7;0.8,0.2,0.1)
A8	(0.3,0.5,0.65,0.9;0.5,0.2,0.1)	(0.3,0.5,0.65,0.9;0.5,0.2,0.1)	(0.3,0.5,0.65,0.9;0.5,0.2,0.1)	(0.1,0.35,0.5,0.7;0.8,0.2,0.1)	(0.1,0.4,0.65,1;0.5,0.2,0.1)	(0.3,0.5,0.65,0.9;0.5,0.2,0.1)	(0.1,0.35,0.5,0.7;0.8,0.2,0.1)	(0.1,0.35,0.5,0.7;0.8,0.2,0.1)
A9	(0.2,0.3,0.4,0.5;0.7,0.2,0.5)	(0.2,0.3,0.4,0.5;0.7,0.2,0.5)	(0.2,0.3,0.4,0.5;0.7,0.2,0.5)	(0.2,0.3,0.4,0.5;0.7,0.2,0.5)	(0.3,0.5,0.65,0.9;0.5,0.2,0.1)	(0.2,0.3,0.4,0.5;0.7,0.2,0.5)	(0.3,0.5,0.65,0.9;0.5,0.2,0.1)	(0.2,0.3,0.4,0.5;0.7,0.2,0.5)
A10	(0.3,0.4,0.5,0.8;0.7,0.2,0.5)	(0.3,0.4,0.5,0.8;0.7,0.2,0.5)	(0.3,0.4,0.5,0.8;0.7,0.2,0.5)	(0.3,0.5,0.65,0.9;0.5,0.2,0.1)	(0.2,0.3,0.4,0.5;0.7,0.2,0.5)	(0.3,0.4,0.5,0.8;0.7,0.2,0.5)	(0.2,0.3,0.4,0.5;0.7,0.2,0.5)	(0.3,0.5,0.65,0.9;0.5,0.2,0.1)
A11	(0.6,0.7,0.75,0.9;0.5,0.2,0.1)	(0.6,0.7,0.75,0.9;0.5,0.2,0.1)	(0.6,0.7,0.75,0.9;0.5,0.2,0.1)	(0.1,0.4,0.65,1;0.5,0.2,0.1)	(0.3,0.4,0.5,0.8;0.7,0.2,0.5)	(0.6,0.7,0.75,0.9;0.5,0.2,0.1)	(0.3,0.4,0.5,0.8;0.7,0.2,0.5)	(0.1,0.4,0.65,1;0.5,0.2,0.1)
A12	(0.1,0.35,0.5,0.7;0.8,0.2,0.1)	(0.1,0.35,0.5,0.7;0.8,0.2,0.1)	(0.1,0.35,0.5,0.7;0.8,0.2,0.1)	(0.5,0.6,0.7,0.8;0.7,0.2,0.5)	(0.6,0.7,0.75,0.9;0.5,0.2,0.1)	(0.1,0.35,0.5,0.7;0.8,0.2,0.1)	(0.6,0.7,0.75,0.9;0.5,0.2,0.1)	(0.2,0.3,0.4,0.5;0.7,0.2,0.5)

Table 4. The third SVTNNs.

	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8
A_1	(0.2,0.3,0.4,0.5;0.7,0.2,0.5)	(0.3,0.5,0.65,0.9;0.5,0.2,0.1)	(0.1,0.4,0.65,1;0.5,0.2,0.1)	(0.5,0.6,0.7,0.8;0.7,0.2,0.5)	(0.1,0.35,0.5,0.7;0.8,0.2,0.1)	(0.6,0.7,0.75,0.9;0.5,0.2,0.1)	(0.3,0.4,0.5,0.8;0.7,0.2,0.5)	(0.1,0.4,0.65,1;0.5,0.2,0.1)
A_2	(0.2,0.3,0.4,0.5;0.7,0.2,0.5)	(0.3,0.4,0.5,0.8;0.7,0.2,0.5)	(0.6,0.7,0.75,0.9;0.5,0.2,0.1)	(0.1,0.35,0.5,0.7;0.8,0.2,0.1)	(0.5,0.6,0.7,0.8;0.7,0.2,0.5)	(0.1,0.4,0.65,1;0.5,0.2,0.1)	(0.2,0.3,0.4,0.5;0.7,0.2,0.5)	(0.5,0.6,0.7,0.8;0.7,0.2,0.5)
A_3	(0.3,0.5,0.65,0.9;0.5,0.2,0.1)	(0.1,0.4,0.65,1;0.5,0.2,0.1)	(0.5,0.6,0.7,0.8;0.7,0.2,0.5)	(0.1,0.35,0.5,0.7;0.8,0.2,0.1)	(0.6,0.7,0.75,0.9;0.5,0.2,0.1)	(0.3,0.5,0.65,0.9;0.5,0.2,0.1)	(0.3,0.5,0.65,0.9;0.5,0.2,0.1)	(0.1,0.35,0.5,0.7;0.8,0.2,0.1)
A_4	(0.5,0.6,0.7,0.8;0.7,0.2,0.5)	(0.1,0.4,0.65,1;0.5,0.2,0.1)	(0.3,0.5,0.65,0.9;0.5,0.2,0.1)	(0.2,0.3,0.4,0.5;0.7,0.2,0.5)	(0.3,0.4,0.5,0.8;0.7,0.2,0.5)	(0.2,0.3,0.4,0.5;0.7,0.2,0.5)	(0.1,0.4,0.65,1;0.5,0.2,0.1)	(0.6,0.7,0.75,0.9;0.5,0.2,0.1)
A_5	(0.1,0.35,0.5,0.7;0.8,0.2,0.1)	(0.5,0.6,0.7,0.8;0.7,0.2,0.5)	(0.1,0.4,0.65,1;0.5,0.2,0.1)	(0.3,0.5,0.65,0.9;0.5,0.2,0.1)	(0.2,0.3,0.4,0.5;0.7,0.2,0.5)	(0.3,0.4,0.5,0.8;0.7,0.2,0.5)	(0.5,0.6,0.7,0.8;0.7,0.2,0.5)	(0.3,0.4,0.5,0.8;0.7,0.2,0.5)
A_6	(0.1,0.35,0.5,0.7;0.8,0.2,0.1)	(0.1,0.35,0.5,0.7;0.8,0.2,0.1)	(0.5,0.6,0.7,0.8;0.7,0.2,0.5)	(0.1,0.35,0.5,0.7;0.8,0.2,0.1)	(0.3,0.4,0.5,0.8;0.7,0.2,0.5)	(0.6,0.7,0.75,0.9;0.5,0.2,0.1)	(0.1,0.35,0.5,0.7;0.8,0.2,0.1)	(0.6,0.7,0.75,0.9;0.5,0.2,0.1)
A_7	(0.6,0.7,0.75,0.9;0.5,0.2,0.1)	(0.6,0.7,0.75,0.9;0.5,0.2,0.1)	(0.1,0.4,0.65,1;0.5,0.2,0.1)	(0.6,0.7,0.75,0.9;0.5,0.2,0.1)	(0.6,0.7,0.75,0.9;0.5,0.2,0.1)	(0.1,0.35,0.5,0.7;0.8,0.2,0.1)	(0.6,0.7,0.75,0.9;0.5,0.2,0.1)	(0.1,0.35,0.5,0.7;0.8,0.2,0.1)
A_8	(0.6,0.7,0.75,0.9;0.5,0.2,0.1)	(0.3,0.4,0.5,0.8;0.7,0.2,0.5)	(0.3,0.5,0.65,0.9;0.5,0.2,0.1)	(0.3,0.4,0.5,0.8;0.7,0.2,0.5)	(0.1,0.35,0.5,0.7;0.8,0.2,0.1)	(0.5,0.6,0.7,0.8;0.7,0.2,0.5)	(0.3,0.4,0.5,0.8;0.7,0.2,0.5)	(0.3,0.5,0.65,0.9;0.5,0.2,0.1)
A_9	(0.3,0.4,0.5,0.8;0.7,0.2,0.5)	(0.2,0.3,0.4,0.5;0.7,0.2,0.5)	(0.2,0.3,0.4,0.5;0.7,0.2,0.5)	(0.2,0.3,0.4,0.5;0.7,0.2,0.5)	(0.5,0.6,0.7,0.8;0.7,0.2,0.5)	(0.1,0.4,0.65,1;0.5,0.2,0.1)	(0.2,0.3,0.4,0.5;0.7,0.2,0.5)	(0.3,0.4,0.5,0.8;0.7,0.2,0.5)
A_{10}	(0.3,0.4,0.5,0.8;0.7,0.2,0.5)	(0.3,0.5,0.65,0.9;0.5,0.2,0.1)	(0.3,0.4,0.5,0.8;0.7,0.2,0.5)	(0.3,0.5,0.65,0.9;0.5,0.2,0.1)	(0.1,0.4,0.65,1;0.5,0.2,0.1)	(0.3,0.5,0.65,0.9;0.5,0.2,0.1)	(0.3,0.5,0.65,0.9;0.5,0.2,0.1)	(0.1,0.4,0.65,1;0.5,0.2,0.1)
A_{11}	(0.2,0.3,0.4,0.5;0.7,0.2,0.5)	(0.1,0.4,0.65,1;0.5,0.2,0.1)	(0.6,0.7,0.75,0.9;0.5,0.2,0.1)	(0.1,0.4,0.65,1;0.5,0.2,0.1)	(0.3,0.5,0.65,0.9;0.5,0.2,0.1)	(0.2,0.3,0.4,0.5;0.7,0.2,0.5)	(0.1,0.4,0.65,1;0.5,0.2,0.1)	(0.1,0.4,0.65,1;0.5,0.2,0.1)
A_{12}	(0.3,0.5,0.65,0.9;0.5,0.2,0.1)	(0.5,0.6,0.7,0.8;0.7,0.2,0.5)	(0.1,0.35,0.5,0.7;0.8,0.2,0.1)	(0.5,0.6,0.7,0.8;0.7,0.2,0.5)	(0.2,0.3,0.4,0.5;0.7,0.2,0.5)	(0.3,0.4,0.5,0.8;0.7,0.2,0.5)	(0.5,0.6,0.7,0.8;0.7,0.2,0.5)	(0.5,0.6,0.7,0.8;0.7,0.2,0.5)

Stage 2 – Establishing criteria weighting utilizing the SVTN-MEREC method.

Then we obtain crisp values between the criteria and alternatives. Then we combine the assessment matrices into a single matrix. Then we normalize the decision matrix using Eq. (9) as shown in Table 5. Then we compute the overall performance of the alternatives using Eq. (10). Then we compute the overall performance of the alternatives by removing each criterion using Eq. (11) as shown in Table 6. Then we compute the summation of the absolute value using Eq. (12). Then we computed the final criteria weights as shown in Fig 1.

Table 5. The normalization matrix.

	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8
A_1	1	1.975	1.717557	1.444444	2.370629	3.25	1.111111	1.451613
A_2	1.324074	1	2.083969	2.574074	1.090909	2.678571	1	1.006452
A_3	2.305556	1.483333	1.19084	2.787037	1.636364	2.821429	1.796296	2.187097

A4	1.787037	1.583333	1.419847	1.25	1	1	2.268519	1.76129
A5	1.898148	1.625	1.450382	1.25	1.027972	1.285714	1.787037	1.245161
A6	2.222222	2.316667	1.48855	1.787037	1.111888	2.178571	1.898148	1.76129
A7	2.268519	1.816667	2.007634	2.527778	1.79021	3.321429	2.37963	2.187097
A8	1.833333	1.616667	1.603053	2.462963	1.678322	2.904762	2.148148	1.496774
A9	1	1.125	1	1	1.440559	1.845238	1.25	1.025806
A10	1.111111	1.225	1.21374	2.157407	1.272727	2.309524	1.722222	1.722581
A11	1.361111	2.008333	1.603053	2.12037	1.027972	2.357143	1.759259	1.303226
A12	2.509259	1.808333	2.587786	1.222222	1.027972	2.154762	1.805556	1

Table 6. The Overall performance matrix.

	C1	**C2**	**C3**	**C4**	**C5**	**C6**	**C7**	**C8**
A1	0.317398	0.271308	0.280942	0.292759	0.258573	0.236177	0.3104	0.292423
A2	0.223761	0.243958	0.190239	0.174237	0.237741	0.171193	0.243958	0.2435
A3	0.342425	0.370496	0.384187	0.330107	0.364315	0.329306	0.358408	0.345824
A4	0.198732	0.207711	0.215728	0.225019	0.241088	0.241088	0.180792	0.199812
A5	0.179226	0.190964	0.199466	0.210477	0.224778	0.2084	0.1838	0.210763
A6	0.304147	0.301351	0.330666	0.318658	0.349542	0.305476	0.314663	0.319617
A7	0.414728	0.427978	0.422037	0.408209	0.428847	0.391568	0.411853	0.41692
A8	0.349741	0.357768	0.358306	0.330642	0.355386	0.319808	0.339535	0.362654
A9	0.115279	0.105691	0.115279	0.115279	0.085261	0.064377	0.097036	0.113212
A10	0.264544	0.257711	0.25836	0.217137	0.255022	0.212139	0.233486	0.233471
A11	0.28556	0.258621	0.274318	0.254804	0.304559	0.247317	0.267873	0.288525
A12	0.250406	0.273325	0.248223	0.30006	0.311648	0.261127	0.273431	0.313483

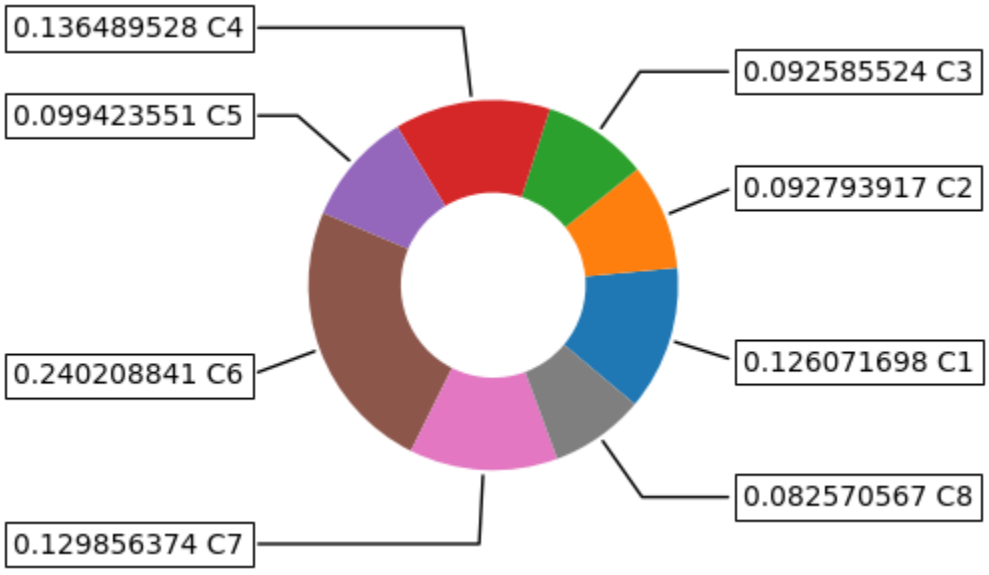

Fig 1. The criteria weights.

Stage 3 – The process of ranking alternatives employing the SVTNSs-ARLON method.

Then we applied the steps of the ARLON method to rank the alternatives and select the best one. Then we used Eqs. (14 and 16) to obtain the first second logarithmic normalization matrix as shown in Table 7 and 8. Then we used Eq. (16) to obtain the combined decision matrix as shown in Table 9. Then we compute the weighted decision matrix using Eq. (17) as shown in Table 10. Then we compute the summations of the weighted combined normalized values for positive and negative criteria using Eqs. (18 and 19). Then we obtain the final rank of alternatives using Eq. (20) as shown in fig 2.

Table 7. The first normalization matrix.

	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8
A_1	0.067406398	0.088274	0.085435	0.078624	0.096608	0.092007	0.073746	0.114597
A_2	0.073483734	0.072868	0.089728	0.091673	0.078996	0.087604	0.071333	0.115587
A_3	0.085490787	0.081793	0.077302	0.093468	0.088197	0.088787	0.084752	0.113489
A_4	0.079975274	0.08327	0.081208	0.075359	0.077022	0.06517	0.090099	0.114074
A_5	0.081281174	0.083858	0.08168	0.075359	0.077648	0.070892	0.084633	0.115012
A_6	0.084693779	0.091886	0.082257	0.083431	0.079428	0.082899	0.086015	0.114074
A_7	0.085140179	0.086382	0.0889	0.091263	0.090236	0.092502	0.091195	0.113489
A_8	0.080529002	0.083742	0.083903	0.090676	0.088771	0.08945	0.08885	0.114514
A_9	0.067406398	0.075534	0.073424	0.070319	0.085305	0.079118	0.076445	0.115536
A_{10}	0.069687412	0.077462	0.077725	0.087685	0.082494	0.084229	0.083787	0.114134
A_{11}	0.074081002	0.088653	0.083903	0.087294	0.077648	0.084693	0.084275	0.114889
A_{12}	0.087323772	0.086278	0.094537	0.074851	0.077648	0.082649	0.08487	0.115605

Table 8. The second normalization matrix.

	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8
A_1	0.071977	0.088274	0.085435	0.078624	0.096608	0.092007	0.073746	0.916777
A_2	0.078466	0.072868	0.089728	0.091673	0.078996	0.087604	0.071333	0.924699
A_3	0.091288	0.081793	0.077302	0.093468	0.088197	0.088787	0.084752	0.90791
A_4	0.085398	0.08327	0.081208	0.075359	0.077022	0.06517	0.090099	0.912594
A_5	0.086793	0.083858	0.08168	0.075359	0.077648	0.070892	0.084633	0.920095
A_6	0.090437	0.091886	0.082257	0.083431	0.079428	0.082899	0.086015	0.912594
A_7	0.090913	0.086382	0.0889	0.091263	0.090236	0.092502	0.091195	0.90791
A_8	0.085989	0.083742	0.083903	0.090676	0.088771	0.08945	0.08885	0.916114
A_9	0.071977	0.075534	0.073424	0.070319	0.085305	0.079118	0.076445	0.924287
A_{10}	0.074413	0.077462	0.077725	0.087685	0.082494	0.084229	0.083787	0.913075
A_{11}	0.079104	0.088653	0.083903	0.087294	0.077648	0.084693	0.084275	0.919109
A_{12}	0.093245	0.086278	0.094537	0.074851	0.077648	0.082649	0.08487	0.924838

Table 9. The combined normalization matrix.

	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8
A_1	0.069672967	0.088274	0.085435	0.078624	0.096608	0.092007	0.073746	0.419908
A_2	0.075954656	0.072868	0.089728	0.091673	0.078996	0.087604	0.071333	0.423537
A_3	0.088365453	0.081793	0.077302	0.093468	0.088197	0.088787	0.084752	0.415847
A_4	0.082664477	0.08327	0.081208	0.075359	0.077022	0.06517	0.090099	0.417992

A5	0.084014289	0.083858	0.08168	0.075359	0.077648	0.070892	0.084633	0.421428
A6	0.087541645	0.091886	0.082257	0.083431	0.079428	0.082899	0.086015	0.417992
A7	0.088003055	0.086382	0.0889	0.091263	0.090236	0.092502	0.091195	0.415847
A8	0.083236825	0.083742	0.083903	0.090676	0.088771	0.08945	0.08885	0.419605
A9	0.069672967	0.075534	0.073424	0.070319	0.085305	0.079118	0.076445	0.423348
A10	0.072030682	0.077462	0.077725	0.087685	0.082494	0.084229	0.083787	0.418213
A11	0.076572008	0.088653	0.083903	0.087294	0.077648	0.084693	0.084275	0.420976
A12	0.090260072	0.086278	0.094537	0.074851	0.077648	0.082649	0.08487	0.4236

Table 10. The weighted normalized matrix.

	C1	**C2**	**C3**	**C4**	**C5**	**C6**	**C7**	**C8**
A1	0.008783789	0.008191	0.00791	0.010731	0.009605	0.022101	0.009576	0.034672
A2	0.009575733	0.006762	0.008308	0.012512	0.007854	0.021043	0.009263	0.034972
A3	0.011140383	0.00759	0.007157	0.012757	0.008769	0.021327	0.011006	0.034337
A4	0.010421651	0.007727	0.007519	0.010286	0.007658	0.015654	0.0117	0.034514
A5	0.010591824	0.007782	0.007562	0.010286	0.00772	0.017029	0.01099	0.034798
A6	0.011036524	0.008526	0.007616	0.011387	0.007897	0.019913	0.01117	0.034514
A7	0.011094695	0.008016	0.008231	0.012456	0.008972	0.02222	0.011842	0.034337
A8	0.010493808	0.007771	0.007768	0.012376	0.008826	0.021487	0.011538	0.034647
A9	0.008783789	0.007009	0.006798	0.009598	0.008481	0.019005	0.009927	0.034956
A10	0.00908103	0.007188	0.007196	0.011968	0.008202	0.020232	0.01088	0.034532
A11	0.009653563	0.008226	0.007768	0.011915	0.00772	0.020344	0.010944	0.03476
A12	0.011379241	0.008006	0.008753	0.010216	0.00772	0.019853	0.011021	0.034977

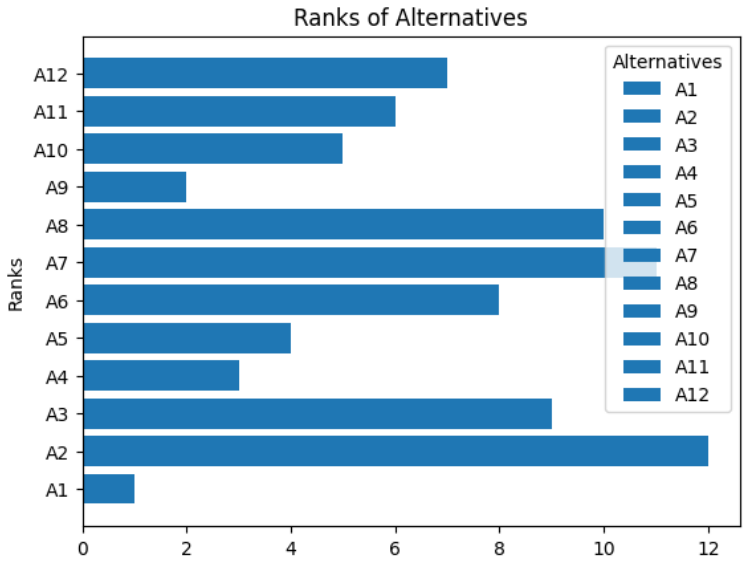

Fig 2. The rank of alternatives.

We conducted the sensitivity analysis between the ranks of the alternatives to show the stability and robustness of the proposed approach. We conducted the sensitivity analysis with two phases. In the first

phase, we change the α value between 0 and 1. In the second phase, we change the β value between 0 and 1. Then we ranked the alternatives.

In the first phase, we rank the alternatives under different cases. In the first case, we show alternative 2 is the best and alternative 1 is the worst. In all cases, we show alternative 2 is the best and alternative 1 is the worst we show the ranks of alternatives are stable under different cases. Fig 3 the ranks of alternatives under phase 1. Fig 4 shows the alternatives values.

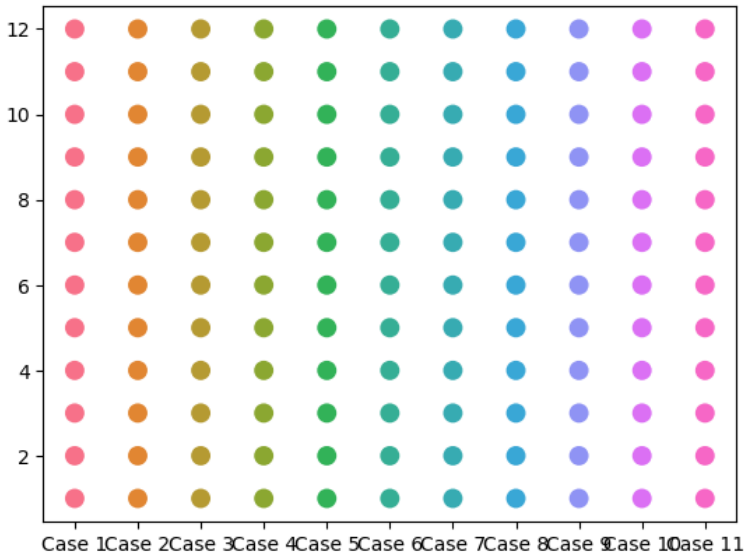

Fig 3. Ranks of alternatives under phase 1.

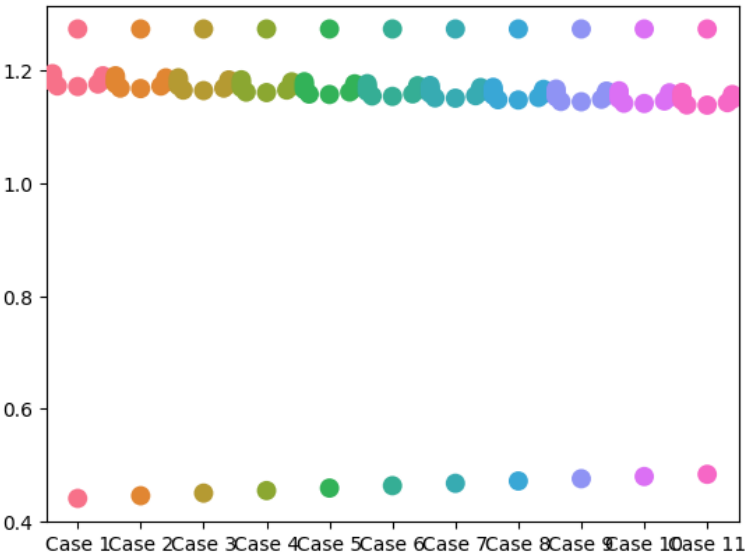

Fig 4. The alternatives values under phase 1.

In the second phase, we rank the alternatives under different cases. In the first case, we show alternative 2 is the best and alternative 1 is the worst. In all cases, we show alternative 2 is the best and alternative 1 is

the worst we show the ranks of alternatives are stable under different cases. Fig 5 shows the alternatives values. Fig 6 the ranks of alternatives under phase 1.

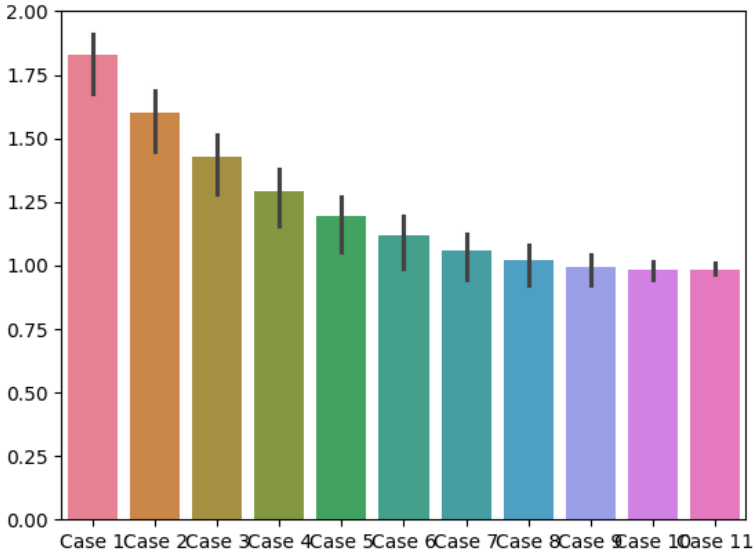

Fig 5. The alternatives values under phase 2.

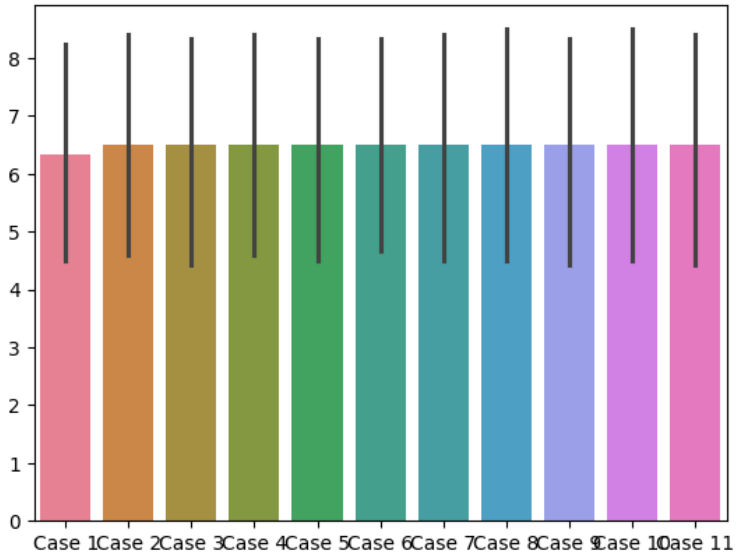

Fig 6. Ranks of alternatives under phase 2.

5. Conclusions

The method of choosing shock absorption performance evaluation of mechanical component springs was clarified by this study. For this goal, the research developed a customized decision support system based on the MAGDM framework. The study emphasized the vital role of shock absorption performance in performance evaluation. The research established a foundational understanding for choosing the best

alternative, especially within the MAGDM paradigm. The study presented the SVTN-MEREC-ARLON method as a novel decision support system employing a tripartite methodology that included expert interaction, criteria significance assessment, and best system determination using a hybrid approach. Validation using real-world case study showed how effective and reliable the suggested method is.

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Fixed Point Theorems in Neutrosophic Fuzzy Metric Space and a Characterization to its Completeness

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Abstract. The present research work aims to establish some basic conventional fixed point theorems such as the Banach fixed point theorem, Edelstein fixed point theorem and Kannan fixed point theorem, for a recently introduced topological space, known as the neutrosophic fuzzy metric space. The Kannan fixed point theorem has also further been classified into two categories namely, $(1/2)-$Kannan and $(1)-$Kannan fixed point theorems. Also the concept of semi-metric in the said space has been introduced, and along with it the results hence obtained are used to characterize the completeness of the mentioned space. To the best of known literatures, this type of work in the said space has not yet been found.

Keywords: Fixed Point (FP); Fuzzy Sets ($F_{\mathfrak{S}}$); Fuzzy Metric Space ($F_{\mathfrak{M}S}$); Neutrosophic Sets ($N_{\mathfrak{S}}$); Neutrosophic Fuzzy Sets ($N_{\mathfrak{F}S}$); Neutrosophic Fuzzy Metric Space ($NF_{\mathfrak{M}S}$); Kannan Contraction ($K_{\mathfrak{C}}$).

1. Introduction

A mathematical paradigm known as $F_{\mathfrak{S}}$ theory addresses sets whose constituents have varying degrees of membership. This divergence from conventional set theory, which holds that an element either belongs to or does not belong to a set, enables more flexible and subtle modeling, especially when category borders are unclear or ambiguous. This field of research was established by [3] a groundbreaking paper, Fuzzy Sets, which was published in 1965. Since then, numerous other scientific disciplines have been impacted by this entirely new area of

inquiry. Since the introduction of $F_{\mathfrak{S}}$ in 1965, many developments have been made. In 1986, for example, [4] presented intuitionistic $F_{\mathfrak{S}}$, and in 1999, [5] proposed $N_{\mathfrak{S}}$. Also, Orthopair $F_{\mathfrak{S}}$ were introduced by [6] in 2023. The introduction of $F_{\mathfrak{M}S}$ by [7] in 1975 and its modification by [8] in 1994, intuitionistic fuzzy topological spaces by [9] in 1997, and intuitionistic $F_{\mathfrak{M}S}$ by [10] in 2004 are just a few of the related concepts that have developed as a result of these foundational contributions.

Nonetheless, a FP of a function is a location that is both within the function's domain and range. It should also be noted that $h(a) = a$ if a is a FP of a function h. In other words, it is the domain element that the function maps to itself. Notably, [1] first proposed this idea in 1912. [2, 16, 18, 19, 41, 45, 47] are some noteworthy works that contributed to the development of this notion after it was established.

Numerous additional mathematical studies have been influenced by the introduction of these ideas and concepts. For example, [11] defined fuzzy metric space in 1984 as the distance between two points and expressed it as a positive fuzzy number; The concept of $N_{\mathfrak{S}}$ was introduced by [12] in 2006 as a generalization of intuitionistic $F_{\mathfrak{S}}$; [14] introduced algebraic operations like modal operators and normalization to intuitionistic $F_{\mathfrak{S}}$ in 2014; [15] investigated the practical applications of $N_{\mathfrak{S}}$ in decision-making contexts; and [13] in 2012 extended the concepts of fuzzy topological space and intuitionistic fuzzy topological space to the case of $N_{\mathfrak{S}}$. In addition to these, a number of researchers investigated other generalizations of fuzzy metric space, including bipolar fuzzy metric space, orthogonal fuzzy metric space, soft fuzzy metric space, intuitionistic fuzzy metric space, and so on [16–26, 47]. Moreover, a number of new ideas pertaining to $N_{\mathfrak{S}}$ were presented by [27], including multi-valued $N_{\mathfrak{S}}$, bipolar $N_{\mathfrak{S}}$, refined $N_{\mathfrak{S}}$, rough $N_{\mathfrak{S}}$, and rough bipolar $N_{\mathfrak{S}}$. Furthermore, [28] used $N_{\mathfrak{S}}$ in combination with multi-criteria decision-making (MCDM) methodologies to solve a real-world wind turbine problem.

[29] presented the idea of neutrosophic metric spaces in 2020. Furthermore, multiple fixed-point results have been developed in this extension [30] with some generalizations of $NF_{\mathfrak{M}S}$ by investigators, which includes orthogonal neutrosophic metric spaces in [31], neutrosophic 2-metric spaces in [32], orthogonal neutrosophic 2-metric spaces in [33], neutrosophic pentagonal metric spaces in [34], neutrosophic b-metric space in [35], then in [48] the definition of the idea of neutrosophic rectangular extended b-metric spaces and establishment of some FP findings it, [49] also proposed a theory regarding neutrosophic triplet v-generalized metric space etc. Notably, $N_{\mathfrak{F}S}$ were also introduced by [36] in 2020. It is easy to see how the idea of fuzzy metric space was developed using the idea of $F_{\mathfrak{S}}$. Similarly, the concept of neutrosophic metric space has been defined by the philosophy of $N_{\mathfrak{S}}$. Given these considerations, the main goal

of this study is to clarify the idea behind $NF_{\mathfrak{MS}}$. For this, the idea of $N_{\mathfrak{FS}}$ has been applied, much like previously mentioned.

More recently, Ghosh et al. [44] introduced $NF_{\mathfrak{MS}}$ (NFMS), a mathematical framework that more closely reflects the complex and uncertain nature of real-world phenomena like medical diagnosis, decision-making under ambiguity, or identifying trends in ambiguous data sets. This encourages the creation of $NF_{\mathfrak{MS}}$ theory in order to improve the models, analyses, and decision-making capabilities in uncertain contexts. FP theorems pertaining to $NF_{\mathfrak{MS}}$ are a very new area of study, and virtually no other mathematician has investigated them. The Banach, Edelstein, and Kannan FP theorems are established in light of the $NF_{\mathfrak{MS}}$ in order to close this gap.

The potential of $NF_{\mathfrak{MS}}$ to more precisely represent uncertainty and indeterminacy in real-world occurrences than conventional fuzzy or crisp sets is what inspired their introduction. $N_{\mathfrak{S}}$ offer a more complex explanation of uncertainty by enabling the representation of elements with three components: truth, indeterminacy, and falsehood. The field of medical diagnosis could serve as an illustration of the necessity of NFMS. Imagine a situation in which a patient exhibits symptoms that are not exclusively suggestive of one illness. On a scale from 0 to 1, traditional $F_{\mathfrak{S}}$ could depict the patient's probability of getting a certain illness. However, if the symptoms are unclear or contradictory, this method might not fully convey the doubt around the diagnosis. Neutosophic sets, on the other hand, can more fully capture the uncertainty by explicitly taking into consideration the degree of truth, indeterminacy, and falsehood connected to every symptom and possible diagnosis. Then, using $NF_{\mathfrak{MS}}$, a metric space may be defined in which the distance between two diagnoses indicates both how similar they are and how ambiguous or indeterminate each one is.

The paper's structure is described as follows. Section 2 presents the basic concepts of $N_{\mathfrak{FS}}$, $F_{\mathfrak{MS}}$, and $NF_{\mathfrak{MS}}$. A few FP theorems in $NF_{\mathfrak{MS}}$ are established in Section 3. Finally, the conclusion is included in Section 4 of this work. Additionally, this research can be expanded to examine $NF_{\mathfrak{MS}}$ in relation to other ideas like approximation theory, optimization theory, best proximity point results, etc. Furthermore, the established theory and results can be linked to ideas such as generalized metric spaces, soft sets, soft equality, soft lattices, etc. [37, 38, 45, 46].

2. **Preliminaries**

To establish the main outcomes the hypothesis are considered as follows:

Theorem 2.1. *[2] Every contraction mapping on a complete metric space possesses a unique FP.*

Definition 2.2. **($F_{\mathfrak{S}}$)** [3, 7, 10] A $F_{\mathfrak{S}}$ $\mathfrak{F}$ in an universal set X is defined as
$\mathfrak{F} = \{< \rho, \mu_{\mathfrak{F}}(\rho) >: 0 \leq \mu_{\mathfrak{F}}(\rho) \leq 1, \rho \in X\}$.
In this context, $\mu_{\mathfrak{F}}(\rho)$ is also known as the membership grade of ρ in $\mathfrak{F}$.

Definition 2.3. **($N_{\mathfrak{S}}$)** [5, 29] A $N_{\mathfrak{S}}$ $\mathfrak{N}$ in an universal set X is defined as $\mathfrak{N} = \{< \rho, (T_{\mathfrak{N}}(\rho), I_{\mathfrak{N}}(\rho), F_{\mathfrak{N}}(\rho)) >: \rho \in X,\ T_{\mathfrak{N}}(\rho), I_{\mathfrak{N}}(\rho), F_{\mathfrak{N}}(\rho) \in]0^-, 1^+[\}$. In this context, $T_{\mathfrak{N}}(\rho), I_{\mathfrak{N}}(\rho), F_{\mathfrak{N}}(\rho)$ are the notations for truth, indeterminacy and falsity membership grades of ρ in $\mathfrak{N}$ respectively. Here, $]0^-, 1^+[$ is another way of representing $[0, 1]$ under the purview of neutrosophic logic.

Definition 2.4. **($N_{\mathfrak{F}S}$)** [36, 44] The definition of a $N_{\mathfrak{F}S}$ $\mathfrak{B}$ in a universal set X is given as,

$$\mathfrak{B} = \{< \rho, (\mu_{\mathfrak{B}}(\rho), T_{\mathfrak{B}}(\rho, \mu), I_{\mathfrak{B}}(\rho, \mu), F_{\mathfrak{B}}(\rho, \mu)) >: \rho \in X, \mu_{\mathfrak{B}}(\rho) \in [0, 1],$$
$$T_{\mathfrak{B}}(\rho, \mu),\ I_{\mathfrak{B}}(\rho, \mu),\ F(\rho, \mu) \in]0^-, 1^+[\}$$

In this context, each membership grade $\mu_{\mathfrak{B}}(\rho)$ is expressed by a truth, indeterminacy and falsity membership grade denoted by $T_{\mathfrak{B}}(\rho, \mu), I_{\mathfrak{B}}(\rho, \mu)$ and $F_{\mathfrak{B}}(\rho, \mu)$ respectively.

Definition 2.5. [7, 8, 29] A function $\star : [0, 1] \times [0, 1] \to [0, 1]$ is called a continuous t-norm (TN) if it satisfies the following conditions, for any $a, b, c, d \in [0, 1]$:

(1) $a \star 1 = a$,
(2) If $a \leq b$ and $c \leq d$, then $a \star c \leq b \star d$,
(3) $\star$ is continuous,
(4) $\star$ is commutative and associative.

Definition 2.6. [7, 8, 29] A function $\diamond : [0, 1] \times [0, 1] \to [0, 1]$ is called a continuous t-conorm (TC) if it satisfies the following conditions, for any $a, b, c, d \in [0, 1]$:

(1) $a \star 0 = a$,
(2) If $a \leq b$ and $c \leq d$, then $a \star c \leq b \star d$,
(3) $\star$ is continuous,
(4) $\star$ is commutative and associative.

Definition 2.7. **($F_{\mathfrak{M}S}$)** [7, 47] A 3-tuple $(G, \mathfrak{S}, \star)$ is said to be a fuzzy metric space if G is a non empty arbitary set, $\star$ is a continuous t-norm and $\mathfrak{S}$ is a $F_{\mathfrak{S}}$ on $G^2 \times (0, \infty)$ satisfying the following conditions, for any $\rho, \sigma, w \in G$ and $s, \kappa > 0$

(1) $\mathfrak{S}(\rho, \sigma, \kappa) > 0$,
(2) $\mathfrak{S}(\rho, \sigma, \kappa) = 1$ iff $\rho = \sigma$,
(3) $\mathfrak{S}(\rho, \sigma, \kappa) = \mathfrak{S}(\sigma, \rho, \kappa)$,
(4) $\mathfrak{S}(\rho, \sigma, \kappa) \star \mathfrak{S}(\sigma, w, s) \leq \mathfrak{S}(\rho, w, \kappa + s)$,
(5) $\mathfrak{S}(\rho, \sigma, .) : (0, \infty) \to [0, 1]$ is continuous.

Definition 2.8. **($NF_{\mathfrak{M}S}$)** [44] A $NF_{\mathfrak{M}S}$ (NFMS) is a seven-tuple $(G, \mathfrak{S}, \mathfrak{A}, \mathfrak{M}, \mathfrak{R}, \star, \diamond)$, given G is an arbitary set, $\star$ a continuous t-norm, $\diamond$ a continuous t-conorm and $\mathfrak{S}, \mathfrak{A}, \mathfrak{M}, \mathfrak{R}$ are $F_{\mathfrak{S}}$ on $G^2 \times (0, \infty)$ which are analogous to the notations $\mu(\rho), T_{\mathfrak{B}}(\rho, \mu), I_{\mathfrak{B}}(\rho, \mu)$ and $F_{\mathfrak{B}}(\rho, \mu)$ in Definition 2.3 satisfying the following conditions, for any $\rho, \sigma, w \in G$ and $s, \kappa > 0$,

(1) $0 \leq \mathfrak{S}(\rho, \sigma, \kappa) \leq 1,\ 0 \leq \mathfrak{A}(\rho, \sigma, \kappa) \leq 1,\ 0 \leq \mathfrak{M}(\rho, \sigma, \kappa) \leq 1,\ 0 \leq \mathfrak{R}(\rho, \sigma, \kappa) \leq 1$,
(2) $\mathfrak{S}(\rho, \sigma, \kappa) + \mathfrak{A}(\rho, \sigma, \kappa) + \mathfrak{M}(\rho, \sigma, \kappa) + \mathfrak{R}(\rho, \sigma, \kappa) \leq 4$,
(3) $\mathfrak{S}(\rho, \sigma, \kappa) = \mathfrak{S}(\sigma, \rho, \kappa)$,
(4) $\mathfrak{S}(\rho, \sigma, \kappa) = 1$ iff $\rho = \sigma$,
(5) $\lim_{t \to \infty} \mathfrak{S}(\rho, \sigma, \kappa) = 1$,
(6) $\mathfrak{S}(\rho, \sigma, \kappa) \star \mathfrak{S}(\sigma, w, s) \leq \mathfrak{S}(\rho, z, \kappa + s)$,
(7) $\mathfrak{S}(\rho, \sigma, .) : \mathbb{R}^+ \cup \{0\} \to (0, 1]$ is continuous,
(8) $\mathfrak{A}(\rho, \sigma, \kappa) = \mathfrak{A}(\sigma, \rho, \kappa)$,
(9) $\mathfrak{A}(\rho, \sigma, \kappa) = 1$ iff $\rho = \sigma$,
(10) $\lim_{t \to \infty} \mathfrak{A}(\rho, \sigma, \kappa) = 1$,
(11) $\mathfrak{A}(\rho, \sigma, \kappa) \star \mathfrak{A}(\sigma, w, s) \leq \mathfrak{A}(\rho, z, \kappa + s)$,
(12) $\mathfrak{A}(\rho, \sigma, .) : \mathbb{R}^+ \cup \{0\} \to (0, 1]$ is continuous,
(13) $\mathfrak{M}(\rho, \sigma, \kappa) = \mathfrak{M}(\sigma, \rho, \kappa)$,
(14) $\mathfrak{M}(\rho, \sigma, \kappa) = 0$ iff $\rho = \sigma$,
(15) $\lim_{t \to \infty} \mathfrak{M}(\rho, \sigma, \kappa) = 0$,
(16) $\mathfrak{M}(\rho, \sigma, \kappa) \diamond \mathfrak{M}(\sigma, w, s) \geq \mathfrak{M}(\rho, z, \kappa + s)$,
(17) $\mathfrak{M}(\rho, \sigma, .) : \mathbb{R}^+ \cup \{0\} \to (0, 1]$ is continuous,
(18) $\mathfrak{R}(\rho, \sigma, \kappa) = \mathfrak{R}(\sigma, \rho, \kappa)$,
(19) $\mathfrak{R}(\rho, \sigma, \kappa) = 0$ iff $\rho = \sigma$,
(20) $\lim_{t \to \infty} \mathfrak{R}(\rho, \sigma, \kappa) = 0$,
(21) $\mathfrak{R}(\rho, \sigma, \kappa) \diamond \mathfrak{R}(\sigma, w, s) \geq \mathfrak{R}(\rho, z, \kappa + s)$,
(22) $\mathfrak{R}(\rho, \sigma, .) : \mathbb{R}^+ \cup \{0\} \to (0, 1]$ is continuous,
(23) $\mathfrak{S}(\rho, \sigma, \kappa) = 0,\ \mathfrak{A}(\rho, \sigma, \kappa) = 0,\ \mathfrak{M}(\rho, \sigma, \kappa) = 1$ and $\mathfrak{R}(\rho, \sigma, \kappa) = 1$ if $\kappa \leq 0$.

Here, the certainty that the distance between ρ and σ is smaller than t is represented by $\mathfrak{S}(\rho, \sigma, \kappa)$, while the degrees of nearness, neutrality, and non-nearness of σ and σ with regard to t are represented by $\mathfrak{A}(\rho, \sigma, \kappa), \mathfrak{M}(\rho, \sigma, \kappa)$, and $\mathfrak{R}(\rho, \sigma, \kappa)$, respectively.

Remark 2.9. From the above definition, it is clear that, for any $\rho, \sigma \in G$, $\mathfrak{S}(\rho, \sigma, .)$ and $\mathfrak{A}(\rho, \sigma, .)$ are non-decreasing functions on $[0, \infty)$, and on the other hand, $\mathfrak{M}(\rho, \sigma, .)$ and $\mathfrak{R}(\rho, \sigma, .)$ are non-increasing functions on $[0, \infty)$.

Remark 2.10. It can be concluded easily from Remark 2.9 that, given $\rho, \sigma \in G$, if $\mathfrak{S}(\rho, \sigma, \kappa),\ \mathfrak{A}(\rho, \sigma, \kappa) > 1 - \kappa$ and $\mathfrak{M}(\rho, \sigma, \kappa),\ \mathfrak{R}(\rho, \sigma, \kappa) < \kappa\ \forall\ \kappa \in (0, \infty)$, then $\rho = \sigma$.

Every neutrosophic fuzzy metric $(\mathfrak{S}, \mathfrak{A}, \mathfrak{M}, \mathfrak{R}, \star, \diamond)$ on a set G induces a metrizable topology $\tau_{(\mathfrak{S},\mathfrak{A},\mathfrak{M},\mathfrak{R})}$ on G which has the family of open sets $\{O(\rho, \epsilon, \kappa) : \rho \in G, \epsilon \in (0,1), \kappa \in (0,\infty)\}$ as its base, where for any $\epsilon \in (0,1)$ and $\kappa > 0$,

$$O(\rho, \epsilon, \kappa) = \{\sigma \in G : \mathfrak{S}(\rho, \sigma, \kappa) > 1 - \epsilon, \mathfrak{A}(\rho, \sigma, \kappa) > 1 - \epsilon, \mathfrak{M}(\rho, \sigma, \kappa) < \epsilon, \mathfrak{R}(\rho, \sigma, \kappa) < \epsilon\}.$$

Definition 2.11. Let $(G, \mathfrak{S}, \mathfrak{A}, \mathfrak{M}, \mathfrak{R}, \star, \diamond)$ be a $NF_{\mathfrak{M}S}$.

(a) A sequence $\{\rho_n\} \in G$ is known to be a Cauchy sequence if for each $t, \alpha > 0$,

$$\lim_{n\to\infty} \mathfrak{S}(\rho_{n+\alpha}, \rho_n, \kappa) = \lim_{n\to\infty} \mathfrak{A}(\rho_{n+\alpha}, \rho_n, \kappa) = 1 \text{ and,}$$
$$\lim_{n\to\infty} \mathfrak{M}(\rho_{n+\alpha}, \rho_n, \kappa) = \lim_{n\to\infty} \mathfrak{R}(\rho_{n+\alpha}, \rho_n, \kappa) = 0.$$

(b) A sequence $\{\rho_n\} \in G$ is convergent to some $\rho \in G$ if $\forall\ \kappa > 0$,

$$\lim_{n\to\infty} \mathfrak{S}(\rho_n, \rho, \kappa) = \lim_{n\to\infty} \mathfrak{A}(\rho_n, \rho, \kappa) = 1 \text{ and,}$$
$$\lim_{n\to\infty} \mathfrak{M}(\rho_n, \rho, \kappa) = \lim_{n\to\infty} \mathfrak{R}(\rho_n, \rho, \kappa) = 0.$$

(The limit may be uniquely ascertained from the specifications (6), (11), (16), and (21) in Definition 2.8, since $\star$ and $\diamond$ are continuous.)

A $NF_{\mathfrak{M}S}$ is considered compact if every sequence contains a convergent subsequence, and complete if and only if every Cauchy sequence is convergent.

We now go over a crucial example involving $NF_{\mathfrak{M}S}$ that will be utilized later, in the last part of the study.

Example 2.12. Let (G, d) be a metric space. Specify the functions,

$$\mathfrak{S}^d_{01}, \mathfrak{A}^d_{01}, \mathfrak{M}^d_{01}, \mathfrak{R}^d_{01} : G \times G \times (0,\infty) \to [0,1] \text{ as,}$$

(1) $\mathfrak{S}^d_{01}(\rho, \sigma, \kappa) = \mathfrak{A}^d_{01}(\rho, \sigma, \kappa) = 1$ if $d(\rho, \sigma) < \kappa$,
(2) $\mathfrak{S}^d_{01}(\rho, \sigma, \kappa) = \mathfrak{A}^d_{01}(\rho, \sigma, \kappa) = 0$ if $d(\rho, \sigma) \geq \kappa$,
(3) $\mathfrak{M}^d_{01}(\rho, \sigma, \kappa) = \mathfrak{R}^d_{01}(\rho, \sigma, \kappa) = 0$ if $d(\rho, \sigma) < \kappa$,
(4) $\mathfrak{M}^d_{01}(\rho, \sigma, \kappa) = \mathfrak{R}^d_{01}(\rho, \sigma, \kappa) = 1$ if $d(\rho, \sigma) \geq \kappa$.

Then, for any continuous t-norm and t-conorm, $\star$ and $\diamond$, respectively, $(\mathfrak{S}^d_{01}, \mathfrak{A}^d_{01}, \mathfrak{M}^d_{01}, \mathfrak{R}^d_{01}, \star, \diamond)$ is a neutrosophic fuzzy metric on G such that the topologies induced by d and $(\mathfrak{S}^d_{01}, \mathfrak{A}^d_{01}, \mathfrak{M}^d_{01}, \mathfrak{R}^d_{01}, \star, \diamond)$ coincide. Additionally, if (G, d) is complete, then it can be said that $(\mathfrak{S}^d_{01}, \mathfrak{A}^d_{01}, \mathfrak{M}^d_{01}, \mathfrak{R}^d_{01}, \star, \diamond)$ is also complete.

Definition 2.13. [39] A semi-metric for a topological space (X, τ) is a function denoted as $d : X \times X \to [0,\infty)$, satisfying the following conditions $\forall\ \rho, \sigma \in X$ and $\mathfrak{A} \subset X$,

(1) $d(\rho, \sigma) = 0 \iff \rho = \sigma$,
(2) $d(\rho, \sigma) = d(\sigma, \rho)$,
(3) $\rho \in \bar{\mathfrak{A}} \iff d(\rho, \mathfrak{A}) = 0$.

Definition 2.14. [39] Let $(X, \mathfrak{S}, \star)$ be a fuzzy metric space. For each $\rho, \sigma \in X$, take $d_{\mathfrak{S}}(\rho, \sigma) = \sup\{\kappa \geq 0 : \mathfrak{S}(\rho, \sigma, \kappa) \leq 1 - \kappa\}$. Then $d_{\mathfrak{S}}$ is a semi-metric for (X, τ_S), satisfying $d_{\mathfrak{S}} \leq 1$ and $d_{\mathfrak{S}}(\rho, \sigma) < \kappa \iff \mathfrak{S}(\rho, \sigma, \kappa) > 1 - \kappa \ \forall \ \kappa > 0$. Therefore a sequence in X is a Cauchy sequence for $d_{\mathfrak{S}}$ iff it is a Cauchy sequence in $(X, \mathfrak{S}, \star)$.

Theorem 2.15. *($K_{\mathfrak{C}}$)***(Kannan [41])** *Let (G, d) be a complete metric space and $\mathfrak{F} : G \to G$ be a mapping such that there exists a constant $c \in (0, 1/2)$ that satisfies, $\forall\ \rho, \sigma \in G$,*

$$d(\mathfrak{F}\rho, \mathfrak{F}\sigma) \leq c[d(\rho, \mathfrak{F}\rho) + d(\sigma, \mathfrak{F}\sigma)],\ \forall\ \rho, \sigma \in G.$$

Then $\mathfrak{F}$ has an unique FP.

Theorem 2.16. (Reich [42], Subrahmanyam [43]) *A metric space (G, d) is complete iff every $K_{\mathfrak{C}}$ on G has a FP.*

Definition 2.17. [39] Let $(G, \mathfrak{S}, \star)$ be a fuzzy metric space. Then a mapping $\mathfrak{F} : G \to G$ is a $(1)-K_{\mathfrak{C}}$ on G if,

$$\min\{\mathfrak{S}(\rho, \mathfrak{F}\rho, \kappa), \mathfrak{S}(\sigma, \mathfrak{F}\sigma, \kappa)\} > 1 - \kappa \implies \mathfrak{S}(\mathfrak{F}\rho, \mathfrak{F}\sigma, c\kappa) > 1 - c\kappa$$

for some $c \in (0, 1),\ \forall\ \kappa > 0$ and $\rho, \sigma \in G$.

Definition 2.18. [39] Let $(G, \mathfrak{S}, \star)$ be a fuzzy metric space. Then a mapping $\mathfrak{F} : G \to G$ is a $(1/2)-K_{\mathfrak{C}}$ on G if,

$$\min\{\mathfrak{S}(\rho, \mathfrak{F}\rho, \kappa), \mathfrak{S}(\sigma, \mathfrak{F}\sigma, \kappa)\} > 1 - \kappa \implies \mathfrak{S}(\mathfrak{F}\rho, \mathfrak{F}\sigma, c\kappa) > 1 - c\kappa$$

for some $c \in (0, 1/2),\ \forall\ \kappa > 0$ and $\rho, \sigma \in G$.

3. Main results

As the first result of the study, we present Banach type FP theorem in a complete $NF_{\mathfrak{M}S}$.

Theorem 3.1. *Let $(G, \mathfrak{S}, \mathfrak{A}, \mathfrak{M}, \mathfrak{R}, \star, \diamond)$ be a complete $NF_{\mathfrak{M}S}$. Let $\mathfrak{F} : G \to G$ be a mapping that satisfies,*

$$\mathfrak{S}(\mathfrak{F}\rho, \mathfrak{F}\sigma, \lambda\kappa) \geq \mathfrak{S}(\rho, \sigma, \kappa), \mathfrak{A}(\mathfrak{F}\rho, \mathfrak{F}\sigma, \lambda\kappa) \geq \mathfrak{A}(\rho, \sigma, \kappa),$$

and

$$\mathfrak{M}(\mathfrak{F}\rho, \mathfrak{F}\sigma, \lambda\kappa) \leq \mathfrak{M}(\rho, \sigma, \kappa), \mathfrak{R}(\mathfrak{F}\rho, \mathfrak{F}\sigma, \lambda\kappa) \leq \mathfrak{R}(\rho, \sigma, \kappa),$$

for any $\rho, \sigma \in G$ and $\lambda \in (0, 1)$. Then F has an unique FP.

Proof. Let $\rho \in G$ and $\rho_n = \mathfrak{F}^n\rho$ $(n \in \mathbb{N})$. By simple induction it can be written that,

$$\mathfrak{S}(\rho_n, \rho_{n+1}, \kappa) \geq \mathfrak{S}(\rho, \rho_1, \kappa/\lambda^n), \mathfrak{A}(\rho_n, \rho_{n+1}, \kappa) \geq \mathfrak{A}(\rho, \rho_1, \kappa/\lambda^n),$$

and

$$\mathfrak{M}(\rho_n, \rho_{n+1}, \kappa) \leq \mathfrak{M}(\rho, \rho_1, \kappa/\lambda^n), \mathfrak{R}(\rho_n, \rho_{n+1}, \kappa) \leq \mathfrak{R}(\rho, \rho_1, \kappa/\lambda^n),$$

for any $n, \kappa > 0$. Thus using the above relations, for any positive integer α it is obtained that,

$$\mathfrak{S}(\rho_n, \rho_{n+\alpha}, \kappa) \geq \mathfrak{S}(\rho_n, \rho_{n+1}, \kappa/\alpha) \star \ldots \star \mathfrak{S}(\rho_{n+\alpha-1}, \rho_{n+\alpha}, \kappa/\alpha) \geq \mathfrak{S}(\rho, \rho_1, \kappa/\alpha\lambda^n) \star \ldots \star \mathfrak{S}(\rho, \rho_1, \kappa/\alpha\lambda^{n+\alpha-1})$$

$$\mathfrak{A}(\rho_n, \rho_{n+\alpha}, \kappa) \geq \mathfrak{A}(\rho_n, \rho_{n+1}, \kappa/\alpha) \star \ldots \star \mathfrak{A}(\rho_{n+\alpha-1}, \rho_{n+\alpha}, \kappa/\alpha) \geq \mathfrak{A}(\rho, \rho_1, \kappa/\alpha\lambda^n) \star \ldots \star \mathfrak{A}(\rho, \rho_1, \kappa/\alpha\lambda^{n+\alpha-1})$$

$$\mathfrak{M}(\rho_n, \rho_{n+\alpha}, \kappa) \leq \mathfrak{M}(\rho_n, \rho_{n+1}, \kappa/\alpha) \diamond \ldots \diamond \mathfrak{M}(\rho_{n+\alpha-1}, \rho_{n+\alpha}, \kappa/\alpha) \leq \mathfrak{M}(\rho, \rho_1, \kappa/\alpha\lambda^n) \diamond \ldots \diamond \mathfrak{M}(\rho, \rho_1, \kappa/\alpha\lambda^{n+\alpha-1})$$

$$\mathfrak{R}(\rho_n, \rho_{n+\alpha}, \kappa) \leq \mathfrak{R}(\rho_n, \rho_{n+1}, \kappa/\alpha) \diamond \ldots \diamond \mathfrak{R}(\rho_{n+\alpha-1}, \rho_{n+\alpha}, \kappa/\alpha) \leq \mathfrak{R}(\rho, \rho_1, \kappa/\alpha\lambda^n) \diamond \ldots \diamond \mathfrak{R}(\rho, \rho_1, \kappa/\alpha\lambda^{n+\alpha-1})$$

Using (5), (10), (15) and (20) from Definition 2.8, it follows that,

$$\lim_{n\to\infty} \mathfrak{S}(\rho_{n+\alpha}, \rho_n, \kappa) \geq \underbrace{1 \star \ldots \star 1}_{\alpha \text{ times}} = 1,$$

$$\lim_{n\to\infty} \mathfrak{A}(\rho_{n+\alpha}, \rho_n, \kappa) \geq \underbrace{1 \star \ldots \star 1}_{\alpha \text{ times}} = 1,$$

$$\lim_{n\to\infty} \mathfrak{M}(\rho_{n+\alpha}, \rho_n, \kappa) \leq \underbrace{0 \diamond \ldots \diamond 0}_{\alpha \text{ times}} = 0,$$

$$\lim_{n\to\infty} \mathfrak{R}(\rho_{n+\alpha}, \rho_n, \kappa) \leq \underbrace{0 \diamond \ldots \diamond 0}_{\alpha \text{ times}} = 0.$$

That is, $\{\rho_n\}$ is a Cauchy sequence, and hence convergent. Let the limit be σ. Thus by (6), (11), (16) and (21) from Definition 2.8, it follows that,

$$\mathfrak{S}(\mathfrak{F}\sigma, \sigma, \kappa) \geq \mathfrak{S}(\mathfrak{F}\sigma, \mathfrak{F}\rho_n, \tfrac{\kappa}{2}) \star \mathfrak{S}(\rho_{n+1}, \sigma, \tfrac{\kappa}{2}) \geq \mathfrak{S}(\sigma, \rho_n, \tfrac{\kappa}{2\lambda}) \star \mathfrak{S}(\rho_{n+1}, \sigma, \tfrac{\kappa}{2}) \to 1 \star 1 = 1,$$
$$\mathfrak{A}(\mathfrak{F}\sigma, \sigma, \kappa) \geq \mathfrak{A}(\mathfrak{F}\sigma, \mathfrak{F}\rho_n, \tfrac{\kappa}{2}) \star \mathfrak{A}(\rho_{n+1}, \sigma, \tfrac{\kappa}{2}) \geq \mathfrak{A}(\sigma, \rho_n, \tfrac{\kappa}{2\lambda}) \star \mathfrak{A}(\rho_{n+1}, \sigma, \tfrac{\kappa}{2}) \to 1 \star 1 = 1,$$
$$\mathfrak{M}(\mathfrak{F}\sigma, \sigma, \kappa) \leq \mathfrak{M}(\mathfrak{F}\sigma, \mathfrak{F}\rho_n, \tfrac{\kappa}{2}) \diamond \mathfrak{M}(\rho_{n+1}, \sigma, \tfrac{\kappa}{2}) \leq \mathfrak{M}(\sigma, \rho_n, \tfrac{\kappa}{2\lambda}) \diamond \mathfrak{M}(\rho_{n+1}, \sigma, \tfrac{\kappa}{2}) \to 0 \diamond 0 = 0,$$
$$\mathfrak{R}(\mathfrak{F}\sigma, \sigma, \kappa) \leq \mathfrak{M}(\mathfrak{F}\sigma, \mathfrak{F}\rho_n, \tfrac{\kappa}{2}) \diamond \mathfrak{R}(\rho_{n+1}, \sigma, \tfrac{\kappa}{2}) \leq \mathfrak{R}(\sigma, \rho_n, \tfrac{\kappa}{2\lambda}) \diamond \mathfrak{R}(\rho_{n+1}, \sigma, \tfrac{\kappa}{2}) \to 0 \diamond 0 = 0.$$

Now by using (4), (9), (14) and (19) from Definition 2.8, it is clear that, $\mathfrak{F}\sigma = \sigma$.

Hence σ is a FP of $\mathfrak{F}$.

For establishing uniqueness, let $\mathfrak{F}w = w$ for some $w \in G$. Then,

$$1 \geq \mathfrak{S}(w, \sigma, \kappa) = \mathfrak{S}(\mathfrak{F}w, \mathfrak{F}\sigma, \kappa) \geq \mathfrak{S}(w, \sigma, \tfrac{\kappa}{\alpha}) = \mathfrak{S}(\mathfrak{F}w, \mathfrak{F}\sigma, \tfrac{\kappa}{\alpha}) \geq \mathfrak{S}(w, \sigma, \tfrac{\kappa}{\alpha^2}) \geq \ldots \geq \mathfrak{S}(w, \sigma, \tfrac{\kappa}{\alpha^n}) \to 1,$$
$$1 \geq \mathfrak{A}(w, \sigma, \kappa) = \mathfrak{A}(\mathfrak{F}w, \mathfrak{F}\sigma, \kappa) \geq \mathfrak{A}(w, \sigma, \tfrac{\kappa}{\alpha}) = \mathfrak{A}(\mathfrak{F}w, \mathfrak{F}\sigma, \tfrac{\kappa}{\alpha}) \geq \mathfrak{A}(w, \sigma, \tfrac{\kappa}{\alpha^2}) \geq \ldots \geq \mathfrak{A}(w, \sigma, \tfrac{\kappa}{\alpha^n}) \to 1,$$
$$0 \leq \mathfrak{M}(w, \sigma, \kappa) = \mathfrak{M}(\mathfrak{F}w, \mathfrak{F}\sigma, \kappa) \leq \mathfrak{M}(w, \sigma, \tfrac{\kappa}{\alpha}) = \mathfrak{M}(\mathfrak{F}w, \mathfrak{F}\sigma, \tfrac{\kappa}{\alpha}) \leq \mathfrak{M}(w, \sigma, \tfrac{\kappa}{\alpha^2}) \leq \ldots \leq \mathfrak{M}(w, \sigma, \tfrac{\kappa}{\alpha^n}) \to 0,$$

$$0 \leq \Re(w,\sigma,\kappa) = \Re(\mathfrak{F}w,\mathfrak{F}\sigma,\kappa) \leq \Re(w,\sigma,\tfrac{\kappa}{\alpha}) = \Re(\mathfrak{F}w,\mathfrak{F}\sigma,\tfrac{\kappa}{\alpha}) \leq \Re(w,\sigma,\tfrac{\kappa}{\alpha^2}) \leq \ldots \leq \Re(w,\sigma,\tfrac{\kappa}{\alpha^n}) \to 0,$$

as $n \to \infty$. Therefore by (4), (9), (14) and (19) from Definition 2.8, it follows that, $\sigma = w$. Hence the FP is unique. □

Lemma 3.2. *If* $\lim_{n\to\infty} \rho_n = x$ *and* $\lim_{n\to\infty} \sigma_n = \sigma$, *then* $\forall\ \kappa \in (0,\infty)$ *and* $\epsilon \in (0,\kappa)$,

$$\mathfrak{S}(\rho,\sigma,\kappa-\epsilon) \leq \lim_{n\to\infty} \inf \mathfrak{S}(\rho_n,\sigma_n,\kappa),\ \mathfrak{A}(\rho,\sigma,\kappa-\epsilon) \leq \lim_{n\to\infty} \inf \mathfrak{A}(\rho_n,\sigma_n,\kappa),$$

and

$$\mathfrak{M}(\rho,\sigma,\kappa-\epsilon) \geq \lim_{n\to\infty} \sup \mathfrak{M}(\rho_n,\sigma_n,\kappa),\ \Re(\rho,\sigma,\kappa-\epsilon) \geq \lim_{n\to\infty} \sup \Re(\rho_n,\sigma_n,\kappa).$$

Proof. By (6), (11), (16) and (21) in Definition 2.8, it can be deduced that,

$$\begin{aligned}
\mathfrak{S}(\rho_n,\sigma_n,\kappa) &\geq \mathfrak{S}(\rho_n,x,\tfrac{\epsilon}{2}) \star \mathfrak{S}(\rho,\sigma,\kappa-\epsilon) \star \mathfrak{S}(\sigma,\sigma_n,\tfrac{\epsilon}{2}),\\
\mathfrak{A}(\rho_n,\sigma_n,\kappa) &\geq \mathfrak{A}(\rho_n,x,\tfrac{\epsilon}{2}) \star \mathfrak{A}(\rho,\sigma,\kappa-\epsilon) \star \mathfrak{A}(\sigma,\sigma_n,\tfrac{\epsilon}{2}),\\
\mathfrak{M}(\rho_n,\sigma_n,\kappa) &\leq \mathfrak{M}(\rho_n,x,\tfrac{\epsilon}{2}) \diamond \mathfrak{M}(\rho,\sigma,\kappa-\epsilon) \diamond \mathfrak{M}(\sigma,\sigma_n,\tfrac{\epsilon}{2}),\\
\Re(\rho_n,\sigma_n,\kappa) &\leq \Re(\rho_n,x,\tfrac{\epsilon}{2}) \diamond \Re(\rho,\sigma,\kappa-\epsilon) \diamond \Re(\sigma,\sigma_n,\tfrac{\epsilon}{2}).
\end{aligned}$$

Thus,

$$\begin{aligned}
\lim_{n\to\infty} \inf \mathfrak{S}(\rho_n,\sigma_n,\kappa) &\geq 1 \star \mathfrak{S}(\rho,\sigma,\kappa-\epsilon) \star 1 = \mathfrak{S}(\rho,\sigma,\kappa-\epsilon),\\
\lim_{n\to\infty} \inf \mathfrak{A}(\rho_n,\sigma_n,\kappa) &\geq 1 \star \mathfrak{A}(\rho,\sigma,\kappa-\epsilon) \star 1 = \mathfrak{A}(\rho,\sigma,\kappa-\epsilon),\\
\lim_{n\to\infty} \sup \mathfrak{M}(\rho_n,\sigma_n,\kappa) &\leq 0 \diamond \mathfrak{M}(\rho,\sigma,\kappa-\epsilon) \diamond 0 = \mathfrak{M}(\rho,\sigma,\kappa-\epsilon),\\
\lim_{n\to\infty} \sup \Re(\rho_n,\sigma_n,\kappa) &\leq 0 \diamond \Re(\rho,\sigma,\kappa-\epsilon) \diamond 0 = \Re(\rho,\sigma,\kappa-\epsilon).
\end{aligned}$$

Therefore,

$$\lim_{n\to\infty} \inf \mathfrak{S}(\rho_n,\sigma_n,\kappa) \geq \mathfrak{S}(\rho,\sigma,\kappa-\epsilon),\ \lim_{n\to\infty} \inf \mathfrak{A}(\rho_n,\sigma_n,\kappa) \geq \mathfrak{A}(\rho,\sigma,\kappa-\epsilon),$$

and

$$\lim_{n\to\infty} \sup \mathfrak{M}(\rho_n,\sigma_n,\kappa) \leq \mathfrak{M}(\rho,\sigma,\kappa-\epsilon),\ \lim_{n\to\infty} \sup \Re(\rho_n,\sigma_n,\kappa) \leq \Re(\rho,\sigma,\kappa-\epsilon).$$

This completes the proof. □

Lemma 3.3. *If* $\lim_{n\to\infty} \rho_n = x$ *and* $\lim_{n\to\infty} \sigma_n = \sigma$, *then* $\forall\ \kappa \in (0,\infty)$ *and* $\epsilon \in (0,\kappa)$,

$$\mathfrak{S}(\rho,\sigma,t+\epsilon) \geq \lim_{n\to\infty} \sup \mathfrak{S}(\rho_n,\sigma_n,\kappa),\ \mathfrak{A}(\rho,\sigma,t+\epsilon) \geq \lim_{n\to\infty} \sup \mathfrak{A}(\rho_n,\sigma_n,\kappa),$$

and

$$\mathfrak{M}(\rho,\sigma,t+\epsilon) \leq \lim_{n\to\infty} \inf \mathfrak{M}(\rho_n,\sigma_n,\kappa),\ \Re(\rho,\sigma,t+\epsilon) \leq \lim_{n\to\infty} \inf \Re(\rho_n,\sigma_n,\kappa).$$

Proof. By (6), (11), (16) and (21) in Definition 2.8, it can be deduced that,

$$\mathfrak{S}(\rho_n,\sigma_n,t+\epsilon)\geq \mathfrak{S}(\rho_n,x,\tfrac{\epsilon}{2})\star\mathfrak{S}(\rho,\sigma,\kappa)\star\mathfrak{S}(\sigma_n,\sigma,\tfrac{\epsilon}{2}),$$
$$\mathfrak{A}(\rho_n,\sigma_n,t+\epsilon)\geq \mathfrak{A}(\rho_n,x,\tfrac{\epsilon}{2})\star\mathfrak{A}(\rho,\sigma,\kappa)\star\mathfrak{A}(\sigma_n,\sigma,\tfrac{\epsilon}{2}),$$
$$\mathfrak{M}(\rho_n,\sigma_n,t+\epsilon)\leq \mathfrak{M}(\rho_n,x,\tfrac{\epsilon}{2})\diamond\mathfrak{M}(\rho,\sigma,\kappa)\diamond\mathfrak{M}(\sigma_n,\sigma,\tfrac{\epsilon}{2}),$$
$$\mathfrak{R}(\rho_n,\sigma_n,t+\epsilon)\leq \mathfrak{R}(\rho_n,x,\tfrac{\epsilon}{2})\diamond\mathfrak{R}(\rho,\sigma,\kappa)\diamond\mathfrak{R}(\sigma_n,\sigma,\tfrac{\epsilon}{2}).$$

Therefore,

$$\mathfrak{S}(\rho,\sigma,t+\epsilon)\geq \lim_{n\to\infty}\sup\mathfrak{S}(\rho_n,\sigma_n,\kappa),\ \mathfrak{A}(\rho,\sigma,t+\epsilon)\geq \lim_{n\to\infty}\sup\mathfrak{A}(\rho_n,\sigma_n,\kappa),$$
and
$$\mathfrak{M}(\rho,\sigma,t+\epsilon)\leq \lim_{n\to\infty}\inf\mathfrak{M}(\rho_n,\sigma_n,\kappa),\ \mathfrak{R}(\rho,\sigma,t+\epsilon)\leq \lim_{n\to\infty}\inf\mathfrak{R}(\rho_n,\sigma_n,\kappa).$$

This completes the proof. □

Corollary 3.4. *If* $\lim_{n\to\infty}\rho_n=\rho$ *and* $\lim_{n\to\infty}\sigma_n=\sigma$, *then* $\forall\ \kappa\in(0,\infty)$,

$$\mathfrak{S}(\rho,\sigma,\kappa)\leq \lim_{n\to\infty}\inf\mathfrak{S}(\rho_n,\sigma_n,\kappa),\ \mathfrak{A}(\rho,\sigma,\kappa)\leq \lim_{n\to\infty}\inf\mathfrak{A}(\rho_n,\sigma_n,\kappa),$$
and
$$\mathfrak{M}(\rho,\sigma,\kappa)\geq \lim_{n\to\infty}\sup\mathfrak{M}(\rho_n,\sigma_n,\kappa),\ \mathfrak{R}(\rho,\sigma,\kappa)\geq \lim_{n\to\infty}\sup\mathfrak{R}(\rho_n,\sigma_n,\kappa).$$

Corollary 3.5. *If* $\lim_{n\to\infty}\rho_n=\rho$ *and* $\lim_{n\to\infty}\sigma_n=\sigma$, *then* $\forall\ \kappa\in(0,\infty)$,

$$\mathfrak{S}(\rho,\sigma,\kappa)\geq \lim_{n\to\infty}\sup\mathfrak{S}(\rho_n,\sigma_n,\kappa),\ \mathfrak{A}(\rho,\sigma,\kappa)\geq \lim_{n\to\infty}\sup\mathfrak{A}(\rho_n,\sigma_n,\kappa),$$
and
$$\mathfrak{M}(\rho,\sigma,\kappa)\leq \lim_{n\to\infty}\inf\mathfrak{M}(\rho_n,\sigma_n,\kappa),\ \mathfrak{R}(\rho,\sigma,\kappa)\leq \lim_{n\to\infty}\inf\mathfrak{R}(\rho_n,\sigma_n,\kappa).$$

Next, as the second result of the study, we present Edelstein type FP theorem in compact $NF_{\mathfrak{M}S}$.

Theorem 3.6. *The compact* $NF_{\mathfrak{M}S}$ $(G,\mathfrak{S},\mathfrak{A},\mathfrak{M},\mathfrak{R},\star,\diamond)$ *is considered. Additionally, let* $\mathfrak{F}: G\to G$ *be a mapping that fulfills,* *(3.1)*

$$\mathfrak{S}(\mathfrak{F}\rho,\mathfrak{F}\sigma,.)>\mathfrak{S}(\rho,\sigma,.)\ ,\ \mathfrak{A}(\mathfrak{F}\rho,\mathfrak{F}\sigma,.)>\mathfrak{A}(\rho,\sigma,.),$$
and
$$\mathfrak{M}(\mathfrak{F}\rho,\mathfrak{F}\sigma,.)<\mathfrak{M}(\rho,\sigma,.)\ ,\ \mathfrak{R}(\mathfrak{F}\sigma,\mathfrak{F}\sigma,.)<\mathfrak{R}(\rho,\sigma,.).$$

for any $\rho,\sigma\in G$ *and* $\rho\neq\sigma$. *Then F possesses a FP, that is unique.*

Proof. Let $\rho\in G$ and $\rho_n=\mathfrak{F}^n\rho$ $(n\in\mathbb{N})$. Suppose $\rho_n\neq\rho_{n+1}\ \forall\ n$ (else, $\mathfrak{F}\rho_n=\rho_n$). As a result, $\rho_n\neq\rho_m$ $(n\neq m)$. Or else, for $m>n$ it is obtained that,

$$\mathfrak{S}(\rho_n,\rho_{n+1},.)=\mathfrak{S}(\rho_m,\rho_{m+1},.)>\mathfrak{S}(\rho_{m-1},\rho_m,.)>\ldots>\mathfrak{S}(\rho_n,\rho_{n+1},.),$$
$$\mathfrak{A}(\rho_n,\rho_{n+1},.)=\mathfrak{A}(\rho_m,\rho_{m+1},.)>\mathfrak{A}(\rho_{m-1},\rho_m,.)>\ldots>\mathfrak{A}(\rho_n,\rho_{n+1},.),$$
$$\mathfrak{M}(\rho_n,\rho_{n+1},.)=\mathfrak{M}(\rho_m,\rho_{m+1},.)<\mathfrak{M}(\rho_{m-1},\rho_m,.)<\ldots<\mathfrak{M}(\rho_n,\rho_{n+1},.),$$
$$\mathfrak{R}(\rho_n,\rho_{n+1},.)=\mathfrak{R}(\rho_m,\rho_{m+1},.)<\mathfrak{R}(\rho_{m-1},\rho_m,.)<\ldots<\mathfrak{R}(\rho_n,\rho_{n+1},.).$$

This is a contradiction. Now as G is compact, there exists a convergent subsequence $\{\rho_{n_i}\}$, for every sequence $\{\rho_n\} \in G$. Let $\lim_i \rho_{n_i} = \sigma$. And suppose $\sigma, \mathfrak{F}\sigma \notin \{\rho_{n_i} : i \in \mathbb{N}\}$ (if it is not the case, a subsequence with such a property has to be chosen).

In accordance with the assumptions above, it can now be written that,

$$\mathfrak{S}(\mathfrak{F}\rho_{n_i}, \mathfrak{F}\sigma, .) > \mathfrak{S}(\rho_{n_i}, \sigma, .) \ , \ \mathfrak{A}(\mathfrak{F}\rho_{n_i}, \mathfrak{F}\sigma, .) > \mathfrak{A}(\rho_{n_i}, \sigma, .),$$
$$\text{and}$$
$$\mathfrak{M}(\mathfrak{F}\rho_{n_i}, \mathfrak{F}\sigma, .) < \mathfrak{M}(\rho_{n_i}, \sigma, .) \ , \ \mathfrak{R}(\mathfrak{F}\rho_{n_i}, \mathfrak{F}\sigma, .) < \mathfrak{R}(\rho_{n_i}, \sigma, .).$$

for any $i \in \mathbb{N}$. Then by Corollary 3.4, for any $\kappa \in (0, \infty)$ it can be obtained that,

$$\lim \mathfrak{S}(\mathfrak{F}\rho_{n_i}, \mathfrak{F}\sigma, \kappa) \geq \lim \mathfrak{S}(\rho_{n_i}, \sigma, \kappa) = \mathfrak{S}(\sigma, \sigma, \kappa) = 1,$$
$$\lim \mathfrak{A}(\mathfrak{F}\rho_{n_i}, \mathfrak{F}\sigma, \kappa) \geq \lim \mathfrak{A}(\rho_{n_i}, \sigma, \kappa) = \mathfrak{A}(\sigma, \sigma, \kappa) = 1,$$
$$\lim \mathfrak{M}(\mathfrak{F}\rho_{n_i}, \mathfrak{F}\sigma, \kappa) \leq \lim \mathfrak{M}(\rho_{n_i}, \sigma, \kappa) = \mathfrak{M}(\sigma, \sigma, \kappa) = 0,$$
$$\lim \mathfrak{R}(\mathfrak{F}\rho_{n_i}, \mathfrak{F}\sigma, \kappa) \leq \lim \mathfrak{R}(\rho_{n_i}, \sigma, \kappa) = \mathfrak{R}(\sigma, \sigma, \kappa) = 0.$$

Hence,

$$\lim \mathfrak{F}\rho_{n_i} = \mathfrak{F}\sigma.$$

Similarly it can be obtained that,

$$\lim \mathfrak{F}^2\rho_{n_i} = \mathfrak{F}^2\sigma.$$

(Recollect that $\mathfrak{F}\rho_{n_i} = \mathfrak{F}\sigma \ \forall \ i \in \mathbb{N}$). Now $\forall \ \kappa \in (0, \infty)$ it is noted that,

$$\begin{aligned}\mathfrak{S}(\rho_{n_i}, \mathfrak{F}\rho_{n_i}, \kappa) &\leq \mathfrak{S}(\mathfrak{F}\rho_{n_i}, \mathfrak{F}^2\rho_{n_i}, \kappa) \leq \ldots \leq \mathfrak{S}(\rho_{n_i}, \mathfrak{F}\rho_{n_i}, \kappa) \leq \mathfrak{S}(\mathfrak{F}\rho_{n_i}, \mathfrak{F}^2\rho_{n_i}, \kappa) \\ &\leq \ldots \leq \ \mathfrak{S}(\mathfrak{F}\rho_{n_{i+1}}, \mathfrak{F}\rho_{n_{i+1}}, \kappa) \\ &\leq \mathfrak{S}(\mathfrak{F}\rho_{n_{i+1}}, \mathfrak{F}^2\rho_{n_{i+1}}, \kappa) \leq \ldots \leq 1,\end{aligned}$$

$$\begin{aligned}\mathfrak{A}(\rho_{n_i}, \mathfrak{F}\rho_{n_i}, \kappa) &\leq \mathfrak{A}(\mathfrak{F}\rho_{n_i}, \mathfrak{F}^2\rho_{n_i}, \kappa) \leq \ldots \leq \mathfrak{A}(\rho_{n_i}, \mathfrak{F}\rho_{n_i}, \kappa) \leq \mathfrak{A}(\mathfrak{F}\rho_{n_i}, \mathfrak{F}^2\rho_{n_i}, \kappa) \\ &\leq \ldots \leq \ \mathfrak{A}(\mathfrak{F}\rho_{n_{i+1}}, \mathfrak{F}\rho_{n_{i+1}}, \kappa) \\ &\leq \mathfrak{A}(\mathfrak{F}\rho_{n_{i+1}}, \mathfrak{F}^2\rho_{n_{i+1}}, \kappa) \leq \ldots \leq 1,\end{aligned}$$

$$\begin{aligned}\mathfrak{M}(\rho_{n_i}, \mathfrak{F}\rho_{n_i}, \kappa) &\geq \mathfrak{M}(\mathfrak{F}\rho_{n_i}, \mathfrak{F}^2\rho_{n_i}, \kappa) \geq \ldots \geq \mathfrak{M}(\rho_{n_i}, \mathfrak{F}\rho_{n_i}, \kappa) \geq \mathfrak{M}(\mathfrak{F}\rho_{n_i}, \mathfrak{F}^2\rho_{n_i}, \kappa) \\ &\geq \ldots \geq \ \mathfrak{M}(\mathfrak{F}\rho_{n_{i+1}}, \mathfrak{F}\rho_{n_{i+1}}, \kappa) \\ &\geq \mathfrak{M}(\mathfrak{F}\rho_{n_{i+1}}, \mathfrak{F}^2\rho_{n_{i+1}}, \kappa) \geq \ldots \geq 0,\end{aligned}$$

$$\begin{aligned}\mathfrak{R}(\rho_{n_i}, \mathfrak{F}\rho_{n_i}, \kappa) &\geq \mathfrak{R}(\mathfrak{F}\rho_{n_i}, \mathfrak{F}^2\rho_{n_i}, \kappa) \geq \ldots \geq \mathfrak{R}(\rho_{n_i}, \mathfrak{F}\rho_{n_i}, \kappa) \geq \mathfrak{R}(\mathfrak{F}\rho_{n_i}, \mathfrak{F}^2\rho_{n_i}, \kappa) \\ &\geq \ldots \geq \ \mathfrak{R}(\mathfrak{F}\rho_{n_{i+1}}, \mathfrak{F}\rho_{n_{i+1}}, \kappa) \\ &\geq \mathfrak{R}(\mathfrak{F}\rho_{n_{i+1}}, \mathfrak{F}^2\rho_{n_{i+1}}, \kappa) \geq \ldots \geq 0.\end{aligned}$$

Therefore, $\{\mathfrak{S}(\rho_{n_i}, \mathfrak{F}\rho_{n_i}, \kappa)\}, \{\mathfrak{A}(\rho_{n_i}, \mathfrak{F}\rho_{n_i}, \kappa)\}, \{\mathfrak{M}(\rho_{n_i}, \mathfrak{F}\rho_{n_i}, \kappa)\}, \{\mathfrak{R}(\rho_{n_i}, \mathfrak{F}\rho_{n_i}, \kappa)\}$, $\{\mathfrak{S}(\mathfrak{F}\rho_{n_i}, \mathfrak{F}^2\rho_{n_i}, \kappa)\}, \{\mathfrak{A}(\mathfrak{F}\rho_{n_i}, \mathfrak{F}^2\rho_{n_i}, \kappa)\}, \{\mathfrak{M}(\mathfrak{F}\rho_{n_i}, \mathfrak{F}^2\rho_{n_i}, \kappa)\}$ and $\{\mathfrak{R}(\mathfrak{F}\rho_{n_i}, \mathfrak{F}^2\rho_{n_i}, \kappa)\}$ are all convergent to a common limit. Thus, by applying Corollary 3.4 and Corollary 3.5 to the above noted inequalities, it is obtained that,

$$\begin{aligned}\mathfrak{S}(\sigma, \mathfrak{F}\sigma, \kappa) \geq \limsup \mathfrak{S}(\rho_{n_i}, \mathfrak{F}\rho_{n_i}, \kappa) = \limsup \mathfrak{S}(\mathfrak{F}\rho_{n_i}, \mathfrak{F}^2\rho_{n_i}, \kappa) &\geq \liminf \mathfrak{S}(\mathfrak{F}\rho_{n_i}, \mathfrak{F}^2\rho_{n_i}, \kappa) \\ &\geq \mathfrak{S}(\mathfrak{F}\sigma, \mathfrak{F}^2\sigma, \kappa),\end{aligned}$$

$$\begin{aligned}\mathfrak{A}(\sigma, \mathfrak{F}\sigma, \kappa) \geq \limsup \mathfrak{A}(\rho_{n_i}, \mathfrak{F}\rho_{n_i}, \kappa) = \limsup \mathfrak{A}(\mathfrak{F}\rho_{n_i}, \mathfrak{F}^2\rho_{n_i}, \kappa) &\geq \liminf \mathfrak{A}(\mathfrak{F}\rho_{n_i}, \mathfrak{F}^2\rho_{n_i}, \kappa) \\ &\geq \mathfrak{A}(\mathfrak{F}\sigma, \mathfrak{F}^2\sigma, \kappa),\end{aligned}$$

$$\begin{aligned}\mathfrak{M}(\sigma, \mathfrak{F}\sigma, \kappa) \leq \liminf \mathfrak{M}(\rho_{n_i}, \mathfrak{F}\rho_{n_i}, \kappa) = \liminf \mathfrak{M}(\mathfrak{F}\rho_{n_i}, \mathfrak{F}^2\rho_{n_i}, \kappa) &\leq \limsup \mathfrak{M}(\mathfrak{F}\rho_{n_i}, \mathfrak{F}^2\rho_{n_i}, \kappa) \\ &\leq \mathfrak{M}(\mathfrak{F}\sigma, \mathfrak{F}^2\sigma, \kappa),\end{aligned}$$

$$\begin{aligned}\mathfrak{R}(\sigma, \mathfrak{F}\sigma, \kappa) \leq \liminf \mathfrak{R}(\rho_{n_i}, \mathfrak{F}\rho_{n_i}, \kappa) = \liminf \mathfrak{R}(\mathfrak{F}\rho_{n_i}, \mathfrak{F}^2\rho_{n_i}, \kappa) &\leq \limsup \mathfrak{R}(\mathfrak{F}\rho_{n_i}, \mathfrak{F}^2\rho_{n_i}, \kappa) \\ &\leq \mathfrak{R}(\mathfrak{F}\sigma, \mathfrak{F}^2\sigma, \kappa).\end{aligned}$$

$\forall\ \kappa \in \mathbb{R}^+$. Now let $y \neq \mathfrak{F}\sigma$. So, using (3.1) it can be claimed that,

$$\mathfrak{S}(\sigma, \mathfrak{F}\sigma, .) < \mathfrak{S}(\mathfrak{F}\sigma, \mathfrak{F}^2\sigma, .)\ ,\ \mathfrak{A}(\sigma, \mathfrak{F}\sigma, .) < \mathfrak{A}(\mathfrak{F}\sigma, \mathfrak{F}^2\sigma, .),$$
$$\text{and}$$
$$\mathfrak{M}(\sigma, \mathfrak{F}\sigma, .) > \mathfrak{M}(\mathfrak{F}\sigma, \mathfrak{F}^2\sigma, .)\ ,\ \mathfrak{R}(\sigma, \mathfrak{F}\sigma, .) > \mathfrak{R}(\mathfrak{F}\sigma, \mathfrak{F}^2\sigma, .).$$

This is a contradiction. This is due to the fact that $\mathfrak{S}$ and $\mathfrak{A}$ are left continuous and non decreasing, and $\mathfrak{M}$ and $\mathfrak{R}$ are right continuous and non increasing.
Hence, $\sigma = \mathfrak{F}\sigma$ is a FP of $\mathfrak{F}$. Uniqueness is immedieate from (3.1). □

We now extend Romaguera's notion of Kannan type contractions for fuzzy metric space under the framework of $NF_{\mathfrak{M}S}$, which is presented as follows, utilizing Definitions 2.17 and 2.18.

Definition 3.7. Let $(G, \mathfrak{S}, \mathfrak{A}, \mathfrak{M}, \mathfrak{R}, \star, \diamond)$ be a $NF_{\mathfrak{M}S}$. Then a mapping $\mathfrak{F} : G \to G$ is a $(1)-K_{\mathfrak{C}}$ on G if,

$$\begin{aligned}\min\{\mathfrak{S}(\rho, \mathfrak{F}\rho, \kappa), \mathfrak{S}(\sigma, \mathfrak{F}\sigma, \kappa)\} > 1-\kappa &\implies \mathfrak{S}(\mathfrak{F}\rho, \mathfrak{F}\sigma, c\kappa) > 1-c\kappa, \\ \min\{\mathfrak{A}(\rho, \mathfrak{F}\rho, \kappa), \mathfrak{A}(\sigma, \mathfrak{F}\sigma, \kappa)\} > 1-\kappa &\implies \mathfrak{A}(\mathfrak{F}\rho, \mathfrak{F}\sigma, c\kappa) > 1-c\kappa, \\ \max\{\mathfrak{M}(\rho, \mathfrak{F}\rho, \kappa), \mathfrak{M}(\sigma, \mathfrak{F}\sigma, \kappa)\} < \kappa &\implies \mathfrak{M}(\mathfrak{F}\rho, \mathfrak{F}\sigma, c\kappa) < c\kappa, \\ \max\{\mathfrak{R}(\rho, \mathfrak{F}\rho, \kappa), \mathfrak{R}(\sigma, \mathfrak{F}\sigma, \kappa)\} < \kappa &\implies \mathfrak{R}(\mathfrak{F}\rho, \mathfrak{F}\sigma, c\kappa) < c\kappa,\end{aligned}$$

for some $c \in (0, 1),\ \forall\ \kappa > 0$ and $\rho, \sigma \in G$.

Definition 3.8. Let $(G, \mathfrak{S}, \mathfrak{A}, \mathfrak{M}, \mathfrak{R}, \star, \diamond)$ be a $NF_{\mathfrak{M}S}$. Then a mapping $\mathfrak{F} : G \to G$ is a $(1/2)-K_{\mathfrak{C}}$ on G if,

$$\begin{aligned}
\min\{\mathfrak{S}(\rho, \mathfrak{F}\rho, \kappa), \mathfrak{S}(\sigma, \mathfrak{F}\sigma, \kappa)\} > 1-\kappa &\implies \mathfrak{S}(\mathfrak{F}\rho, \mathfrak{F}\sigma, c\kappa) > 1-c\kappa,\\
\min\{\mathfrak{A}(\rho, \mathfrak{F}\rho, \kappa), \mathfrak{A}(\sigma, \mathfrak{F}\sigma, \kappa)\} > 1-\kappa &\implies \mathfrak{A}(\mathfrak{F}\rho, \mathfrak{F}\sigma, c\kappa) > 1-c\kappa,\\
\max\{\mathfrak{M}(\rho, \mathfrak{F}\rho, \kappa), \mathfrak{M}(\sigma, \mathfrak{F}\sigma, \kappa)\} < \kappa &\implies \mathfrak{M}(\mathfrak{F}\rho, \mathfrak{F}\sigma, c\kappa) < c\kappa,\\
\max\{\mathfrak{R}(\rho, \mathfrak{F}\rho, \kappa), \mathfrak{R}(\sigma, \mathfrak{F}\sigma, \kappa)\} < \kappa &\implies \mathfrak{R}(\mathfrak{F}\rho, \mathfrak{F}\sigma, c\kappa) < c\kappa,
\end{aligned}$$

for some $c \in (0, 1/2),\ \forall\ \kappa > 0$ and $\rho, \sigma \in G$.

Theorem 3.9. *There is a FP for each $(1)-K_{\mathfrak{C}}$ on a complete $NF_{\mathfrak{M}S}$.*

Proof. Let $(G, \mathfrak{S}, \mathfrak{A}, \mathfrak{M}, \mathfrak{R}, \star, \diamond)$ be a complete $NF_{\mathfrak{M}S}$, and
$\mathfrak{F} : G \to G$ be a $(1)-K_{\mathfrak{C}}$ on G. Then $\exists\ c \in (0, 1)$ such that all the conditions in Definition 3.7 are satisfied.
Take $\kappa_0 > 1$. Then $\forall\ \rho, \sigma \in G$ it can be obtained that,

$$\begin{aligned}
\mathfrak{S}(\rho, \mathfrak{F}\rho, \kappa_0) > 1-\kappa_0 \text{ and } \mathfrak{S}(\sigma, \mathfrak{F}\sigma, \kappa_0) > 1-\kappa_0,\\
\mathfrak{A}(\rho, \mathfrak{F}\rho, \kappa_0) > 1-\kappa_0 \text{ and } \mathfrak{A}(\sigma, \mathfrak{F}\sigma, \kappa_0) > 1-\kappa_0,\\
\mathfrak{M}(\rho, \mathfrak{F}\rho, \kappa_0) < \kappa_0 \text{ and } \mathfrak{M}(\sigma, \mathfrak{F}\sigma, \kappa_0) < \kappa_0,\\
\mathfrak{R}(\rho, \mathfrak{F}\rho, \kappa_0) < \kappa_0 \text{ and } \mathfrak{R}(\sigma, \mathfrak{F}\sigma, \kappa_0) < \kappa_0.
\end{aligned}$$

So by Definition 3.7, it can be written that,

$$\begin{aligned}
\mathfrak{S}(\mathfrak{F}\rho, \mathfrak{F}\sigma, c\kappa_0), \mathfrak{A}(\mathfrak{F}\rho, \mathfrak{F}\sigma, c\kappa_0) > 1-c\kappa_0,\\
\mathfrak{M}(\mathfrak{F}\rho, \mathfrak{F}\sigma, c\kappa_0), \mathfrak{R}(\mathfrak{F}\rho, \mathfrak{F}\sigma, c\kappa_0) < c\kappa_0.
\end{aligned}$$

From,

$$\begin{aligned}
\mathfrak{S}(\rho, \mathfrak{F}\rho, \kappa_0), \mathfrak{S}(\mathfrak{F}\rho, F^2\rho, \kappa_0), \mathfrak{A}(\rho, \mathfrak{F}\rho, \kappa_0), \mathfrak{A}(\mathfrak{F}\rho, F^2\rho, \kappa_0) > 1-\kappa_0,\\
\text{and}\\
\mathfrak{M}(\rho, \mathfrak{F}\rho, \kappa_0), \mathfrak{M}(\mathfrak{F}\rho, F^2\rho, \kappa_0), \mathfrak{R}(\rho, \mathfrak{F}\rho, \kappa_0), \mathfrak{R}(\mathfrak{F}\rho, F^2\rho, \kappa_0) < \kappa_0,
\end{aligned}$$

it can be deduced that,

$$\begin{aligned}
\mathfrak{S}(\mathfrak{F}\rho, \mathfrak{F}^2\rho, c\kappa_0), \mathfrak{A}(\mathfrak{F}\rho, \mathfrak{F}^2\rho, c\kappa_0) > 1-c\kappa_0,\\
\text{and}\\
\mathfrak{M}(\mathfrak{F}\rho, \mathfrak{F}^2\rho, c\kappa_0), \mathfrak{R}(\mathfrak{F}\rho, \mathfrak{F}^2\rho, c\kappa_0) < c\kappa_0.
\end{aligned}$$

And similarly,

$$\begin{aligned}
\mathfrak{S}(\mathfrak{F}\sigma, \mathfrak{F}^2\sigma, c\kappa_0), \mathfrak{A}(\mathfrak{F}\sigma, \mathfrak{F}^2\sigma, c\kappa_0) > 1-c\kappa_0,\\
\text{and}\\
\mathfrak{M}(\mathfrak{F}\sigma, \mathfrak{F}^2\sigma, c\kappa_0), \mathfrak{R}(\mathfrak{F}\sigma, \mathfrak{F}^2\sigma, c\kappa_0) < c\kappa_0.
\end{aligned}$$

Thus, by Definition 3.7 it is inferred that,

$$\mathfrak{S}(\mathfrak{F}^2\rho, \mathfrak{F}^2\sigma, c^2\kappa_0), \mathfrak{A}(\mathfrak{F}^2\rho, \mathfrak{F}^2\sigma, c^2\kappa_0) > 1 - c^2\kappa_0,$$
$$\text{and}$$
$$\mathfrak{M}(\mathfrak{F}^2\rho, \mathfrak{F}^2\sigma, c^2\kappa_0), \mathfrak{R}(\mathfrak{F}^2\rho, \mathfrak{F}^2\sigma, c^2\kappa_0) < c^2\kappa_0.$$

Continuing in this way, $\forall\ n \in \mathbb{N}$,

$$\mathfrak{S}(\mathfrak{F}^n\rho, \mathfrak{F}^n\sigma, c^n\kappa_0), \mathfrak{A}(\mathfrak{F}^n\rho, \mathfrak{F}^n\sigma, c^n\kappa_0) > 1 - c^n\kappa_0,$$
$$\text{and}$$
$$\mathfrak{M}(\mathfrak{F}^n\rho, \mathfrak{F}^n\sigma, c^n\kappa_0), \mathfrak{R}(\mathfrak{F}^n\rho, \mathfrak{F}^n\sigma, c^n\kappa_0) < c^n\kappa_0.$$

Now fix an $\rho_0 \in G$. Define $\rho_n := \mathfrak{F}^n\rho_0\ \forall\ n \in \mathbb{N}$. It is to be shown that $(\rho_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $(G, \mathfrak{S}, \mathfrak{A}, \mathfrak{M}, \mathfrak{R}, \star, \diamond)$.

Certainly, given $\kappa \in \mathbb{R}^+$ and $\epsilon \in (0,1),\ \exists\ n_\epsilon \in \mathbb{N}$ such that $c^n\kappa_0 < min\{\epsilon, \kappa\}\ \forall\ n \geq n_\epsilon$. Now let $m, n \geq n_\epsilon$ and $m > n$. Then, $m = n + k$ for some $k \in \mathbb{N}$, and therefore,

$$\begin{aligned}
\mathfrak{S}(\rho_n, \rho_m, \kappa) &= \mathfrak{S}(\mathfrak{F}^n\rho_0, \mathfrak{F}^n\mathfrak{F}^k\rho_0, \kappa)\\
&\geq \mathfrak{S}(\mathfrak{F}^n\rho_0, \mathfrak{F}^n\mathfrak{F}^k\rho_0, c^n\kappa_0)\\
&> 1 - c^n\kappa_0 > 1 - \epsilon,\\
\mathfrak{A}(\rho_n, \rho_m, \kappa) &= \mathfrak{A}(\mathfrak{F}^n\rho_0, \mathfrak{F}^n\mathfrak{F}^k\rho_0, \kappa)\\
&\geq \mathfrak{A}(\mathfrak{F}^n\rho_0, \mathfrak{F}^n\mathfrak{F}^k\rho_0, c^n\kappa_0)\\
&> 1 - c^n\kappa_0 > 1 - \epsilon,
\end{aligned}$$

$$\begin{aligned}
\mathfrak{M}(\rho_n, \rho_m, \kappa) &= \mathfrak{M}(\mathfrak{F}^n\rho_0, \mathfrak{F}^n\mathfrak{F}^k\rho_0, \kappa)\\
&\leq \mathfrak{M}(\mathfrak{F}^n\rho_0, \mathfrak{F}^n\mathfrak{F}^k\rho_0, c^n\kappa_0)\\
&< c^n\kappa_0 < \epsilon,\\
\mathfrak{R}(\rho_n, \rho_m, \kappa) &= \mathfrak{R}(\mathfrak{F}^n\rho_0, \mathfrak{F}^n\mathfrak{F}^k\rho_0, \kappa)\\
&\leq \mathfrak{R}(\mathfrak{F}^n\rho_0, \mathfrak{F}^n\mathfrak{F}^k\rho_0, c^n\kappa_0)\\
&< c^n\kappa_0 < \epsilon.
\end{aligned}$$

As a result, $(\rho_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $(G, \mathfrak{S}, \mathfrak{A}, \mathfrak{M}, \mathfrak{R}, \star, \diamond)$. So, $\exists\ z \in G$ such that $(\rho_n)_{n\in\mathbb{N}} \to z$ in $\tau_{(\mathfrak{S},\mathfrak{A},\mathfrak{M},\mathfrak{R})}$.

It is now necessary to demonstrate that w is a FP of $\mathfrak{F}$. (A number of details need to be confirmed because the proof is not instantaneous.)

Choose $r, s > 0$ such that $c < s < r < 1$.

By applying induction, in the first step it has to be shown that, $\forall\ k \in \mathbb{N}$, (3.2)

$$\mathfrak{S}(w, \mathfrak{F}w, r^k\kappa_0), \mathfrak{A}(w, \mathfrak{F}w, r^k\kappa_0) \geq 1 - r^k\kappa_0,$$
$$\text{and}$$

$$\mathfrak{M}(w, \mathfrak{F}w, r^k\kappa_0), \mathfrak{R}(w, \mathfrak{F}w, r^k\kappa_0) \leq r^k\kappa_0.$$

Without losing generality suppose $r^k\kappa_0 \leq 1 \ \forall \ k \in \mathbb{N}$. Or else inequality (3.2) is satisfied trivially.

Now for each $k \in \mathbb{N}$ define,

$$T_{k,r,s} := \{\epsilon \in (0,1) : \epsilon + sr^{k-1}\kappa_0 < r^k\kappa_0\}.$$

Assume $k = 1$. Then it is obtained that,

$$\begin{gathered}
\mathfrak{S}(w, \mathfrak{F}w, \kappa_0), \mathfrak{S}(\rho_n, \rho_{n+1}, \kappa_0) > 1 - \kappa_0, \\
\mathfrak{A}(w, \mathfrak{F}w, \kappa_0), \mathfrak{A}(\rho_n, \rho_{n+1}, \kappa_0) > 1 - \kappa_0, \\
\mathfrak{M}(w, \mathfrak{F}w, \kappa_0), \mathfrak{M}(\rho_n, \rho_{n+1}, \kappa_0) < \kappa_0, \\
\mathfrak{R}(w, \mathfrak{F}w, \kappa_0), \mathfrak{R}(\rho_n, \rho_{n+1}, \kappa_0) < \kappa_0.
\end{gathered}$$

So by Remark 2.9 and Definition 3.7, it can be written that, $\forall \ n \in \mathbb{W}$,

$$\begin{gathered}
\mathfrak{S}(\mathfrak{F}w, \rho_{n+1}, s\kappa_0) \geq \mathfrak{S}(\mathfrak{F}w, \rho_{n+1}, c\kappa_0) > 1 - c\kappa_0 > 1 - s\kappa_0, \\
\mathfrak{A}(\mathfrak{F}w, \rho_{n+1}, s\kappa_0) \geq \mathfrak{A}(\mathfrak{F}w, \rho_{n+1}, c\kappa_0) > 1 - c\kappa_0 > 1 - s\kappa_0, \\
\mathfrak{M}(\mathfrak{F}w, \rho_{n+1}, s\kappa_0) \leq \mathfrak{M}(\mathfrak{F}w, \rho_{n+1}, c\kappa_0) < c\kappa_0 < s\kappa_0, \\
\mathfrak{R}(\mathfrak{F}w, \rho_{n+1}, s\kappa_0) \leq \mathfrak{R}(\mathfrak{F}w, \rho_{n+1}, c\kappa_0) < c\kappa_0 < s\kappa_0.
\end{gathered}$$

Since $(\rho_n)_{n\in\mathbb{N}} \to z$, for any $\epsilon \in T_{1,r,s} \ \exists \ n_\epsilon \in \mathbb{N}$ for which $\mathfrak{S}(w, \rho_n, \epsilon), \mathfrak{A}(w, \rho_n, \epsilon) > 1 - \epsilon$ and $\mathfrak{M}(w, \rho_n, \epsilon), \mathfrak{R}(w, \rho_n, \epsilon) < \epsilon$. As a result,

$$\begin{aligned}
\mathfrak{S}(w, \mathfrak{F}w, r\kappa_0) &\geq \mathfrak{S}(w, \rho_{n_\epsilon}, \epsilon) \star \mathfrak{S}(\rho_{n_\epsilon}, \mathfrak{F}w, s\kappa_0) \\
&\geq (1 - \epsilon) \star (1 - s\kappa_0) \\
&\geq (1 - \epsilon) \star (1 - r\kappa_0), \\
\mathfrak{A}(w, \mathfrak{F}w, r\kappa_0) &\geq \mathfrak{A}(w, \rho_{n_\epsilon}, \epsilon) \star \mathfrak{A}(\rho_{n_\epsilon}, \mathfrak{F}w, s\kappa_0) \\
&\geq (1 - \epsilon) \star (1 - s\kappa_0) \\
&\geq (1 - \epsilon) \star (1 - r\kappa_0), \\
\mathfrak{M}(w, \mathfrak{F}w, r\kappa_0) &\leq \mathfrak{M}(w, \rho_{n_\epsilon}, \epsilon) \diamond \mathfrak{M}(\rho_{n_\epsilon}, \mathfrak{F}w, s\kappa_0) \\
&\leq \epsilon \diamond s\kappa_0 \\
&\leq \epsilon \diamond r\kappa_0, \\
\mathfrak{R}(w, \mathfrak{F}w, r\kappa_0) &\leq \mathfrak{R}(w, \rho_{n_\epsilon}, \epsilon) \diamond \mathfrak{R}(\rho_{n_\epsilon}, \mathfrak{F}w, s\kappa_0) \\
&\leq \epsilon \diamond s\kappa_0 \\
&\leq \epsilon \diamond r\kappa_0.
\end{aligned}$$

Taking limit as $\epsilon \to 0$, by continuity of $\star$ and $\diamond$ it can be deduced that,

$$\mathfrak{S}(w, \mathfrak{F}w, r\kappa_0), \mathfrak{A}(w, \mathfrak{F}w, r\kappa_0) \geq 1 - r\kappa_0,$$
and
$$\mathfrak{M}(w, \mathfrak{F}w, r\kappa_0), \mathfrak{R}(w, \mathfrak{F}w, r\kappa_0) \leq r\kappa_0.$$

Now it is to be shown that, if inequality (3.2) holds for $k = j$, then it also holds for $k = j + 1$ (where $j \in \mathbb{N}$).
Putting $k = j$ in inequality (3.2) it is obtained that,

$$\mathfrak{S}(w, \mathfrak{F}w, r^j\kappa_0), \mathfrak{A}(w, \mathfrak{F}w, r^j\kappa_0) > 1 - r^j\kappa_0,$$
and
$$\mathfrak{M}(w, \mathfrak{F}w, r^j\kappa_0), \mathfrak{R}(w, \mathfrak{F}w, r^j\kappa_0) < r^j\kappa_0.$$

Furthermore, since $(\rho_n)_{n\in\mathbb{N}}$ is Cauchy, $\exists\ n_j \in \mathbb{N}$ such that $\forall\ n \geq n_j$,

$$\mathfrak{S}(\rho_n, \rho_{n+1}, r^j\kappa_0), \mathfrak{A}(\rho_n, \rho_{n+1}, r^j\kappa_0) > 1 - r^j\kappa_0,$$
and
$$\mathfrak{M}(\rho_n, \rho_{n+1}, r^j\kappa_0), \mathfrak{R}(\rho_n, \rho_{n+1}, r^j\kappa_0) < r^j\kappa_0.$$

Then by Definition 3.7, it can be said that $\forall\ n \geq n_j$, (3.3)

$$\mathfrak{S}(\mathfrak{F}w, \rho_{n+1}, cr^j\kappa_0), \mathfrak{A}(\mathfrak{F}w, \rho_{n+1}, cr^j\kappa_0) > 1 - cr^j\kappa_0,$$
and
$$\mathfrak{M}(\mathfrak{F}w, \rho_{n+1}, cr^j\kappa_0), \mathfrak{R}(\mathfrak{F}w, \rho_{n+1}, cr^j\kappa_0) < cr^j\kappa_0.$$

Since $s > c$, using inequality (3.3) it can be written that,

$$\mathfrak{S}(\mathfrak{F}w, \rho_{n+1}, sr^j\kappa_0), \mathfrak{A}(\mathfrak{F}w, \rho_{n+1}, sr^j\kappa_0) > 1 - sr^j\kappa_0,$$
and
$$\mathfrak{M}(\mathfrak{F}w, \rho_{n+1}, sr^j\kappa_0), \mathfrak{R}(\mathfrak{F}w, \rho_{n+1}, sr^j\kappa_0) < sr^j\kappa_0.$$

$\forall\ n \geq n_j$. Now let $\epsilon \in T_{j+1,r,s}$. Then $\epsilon + sr^j\kappa_0 < r^{j+1}\kappa_0$ and $\exists\ n_\epsilon > n_j$ such that,

$$\mathfrak{S}(w, \rho_{n_\epsilon}, \epsilon), \mathfrak{A}(w, \rho_{n_\epsilon}, \epsilon) > 1 - \epsilon,$$
and
$$\mathfrak{M}(w, \rho_{n_\epsilon}, \epsilon), \mathfrak{R}(w, \rho_{n_\epsilon}, \epsilon) < \epsilon.$$

Therefore,

$$\begin{aligned}
\mathfrak{S}(w, \mathfrak{F}w, r^{j+1}\kappa_0) &\geq \mathfrak{S}(w, \rho_{n_\epsilon}, \epsilon) \star \mathfrak{S}(\rho_{n_\epsilon}, \mathfrak{F}w, sr^j\kappa_0) \\
&\geq (1-\epsilon) \star (1 - sr^j\kappa_0) \\
&\geq (1-\epsilon) \star (1 - r^{j+1}\kappa_0), \\
\mathfrak{A}(w, \mathfrak{F}w, r^{j+1}\kappa_0) &\geq \mathfrak{A}(w, \rho_{n_\epsilon}, \epsilon) \star \mathfrak{A}(\rho_{n_\epsilon}, \mathfrak{F}w, sr^j\kappa_0) \\
&\geq (1-\epsilon) \star (1 - sr^j\kappa_0) \\
&\geq (1-\epsilon) \star (1 - r^{j+1}\kappa_0), \\
\mathfrak{M}(w, \mathfrak{F}w, r^{j+1}\kappa_0) &\leq \mathfrak{M}(w, \rho_{n_\epsilon}, \epsilon) \diamond \mathfrak{M}(\rho_{n_\epsilon}, \mathfrak{F}w, sr^j\kappa_0) \\
&\leq \epsilon \diamond sr^j\kappa_0 \\
&\leq \epsilon \diamond r^{j+1}\kappa_0, \\
\mathfrak{R}(w, \mathfrak{F}w, r^{j+1}\kappa_0) &\leq \mathfrak{R}(w, \rho_{n_\epsilon}, \epsilon) \diamond \mathfrak{R}(\rho_{n_\epsilon}, \mathfrak{F}w, sr^j\kappa_0) \\
&\leq \epsilon \diamond sr^j\kappa_0 \\
&\leq \epsilon \diamond r^{j+1}\kappa_0.
\end{aligned}$$

Taking limit as $\epsilon \to 0$, by the continuity of $\star$ and $\diamond$ it follows that,

$$\mathfrak{S}(w, \mathfrak{F}w, r^{j+1}\kappa_0), \mathfrak{A}(w, \mathfrak{F}w, r^{j+1}\kappa_0) \geq 1 - r^{j+1}\kappa_0,$$
and
$$\mathfrak{M}(w, \mathfrak{F}w, r^{j+1}\kappa_0), \mathfrak{R}(w, \mathfrak{F}w, r^{j+1}\kappa_0) \leq r^{j+1}\kappa_0.$$

Thus inequality (3.2) also holds for $k = j + 1$.
Now given $\kappa \in (0, \infty)$, $\exists k \in \mathbb{N}$ such that $r^k\kappa_0 < \kappa$. So,

$$\begin{aligned}
\mathfrak{S}(w, \mathfrak{F}w, \kappa) &\geq \mathfrak{S}(w, \mathfrak{F}w, r^k\kappa_0) > 1 - r^k\kappa_0 > 1 - \kappa, \\
\mathfrak{A}(w, \mathfrak{F}w, \kappa) &\geq \mathfrak{A}(w, \mathfrak{F}w, r^k\kappa_0) > 1 - r^k\kappa_0 > 1 - \kappa, \\
\mathfrak{M}(w, \mathfrak{F}w, \kappa) &\leq \mathfrak{M}(w, \mathfrak{F}w, r^k\kappa_0) < r^k\kappa_0 < \kappa, \\
\mathfrak{R}(w, \mathfrak{F}w, \kappa) &\leq \mathfrak{R}(w, \mathfrak{F}w, r^k\kappa_0) < r^k\kappa_0 < \kappa.
\end{aligned}$$

By applying Remark 2.10 here, it follows that,

$$w = \mathfrak{F}w.$$

For uniqueness let $u \in G$ such that $u = \mathfrak{F}u$. Then $\forall\ \kappa \in (0, \infty)$,

$$\begin{aligned}
\min\{\mathfrak{S}(w, \mathfrak{F}w, \kappa), \mathfrak{S}(u, \mathfrak{F}u, \kappa)\} &= 1, \\
\min\{\mathfrak{A}(w, \mathfrak{F}w, \kappa), \mathfrak{A}(u, \mathfrak{F}u, \kappa)\} &= 1, \\
\max\{\mathfrak{M}(w, \mathfrak{F}w, \kappa), \mathfrak{M}(u, \mathfrak{F}u, \kappa)\} &= 0, \\
\max\{\mathfrak{R}(w, \mathfrak{F}w, \kappa), \mathfrak{R}(u, \mathfrak{F}u, \kappa)\} &= 0.
\end{aligned}$$

Hence by Definition 3.7,

$$\mathfrak{S}(\mathfrak{F}w, \mathfrak{F}u, c\kappa), \mathfrak{A}(\mathfrak{F}w, \mathfrak{F}u, c\kappa) > 1 - c\kappa,$$
$$\text{and}$$
$$\mathfrak{M}(\mathfrak{F}w, \mathfrak{F}u, c\kappa), \mathfrak{R}(\mathfrak{F}w, \mathfrak{F}u, c\kappa) < c\kappa.$$

for any $\kappa > 0$. Implies that, $\mathfrak{F}w = \mathfrak{F}u$ or, $w = u$. This completes the proof. □

Corollary 3.10. *There is a unique FP for each* $(1/2)-K_{\mathfrak{C}}$ *on a complete* $NF_{\mathfrak{M}S}$.

Now we are going to discuss an example of $(1)-K_{\mathfrak{C}}$ for $c = 3/5$, which is not a $(1/2)-K_{\mathfrak{C}}$. Indeed, it will also provide a case where Theorem 3.9 can be applied, but not Corollary 3.10.

Example 3.11. Let $G = [0, 9/5]$, and d be the typical metric on G. Then it is evident that, $(G, \mathfrak{S}^d_{01}, \mathfrak{A}^d_{01}, \mathfrak{M}^d_{01}, \mathfrak{R}^d_{01}, \star, \diamond)$ is a complete $NF_{\mathfrak{M}S}$, where $\star$ and $\diamond$ stands for minimum and maximum operators respectively.

Define $\mathfrak{F} : G \to G$ as $\mathfrak{F}\rho = 0 \ \forall \ \rho \in [0, 6/5]$ and $\mathfrak{F}\rho = \rho/3 \ \forall \ \rho \in (6/5, 9/5]$.
Now it has to be established that F is not a $(1/2)-K_{\mathfrak{C}}$.

Take $c \in (0, 1/2)$. Choose $\rho = 0, y = 9/5$ and $t = 3/5c$. Then $\kappa > 1$. So,

$$\min\left\{\mathfrak{S}^d_{01}(\rho, \mathfrak{F}\rho, \kappa), \mathfrak{S}^d_{01}(\sigma, \mathfrak{F}\sigma, \kappa)\right\} > 1 - \kappa,$$
$$\min\left\{\mathfrak{A}^d_{01}(\rho, \mathfrak{F}\rho, \kappa), \mathfrak{A}^d_{01}(\sigma, \mathfrak{F}\sigma, \kappa)\right\} > 1 - \kappa,$$
$$\max\left\{\mathfrak{M}^d_{01}(\rho, \mathfrak{F}\rho, \kappa), \mathfrak{M}^d_{01}(\sigma, \mathfrak{F}\sigma, \kappa)\right\} < \kappa,$$
$$\max\left\{\mathfrak{R}^d_{01}(\rho, \mathfrak{F}\rho, \kappa), \mathfrak{R}^d_{01}(\sigma, \mathfrak{F}\sigma, \kappa)\right\} < \kappa.$$

However,

$$\mathfrak{S}^d_{01}(\mathfrak{F}\rho, \mathfrak{F}\sigma, c\kappa) = \mathfrak{S}^d_{01}(0, 3/5, 3/5) = 0 < 1 - c\kappa,$$
$$\mathfrak{A}^d_{01}(\mathfrak{F}\rho, \mathfrak{F}\sigma, c\kappa) = \mathfrak{A}^d_{01}(0, 3/5, 3/5) = 0 < 1 - c\kappa,$$
$$\mathfrak{M}^d_{01}(\mathfrak{F}\rho, \mathfrak{F}\sigma, c\kappa) = \mathfrak{M}^d_{01}(0, 3/5, 3/5) = 1 > c\kappa,$$
$$\mathfrak{R}^d_{01}(\mathfrak{F}\rho, \mathfrak{F}\sigma, c\kappa) = \mathfrak{R}^d_{01}(0, 3/5, 3/5) = 1 > c\kappa.$$

Therefore, F is not a $(1/2)-K_{\mathfrak{C}}$ on G. It now remains to prove that, F is a $(1)-K_{\mathfrak{C}}$ for $c = 1/2$. To put an end to this, three cases are to be distinguished for any $\rho, \sigma \in G$.
Case 1: $\rho, \sigma \in [0, 6/5]$. Then for any $\kappa \in \mathbb{R}^+$,

$$\mathfrak{S}^d_{01}(\mathfrak{F}\rho, \mathfrak{F}\sigma, c\kappa) = \mathfrak{S}^d_{01}(0, 0, 3\kappa/5) = 1 > 1 - c\kappa,$$
$$\mathfrak{A}^d_{01}(\mathfrak{F}\rho, \mathfrak{F}\sigma, c\kappa) = \mathfrak{A}^d_{01}(0, 0, 3\kappa/5) = 1 > 1 - c\kappa,$$
$$\mathfrak{M}^d_{01}(\mathfrak{F}\rho, \mathfrak{F}\sigma, c\kappa) = \mathfrak{M}^d_{01}(0, 0, 3\kappa/5) = 0 < c\kappa,$$
$$\mathfrak{R}^d_{01}(\mathfrak{F}\rho, \mathfrak{F}\sigma, c\kappa) = \mathfrak{R}^d_{01}(0, 0, 3\kappa/5) = 0 < c\kappa.$$

As a result, it meets every requirement given in Definition 3.7.
Case 2: $\rho \in [0, 6/5], y \in (6/5, 9/5]$. Then if $\kappa > 1$, it can be written that,

$$\sigma/3 \leq 3/5 < 3\kappa/5.$$

Therefore,

$$\begin{aligned}
\mathfrak{S}_{01}^{d}(\mathfrak{F}\rho, \mathfrak{F}\sigma, c\kappa) = \mathfrak{S}_{01}^{d}(0, \sigma/3, 3\kappa/5) = 1 > 1 - c\kappa,\\
\mathfrak{A}_{01}^{d}(\mathfrak{F}\rho, \mathfrak{F}\sigma, c\kappa) = \mathfrak{A}_{01}^{d}(0, \sigma/3, 3\kappa/5) = 1 > 1 - c\kappa,\\
\mathfrak{M}_{01}^{d}(\mathfrak{F}\rho, \mathfrak{F}\sigma, c\kappa) = \mathfrak{M}_{01}^{d}(0, \sigma/3, 3\kappa/5) = 0 < c\kappa,\\
\mathfrak{R}_{01}^{d}(\mathfrak{F}\rho, \mathfrak{F}\sigma, c\kappa) = \mathfrak{R}_{01}^{d}(0, \sigma/3, 3\kappa/5) = 0 < c\kappa.
\end{aligned}$$

And hence, all the conditions in Definition 3.7 are satisfied.

Now take $\kappa \leq 1$ and suppose,

$$\begin{aligned}
\min\left\{\mathfrak{S}_{01}^{d}(\rho, \mathfrak{F}\rho, \kappa), \mathfrak{S}_{01}^{d}(\sigma, \mathfrak{F}\sigma, \kappa)\right\} > 1 - \kappa,\\
\min\left\{\mathfrak{A}_{01}^{d}(\rho, \mathfrak{F}\rho, \kappa), \mathfrak{A}_{01}^{d}(\sigma, \mathfrak{F}\sigma, \kappa)\right\} > 1 - \kappa,\\
\max\left\{\mathfrak{M}_{01}^{d}(\rho, \mathfrak{F}\rho, \kappa), \mathfrak{M}_{01}^{d}(\sigma, \mathfrak{F}\sigma, \kappa)\right\} < \kappa,\\
\max\left\{\mathfrak{R}_{01}^{d}(\rho, \mathfrak{F}\rho, \kappa), \mathfrak{R}_{01}^{d}(\sigma, \mathfrak{F}\sigma, \kappa)\right\} < \kappa.
\end{aligned}$$

Then it follows that,

$$\begin{aligned}
\mathfrak{S}_{01}^{d}(\rho, 0, \kappa) = \mathfrak{S}_{01}^{d}(\sigma, \sigma/3, \kappa) = 1,\\
\mathfrak{A}_{01}^{d}(\rho, 0, \kappa) = \mathfrak{A}_{01}^{d}(\sigma, \sigma/3, \kappa) = 1,\\
\mathfrak{M}_{01}^{d}(\rho, 0, \kappa) = \mathfrak{M}_{01}^{d}(\sigma, \sigma/3, \kappa) = 0,\\
\mathfrak{R}_{01}^{d}(\rho, 0, \kappa) = \mathfrak{R}_{01}^{d}(\sigma, \sigma/3, \kappa) = 0.
\end{aligned}$$

As a result, $\rho < \kappa$ and $2\sigma/3 < \kappa$. This indicates that,

$$\begin{aligned}
\mathfrak{S}_{01}^{d}(\mathfrak{F}\rho, \mathfrak{F}\sigma, c\kappa) = \mathfrak{S}_{01}^{d}(0, \sigma/3, 3\kappa/5) = 1 > 1 - c\kappa,\\
\mathfrak{A}_{01}^{d}(\mathfrak{F}\rho, \mathfrak{F}\sigma, c\kappa) = \mathfrak{A}_{01}^{d}(0, \sigma/3, 3\kappa/5) = 1 > 1 - c\kappa,\\
\mathfrak{M}_{01}^{d}(\mathfrak{F}\rho, \mathfrak{F}\sigma, c\kappa) = \mathfrak{M}_{01}^{d}(0, \sigma/3, 3\kappa/5) = 0 < c\kappa,\\
\mathfrak{R}_{01}^{d}(\mathfrak{F}\rho, \mathfrak{F}\sigma, c\kappa) = \mathfrak{R}_{01}^{d}(0, \sigma/3, 3\kappa/5) = 0 < c\kappa.
\end{aligned}$$

Therefore, all the conditions stated in Definition 3.7 are satisfied.

Case 3: $\rho, \sigma \in (6/5, 9/5]$. Without sacrificing generality, assume $\rho > \sigma$.

Now for $\kappa \in \mathbb{R}^{+} - (0, 1]$, it is attainable that,

$$(\rho - y)/3 < \rho/3 \leq 3/5 < 3\kappa/5.$$

Therefore,

$$\begin{aligned}
\mathfrak{S}_{01}^{d}(\mathfrak{F}\rho, \mathfrak{F}\sigma, c\kappa) = \mathfrak{S}_{01}^{d}(\rho/3, \sigma/3, 3\kappa/5) = 1 > 1 - c\kappa,\\
\mathfrak{A}_{01}^{d}(\mathfrak{F}\rho, \mathfrak{F}\sigma, c\kappa) = \mathfrak{A}_{01}^{d}(\rho/3, \sigma/3, 3\kappa/5) = 1 > 1 - c\kappa,\\
\mathfrak{M}_{01}^{d}(\mathfrak{F}\rho, \mathfrak{F}\sigma, c\kappa) = \mathfrak{M}_{01}^{d}(\rho/3, \sigma/3, 3\kappa/5) = 0 < c\kappa,\\
\mathfrak{R}_{01}^{d}(\mathfrak{F}\rho, \mathfrak{F}\sigma, c\kappa) = \mathfrak{R}_{01}^{d}(\rho/3, \sigma/3, 3\kappa/5) = 0 < c\kappa.
\end{aligned}$$

Thus, all the requirements of Definition 3.7 are met.

At this point, taking $\kappa \leq 1$ and,

$$\begin{aligned}
\min\left\{\mathfrak{S}_{01}^{d}(\rho, \mathfrak{F}\rho, \kappa), \mathfrak{S}_{01}^{d}(\sigma, \mathfrak{F}\sigma, \kappa)\right\} > 1 - \kappa,\\
\min\left\{\mathfrak{A}_{01}^{d}(\rho, \mathfrak{F}\rho, \kappa), \mathfrak{A}_{01}^{d}(\sigma, \mathfrak{F}\sigma, \kappa)\right\} > 1 - \kappa,
\end{aligned}$$

$$\max\left\{\mathfrak{M}_{01}^{d}(\rho,\mathfrak{F}\rho,\kappa),\mathfrak{M}_{01}^{d}(\sigma,\mathfrak{F}\sigma,\kappa)\right\}<\kappa,$$
$$\max\left\{\mathfrak{R}_{01}^{d}(\rho,\mathfrak{F}\rho,\kappa),\mathfrak{R}_{01}^{d}(\sigma,\mathfrak{F}\sigma,\kappa)\right\}<\kappa.$$

It follows that,

$$\mathfrak{S}_{01}^{d}(\rho,\rho/3,\kappa)=\mathfrak{S}_{01}^{d}(\sigma,\sigma/3,\kappa)=1,$$
$$\mathfrak{A}_{01}^{d}(\rho,\rho/3,\kappa)=\mathfrak{A}_{01}^{d}(\sigma,\sigma/3,\kappa)=1,$$
$$\mathfrak{M}_{01}^{d}(\rho,\rho/3,\kappa)=\mathfrak{M}_{01}^{d}(\sigma,\sigma/3,\kappa)=0,$$
$$\mathfrak{R}_{01}^{d}(\rho,\rho/3,\kappa)=\mathfrak{R}_{01}^{d}(\sigma,\sigma/3,\kappa)=0.$$

As a result, $2\rho/3<\kappa$ and $2\sigma/3<\kappa$. Thus,

$$(\rho-\sigma)/3<\rho/3<\kappa/2<3\kappa/5.$$

This indicates that,

$$\mathfrak{S}_{01}^{d}(\mathfrak{F}\rho,\mathfrak{F}\sigma,c\kappa)=\mathfrak{S}_{01}^{d}(\rho/3,\sigma/3,3\kappa/5)=1>1-c\kappa,$$
$$\mathfrak{A}_{01}^{d}(\mathfrak{F}\rho,\mathfrak{F}\sigma,c\kappa)=\mathfrak{A}_{01}^{d}(\rho/3,\sigma/3,3\kappa/5)=1>1-c\kappa,$$
$$\mathfrak{M}_{01}^{d}(\mathfrak{F}\rho,\mathfrak{F}\sigma,c\kappa)=\mathfrak{M}_{01}^{d}(\rho/3,\sigma/3,3\kappa/5)=0<c\kappa,$$
$$\mathfrak{R}_{01}^{d}(\mathfrak{F}\rho,\mathfrak{F}\sigma,c\kappa)=\mathfrak{R}_{01}^{d}(\rho/3,\sigma/3,3\kappa/5)=0<c\kappa.$$

Hence $\mathfrak{F}$ is a $(1)-K_{\mathfrak{C}}$, as well as a $(1/2)-K_{\mathfrak{C}}$ on G.

Remark 3.12. Although the opposite is not always true, it may be inferred from the example above that every $(1/2)-K_{\mathfrak{C}}$ is also a $(1)-K_{\mathfrak{C}}$.

Remark 3.13. For all $k\in(0,1)$, every (k)-$K_{\mathfrak{C}}$ will be a $(1)-K_{\mathfrak{C}}$, but the concerse may not always be true. This can be shown by a similar methodology as in Example 3.11.

We now introduce the notion of semi-metric in $NF_{\mathfrak{M}S}$ and present a characterisation of the space's completeness based on Definition 2.14.

Definition 3.14. Let $(G,\mathfrak{S},\mathfrak{A},\mathfrak{M},\mathfrak{R},\star,\diamond)$ be a $NF_{\mathfrak{M}S}$. For each $\rho,\sigma\in G$ take,

$$d_{\mathfrak{S}}(\rho,\sigma)=\sup\{\kappa\geq 0:\mathfrak{S}(\rho,\sigma,\kappa)\leq 1-\kappa\},$$
$$d_{\mathfrak{A}}(\rho,\sigma)=\sup\{\kappa\geq 0:\mathfrak{A}(\rho,\sigma,\kappa)\leq 1-\kappa\},$$
$$d_{\mathfrak{M}}(\rho,\sigma)=\inf\{\kappa\leq 1:\mathfrak{M}(\rho,\sigma,\kappa)\geq \kappa\},$$
$$d_{\mathfrak{R}}(\rho,\sigma)=\inf\{\kappa\leq 1:\mathfrak{R}(\rho,\sigma,\kappa)\geq \kappa\}.$$

Then $(d_{\mathfrak{S}},d_{\mathfrak{A}},d_{\mathfrak{M}},d_{\mathfrak{R}})$ is semi-metric for $(G,\tau_{(\mathfrak{S},\mathfrak{A},\mathfrak{M},\mathfrak{R})})$ if it satisfies,

(1) $d_{\mathfrak{S}}\leq 1$ and $d_{\mathfrak{S}}(\rho,\sigma)<\kappa \iff \mathfrak{S}(\rho,\sigma,\kappa)>1-\kappa\ \forall\ \kappa>0$,

(2) $d_{\mathfrak{A}}\leq 1$ and $d_{\mathfrak{A}}(\rho,\sigma)<\kappa \iff \mathfrak{A}(\rho,\sigma,\kappa)>1-\kappa\ \forall\ \kappa>0$,

(3) $d_{\mathfrak{M}}\geq 0$ and $d_{\mathfrak{M}}(\rho,\sigma)>1-\kappa \iff \mathfrak{M}(\rho,\sigma,\kappa)<\kappa\ \forall\ \kappa>0$,

(4) $d_{\mathfrak{R}}\geq 0$ and $d_{\mathfrak{R}}(\rho,\sigma)>1-\kappa \iff \mathfrak{R}(\rho,\sigma,\kappa)<\kappa\ \forall\ \kappa>0$.

Consequently, if and only if a sequence in G is a Cauchy sequence in $(G,\mathfrak{S},\mathfrak{A},\mathfrak{M},\mathfrak{R},\star,\diamond)$, then it is a Cauchy sequence for $(d_{\mathfrak{S}},d_{\mathfrak{A}},d_{\mathfrak{M}},d_{\mathfrak{R}})$.

Theorem 3.15. *For a* $NF_{\mathfrak{M}S}$ $(G, \mathfrak{S}, \mathfrak{A}, \mathfrak{M}, \mathfrak{R}, \star, \diamond)$ *the following conditions are equivalent.*

(1) $(G, \mathfrak{S}, \mathfrak{A}, \mathfrak{M}, \mathfrak{R}, \star, \diamond)$ *is complete.*

(2) *A FP exists for each* $(1)-K_{\mathfrak{C}}$ *on* G.

(3) *There is a FP for each* $(1/2)-K_{\mathfrak{C}}$ *on* G.

Proof. By Theorem 3.9, (1) $\implies$ (2).

It is also obvious that, (2) $\implies$ (3).

(3) $\implies$ (1). Let $(G, \mathfrak{S}, \mathfrak{A}, \mathfrak{M}, \mathfrak{R}, \star, \diamond)$ be a fuzzy metric space that is neutrosophic. Make the assumption that $(G, \mathfrak{S}, \mathfrak{A}, \mathfrak{M}, \mathfrak{R}, \star, \diamond)$ is not complete. Then, in $(G, \mathfrak{S}, \mathfrak{A}, \mathfrak{M}, \mathfrak{R}, \star, \diamond)$, there is a Cauchy sequence $(\rho_n)_{n\in\mathbb{N}}$ that does not converge in $\tau_{(\mathfrak{S},\mathfrak{A},\mathfrak{M},\mathfrak{R})}$. Without sacrificing generality, assume that whenever $n \neq m$, $\rho_n \neq \rho_m$.

Take $C := \{\rho_n : n \in \mathbb{N}\}$. Then both C and all the sets of the form $C \setminus \{\sigma\}, y \in G$ are closed for $\tau_{(\mathfrak{S},\mathfrak{A},\mathfrak{M},\mathfrak{R})}$.

Then by condition (iii) in Definition 2.13, for any $y \in G \setminus C$ it can be observed that,

$$d_{\mathfrak{S}}(\sigma, C \setminus \{\sigma\}),\ d_{\mathfrak{A}}(\sigma, C \setminus \{\sigma\}) > 0,$$
$$d_{\mathfrak{M}}(\sigma, C \setminus \{\sigma\}),\ d_{\mathfrak{R}}(\sigma, C \setminus \{\sigma\}) < 1.$$

It can be noted that the sets C and $C \setminus \{\sigma\}$ coincide in this case. Since by Definition 3.14, $(\rho_n)_{n\in\mathbb{N}}$ is also a Cauchy sequence for semi-metrics $d_{\mathfrak{S}}, d_{\mathfrak{A}}, d_{\mathfrak{M}}, d_{\mathfrak{R}}$, $\exists\ n(\sigma) \in \mathbb{N}$ such that, $\forall\ j, k \geq n(\sigma)$,

$$d_{\mathfrak{S}}(\rho_j, \rho_k) < \tfrac{1}{7} d_{\mathfrak{S}}(\sigma, C \setminus \{\sigma\}),$$
$$d_{\mathfrak{A}}(\rho_j, \rho_k) < \tfrac{1}{7} d_{\mathfrak{A}}(\sigma, C \setminus \{\sigma\}),$$
$$d_{\mathfrak{M}}(\rho_j, \rho_k) > 1 - \tfrac{1}{7} d_{\mathfrak{M}}(\sigma, C \setminus \{\sigma\}),$$
$$d_{\mathfrak{R}}(\rho_j, \rho_k) > 1 - \tfrac{1}{7} d_{\mathfrak{R}}(\sigma, C \setminus \{\sigma\}).$$

However for any $\sigma \in C$, there exists $n \in \mathbb{N}$ such that $\sigma := \rho_n$.

Also we have,

$$d_{\mathfrak{S}}(\sigma, C \setminus \{\sigma\}),\ d_{\mathfrak{A}}(\sigma, C \setminus \{\sigma\}) > 0,\ \text{ and }\ d_{\mathfrak{M}}(\sigma, C \setminus \{\sigma\}),\ d_{\mathfrak{R}}(\sigma, C \setminus \{\sigma\}) < 1.$$

Therefore, there exists $n(\sigma) > n$ such that, for any $j, k \geq n(\sigma)$,

$$d_{\mathfrak{S}}(\rho_j, \rho_k) < \tfrac{1}{7} d_{\mathfrak{S}}(\sigma, C \setminus \{\sigma\}),$$
$$d_{\mathfrak{A}}(\rho_j, \rho_k) < \tfrac{1}{7} d_{\mathfrak{A}}(\sigma, C \setminus \{\sigma\}),$$
$$d_{\mathfrak{M}}(\rho_j, \rho_k) > 1 - \tfrac{1}{7} d_{\mathfrak{M}}(\sigma, C \setminus \{\sigma\}),$$
$$d_{\mathfrak{R}}(\rho_j, \rho_k) > 1 - \tfrac{1}{7} d_{\mathfrak{R}}(\sigma, C \setminus \{\sigma\}).$$

Now we define a mapping $\mathfrak{F} : G \to G$ as follows:

$$\mathfrak{F}\sigma = x_{n(\sigma)},\quad y \in G.$$

Then it is obvious that, $\mathfrak{F}$ has no FPs (note that, to be specific, $\rho_n \neq x_{n(\sigma)}$, since $n(\sigma) > n$).

It now remains to show that, $\mathfrak{F}$ is a $(1/2)-K_{\mathfrak{C}}$ on $(G, \mathfrak{S}, \mathfrak{A}, \mathfrak{M}, \mathfrak{R}, \star, \diamond)$.

For this purpose, let $\sigma, w \in G$ such that $\forall\ \kappa > 0$,

$$\begin{gathered}\min\{\mathfrak{S}(\sigma, \mathfrak{F}\sigma, \kappa), \mathfrak{S}(w, \mathfrak{F}w, \kappa)\} > 1 - \kappa,\\ \min\{\mathfrak{A}(\sigma, \mathfrak{F}\sigma, \kappa), \mathfrak{A}(w, \mathfrak{F}w, \kappa)\} > 1 - \kappa,\\ \max\{\mathfrak{M}(\sigma, \mathfrak{F}\sigma, \kappa), \mathfrak{M}(w, \mathfrak{F}w, \kappa)\} < \kappa,\\ \max\{\mathfrak{R}(\sigma, \mathfrak{F}\sigma, \kappa), \mathfrak{R}(w, \mathfrak{F}w, \kappa)\} < \kappa.\end{gathered}$$

Therefore by Definition 3.14,

$$\begin{gathered}d_{\mathfrak{S}}(\sigma, \mathfrak{F}\sigma),\ d_{\mathfrak{S}}(w, \mathfrak{F}w) < \kappa,\\ d_{\mathfrak{A}}(\sigma, \mathfrak{F}\sigma),\ d_{\mathfrak{A}}(w, \mathfrak{F}w) < \kappa,\\ d_{\mathfrak{M}}(\sigma, \mathfrak{F}\sigma),\ d_{\mathfrak{M}}(w, \mathfrak{F}w) > 1 - \kappa,\\ d_{\mathfrak{R}}(\sigma, \mathfrak{F}\sigma),\ d_{\mathfrak{R}}(w, \mathfrak{F}w) > 1 - \kappa.\end{gathered}$$

Now if $n(\sigma) < n(w)$, it is inferred that,

$$\begin{aligned}d_{\mathfrak{S}}(\mathfrak{F}\sigma, \mathfrak{F}w) = d_{\mathfrak{S}}(\rho_{n(\sigma)}, \rho_{n(w)}) &< \frac{1}{7} d_{\mathfrak{S}}(\sigma, C \setminus \{\sigma\})\\ &\leq \frac{1}{7} d_{\mathfrak{S}}(\sigma, \mathfrak{F}\sigma) < \frac{1}{7}\kappa,\\ d_{\mathfrak{A}}(\mathfrak{F}\sigma, \mathfrak{F}w) = d_{\mathfrak{A}}(\rho_{n(\sigma)}, \rho_{n(w)}) &< \frac{1}{7} d_{\mathfrak{A}}(\sigma, C \setminus \{\sigma\})\\ &\leq \frac{1}{7} d_{\mathfrak{A}}(\sigma, \mathfrak{F}\sigma) < \frac{1}{7}\kappa,\\ d_{\mathfrak{M}}(\mathfrak{F}\sigma, \mathfrak{F}w) = d_{\mathfrak{M}}(\rho_{n(\sigma)}, \rho_{n(w)}) &> 1 - \frac{1}{7} d_{\mathfrak{M}}(\sigma, C \setminus \{\sigma\})\\ &\geq 1 - \frac{1}{7} d_{\mathfrak{M}}(\sigma, \mathfrak{F}\sigma) > 1 - \frac{1}{7}\kappa,\\ d_{\mathfrak{R}}(\mathfrak{F}\sigma, \mathfrak{F}w) = d_{\mathfrak{R}}(\rho_{n(\sigma)}, \rho_{n(w)}) &> 1 - \frac{1}{7} d_{\mathfrak{R}}(\sigma, C \setminus \{\sigma\})\\ &\geq 1 - \frac{1}{7} d_{\mathfrak{R}}(\sigma, \mathfrak{F}\sigma) > 1 - \frac{1}{7}\kappa.\end{aligned}$$

As a result by Definition 3.14,

$$\begin{gathered}\mathfrak{S}(\mathfrak{F}\sigma, \mathfrak{F}w, \kappa/7), \mathfrak{A}(\mathfrak{F}\sigma, \mathfrak{F}w, \kappa/7) > 1 - \kappa/7,\\ \mathfrak{M}(\mathfrak{F}\sigma, \mathfrak{F}w, \kappa/7), \mathfrak{R}(\mathfrak{F}\sigma, \mathfrak{F}w, \kappa/7) < \kappa/7.\end{gathered}$$

Similarly, if $n(w) < n(\sigma)$, it is deduced that,

$$\begin{aligned}
d_{\mathfrak{S}}(\mathfrak{F}\sigma, \mathfrak{F}w) = d_{\mathfrak{S}}(\rho_{n(\sigma)}, \rho_{n(w)}) &< \frac{1}{7} d_{\mathfrak{S}}(w, C \setminus \{w\}) \\
&\leq \frac{1}{7} d_{\mathfrak{S}}(w, \mathfrak{F}w) < \frac{1}{7}\kappa, \\
d_{\mathfrak{A}}(\mathfrak{F}\sigma, \mathfrak{F}w) = d_{\mathfrak{A}}(\rho_{n(\sigma)}, \rho_{n(w)}) &< \frac{1}{7} d_{\mathfrak{A}}(w, C \setminus \{w\}) \\
&\leq \frac{1}{7} d_{\mathfrak{A}}(w, \mathfrak{F}w) < \frac{1}{7}\kappa, \\
d_{\mathfrak{M}}(\mathfrak{F}\sigma, \mathfrak{F}w) = d_{\mathfrak{M}}(\rho_{n(\sigma)}, \rho_{n(w)}) &> 1 - \frac{1}{7} d_{\mathfrak{M}}(w, C \setminus \{w\}) \\
&\geq 1 - \frac{1}{7} d_{\mathfrak{M}}(w, \mathfrak{F}w) > 1 - \frac{1}{7}\kappa, \\
d_{\mathfrak{R}}(\mathfrak{F}\sigma, \mathfrak{F}w) = d_{\mathfrak{R}}(\rho_{n(\sigma)}, \rho_{n(w)}) &> 1 - \frac{1}{7} d_{\mathfrak{R}}(w, C \setminus \{w\}) \\
&\geq 1 - \frac{1}{7} d_{\mathfrak{R}}(w, \mathfrak{F}w) > 1 - \frac{1}{7}\kappa.
\end{aligned}$$

Thus, by Definition 3.14,

$$\begin{aligned}
\mathfrak{S}(\mathfrak{F}\sigma, \mathfrak{F}w, \kappa/7), \mathfrak{A}(\mathfrak{F}\sigma, \mathfrak{F}w, \kappa/7) > 1 - \kappa/7, \\
\mathfrak{M}(\mathfrak{F}\sigma, \mathfrak{F}w, \kappa/7), \mathfrak{R}(\mathfrak{F}\sigma, \mathfrak{F}w, \kappa/7) < \kappa/7.
\end{aligned}$$

Hence $\mathfrak{F}$ is a $(1/2)-K_{\mathfrak{C}}$. This completes the proof. □

To conclude this section, we will now demonstrate that Theorem 2.15 can be obtained by a similar methodology to the one given in [40] for Hu's theorem.

Let (G, d) be a metric space. Define $d_{1/2}$ a metric defined on G as follows:

$$d_{1/2}(\rho, \sigma) = \min\{1/2, d(\rho, \sigma)\} \ \forall \ \rho, \sigma \in G$$

The topologies induced by d and $d_{1/2}$ can then be said to coincide. Additionally, if $(G, d_{1/2})$ is complete, then (G, d) is complete. It follows that $(G, \mathfrak{S}_{01}^{d}, \mathfrak{A}_{01}^{d}, \mathfrak{M}_{01}^{d}, \mathfrak{R}_{01}^{d}, \star, \diamond)$ will be complete for every continuous t-norm and t-conorm, $\star$ and $\diamond$, respectively, if $(G, \mathfrak{S}_{01}^{d_{1/2}}, \mathfrak{A}_{01}^{d_{1/2}}, \mathfrak{M}_{01}^{d_{1/2}}, \mathfrak{R}_{01}^{d_{1/2}}, \star, \diamond)$ is also complete (compare Example 2.12).

Proposition 3.16. *[39] A metric space (G, d) is given. Then, a $K_{\mathfrak{C}}$ on $(G, d_{1/2})$ is also a $K_{\mathfrak{C}}$ on (G, d).*

Proposition 3.17. *Given a metric space (G, d) with $d(\rho, \sigma) \leq 1, \ \forall \ \rho, \sigma \in G$. Then for any continuous t-norm and t-conorm, $\star$ and $\diamond$ respectively, every $(1/2)-K_{\mathfrak{C}}$ on $(G, \mathfrak{S}_{01}^{d}, \mathfrak{A}_{01}^{d}, \mathfrak{M}_{01}^{d}, \mathfrak{R}_{01}^{d}, \star, \diamond)$ is a $K_{\mathfrak{C}}$ on (G, d).*

Proof. Let $\mathfrak{F} : G \to G$ be a $(1/2)-K_{\mathfrak{C}}$ on $(G, \mathfrak{S}^d_{01}, \mathfrak{A}^d_{01}, \mathfrak{M}^d_{01}, \mathfrak{R}^d_{01}, \star, \diamond)$. We shall show that,

$$d(\mathfrak{F}\rho, \mathfrak{F}\sigma) \leq c[d(\rho, \mathfrak{F}\rho) + d(\sigma, \mathfrak{F}\sigma)], \ \forall \ \rho, \sigma \in G, \text{ for some } c \in (0, 1/2).$$

In fact, let us assume the contrary. Then, $\exists \ \rho, \sigma \in G$ such that $d(\mathfrak{F}\rho, \mathfrak{F}\sigma) > c[d(\rho, \mathfrak{F}\rho) + d(\sigma, \mathfrak{F}\sigma)]$. Let $L = d(\rho, \mathfrak{F}\rho) + d(\sigma, \mathfrak{F}\sigma)$. Then $L \in [0, 2)$. Choose $\epsilon > 0$ such that $c(L + \epsilon) < 1$ and $d(\mathfrak{F}\rho, \mathfrak{F}\sigma) > c(L + \epsilon)$. Now since,

$$\begin{aligned}
\mathfrak{S}^d_{01}(\rho, \mathfrak{F}\rho, L + \epsilon) &= \mathfrak{S}^d_{01}(\sigma, \mathfrak{F}\sigma, L + \epsilon) = 1, \\
\mathfrak{A}^d_{01}(\rho, \mathfrak{F}\rho, L + \epsilon) &= \mathfrak{A}^d_{01}(\sigma, \mathfrak{F}\sigma, L + \epsilon) = 1, \\
\mathfrak{M}^d_{01}(\rho, \mathfrak{F}\rho, L + \epsilon) &= \mathfrak{M}^d_{01}(\sigma, \mathfrak{F}\sigma, L + \epsilon) = 0, \\
\mathfrak{R}^d_{01}(\rho, \mathfrak{F}\rho, L + \epsilon) &= \mathfrak{R}^d_{01}(\sigma, \mathfrak{F}\sigma, L + \epsilon) = 0,
\end{aligned}$$

and $\mathfrak{F}$ is a $(1/2)-K_{\mathfrak{C}}$, it is deduced that,

$$\begin{aligned}
\mathfrak{S}^d_{01}(\mathfrak{F}\rho, \mathfrak{F}\sigma, c(L + \epsilon)) &> 1 - c(L + \epsilon) > 0, \\
\mathfrak{A}^d_{01}(\mathfrak{F}\rho, \mathfrak{F}\sigma, c(L + \epsilon)) &> 1 - c(L + \epsilon) > 0, \\
\mathfrak{M}^d_{01}(\mathfrak{F}\rho, \mathfrak{F}\sigma, c(L + \epsilon)) &< c(L + \epsilon) < 1, \\
\mathfrak{R}^d_{01}(\mathfrak{F}\rho, \mathfrak{F}\sigma, c(L + \epsilon)) &< c(L + \epsilon) < 1.
\end{aligned}$$

However, on the other hand, from $d(\mathfrak{F}\rho, \mathfrak{F}\sigma) > c(L + \epsilon)$, it can be inferred that,

$$\begin{aligned}
\mathfrak{S}^d_{01}(\mathfrak{F}\rho, \mathfrak{F}\sigma, c(L + \epsilon)) &= \mathfrak{A}^d_{01}(\mathfrak{F}\rho, \mathfrak{F}\sigma, c(L + \epsilon)) = 0, \\
\mathfrak{M}^d_{01}(\mathfrak{F}\rho, \mathfrak{F}\sigma, c(L + \epsilon)) &= \mathfrak{R}^d_{01}(\mathfrak{F}\rho, \mathfrak{F}\sigma, c(L + \epsilon)) = 1.
\end{aligned}$$

This is a contradiction. Hence the proof. □

Proof of Theorem 2.16: Only the 'if' portion has been shown here. Assume that there is a FP for each $K_{\mathfrak{C}}$ on (G, d). On the $NF_{\mathfrak{M}S}$ $(G, \mathfrak{S}, \mathfrak{A}, \mathfrak{M}, \mathfrak{R}, \star, \diamond)$, let F be a $(1/2)-K_{\mathfrak{C}}$. Given that $d_{1/2} \leq 1$, Proposition 3.16 implies that $\mathfrak{F}$ is a $K_{\mathfrak{C}}$ on $(G, d_{1/2})$; hence, Proposition 3.17 asserts that (G, d) is a $K_{\mathfrak{C}}$. T has a FP as a result. Thus, according to Theorem 3.15, $(G, \mathfrak{S}^{d_{1/2}}_{01}, \mathfrak{A}^{d_{1/2}}_{01}, \mathfrak{M}^{d_{1/2}}_{01}, \mathfrak{R}^{d_{1/2}}_{01}, \star, \diamond)$ is complete.
As a result, $(G, \mathfrak{S}^d_{01}, \mathfrak{A}^d_{01}, \mathfrak{M}^d_{01}, \mathfrak{R}^d_{01}, \star, \diamond)$ is complete, and hence, (G, d).

4. **Conclusion**

This study proves important fixed-point results in the context of $NF_{\mathfrak{M}S}$s, such as the Banach fixed-point theorem, the Edelstein fixed-point theorem, and fixed-point results of the Kannan type. The completeness of $NF_{\mathfrak{M}S}$s has been further characterized using the derived Kannan-type results. Furthermore, a number of fundamental ideas have been presented and thoroughly characterized in the context of $NF_{\mathfrak{M}S}$s, including Cauchy sequences, convergence, completeness, and semi-metrics. These theoretical conclusions have been supported by a number of illustrative examples.

The results of this study have potential applications in various fields. In data science, they can be employed to design more robust clustering and classification algorithms under uncertainty and fuzziness. In engineering, these results can aid in the modeling and analysis of systems characterized by indeterminate or imprecise parameters, such as control systems or structural stability. In optimization, the findings provide a theoretical foundation for solving real-world problems involving vague or partial information, such as supply chain management, resource allocation, and scheduling. Furthermore, these results can be applied in medical diagnosis systems to enhance decision-making processes that involve uncertainty and ambiguity.

Looking ahead, the derived results could be extended to explore best proximity results in $NF_{\mathfrak{M}S}$s. Furthermore, the developed concepts have the potential for practical applications in optimization problems across various real-world scenarios. The framework could also be expanded to investigate other conventional fixed-point theorems, such as the Ciric fixed-point theorem, Chatterjea fixed-point theorem, and Suzuki fixed-point theorem, within the realm of $NF_{\mathfrak{M}S}$s.

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Quality Assessment of Innovation and Entrepreneurship Talent Cultivation in Universities from the Perspective of Collaborative Education Under Triangular Neutrosophic Cubic Linguistic Hesitant Fuzzy Set

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Abstract: A technique for representing mathematical models of confusion, ambiguity, confusion, inconsistency, and inconsistency, neutrophilic logic is built on non-standard analysis. While truth membership, indeterminacy membership, and falsity membership are independent, indeterminacy is explicitly quantified in the Neutrosophic set. In many circumstances where we deal with conflicting and partial information, this is crucial. Differential equations are widely used in science and engineering to represent issues, and one of the newest areas of study is the study of differential equations with uncertainty. We used the Triangular Neutrosophic Cubic Linguistic Hesitant Fuzzy Set with the decision-making methods. Two methods are used in this study. The RANCOM method is used to compute the criteria weights of Innovation and Entrepreneurship Talent Cultivation. The PROMETHEE method is used to rank the alternatives. nine criteria and ten alternatives are evaluated by three experts. A case study is conducted to show the validity of the proposed approach.

Keywords: Triangular Neutrosophic Cubic Linguistic Hesitant Fuzzy; Innovation and Entrepreneurship; Collaborative Education.

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1. Introduction

A novel similarity metric between Atanassov's intuitionistic fuzzy values was introduced by Chen et al. To obtain the aggregated decision matrix of each decision-maker, Chen et al.' suggested method aggregates each decision-maker's decision matrix and attribute weights using the evidential reasoning methodology. The intuitionistic fuzzy Archimedean Heronian aggregation (IFAHA) operator was proposed by Liu et al. Some characteristics and specific situations of these suggested operators were defined by Liu et al. Based on the suggested I2LGA operator, Liu and Chen presented a novel MAGDM technique using the I2LI. Each alternative's weighted positive

and negative scores were determined by Chen et al. Using the new scoring function and the linear programming (LP) methodology, Wang and Chen presented a novel MADM approach[1], [2].

Decision-makers find it difficult to provide an accurate assessment of complicated issues in real-world decision-making situations because of both subjective and objective elements. Decision-makers typically have some reluctance to evaluate the ambiguous and uncertain quantities. One of the extensions of fuzzy set theory is intuitionistic fuzzy sets (IFS)[3], [4]. It has been discovered that IFS, which was initially presented by Atanassov, is consistent with handling vagueness among several higher order fuzzy sets. When there is insufficient information available to describe the impreciseness using the traditional fuzzy set, the idea of IFS can be seen as a suitable solution. Later, Liu and Yuan combined intuitionistic fuzzy sets with triangle fuzzy numbers to create triangular intuitionistic fuzzy sets[5], [6].

The triangle intuitionistic fuzzy set's primary feature is that its membership function and non-membership function have triangular fuzzy values instead of exact ones. As a generalization of the classic set, fuzzy set, and intuitionistic fuzzy set, Wang et al. Recently, I recently published a single-valued neutrosophic set, which is a subclass of a neutrosophic set introduced by Smarandache[7], [8]. In addition to dealing with inconsistent, indeterminate, and incomplete information, the single-valued neutrosophic set can independently express truth-membership degree, indeterminacy-membership degree, and falsity-membership degree. Because of the imperfections in the knowledge that humans acquire or perceive from the outside world, all the elements that are defined by the single-valued neutrosophic set are very appropriate for human thought.

For instance, if the proposition is "Movie X would be hit," the human brain is undoubtedly unable to produce exact yes/no responses since indeterminacy is the area where one is unsure of the proposition's value between truth and falsity. It goes without saying that the intuitionistic fuzzy set is unable to handle and express indeterminacy and inconsistent information, whereas the neutrosophic components are better suited to do so. As a result, the single-valued neutrosophic set has developed quickly and has many uses[9], [10]. A few persuasive techniques were used to suggest the Hesitant Fuzzy Set and the Statement of Fuzzy Set, which permit an element to belong to a precise set.

Fuzzy sets and intuitionistic fuzzy sets are generalizations of cubic sets, which were developed by Jun et al. They include two representations: one for the degree of membership and another for the degree of non-membership. Non-membership is considered over the regular fuzzy set, whilst the membership function is held in the form of an interval.

2. Triangular neutrosophic cubic linguistic hesitant fuzzy numbers

This section shows the definition of the triangular neutrosophic cubic linguistic hesitant fuzzy numbers (TNCLHFN)[11], [12]. Let two TNCLHFNs such as:

$$A_1 = \begin{Bmatrix} S_{\theta_1}, [u_1, v_1, x_1]; ([B_1^-, B_1^+]; B_1) \\ ([C_1^-, C_1^+]; C_1), ([D_1^-, D_1^+]; D_1) \end{Bmatrix} \tag{1}$$

$$A_2 = \begin{Bmatrix} S_{\theta_2}, [u_2, v_2, x_2]; ([B_2^-, B_2^+]; B_2) \\ ([C_2^-, C_2^+]; C_2), ([D_2^-, D_2^+]; D_2) \end{Bmatrix} \tag{2}$$

$$A_1 + A_2 = \begin{Bmatrix} S_{\theta_{a_1+a_2}}, [u_1 + u_2, v_1 + v_2, x_1 + x_2], \\ \left([\min(B_1^-, B_2^-), \min(B_1^+, B_1^+)], \max(B_1, B_2)\right), \\ \left([\min(C_1^-, C_2^-), \min(C_1^+, C_1^+)], \max(C_1, C_2)\right), \\ \left([\min(D_1^-, D_2^-), \min(D_1^+, D_1^+)], \max(D_1, D_2)\right) \end{Bmatrix} \tag{3}$$

$$A_1 - A_2 = \begin{Bmatrix} S_{\theta_{a_1-a_2}}, [u_1 - u_2, v_1 - v_2, x_1 - x_2], \\ \left([\min(B_1^-, B_2^-), \min(B_1^+, B_1^+)], \max(B_1, B_2)\right), \\ \left([\min(C_1^-, C_2^-), \min(C_1^+, C_1^+)], \max(C_1, C_2)\right), \\ \left([\min(D_1^-, D_2^-), \min(D_1^+, D_1^+)], \max(D_1, D_2)\right) \end{Bmatrix} \tag{4}$$

$$A_1 A_2 = \begin{Bmatrix} S_{\theta_{a_1 a_2}}, [u_1 u_2, v_1 v_2, x_1 x_2], \\ \left([\min(B_1^-, B_2^-), \min(B_1^+, B_1^+)], \max(B_1, B_2)\right), \\ \left([\min(C_1^-, C_2^-), \min(C_1^+, C_1^+)], \max(C_1, C_2)\right), \\ \left([\min(D_1^-, D_2^-), \min(D_1^+, D_1^+)], \max(D_1, D_2)\right) \end{Bmatrix} \tag{5}$$

$$\omega A_1 = \begin{Bmatrix} S_{\theta_{\omega a_1}}, [\omega u_1, \omega v_1, \omega x_1], \\ ([B_1^-, B_1^+], B_1), \\ ([C_1^-, C_1^+], C_1), \\ ([D_1^-, D_1^+], D_1) \end{Bmatrix} \tag{6}$$

$$A_1^\omega = \begin{Bmatrix} S_{\theta_{a_1^\omega}}, [u_1^\omega, v_1^\omega, x_1^\omega], \\ ([(B_1^-)^\omega, (B_1^+)^\omega], 1 - (1 - B_1)^\omega), \\ ([(C_1^-)^\omega, (C_1^+)^\omega], 1 - (1 - C_1)^\omega), \\ ([(D_1^-)^\omega, (D_1^+)^\omega], 1 - (1 - D_1)^\omega) \end{Bmatrix} \tag{7}$$

$$\sum_{j=1}^{n} w_j n_j \begin{Bmatrix} S\prod_{j=1}^{n} a_j, \left[\sum_{j=1}^{n} u_1^{w_j}, \sum_{j=1}^{n} v_1^{w_j}, \sum_{j=1}^{n} x_1^{w_j}\right], \\ \left(\begin{bmatrix} 1-\prod_{j=1}^{n}(1-B_1^-)^{w_j}, \\ 1-\prod_{j=1}^{n}(1-B_1^+)^{w_j} \end{bmatrix}, 1-\prod_{j=1}^{n}(1-B_1)^{w_j}\right), \\ \left(\begin{bmatrix} \prod_{j=1}^{n}(C_1^-)^{w_j}, \prod_{j=1}^{n}(C_1^+)^{w_j} \end{bmatrix}, \prod_{j=1}^{n}(C_1)^{w_j}\right), \\ \left(\begin{bmatrix} \prod_{j=1}^{n}(D_1^-)^{w_j}, \prod_{j=1}^{n}(D_1^+)^{w_j} \end{bmatrix}, \prod_{j=1}^{n}(D_1)^{w_j}\right) \end{Bmatrix} \tag{8}$$

3. A hybrid TNCLHFNs-RANCOM-PROMETHEE method for solving MCDM problems

This section shows the steps of the proposed approach. RANCOM is used to compute the criteria weights. The PROMETHEE method is used to rank the alternatives. These methods are used under the neutrosophic sets to deal with the uncertainty and vague information.

The RANCOM Method

The steps of this method are organized as follows[13], [14]:

1. Construct the decision matrix

The decision matrix is built based on the opinions of the experts and decision makers. These matrices are used the neutrosophic numbers.

2. Apply the score function

The score function is applied to obtain crisp values in the decision matrix.

3. Aggrege the decision matrix

The decision matrix values are aggregated using the average method.

4. Compute the rank of the criteria

5. Generate the ranking comparison matrix between the criteria.

$$q_{kj} = \begin{bmatrix} q_{11} & \cdots & q_{1n} \\ \vdots & \ddots & \vdots \\ q_{n1} & \cdots & q_{nn} \end{bmatrix} \tag{9}$$

Where

$$q_{kj} = \begin{cases} 1, & if\ S(Q_j) < S(Q_k), \\ 0.5, & if\ S(Q_j) = S(Q_k), \\ 0, & if\ S(Q_j) > S(Q_k) \end{cases} \tag{10}$$

6. Compute the summed criteria weight

$$e_j = \sum_{k=1}^{n} q_{kj} \tag{11}$$

7. Compute subjective weights

$$w_j = \frac{e_j}{\sum_{j=1}^{n} e_j} \tag{12}$$

The PROMETHEE Method

This method is used to rank the alternatives.

1. Normalize the decision matrix

$$n_{ij} = \frac{x_{ij} - \min(x_{ij})}{\max(x_{ij}) - \min(x_{ij})} \tag{13}$$

$$n_{ij} = \frac{max - x_{ij}}{\max(x_{ij}) - \min(x_{ij})} \tag{14}$$

2. Compute the relative difference between the alternatives.

3. Compute the preference function

$$p_j(a,b) = 0 \quad if\ n_{aj} \le n_{bj} \tag{15}$$

$$p_j(a,b) = \left(n_{aj} - n_{bj}\right)\ if\ n_{aj} > n_{bj} \tag{16}$$

4. Combined the preferences functions

$$y(a,b) = \left[\sum_{j=1}^{n} w_j\, p_j(a,b)\right] / \sum_{j=1}^{n} w_j \tag{17}$$

5. Compute the leaving and entering outranking flows

$$F^{+} = \frac{1}{m-1} \sum_{b=1}^{m} y(a,b) \tag{18}$$

$$F^{-} = \frac{1}{m-1} \sum_{b=1}^{m} y(a,b) \tag{19}$$

6. Compute the net outranking flow

$$F(a) = F^{+}(a) - F^{-}(a) \tag{20}$$

7. Ranking the alternatives.

4. Case Study

This section shows the results of the proposed approach to computing the criteria weights and rank the alternatives. This study uses 9 criteria and 10 alternatives to be evaluated: Practical Training, International Exchange, Interdisciplinary Learning Opportunities

Soft Skills, Employment Outcomes, Personalized Learning, Entrepreneurship, Technology Integration in Education, Industry-Academia Collaboration. These criteria are evaluated using three experts and decision makers. The alternatives are University-Industry Joint Training Programs, Entrepreneurial Incubators, Community-Engaged, Work-Integrated Learning, Project-Based Collaborative Learning, International Exchange, Dual-Degree and Interdisciplinary Programs, Research-Based Collaborative Education, AI-Powered Personalized Learning Systems, Lifelong Learning

Then we use the neutrosophic numbers to evaluate ethe criteria and alternatives as shown in Table 1. Then we apply the score function to obtain crisp values. Then we combined these values into a single matrix. We combined these values using the average method. Then we applied the steps to compute the criteria weights and rank the alternatives.

Table 1. The experts 'opinions.

	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8	C_9
A_1	{S2,[0.4,0.5,0.6],([0.2,0.4],0.3),([0.2,0.5],0.3),([0.1,0.12],0.11)}	{S3,[0.1,0.2,0.3],([0.3,0.6],0.1),([0.2,0.4],0.2),([0.5,0.8],0.4)}	{S1,[0.5,0.4,0.7],([0.7,0.9],0.3),([0.3,0.8],0.6),([0.4,0.7],0.5)}	{S1,[0.1,0.3,0.5],([0.2,0.7],0.3),([0.4,0.6],0.1),([0.4,0.7],0.3)}	{S1,[0.1,0.3,0.5],([0.2,0.7],0.3),([0.4,0.6],0.1),([0.4,0.7],0.3)}	{S2,[0.4,0.5,0.6],([0.2,0.4],0.3),([0.2,0.5],0.3),([0.1,0.12],0.11)}	{S2,[0.2,0.4,0.4],([0.3,0.6],0.2),([0.5,0.7],0.5),([0.2,0.7],0.1)}	{S1,[0.3,0.4,0.6],([0.1,0.4],0.2),([0.3,0.5],0.5),([0.4,0.6],0.1)}	{S2,[0.4,0.5,0.6],([0.2,0.4],0.3),([0.2,0.5],0.3),([0.1,0.12],0.11)}
A_2	{S2,[0.7,0.8,0.9],([0.2,0.4],0.1),([0.3,0.5],0.5),([0.1,0.3],0.6)}	{S1,[0.1,0.3,0.5],([0.2,0.7],0.3),([0.4,0.6],0.1),([0.4,0.7],0.3)}	{S2,[0.4,0.5,0.6],([0.2,0.4],0.3),([0.2,0.5],0.3),([0.1,0.12],0.11)}	{S3,[0.1,0.2,0.3],([0.3,0.6],0.1),([0.2,0.4],0.2),([0.5,0.8],0.4)}	{S1,[0.5,0.4,0.7],([0.7,0.9],0.3),([0.3,0.8],0.6),([0.4,0.7],0.5)}	{S1,[0.1,0.3,0.5],([0.2,0.7],0.3),([0.4,0.6],0.1),([0.4,0.7],0.3)}	{S1,[0.1,0.3,0.5],([0.2,0.7],0.3),([0.4,0.6],0.1),([0.4,0.7],0.3)}	{S2,[0.7,0.8,0.9],([0.2,0.4],0.1),([0.3,0.5],0.5),([0.1,0.3],0.6)}	{S3,[0.1,0.2,0.3],([0.3,0.6],0.1),([0.2,0.4],0.2),([0.5,0.8],0.4)}
A_3	{S2,[0.2,0.4,0.4],([0.3,0.6],0.2),([0.5,0.7],0.5),([0.2,0.7],0.1)}	{S1,[0.5,0.4,0.7],([0.7,0.9],0.3),([0.3,0.8],0.6),([0.4,0.7],0.5)}	{S1,[0.5,0.4,0.7],([0.7,0.9],0.3),([0.3,0.8],0.6),([0.4,0.7],0.5)}	{S1,[0.1,0.3,0.5],([0.2,0.7],0.3),([0.4,0.6],0.1),([0.4,0.7],0.3)}	{S3,[0.1,0.2,0.3],([0.3,0.6],0.1),([0.2,0.4],0.2),([0.5,0.8],0.4)}	{S1,[0.5,0.4,0.7],([0.7,0.9],0.3),([0.3,0.8],0.6),([0.4,0.7],0.5)}	{S1,[0.5,0.4,0.7],([0.7,0.9],0.3),([0.3,0.8],0.6),([0.4,0.7],0.5)}	{S2,[0.2,0.4,0.4],([0.3,0.6],0.2),([0.5,0.7],0.5),([0.2,0.7],0.1)}	{S1,[0.5,0.4,0.7],([0.7,0.9],0.3),([0.3,0.8],0.6),([0.4,0.7],0.5)}
A_4	{S1,[0.1,0.3,0.5],([0.2,0.7],0.3),([0.4,0.6],0.1),([0.4,0.7],0.3)}	{S3,[0.1,0.2,0.3],([0.3,0.6],0.1),([0.2,0.4],0.2),([0.5,0.8],0.4)}	{S1,[0.1,0.3,0.5],([0.2,0.7],0.3),([0.4,0.6],0.1),([0.4,0.7],0.3)}	{S2,[0.4,0.5,0.6],([0.2,0.4],0.3),([0.2,0.5],0.3),([0.1,0.12],0.11)}	{S2,[0.4,0.5,0.6],([0.2,0.4],0.3),([0.2,0.5],0.3),([0.1,0.12],0.11)}	{S3,[0.1,0.2,0.3],([0.3,0.6],0.1),([0.2,0.4],0.2),([0.5,0.8],0.4)}	{S3,[0.1,0.2,0.3],([0.3,0.6],0.1),([0.2,0.4],0.2),([0.5,0.8],0.4)}	{S1,[0.1,0.3,0.5],([0.2,0.7],0.3),([0.4,0.6],0.1),([0.4,0.7],0.3)}	{S1,[0.1,0.3,0.5],([0.2,0.7],0.3),([0.4,0.6],0.1),([0.4,0.7],0.3)}
A_5	{S1,[0.5,0.4,0.7],([0.7,0.9],0.3),([0.3,0.8],0.6),([0.4,0.7],0.5)}	{S2,[0.4,0.5,0.6],([0.2,0.4],0.3),([0.2,0.5],0.3),([0.1,0.12],0.11)}	{S1,[0.5,0.4,0.7],([0.7,0.9],0.3),([0.3,0.8],0.6),([0.4,0.7],0.5)}	{S1,[0.1,0.3,0.5],([0.2,0.7],0.3),([0.4,0.6],0.1),([0.4,0.7],0.3)}	{S1,[0.1,0.3,0.5],([0.2,0.7],0.3),([0.4,0.6],0.1),([0.4,0.7],0.3)}	{S2,[0.4,0.5,0.6],([0.2,0.4],0.3),([0.2,0.5],0.3),([0.1,0.12],0.11)}	{S2,[0.4,0.5,0.6],([0.2,0.4],0.3),([0.2,0.5],0.3),([0.1,0.12],0.11)}	{S1,[0.5,0.4,0.7],([0.7,0.9],0.3),([0.3,0.8],0.6),([0.4,0.7],0.5)}	{S2,[0.2,0.4,0.4],([0.3,0.6],0.2),([0.5,0.7],0.5),([0.2,0.7],0.1)}
A_6	{S3,[0.1,0.2,0.3],([0.3,0.6],0.1),([0.2,0.4],0.2),([0.5,0.8],0.4)}	{S1,[0.1,0.3,0.5],([0.2,0.7],0.3),([0.4,0.6],0.1),([0.4,0.7],0.3)}	{S3,[0.1,0.2,0.3],([0.3,0.6],0.1),([0.2,0.4],0.2),([0.5,0.8],0.4)}	{S1,[0.5,0.4,0.7],([0.7,0.9],0.3),([0.3,0.8],0.6),([0.4,0.7],0.5)}	{S1,[0.5,0.4,0.7],([0.7,0.9],0.3),([0.3,0.8],0.6),([0.4,0.7],0.5)}	{S1,[0.1,0.3,0.5],([0.2,0.7],0.3),([0.4,0.6],0.1),([0.4,0.7],0.3)}	{S1,[0.3,0.4,0.6],([0.1,0.4],0.2),([0.3,0.5],0.5),([0.4,0.6],0.1)}	{S3,[0.1,0.2,0.3],([0.3,0.6],0.1),([0.2,0.4],0.2),([0.5,0.8],0.4)}	{S2,[0.7,0.8,0.9],([0.2,0.4],0.1),([0.3,0.5],0.5),([0.1,0.3],0.6)}
A_7	{S2,[0.4,0.5,0.6],([0.2,0.	{S1,[0.5,0.4,0.7],([0.7,0.	{S2,[0.4,0.5,0.6],([0.2,0.	{S3,[0.1,0.2,0.3],([0.3,0.	{S3,[0.1,0.2,0.3],([0.3,0.	{S1,[0.5,0.4,0.7],([0.7,0.	{S2,[0.4,0.5,0.6],([0.2,0.	{S2,[0.4,0.5,0.6],([0.2,0.	{S1,[0.3,0.4,0.6],([0.1,0.

	4],0.3),([0.2,0.5],0.3),([0.1,0.12],0.11)}	9],0.3),([0.3,0.8],0.6),([0.4,0.7],0.5)}	4],0.3),([0.2,0.5],0.3),([0.1,0.12],0.11)}	6],0.1),([0.2,0.4],0.2),([0.5,0.8],0.4)}	6],0.1),([0.2,0.4],0.2),([0.5,0.8],0.4)}	9],0.3),([0.3,0.8],0.6),([0.4,0.7],0.5)}	4],0.3),([0.2,0.5],0.3),([0.1,0.12],0.11)}	4],0.3),([0.2,0.5],0.3),([0.1,0.12],0.11)}	4],0.2),([0.3,0.5],0.5),([0.4,0.6],0.1)}
A8	{S1,[0.3,0.4,0.6],([0.1,0.4],0.2),([0.3,0.5],0.5),([0.4,0.6],0.1)}	{S3,[0.1,0.2,0.3],([0.3,0.6],0.1),([0.2,0.4],0.2),([0.5,0.8],0.4)}	{S2,[0.2,0.4,0.4],([0.3,0.6],0.2),([0.5,0.7],0.5),([0.2,0.7],0.1)}	{S2,[0.4,0.5,0.6],([0.2,0.4],0.3),([0.2,0.5],0.3),([0.1,0.12],0.11)}	{S2,[0.4,0.5,0.6],([0.2,0.4],0.3),([0.2,0.5],0.3),([0.1,0.12],0.11)}	{S3,[0.1,0.2,0.3],([0.3,0.6],0.1),([0.2,0.4],0.2),([0.5,0.8],0.4)}	{S3,[0.1,0.2,0.3],([0.3,0.6],0.1),([0.2,0.4],0.2),([0.5,0.8],0.4)}	{S1,[0.3,0.4,0.6],([0.1,0.4],0.2),([0.3,0.5],0.5),([0.4,0.6],0.1)}	{S2,[0.4,0.5,0.6],([0.2,0.4],0.3),([0.2,0.5],0.3),([0.1,0.12],0.11)}
A9	{S2,[0.7,0.8,0.9],([0.2,0.4],0.1),([0.3,0.5],0.5),([0.1,0.3],0.6)}	{S2,[0.4,0.5,0.6],([0.2,0.4],0.3),([0.2,0.5],0.3),([0.1,0.12],0.11)}	{S2,[0.7,0.8,0.9],([0.2,0.4],0.1),([0.3,0.5],0.5),([0.1,0.3],0.6)}	{S2,[0.7,0.8,0.9],([0.2,0.4],0.1),([0.3,0.5],0.5),([0.1,0.3],0.6)}	{S1,[0.1,0.3,0.5],([0.2,0.7],0.3),([0.4,0.6],0.1),([0.4,0.7],0.3)}	{S2,[0.4,0.5,0.6],([0.2,0.4],0.3),([0.2,0.5],0.3),([0.1,0.12],0.11)}	{S1,[0.5,0.4,0.7],([0.7,0.9],0.3),([0.3,0.8],0.6),([0.4,0.7],0.5)}	{S2,[0.7,0.8,0.9],([0.2,0.4],0.1),([0.3,0.5],0.5),([0.1,0.3],0.6)}	{S3,[0.1,0.2,0.3],([0.3,0.6],0.1),([0.2,0.4],0.2),([0.5,0.8],0.4)}
A10	{S2,[0.2,0.4,0.4],([0.3,0.6],0.2),([0.5,0.7],0.5),([0.2,0.7],0.1)}	{S1,[0.1,0.3,0.5],([0.2,0.7],0.3),([0.4,0.6],0.1),([0.4,0.7],0.3)}	{S1,[0.5,0.4,0.7],([0.7,0.9],0.3),([0.3,0.8],0.6),([0.4,0.7],0.5)}	{S3,[0.1,0.2,0.3],([0.3,0.6],0.1),([0.2,0.4],0.2),([0.5,0.8],0.4)}	{S2,[0.4,0.5,0.6],([0.2,0.4],0.3),([0.2,0.5],0.3),([0.1,0.12],0.11)}	{S1,[0.3,0.4,0.6],([0.1,0.4],0.2),([0.3,0.5],0.5),([0.4,0.6],0.1)}	{S1,[0.1,0.3,0.5],([0.2,0.7],0.3),([0.4,0.6],0.1),([0.4,0.7],0.3)}	{S2,[0.2,0.4,0.4],([0.3,0.6],0.2),([0.5,0.7],0.5),([0.2,0.7],0.1)}	{S1,[0.5,0.4,0.7],([0.7,0.9],0.3),([0.3,0.8],0.6),([0.4,0.7],0.5)}
	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8	C_9
A1	{S1,[0.3,0.4,0.6],([0.1,0.4],0.2),([0.3,0.5],0.5),([0.4,0.6],0.1)}	{S3,[0.1,0.2,0.3],([0.3,0.6],0.1),([0.2,0.4],0.2),([0.5,0.8],0.4)}	{S1,[0.5,0.4,0.7],([0.7,0.9],0.3),([0.3,0.8],0.6),([0.4,0.7],0.5)}	{S1,[0.1,0.3,0.5],([0.2,0.7],0.3),([0.4,0.6],0.1),([0.4,0.7],0.3)}	{S1,[0.3,0.4,0.6],([0.1,0.4],0.2),([0.3,0.5],0.5),([0.4,0.6],0.1)}	{S2,[0.4,0.5,0.6],([0.2,0.4],0.3),([0.2,0.5],0.3),([0.1,0.12],0.11)}	{S2,[0.2,0.4,0.4],([0.3,0.6],0.2),([0.5,0.7],0.5),([0.2,0.7],0.1)}	{S1,[0.3,0.4,0.6],([0.1,0.4],0.2),([0.3,0.5],0.5),([0.4,0.6],0.1)}	{S2,[0.4,0.5,0.6],([0.2,0.4],0.3),([0.2,0.5],0.3),([0.1,0.12],0.11)}
A2	{S2,[0.4,0.5,0.6],([0.2,0.4],0.3),([0.2,0.5],0.3),([0.1,0.12],0.11)}	{S1,[0.3,0.4,0.6],([0.1,0.4],0.2),([0.3,0.5],0.5),([0.4,0.6],0.1)}	{S2,[0.4,0.5,0.6],([0.2,0.4],0.3),([0.2,0.5],0.3),([0.1,0.12],0.11)}	{S3,[0.1,0.2,0.3],([0.3,0.6],0.1),([0.2,0.4],0.2),([0.5,0.8],0.4)}	{S1,[0.5,0.4,0.7],([0.7,0.9],0.3),([0.3,0.8],0.6),([0.4,0.7],0.5)}	{S1,[0.1,0.3,0.5],([0.2,0.7],0.3),([0.4,0.6],0.1),([0.4,0.7],0.3)}	{S1,[0.1,0.3,0.5],([0.2,0.7],0.3),([0.4,0.6],0.1),([0.4,0.7],0.3)}	{S2,[0.2,0.4,0.4],([0.3,0.6],0.2),([0.5,0.7],0.5),([0.2,0.7],0.1)}	{S3,[0.1,0.2,0.3],([0.3,0.6],0.1),([0.2,0.4],0.2),([0.5,0.8],0.4)}
A3	{S3,[0.1,0.2,0.3],([0.3,0.6],0.1),([0.2,0.4],0.2),([0.5,0.8],0.4)}	{S1,[0.3,0.4,0.6],([0.1,0.4],0.2),([0.3,0.5],0.5),([0.4,0.6],0.1)}	{S1,[0.5,0.4,0.7],([0.7,0.9],0.3),([0.3,0.8],0.6),([0.4,0.7],0.5)}	{S1,[0.1,0.3,0.5],([0.2,0.7],0.3),([0.4,0.6],0.1),([0.4,0.7],0.3)}	{S2,[0.2,0.4,0.4],([0.3,0.6],0.2),([0.5,0.7],0.5),([0.2,0.7],0.1)}	{S1,[0.3,0.4,0.6],([0.1,0.4],0.2),([0.3,0.5],0.5),([0.4,0.6],0.1)}	{S1,[0.1,0.3,0.5],([0.2,0.7],0.3),([0.4,0.6],0.1),([0.4,0.7],0.3)}	{S1,[0.1,0.3,0.5],([0.2,0.7],0.3),([0.4,0.6],0.1),([0.4,0.7],0.3)}	{S1,[0.5,0.4,0.7],([0.7,0.9],0.3),([0.3,0.8],0.6),([0.4,0.7],0.5)}
A4	{S1,[0.5,0.4,0.7],([0.7,0.9],0.3),([0.3,0.8],0.6),([0.4,0.7],0.5)}	{S2,[0.7,0.8,0.9],([0.2,0.4],0.1),([0.3,0.5],0.5),([0.1,0.3],0.6)}	{S3,[0.1,0.2,0.3],([0.3,0.6],0.1),([0.2,0.4],0.2),([0.5,0.8],0.4)}	{S2,[0.4,0.5,0.6],([0.2,0.4],0.3),([0.2,0.5],0.3),([0.1,0.12],0.11)}	{S1,[0.3,0.4,0.6],([0.1,0.4],0.2),([0.3,0.5],0.5),([0.4,0.6],0.1)}	{S2,[0.4,0.5,0.6],([0.2,0.4],0.3),([0.2,0.5],0.3),([0.1,0.12],0.11)}	{S3,[0.1,0.2,0.3],([0.3,0.6],0.1),([0.2,0.4],0.2),([0.5,0.8],0.4)}	{S1,[0.5,0.4,0.7],([0.7,0.9],0.3),([0.3,0.8],0.6),([0.4,0.7],0.5)}	{S1,[0.1,0.3,0.5],([0.2,0.7],0.3),([0.4,0.6],0.1),([0.4,0.7],0.3)}
A5	{S1,[0.1,0.3,0.5],([0.2,0.7],0.3),([0.4,0.6],0.1),([0.4,0.7],0.3)}	{S1,[0.3,0.4,0.6],([0.1,0.4],0.2),([0.3,0.5],0.5),([0.4,0.6],0.1)}	{S1,[0.1,0.3,0.5],([0.2,0.7],0.3),([0.4,0.6],0.1),([0.4,0.7],0.3)}	{S1,[0.3,0.4,0.6],([0.1,0.4],0.2),([0.3,0.5],0.5),([0.4,0.6],0.1)}	{S2,[0.4,0.5,0.6],([0.2,0.4],0.3),([0.2,0.5],0.3),([0.1,0.12],0.11)}	{S3,[0.1,0.2,0.3],([0.3,0.6],0.1),([0.2,0.4],0.2),([0.5,0.8],0.4)}	{S2,[0.4,0.5,0.6],([0.2,0.4],0.3),([0.2,0.5],0.3),([0.1,0.12],0.11)}	{S3,[0.1,0.2,0.3],([0.3,0.6],0.1),([0.2,0.4],0.2),([0.5,0.8],0.4)}	{S2,[0.2,0.4,0.4],([0.3,0.6],0.2),([0.5,0.7],0.5),([0.2,0.7],0.1)}
A6	{S3,[0.1,0.2,0.3],([0.3,0.6],0.1),([0.2,0.4],0.2),([0.5,0.8],0.4)}	{S2,[0.4,0.5,0.6],([0.2,0.4],0.3),([0.2,0.5],0.3),([0.1,0.12],0.11)}	{S1,[0.3,0.4,0.6],([0.1,0.4],0.2),([0.3,0.5],0.5),([0.4,0.6],0.1)}	{S2,[0.4,0.5,0.6],([0.2,0.4],0.3),([0.2,0.5],0.3),([0.1,0.12],0.11)}	{S3,[0.1,0.2,0.3],([0.3,0.6],0.1),([0.2,0.4],0.2),([0.5,0.8],0.4)}	{S1,[0.5,0.4,0.7],([0.7,0.9],0.3),([0.3,0.8],0.6),([0.4,0.7],0.5)}	{S1,[0.3,0.4,0.6],([0.1,0.4],0.2),([0.3,0.5],0.5),([0.4,0.6],0.1)}	{S2,[0.4,0.5,0.6],([0.2,0.4],0.3),([0.2,0.5],0.3),([0.1,0.12],0.11)}	{S2,[0.7,0.8,0.9],([0.2,0.4],0.1),([0.3,0.5],0.5),([0.1,0.3],0.6)}

A_7	{S2,[0.4,0.5,0.6],([0.2,0.4],0.3),([0.2,0.5],0.3),([0.1,0.12],0.11)}	{S3,[0.1,0.2,0.3],([0.3,0.6],0.1),([0.2,0.4],0.2),([0.5,0.8],0.4)}	{S2,[0.4,0.5,0.6],([0.2,0.4],0.3),([0.2,0.5],0.3),([0.1,0.12],0.11)}	{S3,[0.1,0.2,0.3],([0.3,0.6],0.1),([0.2,0.4],0.2),([0.5,0.8],0.4)}	{S1,[0.5,0.4,0.7],([0.7,0.9],0.3),([0.3,0.8],0.6),([0.4,0.7],0.5)}	{S1,[0.1,0.3,0.5],([0.2,0.7],0.3),([0.4,0.6],0.1),([0.4,0.7],0.3)}	{S3,[0.1,0.2,0.3],([0.3,0.6],0.1),([0.2,0.4],0.2),([0.5,0.8],0.4)}	{S1,[0.3,0.4,0.6],([0.1,0.4],0.2),([0.3,0.5],0.5),([0.4,0.6],0.1)}	{S1,[0.3,0.4,0.6],([0.1,0.4],0.2),([0.3,0.5],0.5),([0.4,0.6],0.1)}
A_8	{S1,[0.3,0.4,0.6],([0.1,0.4],0.2),([0.3,0.5],0.5),([0.4,0.6],0.1)}	{S1,[0.5,0.4,0.7],([0.7,0.9],0.3),([0.3,0.8],0.6),([0.4,0.7],0.5)}	{S3,[0.1,0.2,0.3],([0.3,0.6],0.1),([0.2,0.4],0.2),([0.5,0.8],0.4)}	{S1,[0.5,0.4,0.7],([0.7,0.9],0.3),([0.3,0.8],0.6),([0.4,0.7],0.5)}	{S1,[0.1,0.3,0.5],([0.2,0.7],0.3),([0.4,0.6],0.1),([0.4,0.7],0.3)}	{S1,[0.5,0.4,0.7],([0.7,0.9],0.3),([0.3,0.8],0.6),([0.4,0.7],0.5)}	{S1,[0.5,0.4,0.7],([0.7,0.9],0.3),([0.3,0.8],0.6),([0.4,0.7],0.5)}	{S2,[0.7,0.8,0.9],([0.2,0.4],0.1),([0.3,0.5],0.5),([0.1,0.3],0.6)}	{S2,[0.4,0.5,0.6],([0.2,0.4],0.3),([0.2,0.5],0.3),([0.1,0.12],0.11)}
A_9	{S2,[0.7,0.8,0.9],([0.2,0.4],0.1),([0.3,0.5],0.5),([0.1,0.3],0.6)}	{S1,[0.1,0.3,0.5],([0.2,0.7],0.3),([0.4,0.6],0.1),([0.4,0.7],0.3)}	{S1,[0.5,0.4,0.7],([0.7,0.9],0.3),([0.3,0.8],0.6),([0.4,0.7],0.5)}	{S1,[0.1,0.3,0.5],([0.2,0.7],0.3),([0.4,0.6],0.1),([0.4,0.7],0.3)}	{S1,[0.1,0.3,0.5],([0.2,0.7],0.3),([0.4,0.6],0.1),([0.4,0.7],0.3)}	{S1,[0.1,0.3,0.5],([0.2,0.7],0.3),([0.4,0.6],0.1),([0.4,0.7],0.3)}	{S1,[0.1,0.3,0.5],([0.2,0.7],0.3),([0.4,0.6],0.1),([0.4,0.7],0.3)}	{S2,[0.2,0.4,0.4],([0.3,0.6],0.2),([0.5,0.7],0.5),([0.2,0.7],0.1)}	{S3,[0.1,0.2,0.3],([0.3,0.6],0.1),([0.2,0.4],0.2),([0.5,0.8],0.4)}
A_{10}	{S2,[0.2,0.4,0.4],([0.3,0.6],0.2),([0.5,0.7],0.5),([0.2,0.7],0.1)}	{S1,[0.1,0.3,0.5],([0.2,0.7],0.3),([0.4,0.6],0.1),([0.4,0.7],0.3)}	{S1,[0.1,0.3,0.5],([0.2,0.7],0.3),([0.4,0.6],0.1),([0.4,0.7],0.3)}	{S3,[0.1,0.2,0.3],([0.3,0.6],0.1),([0.2,0.4],0.2),([0.5,0.8],0.4)}	{S2,[0.4,0.5,0.6],([0.2,0.4],0.3),([0.2,0.5],0.3),([0.1,0.12],0.11)}	{S1,[0.3,0.4,0.6],([0.1,0.4],0.2),([0.3,0.5],0.5),([0.4,0.6],0.1)}	{S2,[0.2,0.4,0.4],([0.3,0.6],0.2),([0.5,0.7],0.5),([0.2,0.7],0.1)}	{S1,[0.1,0.3,0.5],([0.2,0.7],0.3),([0.4,0.6],0.1),([0.4,0.7],0.3)}	{S1,[0.5,0.4,0.7],([0.7,0.9],0.3),([0.3,0.8],0.6),([0.4,0.7],0.5)}
	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8	C_9
A_1	{S2,[0.4,0.5,0.6],([0.2,0.4],0.3),([0.2,0.5],0.3),([0.1,0.12],0.11)}	{S3,[0.1,0.2,0.3],([0.3,0.6],0.1),([0.2,0.4],0.2),([0.5,0.8],0.4)}	{S1,[0.5,0.4,0.7],([0.7,0.9],0.3),([0.3,0.8],0.6),([0.4,0.7],0.5)}	{S1,[0.1,0.3,0.5],([0.2,0.7],0.3),([0.4,0.6],0.1),([0.4,0.7],0.3)}	{S2,[0.2,0.4,0.4],([0.3,0.6],0.2),([0.5,0.7],0.5),([0.2,0.7],0.1)}	{S2,[0.4,0.5,0.6],([0.2,0.4],0.3),([0.2,0.5],0.3),([0.1,0.12],0.11)}	{S2,[0.2,0.4,0.4],([0.3,0.6],0.2),([0.5,0.7],0.5),([0.2,0.7],0.1)}	{S1,[0.3,0.4,0.6],([0.1,0.4],0.2),([0.3,0.5],0.5),([0.4,0.6],0.1)}	{S2,[0.4,0.5,0.6],([0.2,0.4],0.3),([0.2,0.5],0.3),([0.1,0.12],0.11)}
A_2	{S2,[0.7,0.8,0.9],([0.2,0.4],0.1),([0.3,0.5],0.5),([0.1,0.3],0.6)}	{S1,[0.3,0.4,0.6],([0.1,0.4],0.2),([0.3,0.5],0.5),([0.4,0.6],0.1)}	{S2,[0.4,0.5,0.6],([0.2,0.4],0.3),([0.2,0.5],0.3),([0.1,0.12],0.11)}	{S3,[0.1,0.2,0.3],([0.3,0.6],0.1),([0.2,0.4],0.2),([0.5,0.8],0.4)}	{S1,[0.5,0.4,0.7],([0.7,0.9],0.3),([0.3,0.8],0.6),([0.4,0.7],0.5)}	{S1,[0.1,0.3,0.5],([0.2,0.7],0.3),([0.4,0.6],0.1),([0.4,0.7],0.3)}	{S1,[0.1,0.3,0.5],([0.2,0.7],0.3),([0.4,0.6],0.1),([0.4,0.7],0.3)}	{S2,[0.7,0.8,0.9],([0.2,0.4],0.1),([0.3,0.5],0.5),([0.1,0.3],0.6)}	{S3,[0.1,0.2,0.3],([0.3,0.6],0.1),([0.2,0.4],0.2),([0.5,0.8],0.4)}
A_3	{S2,[0.2,0.4,0.4],([0.3,0.6],0.2),([0.5,0.7],0.5),([0.2,0.7],0.1)}	{S1,[0.3,0.4,0.6],([0.1,0.4],0.2),([0.3,0.5],0.5),([0.4,0.6],0.1)}	{S1,[0.5,0.4,0.7],([0.7,0.9],0.3),([0.3,0.8],0.6),([0.4,0.7],0.5)}	{S1,[0.1,0.3,0.5],([0.2,0.7],0.3),([0.4,0.6],0.1),([0.4,0.7],0.3)}	{S2,[0.2,0.4,0.4],([0.3,0.6],0.2),([0.5,0.7],0.5),([0.2,0.7],0.1)}	{S2,[0.2,0.4,0.4],([0.3,0.6],0.2),([0.5,0.7],0.5),([0.2,0.7],0.1)}	{S1,[0.5,0.4,0.7],([0.7,0.9],0.3),([0.3,0.8],0.6),([0.4,0.7],0.5)}	{S2,[0.2,0.4,0.4],([0.3,0.6],0.2),([0.5,0.7],0.5),([0.2,0.7],0.1)}	{S1,[0.5,0.4,0.7],([0.7,0.9],0.3),([0.3,0.8],0.6),([0.4,0.7],0.5)}
A_4	{S1,[0.1,0.3,0.5],([0.2,0.7],0.3),([0.4,0.6],0.1),([0.4,0.7],0.3)}	{S2,[0.7,0.8,0.9],([0.2,0.4],0.1),([0.3,0.5],0.5),([0.1,0.3],0.6)}	{S3,[0.1,0.2,0.3],([0.3,0.6],0.1),([0.2,0.4],0.2),([0.5,0.8],0.4)}	{S2,[0.4,0.5,0.6],([0.2,0.4],0.3),([0.2,0.5],0.3),([0.1,0.12],0.11)}	{S1,[0.3,0.4,0.6],([0.1,0.4],0.2),([0.3,0.5],0.5),([0.4,0.6],0.1)}	{S2,[0.7,0.8,0.9],([0.2,0.4],0.1),([0.3,0.5],0.5),([0.1,0.3],0.6)}	{S3,[0.1,0.2,0.3],([0.3,0.6],0.1),([0.2,0.4],0.2),([0.5,0.8],0.4)}	{S1,[0.1,0.3,0.5],([0.2,0.7],0.3),([0.4,0.6],0.1),([0.4,0.7],0.3)}	{S1,[0.1,0.3,0.5],([0.2,0.7],0.3),([0.4,0.6],0.1),([0.4,0.7],0.3)}
A_5	{S1,[0.5,0.4,0.7],([0.7,0.9],0.3),([0.3,0.8],0.6),([0.4,0.7],0.5)}	{S2,[0.2,0.4,0.4],([0.3,0.6],0.2),([0.5,0.7],0.5),([0.2,0.7],0.1)}	{S1,[0.1,0.3,0.5],([0.2,0.7],0.3),([0.4,0.6],0.1),([0.4,0.7],0.3)}	{S1,[0.3,0.4,0.6],([0.1,0.4],0.2),([0.3,0.5],0.5),([0.4,0.6],0.1)}	{S2,[0.4,0.5,0.6],([0.2,0.4],0.3),([0.2,0.5],0.3),([0.1,0.12],0.11)}	{S1,[0.3,0.4,0.6],([0.1,0.4],0.2),([0.3,0.5],0.5),([0.4,0.6],0.1)}	{S2,[0.4,0.5,0.6],([0.2,0.4],0.3),([0.2,0.5],0.3),([0.1,0.12],0.11)}	{S1,[0.5,0.4,0.7],([0.7,0.9],0.3),([0.3,0.8],0.6),([0.4,0.7],0.5)}	{S2,[0.2,0.4,0.4],([0.3,0.6],0.2),([0.5,0.7],0.5),([0.2,0.7],0.1)}
A_6	{S3,[0.1,0.2,0.3],([0.3,0.6],0.1),([0.2,0.4],0.2),([	{S1,[0.1,0.3,0.5],([0.2,0.7],0.3),([0.4,0.6],0.1),([	{S1,[0.5,0.4,0.7],([0.7,0.9],0.3),([0.3,0.8],0.6),([	{S2,[0.7,0.8,0.9],([0.2,0.4],0.1),([0.3,0.5],0.5),([	{S2,[0.4,0.5,0.6],([0.2,0.4],0.3),([0.2,0.5],0.3),([	{S2,[0.4,0.5,0.6],([0.2,0.4],0.3),([0.2,0.5],0.3),([	{S1,[0.3,0.4,0.6],([0.1,0.4],0.2),([0.3,0.5],0.5),([	{S3,[0.1,0.2,0.3],([0.3,0.6],0.1),([0.2,0.4],0.2),([	{S2,[0.7,0.8,0.9],([0.2,0.4],0.1),([0.3,0.5],0.5),([

	0.5,0.8],0.4)}	0.4,0.7],0.3)}	0.4,0.7],0.5)}	0.1,0.3],0.6)}	0.1,0.12],0.11)}	0.1,0.12],0.11)}	0.4,0.6],0.1)}	0.5,0.8],0.4)}	0.1,0.3],0.6)}
A7	{S2,[0.4,0.5,0.6],([0.2,0.4],0.3),([0.2,0.5],0.3),([0.1,0.12],0.11)}	{S1,[0.5,0.4,0.7],([0.7,0.9],0.3),([0.3,0.8],0.6),([0.4,0.7],0.5)}	{S1,[0.1,0.3,0.5],([0.2,0.7],0.3),([0.4,0.6],0.1),([0.4,0.7],0.3)}	{S2,[0.2,0.4,0.4],([0.3,0.6],0.2),([0.5,0.7],0.5),([0.2,0.7],0.1)}	{S3,[0.1,0.2,0.3],([0.3,0.6],0.1),([0.2,0.4],0.2),([0.5,0.8],0.4)}	{S3,[0.1,0.2,0.3],([0.3,0.6],0.1),([0.2,0.4],0.2),([0.5,0.8],0.4)}	{S2,[0.4,0.5,0.6],([0.2,0.4],0.3),([0.2,0.5],0.3),([0.1,0.12],0.11)}	{S2,[0.4,0.5,0.6],([0.2,0.4],0.3),([0.2,0.5],0.3),([0.1,0.12],0.11)}	{S1,[0.3,0.4,0.6],([0.1,0.4],0.2),([0.3,0.5],0.5),([0.4,0.6],0.1)}
A8	{S1,[0.3,0.4,0.6],([0.1,0.4],0.2),([0.3,0.5],0.5),([0.4,0.6],0.1)}	{S3,[0.1,0.2,0.3],([0.3,0.6],0.1),([0.2,0.4],0.2),([0.5,0.8],0.4)}	{S2,[0.2,0.4,0.4],([0.3,0.6],0.2),([0.5,0.7],0.5),([0.2,0.7],0.1)}	{S1,[0.1,0.3,0.5],([0.2,0.7],0.3),([0.4,0.6],0.1),([0.4,0.7],0.3)}	{S1,[0.5,0.4,0.7],([0.7,0.9],0.3),([0.3,0.8],0.6),([0.4,0.7],0.5)}	{S1,[0.5,0.4,0.7],([0.7,0.9],0.3),([0.3,0.8],0.6),([0.4,0.7],0.5)}	{S3,[0.1,0.2,0.3],([0.3,0.6],0.1),([0.2,0.4],0.2),([0.5,0.8],0.4)}	{S1,[0.3,0.4,0.6],([0.1,0.4],0.2),([0.3,0.5],0.5),([0.4,0.6],0.1)}	{S2,[0.4,0.5,0.6],([0.2,0.4],0.3),([0.2,0.5],0.3),([0.1,0.12],0.11)}
A9	{S2,[0.7,0.8,0.9],([0.2,0.4],0.1),([0.3,0.5],0.5),([0.1,0.3],0.6)}	{S2,[0.4,0.5,0.6],([0.2,0.4],0.3),([0.2,0.5],0.3),([0.1,0.12],0.11)}	{S2,[0.7,0.8,0.9],([0.2,0.4],0.1),([0.3,0.5],0.5),([0.1,0.3],0.6)}	{S2,[0.7,0.8,0.9],([0.2,0.4],0.1),([0.3,0.5],0.5),([0.1,0.3],0.6)}	{S1,[0.1,0.3,0.5],([0.2,0.7],0.3),([0.4,0.6],0.1),([0.4,0.7],0.3)}	{S1,[0.1,0.3,0.5],([0.2,0.7],0.3),([0.4,0.6],0.1),([0.4,0.7],0.3)}	{S1,[0.5,0.4,0.7],([0.7,0.9],0.3),([0.3,0.8],0.6),([0.4,0.7],0.5)}	{S2,[0.7,0.8,0.9],([0.2,0.4],0.1),([0.3,0.5],0.5),([0.1,0.3],0.6)}	{S3,[0.1,0.2,0.3],([0.3,0.6],0.1),([0.2,0.4],0.2),([0.5,0.8],0.4)}
A10	{S2,[0.2,0.4,0.4],([0.3,0.6],0.2),([0.5,0.7],0.5),([0.2,0.7],0.1)}	{S1,[0.1,0.3,0.5],([0.2,0.7],0.3),([0.4,0.6],0.1),([0.4,0.7],0.3)}	{S1,[0.5,0.4,0.7],([0.7,0.9],0.3),([0.3,0.8],0.6),([0.4,0.7],0.5)}	{S3,[0.1,0.2,0.3],([0.3,0.6],0.1),([0.2,0.4],0.2),([0.5,0.8],0.4)}	{S2,[0.4,0.5,0.6],([0.2,0.4],0.3),([0.2,0.5],0.3),([0.1,0.12],0.11)}	{S1,[0.3,0.4,0.6],([0.2,0.5],0.5),([0.4,0.6],0.1)}	{S1,[0.1,0.3,0.5],([0.2,0.7],0.3),([0.4,0.6],0.1),([0.4,0.7],0.3)}	{S2,[0.2,0.4,0.4],([0.3,0.6],0.2),([0.5,0.7],0.5),([0.2,0.7],0.1)}	{S1,[0.5,0.4,0.7],([0.7,0.9],0.3),([0.3,0.8],0.6),([0.4,0.7],0.5)}

Then we rank the criteria. Eq. (9) and Eq. (10) are used to generate ranking comparison matrix. Then we compute the summed criteria weight suing Eq. (11). Then we compute subjective weights using Eq. (12) as shown in Fig 1.

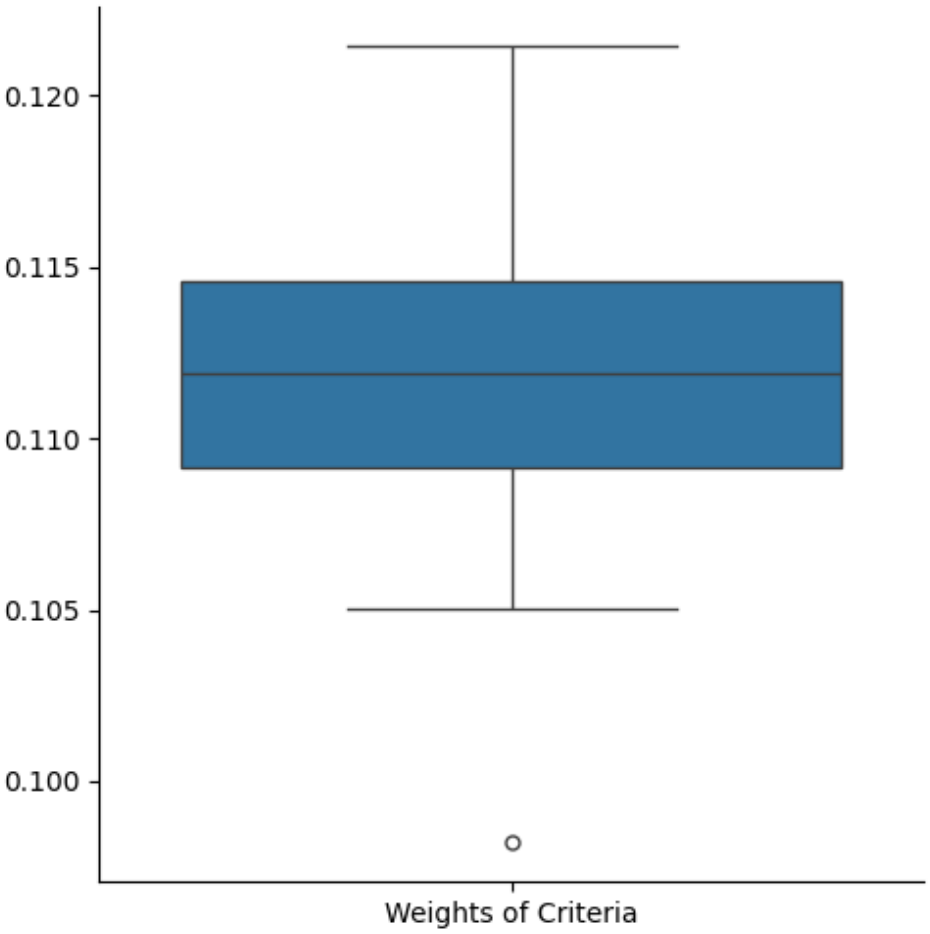

Fig 1. The criteria importance.

The PROMETHEE Method

Eq. (13) is used to normalize the decision matrix using Eqs. (13 and 14) as shown in Table 2. Then we compute the relative difference between the alternatives as shown in Fig 2. Then we compute the preference function between the alternatives as shown in Fig 3. Then we combined these preference functions into a single matrix. Then we compute the leaving and entering flows. Then we compute the net outranking flow. Then we ranked the alternatives as shown in Fig 4.

Table 2. The normalization matrix.

	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8	C_9
A_1	0.119385	0.327731	1	0.710526	0.360185	0	0.574468	0.273342	0
A_2	0.406619	0.282913	0	0.676692	1	0.45	0.446809	0.548038	0.333333
A_3	0.524823	0.610644	1	0.710526	0.411111	0.741667	1	0.47226	1
A_4	0.723404	0.473389	0.338889	0	0.187037	0.347619	0.425532	0.691475	0.35
A_5	1	0.191877	0.566667	0.616541	0.116667	0.263095	0	1	0.45
A_6	0.425532	0.176471	0.537963	1	0.444444	0.578571	0.358156	0.188092	0.477778
A_7	0	1	0.116667	0.755639	0.555556	0.721429	0.141844	0	0.280556

A8	0.358156	0.663866	0.411111	0.913534	0.45	1	0.70922	0.369418	0
A9	0.609929	0	0.651852	0.883459	0.35	0.3	1	0.548038	0.333333
A10	0.574468	0.352941	0.783333	0.676692	0	0.360714	0.489362	0.47226	1

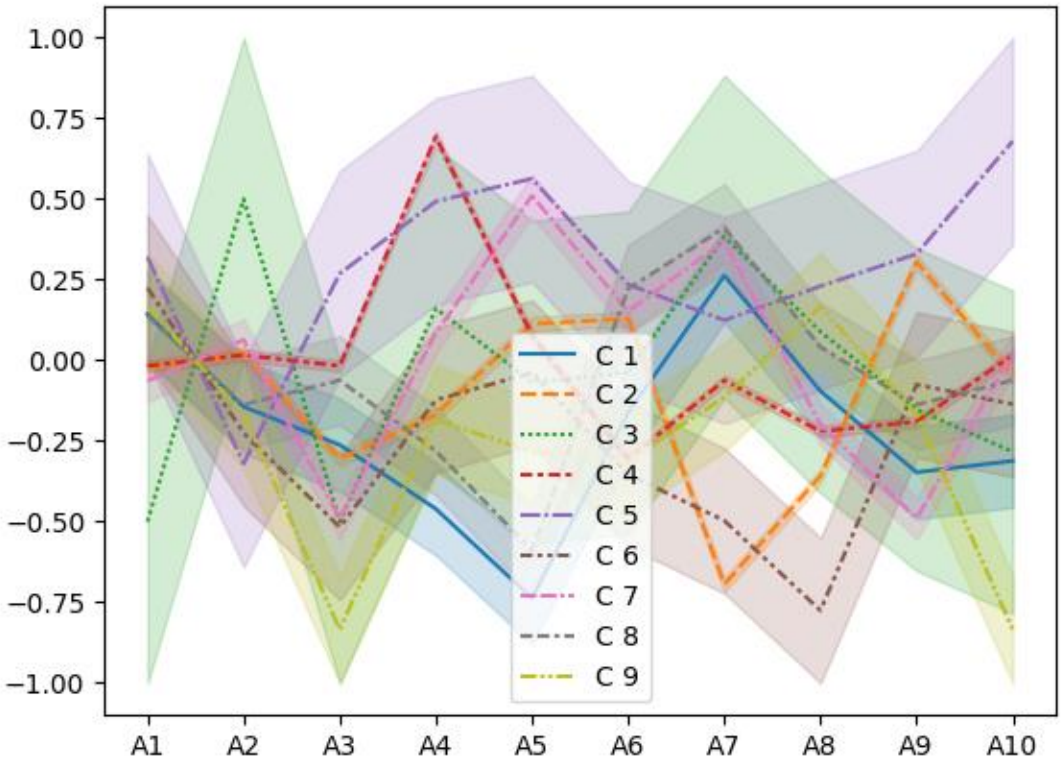

Fig 2-a. The difference between the alternatives.

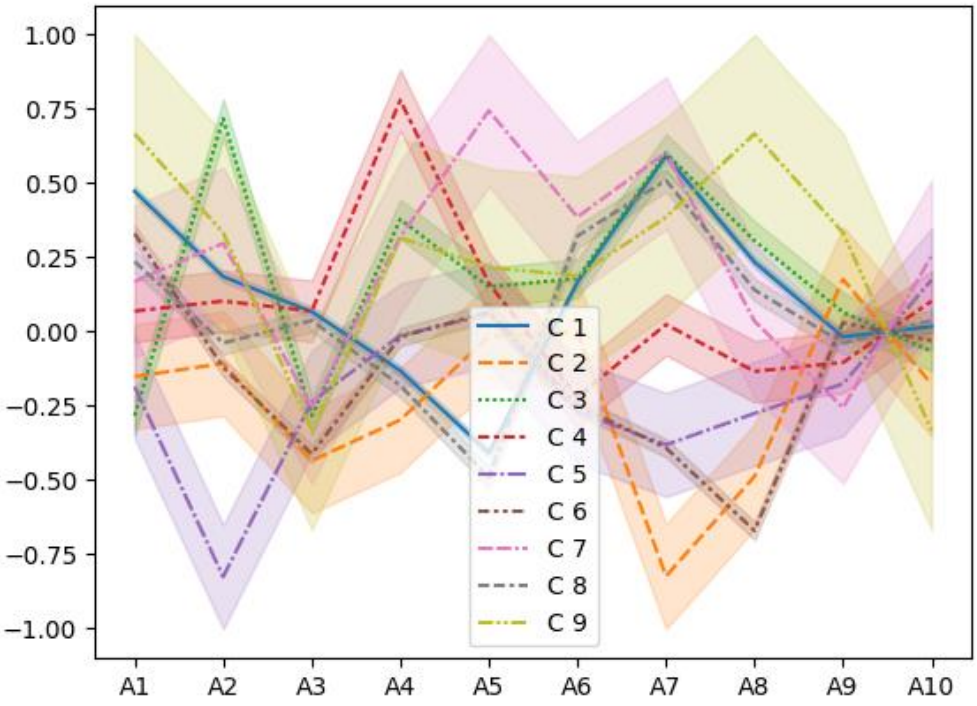

Fig 2-b. The difference between the alternatives.

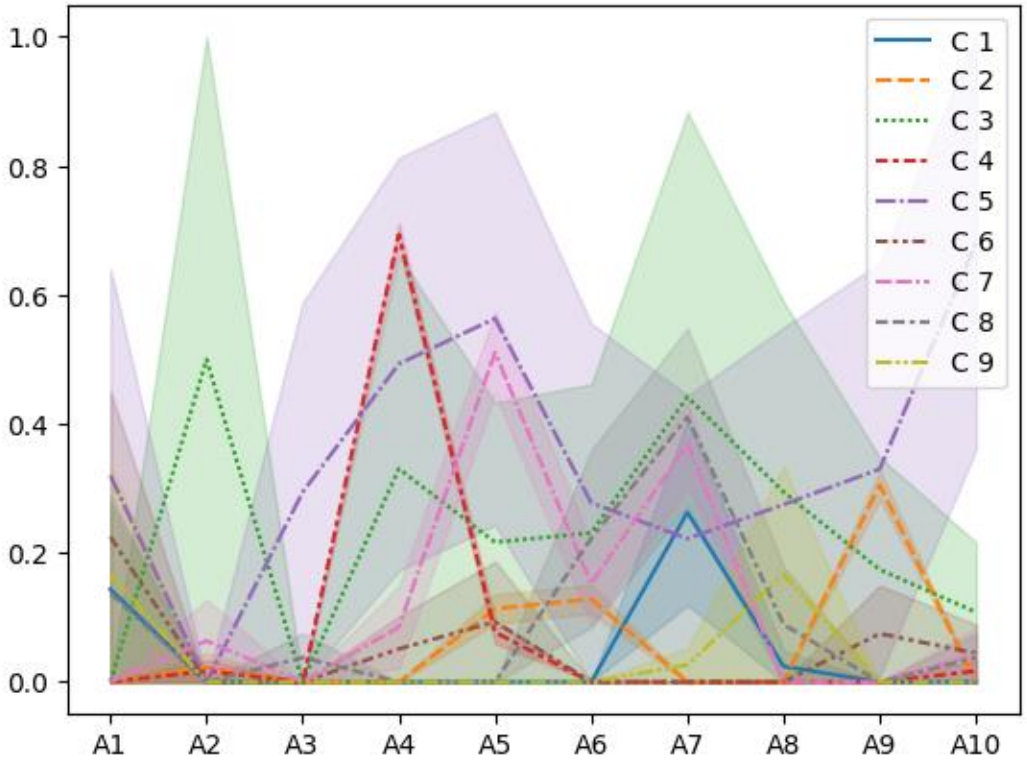

Fig 3-a. The preference values.

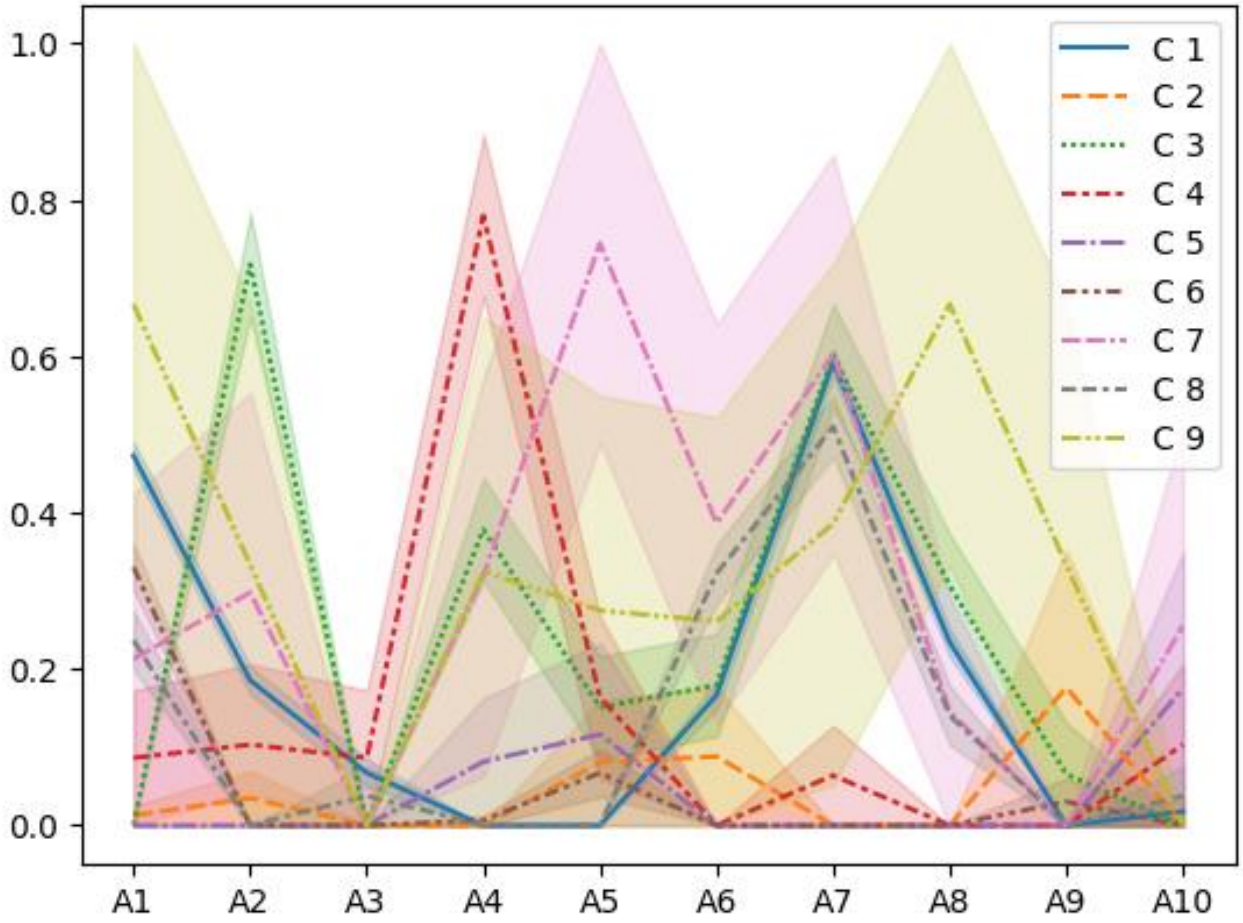

Fig 3-b. The preference values.

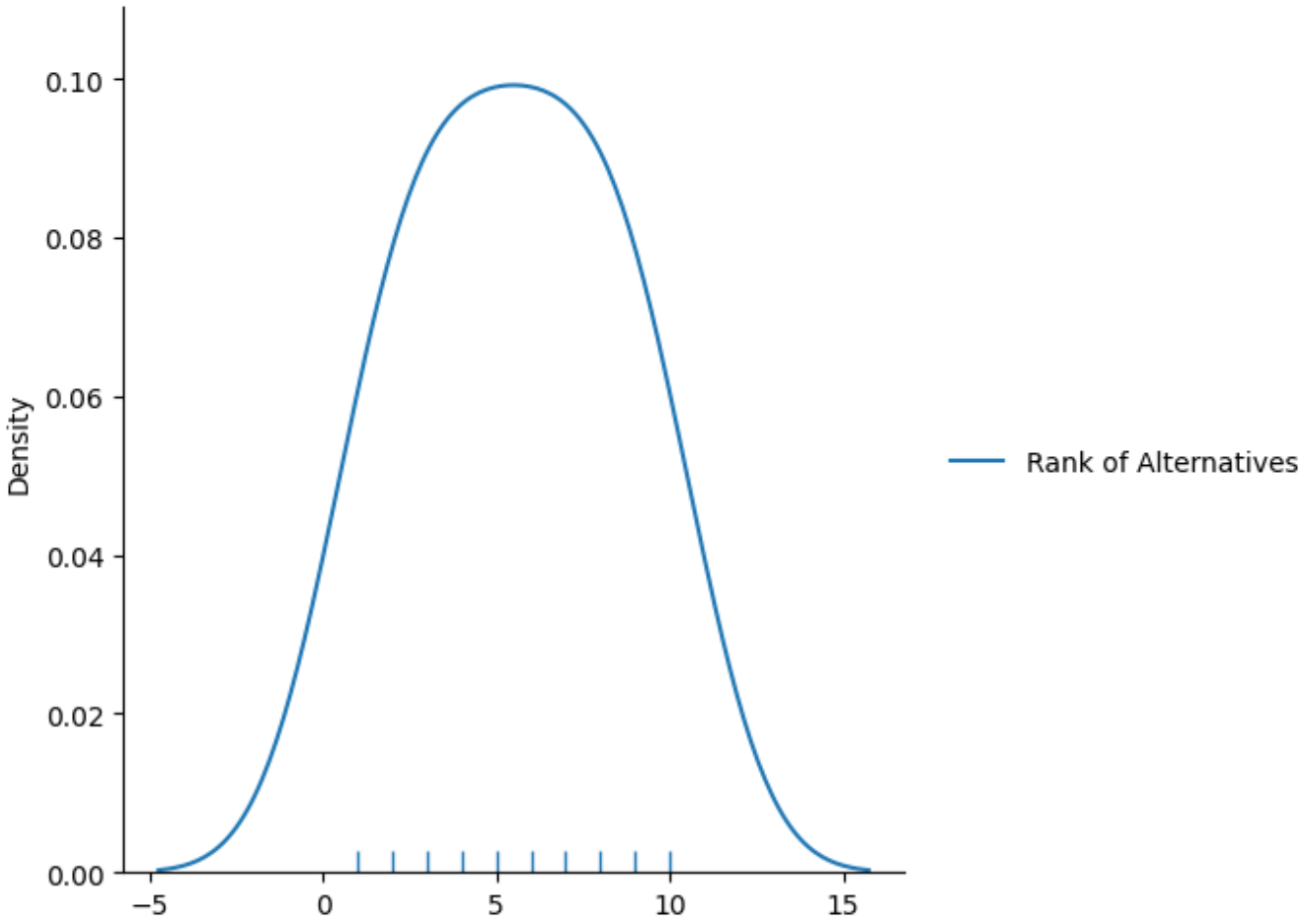

Fig 4. The rank of alternatives.

Then we applied the sensitivity analysis between the ranks of the alternatives. We conducted ten cases in the criteria weights to show the rank of the alternatives under different cases as shown in fig 5. Then we applied the proposed approach to obtain the rank of alternatives.

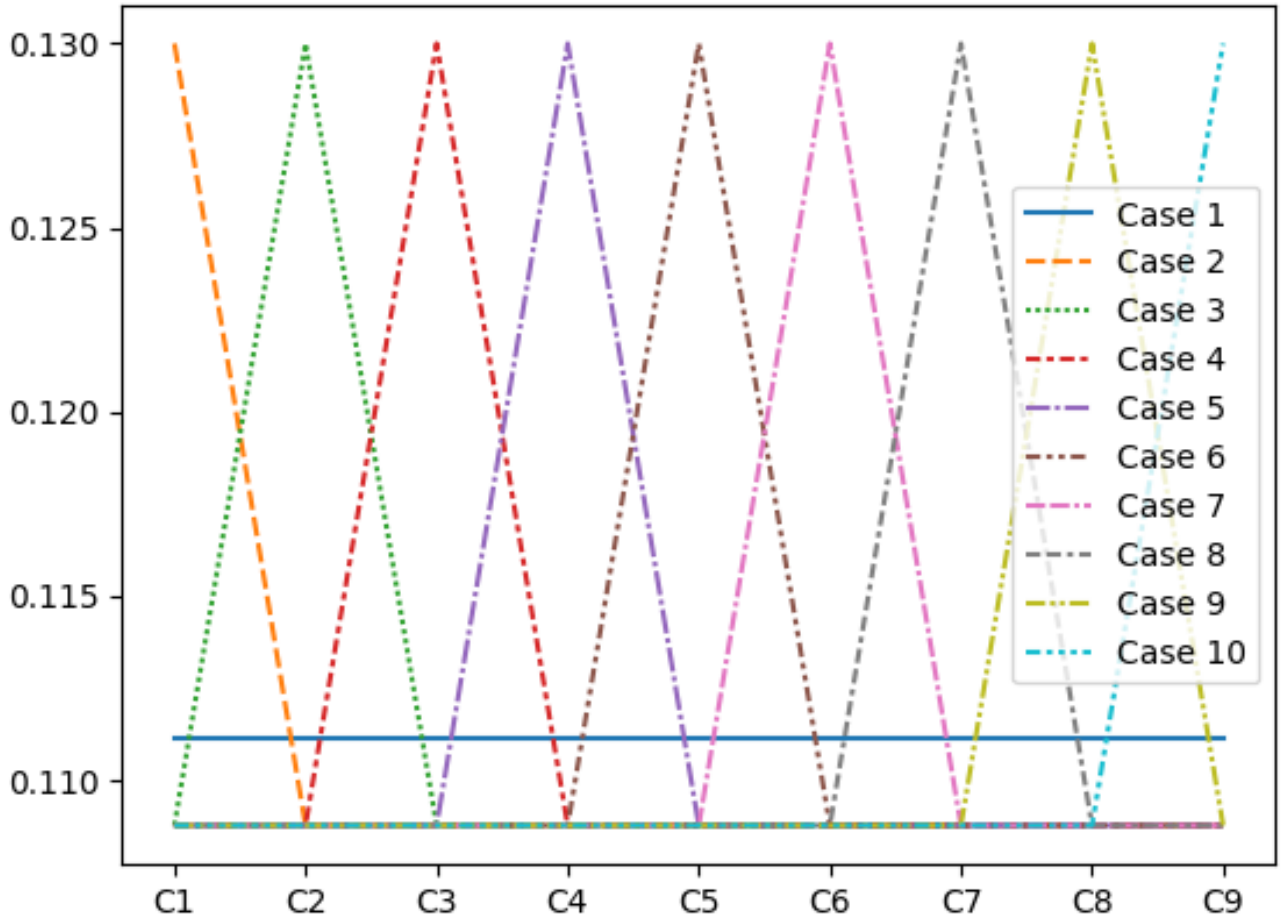

Fig 5. The various criteria weights.

In the criteria weights, we put all criteria with the same weight, then we rank the alternatives under this case, we show the alternative 1 is the best and alternative 3 is the worst. In the second case we increase the first criterion weight with 13%, then we put all other criteria with the same weight. In this case, we show alternative 1 is the best and alternative 3 is the worst. In the third case we increase the second criterion weight with 13%, then we put all other criteria with the same weight. In this case, we show alternative 1 is the best and alternative 3 is the worst. In the fourth case we increase the third criterion weight with 13%, then we put all other criteria with the same weight. In this case, we show alternative 1 is the best and alternative 3 is the worst. Fig 6 shows the net ranking of the alternatives. Fig 7 shows rank of the alternatives.

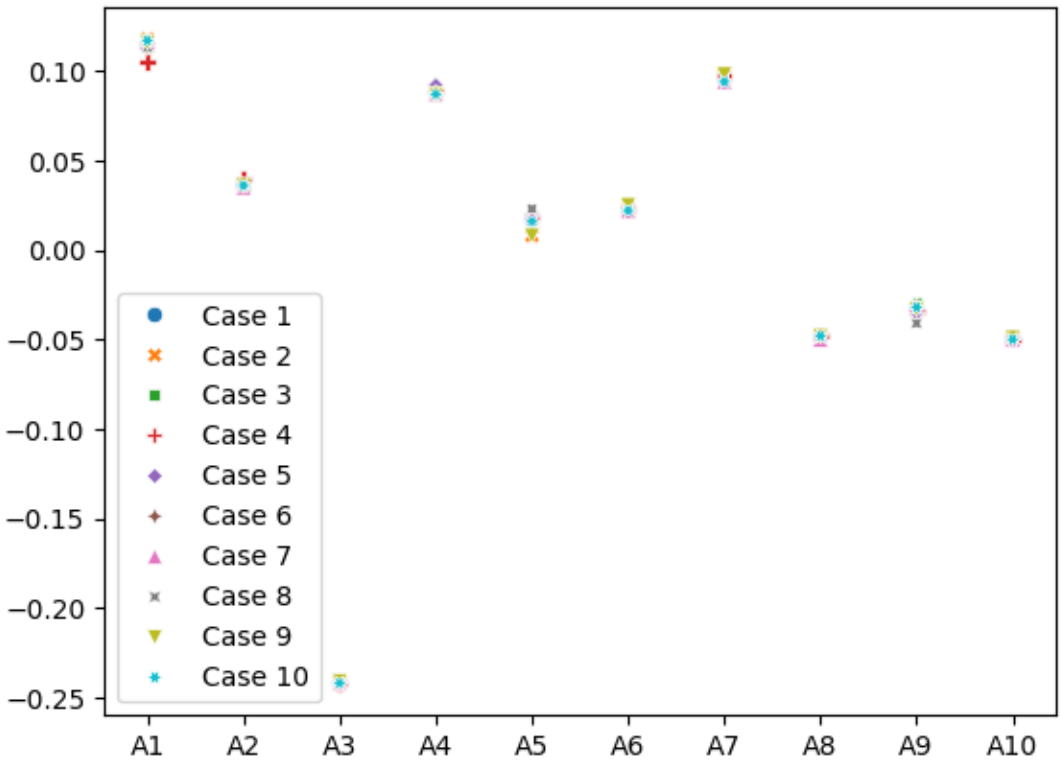

Fig 6. The net ranking of the alternatives.

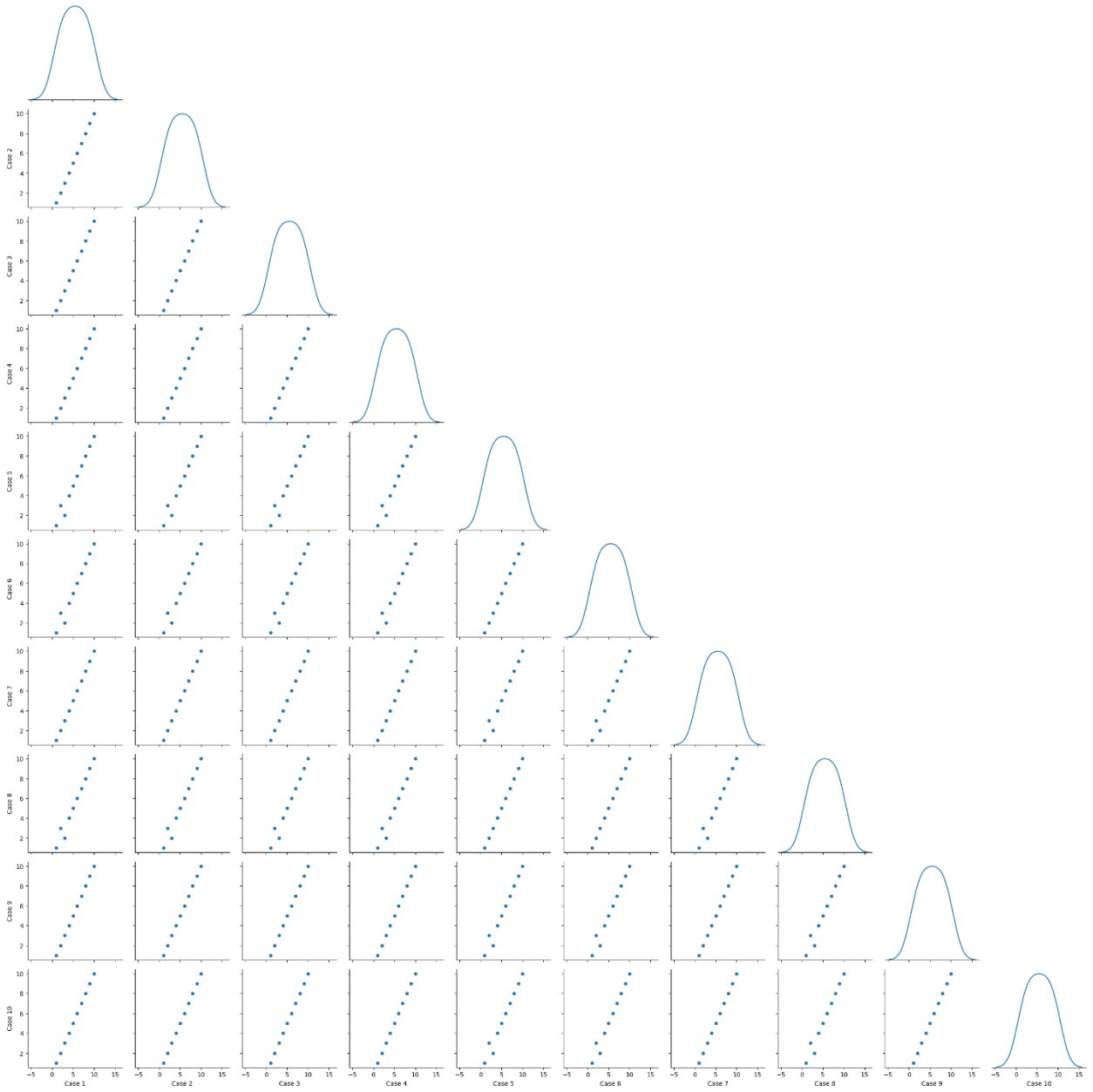

Fig 7. The various ranks.

5. Conclusions

In this study, a decision-making model is developed for evaluation of Innovation and Entrepreneurship Talent Cultivation. Two MCDM methods are used such as RANCOM method to compute the criteria weights and the PROMETHEE method to rank the alternatives. These methods are used under the Triangular Neutrosophic Cubic Linguistic Hesitant Fuzzy Set to deal with uncertainty and vague information. 9 criteria and 10 alternatives are evaluated by three experts and decision makers. The results show alternative 1 is the best and alternative 3 is the

worst. The sensitivity analysis was conducted with ten cases to show the ranks of the alternatives. The results show the ranks of the alternative are stable in different cases.

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Application of the Neutrosophic Fuzzy Sets in Profession Determination

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Abstract: We present several operations, such as algebraic sum, algebraic product, and arithmetic mean, over the neutrosophic sets extended with moment in time and sketch numbers. Through mathematical illustrations, we explore various properties and apply this concept to professional decision-making. Existing neutrosophic set (NS) and fuzzy set theories lack the capability to dynamically account for time and evolving circumstances. Many approaches fail to provide satisfactory solutions for handling evolving decision criteria or dynamic uncertainties. While operations like algebraic sum, algebraic product, and arithmetic mean exist for neutrosophic sets, their adaptation and extension to dynamic and temporal dimensions (moment in time and sketch numbers) remain underexplored. Additionally, there is a lack of studies applying neutrosophic set extensions in critical domains such as professional decision-making, which involves dynamic and complex evaluation processes. Many existing models do not adequately demonstrate practical applicability with robust mathematical illustrations. The introduction of Interval-Valued Temporal Neutrosophic Fuzzy Sets (IVTNFS), incorporating advanced operations and additional dimensions, provides a stronger framework for addressing evolving uncertainties in decision-making.

Keywords: Intuitionistic-FS (IFS), Temporal-IFS (TIFS), Neutrosophic Set (NS), Interval Valued-IFS (IVIFS) and IV-Temporal NFS (IVTNFS).

1. Introduction

L.A. Zadeh [40] introduced fuzzy set (FS) structure to scientifically express uncertainty in the year 1965. This abstract concept aimed to address the challenge of assigning a degree of membership to every component of a specified set. This, inside turn, laid the foundation for FS theory. Zadeh established the notion of a vague set, which generalizes the membership function of a set, allowing the degree of membership of an component to be more flexible than just "true" or "false." A vague set is mathematically constructed by assigning each element in the universe a value that represents its degree of membership in the fuzzy set. This value corresponds to the extent to which the element aligns with the conception

represent through the vague set. Membership values are often expressed as real numbers within the closed interval [0, 1].

The word fuzzy sense was introduced by Zadeh [40] in 1962. His work progressed further in 1965 with the formalization of this mathematical concept, now known as Fuzzy Set Theory. Later, Krassimir T. Atanassov [2] introduced IFS, which extend fuzzy sets by incorporating together a membership degree with a non-membership degree. Since their introduction, intuitionistic fuzzy sets have garnered significant attention from researchers, leading to their application in various domains such as model discovery.

A wide-ranging IFSs - GIFS was proposed by Samanta & Mondal [16] beneath the condition to facilitate the sum of the membership and non-membership degrees does not exceed one. Additionally, Gargov and K.T. Atanassov [5] developed interval-valued IFSs, second-type IFSs, and temporal IFSs. Further extensions include rough IFSs proposed by Rizvi et al. [25] and IF soft sets introduced by Majiet et al. [14]. In 2015, R. Srinivasan and K. Rajesh [18] developed an extension of IVIFS, studying its fundamental operations and operators. They further introduced various distance measures over IVIFSST in 2017 and the second type of temporal IFS [19], applying the concept to real-life situations in 2019. Later, K. Rajesh [20] introduced Certain Level Operators over TIFS in 2022.

There are two types of time: instantaneous time and interval time. An instant represents a single moment, whereas an interval denotes the duration among two points in occasion. A chronological representation can be based on one or together of these types. If the granularity of the moment aspect is adequately refined, or else if start and end points of a closed interval are equal, then a time instant can be considered a time interval. While most existing research on time modeling focuses on defining distinct and precise temporal information, in reality, temporal information is often uncertain and ambiguous.

Smarandache [29] established the notion of Neutrosophic Sets (NS) and its properties in 1999. Later, in 2005, he proposed Neutrosophic Sets with Intervals [28], where each element's uncertain membership status is represented using three components: truth, indeterminacy, and falsity. In 2023, Smarandache [30] introduce the idea of Excellent Anxious Soft Sets and Fuzzy Expansion of Excellent Anxious Soft Sets, along with developments in Revolutionary Topologies, New Types of Topologies, and Neutrosophic Topologies [36]. In 2024, he further introduced Appurtenance & Insertion Equations to construct neutrosophic operations essential in neutrosophic statistics [27].

In 2021, Deli, Uluçay, and Polat [9] introduced N-Valued Neutrosophic Trapezoidal Numbers with similarity measures and applications in MCDM problems. Riad K with Al-Hamido [24] explored Neutrosophic Algebraic Structure II in 2024, following Smarandache's [27] introduction of Neutrosophic

Two-Fold Algebra that same year. In 2024, K. Rajesh [21, 22] and M.C.J. Anand et al. introduce the idea of IVNFSs, incorporating both temporal and interval aspects along with various operations. In 2025, Khalid and Saeed [12] developed an approach for hybridizing N-Subalgebra with quantified neutrosophic sets using G-Algebra. Ram and Singh [23] introduced Neutrosophic Automata and Its Algebraic Properties in 2025. These concepts have since been applied in diverse fields, including preference structures, relational databases, consistency theory, expert systems, and more.

Traditional neutrosophic set extensions that rely solely on membership and non-membership degrees often fail to adequately capture dynamic uncertainty. However, the IVTNFS plays a more significant role by incorporating time-based extensions into neutrosophic sets, yielding more accurate results. This advancement is a key motivation for further developing neutrosophic fuzzy sets.

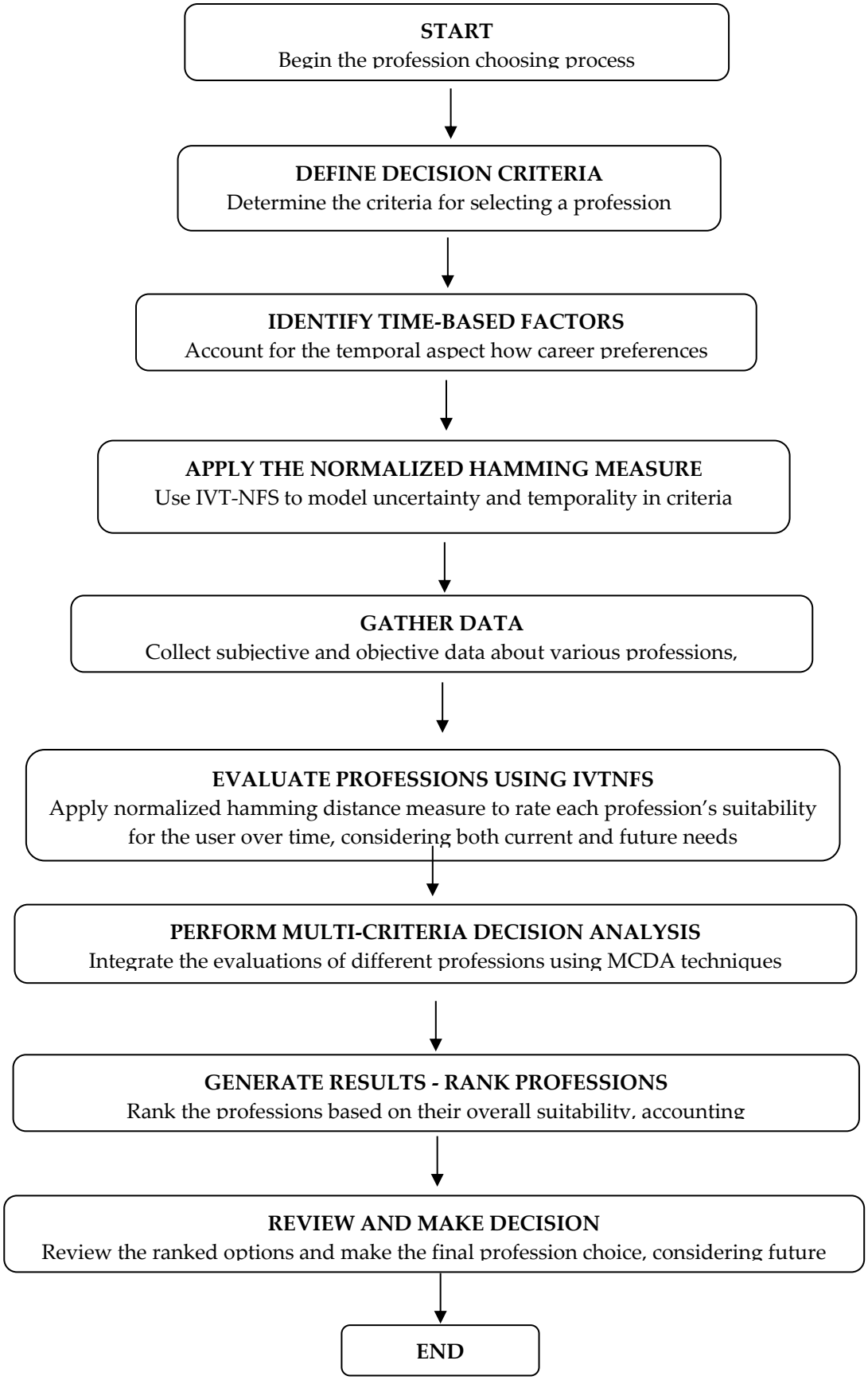
Motivation of this work:

Many real-world problems (e.g., career determination, image processing, and pattern recognition) involve uncertainty, vagueness, and incomplete information. Existing models struggle to handle these complexities effectively. Traditional fuzzy sets and neutrosophic sets are valuable but insufficient for scenarios that demand a dynamic representation of uncertainty over time or additional factors. Decision-making often requires incorporating the element of time, where preferences and conditions evolve. The proposed moment-based extension of neutrosophic sets adds temporal relevance and adaptability, making it a more realistic and practical tool. The extended framework has significant potential in solving real-life problems such as professional career selection, image processing, and decision-making, which are critical and impactful fields.

Novelties of the work:

This work introduces a novel thought of intervals with the temporal neutrosophic fuzzy sets IVTNFS, which incorporates a temporal dimension to better represent dynamic changes in real-world scenarios. The development and application of algebraic sum, algebraic product, and arithmetic mean operations over IVTNFS provide innovative tools for handling complex decision-making scenarios. Mathematical demonstrations and examples are provided to validate the efficiency of the proposed approach in real-life contexts, showcasing its practicality. The proposed framework is applied to career determination, demonstrating its potential to handle uncertainty and dynamic factors in significant life decisions. This can be further extended to areas like image processing and pattern recognition. By leveraging the temporal aspect and advanced mathematical operations, this work offers a refined methodology for achieving precise and reliable decisions in complex and evolving environments.

This research paper is going to organize while tag along: in the first segment we present a few essential definitions which determination be use in the 2nd section. In the third segment, we introduced a few operations like, addition, multiplication, arithmetic mean operation and geometric mean operation over IVTNFS and also studied several properties of IVTNFS finally we conclude this research.


START
Begin the profession choosing process
DEFINE DECISION CRITERIA
Determine the criteria for selecting a profession
IDENTIFY TIME-BASED FACTORS
Account for the temporal aspect how career preferences
APPLY THE NORMALIZED HAMMING MEASURE
Use IVT-NFS to model uncertainty and temporality in criteria
GATHER DATA
Collect subjective and objective data about various professions,
EVALUATE PROFESSIONS USING IVTNFS
Apply normalized hamming distance measure to rate each profession's suitability for the user over time, considering both current and future needs
PERFORM MULTI-CRITERIA DECISION ANALYSIS
Integrate the evaluations of different professions using MCDA techniques
GENERATE RESULTS - RANK PROFESSIONS
Rank the professions based on their overall suitability, accounting
REVIEW AND MAKE DECISION
Review the ranked options and make the final profession choice, considering future
END

2. Preliminary

Definition: 2.1 [2] Allow us consider K exist located in a filled sets. An Intuitionistic Fuzzy Sets (IFS) Y in K which is precise by K.T. Atanassov as the subsequent formula.

$$Y = \{< e, \sigma_Y(e), \varphi_Y(e) > \mid e \in K\}$$

Wherever the affiliation function denoted as σ_Y map from K to [0, 1] and the non-affiliation function denoted as φ_Y map from K to [0, 1] respectively with $0 \leq \sigma_Y(e) + \varphi_Y(e) \leq 1$ and for each components $e \in$ K.

Definition: 2.2 [5] Allow us consider K exist located in a filled sets. An intuitionistic fuzzy sets with time moments (TIFS) Y in K which is defined by K.T. Atanassov as the subsequent formula.

$$Y = \{< (e, s), \sigma_Y(e, s), \varphi_Y(e, s) > \mid (e, s) \in K \times L\}$$

Wherever the affiliation function denoted as σ_Y map from K to [0, 1] and the non-affiliation function denoted as φ_Y map from K to [0, 1] respectively with the condition $0 \leq \sigma_Y(e, s) + \varphi_Y(e, s) \leq 1$ and for each components $e \in$ K.

Definition: 2.3 [4] Let us consider K be located in a non-unfilled set. A fuzzy set with non membership values and intervals (name as IVIFS) Y in K which is defined by K.T. Atanassov and Gargov as the subsequent formula.

$$Y = \{< (e, s), \alpha_Y(e, s), \beta_Y(e, s) > \mid (e, s) \in K \times L\}$$

Everywhere $\alpha_Y: K \to [0, 1], \beta_Y: K \to [0, 1]$. The interludes $\alpha_Y(e, s)$ as well as $\beta_Y(e, s)$ symbolise the gradation of affiliation and gradation of non-affiliation of the component K to a set R, everywhere $\alpha_Y(e, s) = [\alpha_{YL}(e, s), \alpha_{YU}(e, s)]$ and $\beta_Y(e, s) = [\beta_{YL}(e, s), \beta_{YU}(e, s)]$ using the circumstance that $\alpha_{YU}(e, s) + \beta_{YU}(e, s) \leq 1$ for all $e \in K$.

Definition: 2.4 [33] Let us consider K be located in a non-unfilled set. A Neutrosophic Sets (NS) Y voguish K is pigeon-holed by an accuracy, indeterminacy & inaccuracy affiliation utility which exist, correspondingly represented as T_Y, I_Y as well F_Y in addition it is designated such as the succeeding formula

$$Y = \{< e, T_Y(e), I_Y(e), F_Y(e) > \mid e \in K\}$$

The values of $T_Y(e), I_Y(e)$ & $F_Y(e)$ be presentusual or unusual subcategories of $]0^-, 1^+[$, .i.e., $T_Y(e): K \to]0^-, 1^+[, I_Y(e): K \to]0^-, 1^+[$ as well as $F_Y(e): K \to]0^-, 1^+[$ certainly not constraint is realistic on the totality of $T_Y(e), I_Y(e)$ & $F_Y(e)$, with

$$0^- \leq T_Y(e) + I_Y(e) + F_Y(e) \leq 3^+$$

Where $T_Y(e), I_Y(e)$ also $F_Y(e)$ is termed as neutrosophic integers.

Definition: 2.5 [34] Let us consider K be located in a non-unfilled set. A Neutrosophic Fuzzy Sets (NFS) Y voguish K is pigeon-holed by an accuracy, indeterminacy & inaccuracy affiliation utility which exist, correspondingly represented as M_Y, T_Y, I_Y as well F_Y in addition it is designated such as the succeeding formula

$$Y = \{< e, M_Y(e), T_Y(e), I_Y(e), F_Y(e) > | e \in K\}$$

The values of $M_Y(e), T_Y(e), I_Y(e)$ & $F_Y(e)$ be present usual or unusual sub categories of $]0^-, 1^+[$ i.e., $M_Y(e) : K \to]0^-, 1^+[, T_Y(e) : K \to]0^-, 1^+[, I_Y(e) : K \to]0^-, 1^+[$ as well as $F_Y(e) : K \to]0^-, 1^+[$ certainly not constraint is realistic on the totality of $M_Y(e), T_Y(e), I_Y(e)$ &

$F_Y(e)$, therefore

$$0^- \leq M_Y(e) + T_Y(e) + I_Y(e) + F_Y(e) \leq 3^+$$

Where $M_Y(e), T_Y(e), I_Y(e)$ also $F_Y(e)$ is termed as neutrosophic integers.

Definition: 2.6 [42] Let us consider K be located in a non-unfilled set. A Neutrosophic Sets in intervals (IVNS) Y voguish K is pigeon-holed by an accuracy, indeterminacy &inaccuracy affiliation utility which exist, correspondingly represented as T_Y, I_Y as well F_Y in addition it is designated such as the succeeding formula

$$\mathrm{Y} = \{< e, [\mathrm{T_{YL}(e)}, \mathrm{T_{YU}(e)}], [\mathrm{I_{YL}(e)}, \mathrm{I_{YU}(e)}], [\mathrm{F_{YL}(e)}, \mathrm{F_{YU}(e)}] > | \mathrm{e} \in \mathrm{K}\}$$

Everywhere the utilities $T_Y(e), I_Y(e)$ & $F_Y(e) : K \to]0^-, 1^+[$ with the condition

$0^- \leq \mathrm{T_{YU}(e)} + \mathrm{I_{YU}(e)} + \mathrm{F_{YU}(e)} \leq 3^+$.

Definition: 2.7 [24] Let us consider K be located in a non-unfilled set. A Temporal Neutrosophic Sets in intervals (IVTNFS) Y voguish K is pigeon-holed by an accuracy, indeterminacy & inaccuracy affiliation utility which exist, correspondingly represented as $\mathrm{T_Y}, \mathrm{I_Y}$ as well $\mathrm{F_Y}$ in addition it is designated such as the succeeding formula

$$\mathrm{Y} = \{< (\mathrm{e}, \mathrm{s}), [\mathrm{M_{YL}(e,s)}, \mathrm{M_{YU}(e,s)}], [\mathrm{T_{YL}(e,s)}, \mathrm{T_{YU}(e,s)}], [\mathrm{I_{YL}(e,s)}, \mathrm{I_{YU}(e,s)}], [\mathrm{F_{YL}(e,s)}, \mathrm{F_{YU}(e,s)}] > | e \in K \,\&\, s \in L\}$$

Everywhere the utilities $T_Y(e), I_Y(e)$ & $F_Y(e) : K \to]0^-, 1^+[$ with the time moments and satisfy the following condition

$$0^- \leq \mathrm{M_{YU}(e,s)} + \mathrm{T_{YU}(e,s)} + \mathrm{I_{YU}(e,s)} + \mathrm{F_{YU}(e,s)} \leq 3^+.$$

Definition: 2.8 [24] Let us consider Y be any IVTNFSs on K, then the accompaniment of Y is designated as $\bar{Y}$ further it is well-defined partakes the succeeding formula

$$\bar{Y} = \{< (\mathrm{e}, \mathrm{s}), [\mathrm{M_{YL}(e,s)}, \mathrm{M_{YU}(e,s)}], [\mathrm{F_{YL}(e,s)}, \mathrm{F_{YU}(e,s)}], [1 - \mathrm{I_{YU}(e,s)}, 1 - \mathrm{I_{YU}(e,s)}], [\mathrm{T_{YL}(e,s)}, \mathrm{T_{YU}(e,s)}] > | e \in K \,\&\, s \in L\}$$

3. Various operations over the branch of Neutrosophic fuzzy sets with intervals

We introduce some new operations over IVTNFS through arithmetical example and in addition establish a number of his property.

Definition: 3.1 Allow M, N are the wing of Neutrosophic Fuzzy Sets then, we define algebraic sum in the following form

$$M+N=\{\langle(e,s),((R_{ML}(e,s)+R_{NL}(e,s)-\big(R_{ML}(e,s).R_{NL}(e,s)\big),$$
$$(R_{MU}(e,s)+R_{NU}(e,s)-\big(R_{MU}(e,s).R_{NU}(e,s)\big)),$$
$$\{((T_{ML}(e,s)+T_{NL}(e,s)-\big(T_{ML}(e,s).T_{NL}(e,s)\big),$$
$$((T_{MU}(e,s)+T_{NU}(e,s)-\big(T_{MU}(e,s).T_{NU}(e,s)\big)),$$
$$((I_{ML}(e,s)+I_{NL}(e,s)-\big(I_{ML}(e,s).I_{NL}(e,s)\big),$$
$$(I_{MU}(e,s)+I_{NU}(e,s)-\big(I_{MU}(e,s),I_{NU}(e,s)\big)),$$
$$((F_{ML}(e,s)+F_{NL}(e,s)-\big(F_{ML}(e,s).F_{NL}(e,s)\big),$$
$$(F_{MU}(e,s)+F_{NU}(e,s)-\big(F_{MU}(e,s).F_{NU}(e,s)\big))\}>|(e,s)\in K\times L\}$$

Definition: 3.2 Let M, N are in the expansion of Neutrosophic Fuzzy Sets then, the algebraic product is describe as the succeeding formula

$$M\cdot N=\{\langle(e,s),(\,R_{ML}(e,s).R_{NL}(e,s),R_{MU}(e,s).R_{NU}(e,s)),$$
$$\{(T_{ML}(e,s).T_{NL}(e,s),T_{MU}(e,s).T_{NU}(e,s)),(I_{ML}(e,s).I_{NL}(e,s),$$
$$I_{MU}(e,s).I_{NU}(e,s)),(F_{ML}(e,s)+F_{NL}(e,s)-F_{ML}(e,s).F_{NL}(e,s),$$
$$F_{MU}(e,s)+F_{NU}(e,s)-F_{MU}(e,s).F_{NU}(e,s)\}\rangle|(e,s)\in K\times L\}$$

Definition: 3.3 Let M, N are in the extension of Neutrosophic Fuzzy Sets then, the arithmetic mean operation is describe as the succeeding formula

$$M\,@\,N=\left\{\langle(e,s),\left[\frac{R_{ML}(e,s)+R_{NL}(e,s)}{2},\frac{R_{MU}(e,s)+R_{NU}(e,s)}{2}\right]\right.$$
$$\left[\frac{T_{ML}(e,s)+T_{NL}(e,s)}{2},\frac{T_{MU}(e,s)+T_{NU}(e,s)}{2}\right],$$
$$\left[\frac{I_{ML}(e,s)+I_{NL}(e,s)}{2},\frac{I_{MU}(e,s)+I_{NU}(e,s)}{2}\right],$$
$$\left.\left[\frac{F_{ML}(e,s)+F_{NL}(e,s)}{2},\frac{F_{MU}(e,s)+F_{NU}(e,s)}{2}\right]\rangle\,|(e,s)\in K\times L\right\}$$

Definition: 3.4 Let us consider M, N are in the extension of Neutrosophic Fuzzy Sets then, the geometric mean operation is define as the following form

$$M\,\$\,N=\left\{\langle(e,s),\left[\sqrt{R_{ML}(e,s)\cdot R_{NL}(e,s)},\sqrt{R_{MU}(e,s)\cdot R_{NU}(e,s)}\right],\right.$$
$$\left\{\left[\sqrt{T_{ML}(e,s)\cdot T_{NL}(e,s)},\sqrt{T_{MU}(e,s)\cdot T_{NU}(e,s)}\right],\right.$$
$$\left[\sqrt{I_{ML}(e,s)\cdot I_{NL}(e,s)},\sqrt{I_{MU}(e,s)\cdot I_{NU}(e,s)}\right],$$

$$\left[\sqrt{F_{ML}(e,s)\cdot F_{NL}(e,s)},\sqrt{F_{MU}(e,s)\cdot F_{NU}(e,s)}\right]\}\rangle|(e,s)\in K\times L\}$$

Example: 3.5 Let us consider an IVTNFSs M and N over K given by

$$M=\{<a,[0.2,0.4],[0.6,0.3],[0.3,0.5],[0.2.0.3]>,<b,[0.6,0.8],[0.5,0.6],[0.1,0.4],[0.4,0.7]>\}$$

And

$$N=\{<a,[0.3,0.5],[0.4,0.2],[0.4,0.5],[0.2.0.6]>,<b,[0.3,0.4],[0.3,0.6],[0.1,0.8],[0.5,0.4]>\}$$

Then,

$M+N=\{\langle a,[0.44,0.7],[0.76,0.44],[0.58,0.75],[0.36,0.72]\rangle,$

$\langle b,[0.72,0.88],[0.65,0.84],[0.19,0.88],[0.7,0.82]\rangle\}$

$M\cdot N=\{\langle a,[0.06,0.2],[0.24,0.06],[0.12,0.25],[0.36,0.72]\rangle,$

$\langle b,[0.18,0.32],[0.15,0.36],[0.01,0.32],[0.7,0.82]\rangle\}$

$M@N=\{\langle a,[0.25,0.45],[0.5,0.25],[0.35,0.5],[0.2,0.45]\rangle,\langle b,[0.45,0.6],[0.4,0.6],[0.1,0.6],[0.45,0.55]\rangle\}$

$M\$N$

$=\{\langle a,[0.24,0.44],[0.49,0.24],[0.34,0.5],[0.2,0.42]\rangle,\langle b,[0.42,0.57],[0.39,0.6],[0.1,0.57],[0.45,0.53]\rangle\}$

Therefore M and N satisfied the interval valued temporal neutrosophic fuzzy sets condition

Algorithm: 3.6

STEP 1: $Y=\{<e,\alpha_Y(e)>|e\in K\}$ The affiliation function lies between (0 - 1), then we move next step. (Then the function R is fuzzy sets**.**)

STEP 2: $Y=\left\{<e,\alpha_Y(e),\beta_Y(e)>|e\in K\right\}$ The affiliation & non-affiliation function lies between (0 - 1) & $0\le\alpha_Y(e)+\beta_Y(e)\le 1$ Then, we move next step. (Then the function E is Intuitionistic fuzzy sets.)

STEP 3: $Y=\left\{<e,\alpha_Y(e),\beta_Y(e)>|e\in K\right\}$ the addition of upper limit of affiliation and non-affiliation values is 1 then, we move the next step. (Then the function E is said to be IVIFS.)

STEP 4: $Y=\left\{<(e,s),\alpha_Y(e,s),\beta_Y(e,s)>|(e,s)\in K\times L\right\}$ & the addition of affiliation value, Truth, Indeterminacy and falsity value is lies between zero to one, then the function E is said to be IVTIFS. We move the next step.

STEP 5: $Y=\{<(e,s),[M_{YL}(e,s),M_{YU}(e,s)],[T_{YL}(e,s),T_{YU}(e,s)],[I_{YL}(e,s),I_{YU}(e,s)],$

$[F_{YL}(e,s),F_{YU}(e,s)]>|e\in K\ \&\ s\in L\}$ with the sum of all upper bound values of affiliation, Truth, Indeterminacy and falsity is lies between zero to one, then the function R is called IVTNFS.

Theorem 3.7 Allow M and N be a IVTNFSs on K. Then and there, we partake the resulting

(i) $\boldsymbol{M@N=N@M}$,

(ii) $\boldsymbol{M\$N=N\$M}$.

Proof:

For M, N ∈ IVTNFS, then the arithmetic and geometric mean procedure are defined as

$$M\ @\ N = \Bigg\{\langle (e,s), \left[\frac{R_{ML}(e,s)+R_{NL}(e,s)}{2}, \frac{R_{MU}(e,s)+R_{NU}(e,s)}{2}\right] \left[\frac{T_{ML}(e,s)+T_{NL}(e,s)}{2}, \frac{T_{MU}(e,s)+T_{NU}(e,s)}{2}\right], \left[\frac{I_{ML}(e,s)+I_{NL}(e,s)}{2}, \frac{I_{MU}(e,s)+I_{NU}(e,s)}{2}\right], \left[\frac{F_{ML}(e,s)+F_{NL}(e,s)}{2}, \frac{F_{MU}(e,s)+F_{NU}(e,s)}{2}\right]\rangle \mid (e,s) \in K \times L\}$$

$$M\ \$\ N = \Bigg\{\langle (e,s), \left[\sqrt{R_{ML}(e,s)\cdot R_{NL}(e,s)}, \sqrt{R_{MU}(e,s)\cdot R_{NU}(e,s)}\right], \left\{\left[\sqrt{T_{ML}(e,s)\cdot T_{NL}(e,s)}, \sqrt{T_{MU}(e,s)\cdot T_{NU}(e,s)}\right], \left[\sqrt{I_{ML}(e,s)\cdot I_{NL}(e,s)}, \sqrt{I_{MU}(e,s)\cdot I_{NU}(e,s)}\right], \left[\sqrt{F_{ML}(e,s)\cdot F_{NL}(e,s)}, \sqrt{F_{MU}(e,s)\cdot F_{NU}(e,s)}\right]\right\}\rangle \mid (e,s) \in K \times L\}$$

(i) $M\ @\ N\ = M\ @\ N$

$$M\ @\ N = \Bigg\{\langle (e,s), \left[\frac{R_{ML}(e,s)+R_{NL}(e,s)}{2}, \frac{R_{MU}(e,s)+R_{NU}(e,s)}{2}\right] \left[\frac{T_{ML}(e,s)+T_{NL}(e,s)}{2}, \frac{T_{MU}(e,s)+T_{NU}(e,s)}{2}\right], \left[\frac{I_{ML}(e,s)+I_{NL}(e,s)}{2}, \frac{I_{MU}(e,s)+I_{NU}(e,s)}{2}\right], \left[\frac{F_{ML}(e,s)+F_{NL}(e,s)}{2}, \frac{F_{MU}(e,s)+F_{NU}(e,s)}{2}\right]\rangle \mid (e,s) \in K \times L\}$$

$$= \Bigg\{\langle (e,s), \left[\frac{R_{NL}(e,s)+R_{ML}(e,s)}{2}, \frac{R_{NU}(e,s)+R_{MU}(e,s)}{2}\right] \left[\frac{T_{NL}(e,s)+T_{ML}(e,s)}{2}, \frac{T_{NU}(e,s)+T_{MU}(e,s)}{2}\right], \left[\frac{I_{NL}(e,s)+I_{ML}(e,s)}{2}, \frac{I_{NU}(e,s)+I_{MU}(e,s)}{2}\right], \left[\frac{F_{NL}(e,s)+F_{ML}(e,s)}{2}, \frac{F_{NU}(e,s)+F_{MU}(e,s)}{2}\right]\rangle \mid (e,s) \in K \times L\}$$

$$= N\ @\ M$$

which is prooved the part (i) of the theorem.

(ii) $M\ \$\ N\ = M\ \$N$

$$M\ \$\ N = \Bigg\{\langle (e,s), \left[\sqrt{R_{ML}(e,s)\cdot R_{NL}(e,s)}, \sqrt{R_{MU}(e,s)\cdot R_{NU}(e,s)}\right], \left\{\left[\sqrt{T_{ML}(e,s)\cdot T_{NL}(e,s)}, \sqrt{T_{MU}(e,s)\cdot T_{NU}(e,s)}\right],$$

$$\left[\sqrt{I_{ML}(e,s)\cdot I_{NL}(e,s)},\sqrt{I_{MU}(e,s)\cdot I_{NU}(e,s)}\right],$$
$$\left[\sqrt{F_{ML}(e,s)\cdot F_{NL}(e,s)},\sqrt{F_{MU}(e,s)\cdot F_{NU}(e,s)}\right]\Big\}\rangle|(e,s)\in K\times L\}$$

$$= \Big\{\langle(e,s),\left[\sqrt{R_{NL}(e,s)\cdot R_{ML}(e,s)},\sqrt{R_{NU}(e,s)\cdot R_{MU}(e,s)}\right],$$
$$\Big\{\left[\sqrt{T_{NL}(e,s)\cdot T_{ML}(e,s)},\sqrt{T_{NU}(e,s)\cdot T_{MU}(e,s)}\right],$$
$$\left[\sqrt{I_{NL}(e,s)\cdot I_{ML}(e,s)},\sqrt{I_{NU}(e,s)\cdot I_{MU}(e,s)}\right],$$
$$\left[\sqrt{F_{NL}(e,s)\cdot F_{ML}(e,s)},\sqrt{F_{NU}(e,s)\cdot F_{MU}(e,s)}\right]\Big\}\rangle|(e,s)\in K\times L\}$$

$= N \$ M$

Which is complete the evidence of $M @ N = N @ M$ and $M \$ N = N \$ M$.

Theorem: 3.8 Let us choose M also N be a IVTNFSs on K. Then, we partake the resulting

(i) $(\mathbf{M^c@N^c})^c = \mathbf{M @ N}$,

(ii) $(\mathbf{M^c\$N^c})^c \neq \mathbf{M \$ N}$.

Proof: Consider

$$M = \{< (e,s),[M_{ML}(e,s),M_{MU}(e,s)],[T_{ML}(e,s),T_{MU}(e,s)],$$
$$[I_{ML}(e,s),I_{MU}(e,s)],[F_{ML}(e,s),F_{MU}(e,s)] > |e\in K \,\&\, s\in L\}$$

And

$$N = \{< (e,s),[M_{NL}(e,s),M_{NU}(e,s)],[T_{NL}(e,s),T_{NU}(e,s)],$$
$$[I_{NL}(e,s),I_{NU}(e,s)],[F_{NL}(e,s),F_{NU}(e,s)] > |e\in K \,\&\, s\in L\}$$

Then, the complement of M, N and arithmetic operation as follows

$$M^c = \{< (e,s),[M_{ML}(e,s),M_{MU}(e,s)],[F_{ML}(e,s),F_{MU}(e,s)],$$
$$[1-I_{MU}(e,s),1-I_{ML}(e,s)],[T_{ML}(e,s),T_{MU}(e,s)] > |e\in K \,\&\, s\in L\}$$
$$N^c = \{< (e,s),[M_{NL}(e,s),M_{NU}(e,s)],[F_{NL}(e,s),F_{NU}(e,s)],$$
$$[1-I_{NL}(e,s),1-I_{NU}(e,s)],[T_{NL}(e,s),T_{NU}(e,s)] > |e\in K \,\&\, s\in L\}$$

Now,

$$M^c@N^c = \Big\{\langle(e,s),\left[\frac{M_{ML}(e,s)+M_{NL}(e,s)}{2},\frac{M_{MU}(e,s)+M_{NU}(e,s)}{2}\right]$$
$$\left[\frac{F_{ML}(e,s)+F_{NL}(e,s)}{2},\frac{F_{MU}(e,s)+F_{NU}(e,s)}{2}\right],$$
$$\left[\frac{1-I_{MU}(e,s)+1-I_{NU}(e,s)}{2},\frac{1-I_{ML}(e,s)+1-I_{NL}(e,s)}{2}\right],$$
$$\left[\frac{T_{ML}(e,s)+T_{NL}(e,s)}{2},\frac{T_{MU}(e,s)+T_{NU}(e,s)}{2}\right]\rangle\,|(e,s)\in K\times L\}$$

$$M^c@N^c = \Big\{\langle(e,s),\left[\frac{M_{ML}(e,s)+M_{NL}(e,s)}{2},\frac{M_{MU}(e,s)+M_{NU}(e,s)}{2}\right]$$

$$\left[\frac{F_{ML}(e,s)+F_{NL}(e,s)}{2},\frac{F_{MU}(e,s)+F_{NU}(e,s)}{2}\right],$$
$$\left[\frac{I_{ML}(e,s)+I_{NL}(e,s)}{2},\frac{I_{MU}(e,s)+I_{NU}(e,s)}{2}\right],$$
$$\left[\frac{T_{ML}(e,s)+T_{NL}(e,s)}{2},\frac{T_{MU}(e,s)+T_{NU}(e,s)}{2}\right]\rangle \mid (e,s)\in K\times L\}$$

$$(M^c@N^c)^c=\left\{\langle(e,s),\left[\frac{R_{NL}(e,s)+R_{ML}(e,s)}{2},\frac{R_{NU}(e,s)+R_{MU}(e,s)}{2}\right]\right.$$
$$\left[\frac{F_{NL}(e,s)+F_{ML}(e,s)}{2},\frac{F_{NU}(e,s)+F_{MU}(e,s)}{2}\right],$$
$$\left[\frac{1-I_{MU}(e,s)+1-I_{NU}(e,s)}{2},\frac{1-I_{ML}(e,s)+1-I_{NL}(e,s)}{2}\right],$$
$$\left.\left[\frac{T_{NL}(e,s)+T_{ML}(e,s)}{2},\frac{T_{NU}(e,s)+T_{MU}(e,s)}{2}\right]\rangle \mid (e,s)\in K\times L\right\}$$

$$=\left\{\langle(e,s),\left[\frac{R_{NL}(e,s)+R_{ML}(e,s)}{2},\frac{R_{NU}(e,s)+R_{MU}(e,s)}{2}\right]\right.$$
$$\left[\frac{T_{NL}(e,s)+T_{ML}(e,s)}{2},\frac{T_{NU}(e,s)+T_{MU}(e,s)}{2}\right],$$
$$\left[\frac{I_{NL}(e,s)+I_{ML}(e,s)}{2},\frac{I_{NU}(e,s)+I_{MU}(e,s)}{2}\right],$$
$$\left.\left[\frac{F_{NL}(e,s)+F_{ML}(e,s)}{2},\frac{F_{NU}(e,s)+F_{MU}(e,s)}{2}\right]\rangle \mid (e,s)\in K\times L\right\}$$

$$=M\ @\ N$$

Therefore, $(M^c@N^c)^c=M\ @\ N$.

Easy to verify the second part of the proof $(M^c\$N^c)^c\neq M\ \$\ N$.

4. Application of the Neutrosophic Fuzzy Sets in Profession determination

Profession choice is one of the most important challenges students face in making decisions about their future. It plays a crucial role in shaping their professional paths within the field of education. When students select a course, they initiate their academic journey and work towards finding a suitable career. Upon entering university, students take on greater responsibility and often prioritize choosing the right course over selecting a school. Their career and course decisions are influenced by various factors, such as academic interests and career aspirations, leading to diverse experiences and knowledge. In such cases, uncertainty arises, making decision-making more complex. To address this uncertainty, neutrosophic

fuzzy set theory can be effectively utilized to analyze and identify problems in students' decision-making processes.

From the sample collection, we provide the shortest academic and professional usage of the $d_{NH}(M,N)$ measure of IVTNFS. The Student set is S = {First Student, Second Student, Third Student, Fourth Student}, the Occupation set is C = {Data Analysis, Data Science, Software developer & ML Engineer} and the subject set is Sub = {Physics, Electronics, Mathematics & Computer Science}, which is a group of work-related courses. Table 1 shows the requirements for majors and other courses taught according to IVTNFS values. Each performance is defined by four numbers, R_M, T_M, I_M and F_M, which are the membership existence, true, uncertain and false membership utility.

	Physics	**Electronics**	**Mathematics**	**Computer science**
Data Analysis	[0.3,0.3],[0.5,0.4], [0.6,0.3], [0.5,0.7]	[0.4,0.5],[0.3,0.4], [0.7,0.3], [0.2,0.3]	[0.5, 0.5], [0.3, 0.5], [0.6, 0.7], [0.4, 0.3]	[0.4, 0.6],[0.3,0.2], [0.5, 0.4], [0.5,0.6]
Data Science	[0.4,0.6],[0.5,0.4], [0.3, 0.5],[0.3,0.7]	[0.3,0.4],[0.4,0.6], [0.5,0.5], [0.9,0.5]	[0.4, 0.4], [0.4, 0.5], [0.6, 0.5], [0.5, 0.6]	[0.7, 0.9],[0.4,0.3], [0.5, 0.3], [0.4,0.4]
Software developer	[0.3,0.6],[0.2,0.4], [0.6,0.3], [0.5,0.8]	[0.5,0.4],[0.6,0.3], [0.4, 0.3],[0.7,0.5]	[0.5, 0.6], [0.4, 0.5], [0.3, 0.4], [0.3, 0.4]	[0.4,0.6], [0.7,0.5], [0.4,0.5], [0.4,0.2]
ML Engineer	[0.6,0.5],[0.6,0.3], [0.5,0.4], [0.7,0.9]	[0.6,0.3],[0.2,0.7], [0.5,0.3], [0.4,0.5]	[0.2, 0.3], [0.5, 0.6], [0.5, 0.4], [0.4, 0.7]	[0.3, 0.5],[0.4,0.7], [0.6,0.6], [0.3,0.2]

Table 1: Professions vs Subjects

The results obtained by the students after several tests are shown in Table 2

	Physics	**Electronics**	**Mathematics**	**Computer science**
First Student	[0.7,0.3],[0.5,0.4], [0.5,0.5], [0.3,0.5]	[0.6,0.7],[0.2,0.3], [0.6,0.5], [0.4,0.7]	[0.5, 0.4], [0.6, 0.3], [0.3, 0.5], [0.4, 0.8]	[0.5,0.4], [0.6,0.3], [0.7,0.3], [0.4,0.5]
Second Student	[0.5,0.4],[0.3,0.6], [0.7,0.4], [0.4,0.6]	[0.3,0.4],[0.5,0.3], [0.5,0.3], [0.5,0.3]	[0.5, 0.3], [0.3, 0.3], [0.4, 0.3], [0.5, 0.1]	[0.6,0.3], [0.2,0.7], [0.3,0.3], [0.4,0.5]
Third Student	[0.1,0.3],[0.6,0.3], [0.5,0.3], [0.4,0.6]	[0.5,0.4],[0.7,0.8], [0.7,0.2], [0.4,0.9]	[0.2, 0.7], [0.8, 0.6], [0.3, 0.5], [0.1, 0.3]	[0.2,0.6], [0.5,0.4], [0.6,0.5], [0.6,0.6]
Fourth Student	[0.6,0.5],[0.6,0.3], [0.5,0.4], [0.7,0.9]	[0.7,0.4],[0.5,0.6], [0.5,0.4], [0.3,0.6]	[0.6, 0.5], [0.6, 0.3], [0.4, 0.4], [0.6, 0.6]	[0.3,0.4], [0.4,0.9], [0.3,0.5], [0.6,0.5]

Table 2: Students vs Subjects

Result: Using Hamming normalized distance to calculate with the study, we get the following message with the distance for all students and all subjects.

IVTNFS	Data Analysis	Data Science	Software Developer	ML Engineer
First Student	0.1688	**0.1625**	0.1906	0.1750
Second Student	0.1594	0.1750	**0.1563**	0.2000
Third Student	**0.1781**	0.2344	0.1875	0.1969
Fourth Student	0.1719	0.1844	0.1688	**0.1594**

Table 3: shortest distance among Students vs Professions

In Table 3, we conclude that the shortest distance provides accurate location determination among students and professions by using the normalized Hamming distance measure.

Discussion: The first student chooses the Data Science Department, the second student chooses the Software Development Department, the third student chooses the Data Analytics Department, and the fourth student chooses the Machine Learning Engineer Department. We conclude that the use of normalized Hamming distance measure in IVTNFS act as a significant role in obtaining shortest distance. Suppose the distance between the student and the job is the shortest, the student chooses that job.

5. Conclusion

In this paper, we introduced few operations over the extended type of neutrosophic sets, namely IVTNFS, and studied few properties with numerical examples. Further we applied this IVTNFS concept in the determination of their Professions.

Future Work: We will study the concepts of similarity measure and also study the properties of various distance measures among the intervals temporal neutrosophic fuzzy sets with members. Further we can apply the above mentioned concepts in the real-life application of IVTNFS, such as pattern recognition, supplier selection, and image processing, and so on.

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An Analysis of Bipolar Trapezoidal Neutrosophic Sets (BTNSs) for Examining the Income-Increasing and Poverty-Reduction Effects of Ecological Compensation in the Huai River Basin of Anhui Province: A Comprehensive Study

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Abstract: This study proposes the decision-making approach for evaluation of ecological compensation in the Huai river basin of Anhui province. We used the neutrosophic sets to deal with the uncertainty information. In both the trapezoidal and bipolar neutrosophic environments, decision makers supply all the information needed to make decisions in complex multicriteria decision-making problems. This work suggests a novel, efficient method to handle these situations. We used two methods to evaluate the criteria and alternatives such as LMAW and MARCOS. We used the LMAW method to compute the criteria weights and the MARCOS method to rank the alternatives. Nine criteria and eleven alternatives are evaluated in this study. The sensitivity analysis is conducted to show the stability of the results. The outcomes show the rank of the alternative is stable.

Keywords: Bipolar Trapezoidal Neutrosophic Sets (BTNSs); Ecological Compensation; Huai River Basin of Anhui Province; Uncertainty. MARCOS Method; LMAW Method.

1. Introduction and Literature Review

Two-valued (Boolean) logic is the foundation for the membership value (truth value) of the classical set element. This usually doesn't offer a bit of possible information to catch the questionable data in the real world. Based on the extension of binary logic, a type of multivalue logic where the actual value might be any real integer between 0 and 1, inclusive, Zadeh [1] proposed the membership value of a classical set element in 1965. A fuzzy set (FS) is a collection of pairs that include every element in the classical set and its true value in relation to the membership function. To manage education and relationships in schools, Hoskova-Mayerova and Maturo [2] have attempted to make sure that fuzzy sets can be utilized. The writers of Nguyen et al. [3], Peng, and Garg [4] talked about several kinds of similarity measures, including multiparametric and exponential fuzzy units, and noted that similarity measures can be used for

issues like pattern recognition and decision making. Chakravarty [5] has developed an axiomatic approach to measuring multidimensional poverty using fuzzy sets. In fuzzy soft set theory, Zimmermann proposed triangular and trapezoidal fuzzy numbers.

According to him, trapezoidal membership functions are frequently employed for the purposes of ease of data collecting and computational efficiency. Despite its validity in expressing ambiguous evaluation data, the fuzzy set is unable to handle several complicated real-world problems. Given the current situation, it is impossible to avoid the bipolar sense of the attribute's appraisal. Lee [6] created a bipolar-valued fuzzy set (BFS) by classifying a set element with both positive and negative fuzzy information. Dutta and Doley [7] have demonstrated that the composition of bipolar-valued fuzzy sets can be used to make a scientific diagnosis in an efficient manner.

To handle a commercial enterprise person’s concern that fall under the category of multi characteristic group decision making in the context of cubic bipolar fuzzy facts, Riaz and Tehrim [8] provided the averaging aggregation operators. The idea of intuitionistic fuzzy sets with only truth and falsity membership values was first presented by Atanassov[9]. Smarandache [10] discovered in 1998 that the intuitionistic fuzzy sets' membership values may be separated into three categories: truth, indeterminacy, and falsehood. It results in Smarandache's neutrosophic set (NS) shape. It is another significant turning point after the sense of a murky environment with problems associated with ambiguity. By functionalizing the Smarandache's neutrosophic set for certain situations,

Wang et al. [11] developed the concept of a single-valued neutrosophic set. It is commonly recognized that the single-valued neutrosophic set is the most practical way to implement the concept of this paradigm to get rid of the challenges brought on by the values in the neutrosophic set's construction. Ye [12] investigated the trapezoidal sense of the neutrosophic set element's membership value. He was inspired by this to lay the groundwork for a trapezoidal neutrosophic set (TNS). To help an investment business invest their money in the most efficient manner, Pramanik and Mallick [13]developed the TODIM (an acronym in Portuguese for interactive and a couple of attribute choice making) approach, which is fully based on the trapezoidal neutrosophic environment.

The bipolar trapezoidal neutrosophic set, which consists of both bipolar and trapezoidal neutrosophic variables, is the reason behind this study [14]. This set offers a broad overview of several of the neutrosophic set's generalized forms. Furthermore, to integrate bipolar trapezoidal neutrosophic information in uncertain conditions, the research introduces a few aggregating operations. The theory of bipolar trapezoidal neutrosophic aggregation operators is developed and enhanced by these operators' unique ability to realize the synthesis of bipolar trapezoidal neutrosophic information. Furthermore, the suggested (Dombi) aggregation operators facilitate the establishment of many criteria decision making in the bipolar trapezoidal neutrosophic context since they are useful instruments for decision-making.

Accurately implementing the evaluation of ecological compensation process is extremely difficult due to the unavoidable presence of imprecise data, human judgment ambiguity, and uncertainty among different parameters. The evaluation of the ecological compensation process has long been seen as a multi-criteria decision-making (MCDM) issue[15], [16]. It is evident that the selection criteria for alternatives have shifted from quantitative to qualitative, which has led to a change in the methodologies available for evaluation of ecological compensation. Traditional MCDM techniques make it simple to select alternative based on quantitative factors. They offer a useful framework for handling the evaluation of ecological compensation issues when the objectives of the decision-makers are at odds.

Other techniques that facilitate decision-making in the event of imprecise and ambiguous information have begun to be employed because of the move to the use of qualitative factors in evaluation of ecological compensation[17], [18]. The application of neutrosophic set theory is the basis of these techniques. Because some real-world scenarios are not well defined and it might be challenging to specify the set's limits, the idea of neutrosophic sets' application is more akin to human thought.

This is especially true when choosing alternatives based on qualitative standards, as in the case of evaluation of ecological compensation. Different MCDM techniques, whether in the classic, neutrosophic, or rough form, should be used in the evaluation of ecological compensation process to develop a partnership with the alternative who best meets the established business objectives. Since there are many applications for the MCDM technique in evaluation of ecological compensation.

2. Bipolar trapezoidal neutrosophic set (BTNS)

This section shows the definitions of BTNS [14].

Definition 1

We can define the BTNS such as:

$$B = \left\{\left((\alpha^+, \beta^+, \gamma^+), (\delta^-, \varepsilon^-, \theta^-)\right)\right\}$$

$$B = \left\{\left(\left(\begin{pmatrix} a_B^+(x), \\ b_B^+(x), \\ c_B^+(x), \\ d_B^+(x) \end{pmatrix}, \begin{pmatrix} e_B^+(x), \\ f_B^+(x), \\ g_B^+(x), \\ h_B^+(x) \end{pmatrix}, \begin{pmatrix} l_B^+(x), \\ m_B^+(x), \\ n_B^+(x), \\ p_B^+(x) \end{pmatrix}\right), \left(\begin{pmatrix} a_B^-(x), \\ b_B^-(x), \\ c_B^-(x), \\ d_B^-(x) \end{pmatrix}, \begin{pmatrix} e_B^-(x), \\ f_B^-(x), \\ g_B^-(x), \\ h_B^-(x) \end{pmatrix}, \begin{pmatrix} l_B^-(x), \\ m_B^-(x), \\ n_B^-(x), \\ p_B^-(x) \end{pmatrix}\right)\right\} \quad (1)$$

Example

BTNS can be defined as examples such as:

$$B = \left\{ \left(\begin{pmatrix} \begin{pmatrix} 0.2, \\ 0.3, \\ 0.4, \\ 0.7 \end{pmatrix}, \\ \begin{pmatrix} 0.4, \\ 0.4, \\ 0.5, \\ 1 \end{pmatrix}, \\ \begin{pmatrix} 0.2, \\ 0.2, \\ 0.2, \\ 0.2 \end{pmatrix} \end{pmatrix}, \begin{pmatrix} \begin{pmatrix} -0.5, \\ -0.4, \\ -0.3, \\ 0 \end{pmatrix}, \\ \begin{pmatrix} 0, \\ 0, \\ 0, \\ 0 \end{pmatrix}, \\ \begin{pmatrix} -1, \\ -0.6, \\ -0.2, \\ -0.1 \end{pmatrix} \end{pmatrix} \right\} \tag{2}$$

Definition 2

This definition can define the operations of bipolar transposal neutrosophic numbers such as:

$$B_1 = \left\{ \left(\left(\begin{pmatrix} a_{B_1}^{+}(x), \\ b_{B_1}^{+}(x), \\ c_{B_1}^{+}(x), \\ d_{B_1}^{+}(x) \end{pmatrix}, \begin{pmatrix} e_{B_1}^{+}(x), \\ f_{B_1}^{+}(x), \\ g_{B_1}^{+}(x), \\ h_{B_1}^{+}(x) \end{pmatrix}, \begin{pmatrix} l_{B_1}^{+}(x), \\ m_{B_1}^{+}(x), \\ n_{B_1}^{+}(x), \\ p_{B_1}^{+}(x) \end{pmatrix} \right), \left(\begin{pmatrix} a_{B_1}^{-}(x), \\ b_{B_1}^{-}(x), \\ c_{B_1}^{-}(x), \\ d_{B_1}^{-}(x) \end{pmatrix}, \begin{pmatrix} e_{B_1}^{-}(x), \\ f_{B_1}^{-}(x), \\ g_{B_1}^{-}(x), \\ h_{B_1}^{-}(x) \end{pmatrix}, \begin{pmatrix} l_{B_1}^{-}(x), \\ m_{B_1}^{-}(x), \\ n_{B_1}^{-}(x), \\ p_{B_1}^{-}(x) \end{pmatrix} \right) \right) \right\}$$

$$B_2 = \left\{ \left(\left(\begin{pmatrix} a_{B_2}^+(x), \\ b_{B_2}^+(x), \\ c_{B_2}^+(x), \\ d_{B_{12}}^+(x) \end{pmatrix}, \begin{pmatrix} e_{B_2}^+(x), \\ f_{B_2}^+(x), \\ g_{B_2}^+(x), \\ h_{B_2}^+(x) \end{pmatrix}, \begin{pmatrix} l_{B_2}^+(x), \\ m_{B_2}^+(x), \\ n_{B_2}^+(x), \\ p_{B_2}^+(x) \end{pmatrix} \right), \left(\begin{pmatrix} a_{B_2}^-(x), \\ b_{B_2}^-(x), \\ c_{B_2}^-(x), \\ d_{B_2}^-(x) \end{pmatrix}, \begin{pmatrix} e_{B_2}^-(x), \\ f_{B_2}^-(x), \\ g_{B_2}^-(x), \\ h_{B_2}^-(x) \end{pmatrix}, \begin{pmatrix} l_{B_2}^-(x), \\ m_{B_2}^-(x), \\ n_{B_2}^-(x), \\ p_{B_2}^-(x) \end{pmatrix} \right) \right) \right\}$$

$$B_1 \sqsubseteq B_2 \; if \; \left\{ \begin{array}{l} \begin{pmatrix} a_{B_1}^+(x) \le a_{B_2}^+(x), b_{B_1}^+(x) \le b_{B_2}^+(x), \\ c_{B_1}^+(x) \le c_{B_2}^+(x), d_{B_1}^+(x) \le d_{B_2}^+(x) \end{pmatrix}, \\ \begin{pmatrix} e_{B_1}^+(x) \ge e_{B_2}^+(x), f_{B_1}^+(x) \ge f_{B_2}^+(x), \\ g_{B_1}^+(x) \ge g_{B_2}^+(x), h_{B_1}^+(x) \ge h_{B_2}^+(x) \end{pmatrix}, \\ \begin{pmatrix} l_{B_1}^+(x) \ge l_{B_2}^+(x), m_{B_1}^+(x) \ge m_{B_2}^+(x), \\ n_{B_1}^+(x) \ge n_{B_2}^+(x), p_{B_1}^+(x) \ge p_{B_2}^+(x) \end{pmatrix}, \\ \begin{pmatrix} a_{B_1}^-(x) \ge a_{B_2}^-(x), b_{B_1}^-(x) \ge b_{B_2}^-(x), \\ c_{B_1}^-(x) \ge c_{B_2}^-(x), d_{B_1}^-(x) \ge d_{B_2}^-(x) \end{pmatrix}, \\ \begin{pmatrix} e_{B_1}^-(x) \le e_{B_2}^-(x), f_{B_1}^-(x) \le f_{B_2}^-(x), \\ g_{B_1}^-(x) \le g_{B_2}^-(x), h_{B_1}^-(x) \le h_{B_2}^-(x) \end{pmatrix}, \\ l_{B_1}^-(x) \le l_{B_2}^-(x), m_{B_1}^-(x) \le m_{B_2}^-(x), \\ n_{B_1}^-(x) \le n_{B_2}^-(x), p_{B_1}^-(x) \le p_{B_2}^-(x) \end{array} \right\} \tag{3}$$

$$B_1 \cap B_2 \; if \; \left\{ \begin{array}{l} \begin{pmatrix} min\{a_{B_1}^+(x), a_{B_2}^+(x)\}, \min\{b_{B_1}^+(x), b_{B_2}^+(x)\}, \\ \min\{c_{B_1}^+(x), c_{B_2}^+(x)\}, \min\{d_{B_1}^+(x), d_{B_2}^+(x)\} \end{pmatrix}, \\ \begin{pmatrix} \max\{e_{B_1}^+(x), e_{B_2}^+(x)\}, \max\{f_{B_1}^+(x), f_{B_2}^+(x)\}, \\ \max\{g_{B_1}^+(x), g_{B_2}^+(x)\}, \max\{h_{B_1}^+(x) \ge h_{B_2}^+(x)\} \end{pmatrix}, \\ \begin{pmatrix} \max\{l_{B_1}^+(x), l_{B_2}^+(x)\}, \max\{m_{B_1}^+(x), m_{B_2}^+(x)\}, \\ \max\{n_{B_1}^+(x), n_{B_2}^+(x)\}, \max\{p_{B_1}^+(x), p_{B_2}^+(x)\} \end{pmatrix}, \\ \begin{pmatrix} \max\{a_{B_1}^-(x), a_{B_2}^-(x)\}, \max\{b_{B_1}^-(x), b_{B_2}^-(x)\}, \\ \max\{c_{B_1}^-(x), c_{B_2}^-(x)\}, \max\{d_{B_1}^-(x), d_{B_2}^-(x)\} \end{pmatrix}, \\ \begin{pmatrix} \min\{e_{B_1}^-(x), e_{B_2}^-(x)\}, \min\{f_{B_1}^-(x), f_{B_2}^-(x)\}, \\ \min\{g_{B_1}^-(x), g_{B_2}^-(x)\}, \min\{h_{B_1}^-(x), h_{B_2}^-(x)\} \end{pmatrix}, \\ \begin{pmatrix} \min\{l_{B_1}^-(x), l_{B_2}^-(x)\}, \min\{m_{B_1}^-(x), m_{B_2}^-(x)\}, \\ \min\{n_{B_1}^-(x), n_{B_2}^-(x)\}, \min\{p_{B_1}^-(x), p_{B_2}^-(x)\} \end{pmatrix} \end{array} \right\} \tag{4}$$

$$B_1 \cup B_2 \; if \left\{ \begin{array}{l} \left(\begin{matrix} max\{a_{B_1}^{+}(x), a_{B_2}^{+}(x)\}, max\{b_{B_1}^{+}(x), b_{B_2}^{+}(x)\}, \\ max\{c_{B_1}^{+}(x), c_{B_2}^{+}(x)\}, max\{d_{B_1}^{+}(x), d_{B_2}^{+}(x)\} \end{matrix} \right), \\ \left(\begin{matrix} \min\{e_{B_1}^{+}(x), e_{B_2}^{+}(x)\}, \min\{f_{B_1}^{+}(x), f_{B_2}^{+}(x)\}, \\ \min\{g_{B_1}^{+}(x), g_{B_2}^{+}(x)\}, \min\{h_{B_1}^{+}(x) \geq h_{B_2}^{+}(x)\} \end{matrix} \right), \\ \left(\begin{matrix} \min\{l_{B_1}^{+}(x), l_{B_2}^{+}(x)\}, \min\{m_{B_1}^{+}(x), m_{B_2}^{+}(x)\}, \\ \min\{n_{B_1}^{+}(x), n_{B_2}^{+}(x)\}, \min\{p_{B_1}^{+}(x), p_{B_2}^{+}(x)\} \end{matrix} \right), \\ \left(\begin{matrix} \min\{a_{B_1}^{-}(x), a_{B_2}^{-}(x)\}, \min\{b_{B_1}^{-}(x), b_{B_2}^{-}(x)\}, \\ \min\{c_{B_1}^{-}(x), c_{B_2}^{-}(x)\}, \min\{d_{B_1}^{-}(x), d_{B_2}^{-}(x)\} \end{matrix} \right), \\ \left(\begin{matrix} max\{e_{B_1}^{-}(x), e_{B_2}^{-}(x)\}, max\{f_{B_1}^{-}(x), f_{B_2}^{-}(x)\}, \\ max\{g_{B_1}^{-}(x), g_{B_2}^{-}(x)\}, max\{h_{B_1}^{-}(x), h_{B_2}^{-}(x)\} \end{matrix} \right), \\ \left(\begin{matrix} max\{l_{B_1}^{-}(x), l_{B_2}^{-}(x)\}, max\{m_{B_1}^{-}(x), m_{B_2}^{-}(x)\}, \\ max\{n_{B_1}^{-}(x), n_{B_2}^{-}(x)\}, max\{p_{B_1}^{-}(x), p_{B_2}^{-}(x)\} \end{matrix} \right) \end{array} \right\} \quad (5)$$

3. MCDM Approach

This section shows the steps of the proposed approach under the BTNS to compute the criteria weights and rank the alternatives.

3.1 Logarithm Methodology of Additive Weights (LMAW)[19]

1. Evaluate the criteria by a set of experts and decision makers.

2. Prioritize the criteria

The ranking of the criteria is created using the experts and decision makers. They are adding the highest value and lowest value to the criteria.

3. Define the absolute anti-ideal point.

This anti-ideal point is defined by the minimum value from the priority vector A_j.

4. Compute the relation between the criteria

The relation between elements of the criteria is computed such as:

$$n_j = \frac{B_j}{A_j} \quad (6)$$

5. Compute the weight coefficient such as:

$$w_j = \frac{\log(n_j)}{\log\left(\prod_{j=1}^{n} n_j\right)} \quad (7)$$

6. Rank the criteria

3.2 The MARCOS Method

This section shows the steps of the MARCOS method under the BTNS to rank alternatives.

1. Build the decision matrix.

The decision matrix is built with the ideal AI and anti-ideal solution AAI

$$AAI = \min x_{ij} for\ positive\ criteria\ and \max x_{ij} for\ negative\ criteria \tag{8}$$

$$AAI = \max x_{ij} for\ positive\ criteria\ and \min x_{ij} for\ negative\ criteria \tag{9}$$

2. Normalize the decision matrix

$$q_{ij} = \frac{x_{ai}}{x_{ij}}\ for\ negative\ criteria \tag{10}$$

$$q_{ij} = \frac{x_{ij}}{x_{ai}}\ for\ positive\ criteria \tag{11}$$

3. Compute the weighted matrix

$$u_{ij} = q_{ij} w_j \tag{12}$$

4. Compute the utility degree

$$K_i^- = \frac{S_i}{S_{aai}} \tag{13}$$

$$K_i^+ = \frac{S_i}{S_{ai}} \tag{14}$$

$$S_i = \sum_{i=1}^{n} u_{ij} \tag{15}$$

5. Compute the utility function

$$f(K_i) = \frac{K_i^+ + K_i^-}{1 + \frac{1-f(K_i^+)}{f(K_i^+)} + \frac{1-f(K_i^-)}{f(K_i^-)}} \tag{16}$$

$$f(K_i^+) = \frac{K_i^-}{K_i^+ + K_i^-} \tag{17}$$

$$f(K_i^-) = \frac{K_i^+}{K_i^+ + K_i^-} \tag{18}$$

6. Ran alternatives.

4. Application

The eco-compensation policy in the Huaihe River Basin of Anhui Province has significantly contributed to income growth and poverty reduction. By promoting ecological restoration and sustainable development, the policy incentivizes local residents to engage in eco-friendly practices such as afforestation, wetland conservation, and pollution control. These initiatives not only improve environmental quality but also create job opportunities and increase household income, particularly for rural and impoverished communities. The compensation funds provide

financial support for farmers and fishermen transitioning to sustainable livelihoods, reducing their economic vulnerability. Overall, the eco-compensation mechanism effectively combines environmental protection with poverty alleviation, fostering a harmonious relationship between ecological preservation and socioeconomic development in the region. This section proposes the decision-making approach for evaluation of ecological compensation in the Huai river basin of Anhui province. This section shows the application of the proposed approach to compute the criteria weighs and rank the alternatives. This study is evaluated the criteria and alternatives by the three experts and decision makers. Three experts are gathered with nine criteria and eleven alternatives as : Climate Resilience, Sustainable Land Use, Biodiversity Conservation, Water Quality Improvement, Evaluation System, Stakeholder Participation, Economic Impact, Water Conservation, Compensation Mechanism Efficiency. The alternatives are: Watershed Protection, Pollution Control, Ecological Redline Policy Enforcement, Carbon Credit Trading, Sustainable Fisheries, Green Finance, Community-Based River Management, Cross-Province Water Quality, Public Awareness, Payment for Ecosystem Services, Eco-Friendly Agricultural Subsidies

LMAW Results

Let experts evaluate the criteria by using the BTNNs. Then they are evaluated by the criteria to rank them by their opinions. Then we defined the absolute anti-ideal point. Then we compute the relation between the criteria using Eq. (6). Then we compute the weights of criteria using Eq. (7) as shown in Fig 1.

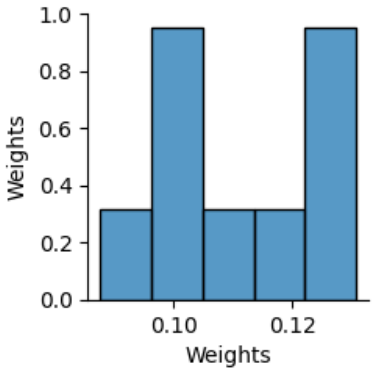

Fig 1. The weights of criteria.

The MARCOS Results

We built the decision matrix using the BTNNs between the criteria and alternative as shown in tables 1-3. Then we normalize the decision matrix using Eq. (10) as shown in table 4. Then we obtained weighted matrix using Eq. (12) as shown in table 5. Then we compute the utility degree using Eqs. (13,14, and 15). Then we compute the utility function using Eqs. (16,17, and 18). Then we rank the alternatives as shown in Fig 2.

Table 1. The first BTNN matrix

	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8	C_9
A_1	{((0.2,0.3,0.4,0.7),(0.4,0.4,0.5,1),(0.2,0.2,0.2,0.2)),((-0.5,-0.4,-0.3,0),(0,0,0,0),(-1,-0.6,-0.2,-0.1))}	{((0.1,0.3,0.5,0.9),(0,0.2,0.4,0.6),(1,1,1,1)),((-0.7,-0.5,-0.3,-0.1),(-1,-1,-0.8,-0.8),(-0.2,0.1,0,0))}	{((0.1,0.2,0.2,0.3),(0.8,0.8,1,1),(0,0,0.6,1)),((-1,-1,-1,-0.2),(-0.6,-0.6,0,0),(-1,-1,-0.5,-0.3))}	{((0.4,0.5,0.5,0.6),(0.8,0.8,1,1),(0,0,0.6,1)),((-1,-1,-0.9,-0.9),(-0.2,-0.2,-0.1,-0.1),(-1,-1,-0.4,-0.4))}	{((0.1,0.5,0.6,1),(0.3,0.3,0.5,0.5),(0,0.2,0.2,0.7)),((-0.8,-0.2,-0.2,-0.1),(-1,-0.1,-0.1,0),(-0.5,0.5,0,0))}	{((0.2,0.5,0.5,0.9),(0.5,0.5,1,1),(0.1,0.1,0.1,0.1)),((-0.3,-0.3,-0.1,-0.1),(-0.5,-0.2,-0.2,0),(-1,-1,-0.4,-0.4))}	{((0.3,0.5,0.6,0.7),(0.2,0.3,0.8,0.8),(0.1,0.4,0.4,0.4)),((-0.5,-0.5,-0.1,-0.1),(-0.9,-0.1,-0.1,-0.1),(-0.9,-0.9,-0.2,-0.1))}	{((0.1,0.2,0.2,0.3),(0.8,0.8,1,1),(0,0,0.6,1)),((-1,-1,-1,-0.2),(-0.6,-0.6,0,0),(-1,-1,-0.5,-0.3))}	{((0.1,0.3,0.5,0.9),(0,0.2,0.4,0.6),(1,1,1,1)),((-0.7,-0.5,-0.3,-0.1),(-1,-1,-0.8,-0.8),(-0.2,-0.1,0,0))}
A_2	{((0.3,0.5,0.6,0.7),(0.2,0.3,0.8,0.8),(0.1,0.4,0.4,0.4)),((-0.5,-0.5,-0.1,-0.1),(-0.9,-0.1,-0.1,-0.1),(-0.9,-0.9,-0.2,-0.1))}	{((0.3,0.5,0.6,0.7),(0.2,0.3,0.8,0.8),(0.1,0.4,0.4,0.4)),((-0.5,-0.5,-0.1,-0.1),(-0.9,-0.1,-0.1,-0.1),(-0.9,-0.9,-0.2,-0.1))}	{((0.2,0.5,0.5,0.9),(0.5,0.5,1,1),(0.1,0.1,0.1,0.1)),((-0.3,-0.3,-0.1,-0.1),(-0.5,-0.2,-0.2,0),(-1,-1,-0.4,-0.4))}	{((0.1,0.5,0.6,1),(0.3,0.3,0.5,0.5),(0,0.2,0.2,0.7)),((-0.8,-0.2,-0.2,-0.1),(-1,-0.1,-0.1,0),(-0.5,0.5,0,0))}	{((0.4,0.5,0.5,0.6),(0.8,0.8,1,1),(0,0,0.6,1)),((-1,-1,-0.9,-0.9),(-0.2,-0.2,-0.1,-0.1),(-1,-1,-0.4,-0.4))}	{((0.1,0.2,0.2,0.3),(0.8,0.8,1,1),(0,0,0.6,1)),((-1,-1,-1,-0.2),(-0.6,-0.6,0,0),(-1,-1,-0.5,-0.3))}	{((0.2,0.3,0.4,0.7),(0.4,0.4,0.5,1),(0.2,0.2,0.2,0.2)),((-0.5,-0.4,-0.3,0),(0,0,0,0),(-1,-0.6,-0.2,-0.1))}	{((0.2,0.3,0.4,0.7),(0.4,0.4,0.5,1),(0.2,0.2,0.2,0.2)),((-0.5,-0.4,-0.3,0),(0,0,0,0),(-1,-0.6,-0.2,-0.1))}	{((0.4,0.5,0.5,0.6),(0.8,0.8,1,1),(0,0,0.6,1)),((-1,-1,-0.9,-0.9),(-0.2,-0.2,-0.1,-0.1),(-1,-1,-0.4,-0.4))}	{((0.4,0.5,0.5,0.6),(0.8,0.8,1,1),(0,0,0.6,1)),((-1,-1,-0.9,-0.9),(-0.2,-0.2,-0.1,-0.1),(-1,-1,-0.4,-0.4))}
A_3	{((0.2,0.5,0.5,0.9),(0.5,0.5,1,1),(0.1,0.1,0.1,0.1)),((-0.3,-0.3,-0.1,-0.1),(-0.5,-0.2,-0.2,0),(-1,-1,-0.4,-0.4))}	{((0.2,0.3,0.4,0.7),(0.4,0.4,0.5,1),(0.2,0.2,0.2,0.2)),((-0.5,-0.4,-0.3,0),(0,0,0,0),(-1,-0.6,-0.2,-0.1))}	{((0.4,0.5,0.5,0.6),(0.8,0.8,1,1),(0,0,0.6,1)),((-1,-1,-0.9,-0.9),(-0.2,-0.2,-0.1,-0.1),(-1,-1,-0.4,-0.4))}	{((0.1,0.5,0.6,1),(0.3,0.3,0.5,0.5),(0,0.2,0.2,0.7)),((-0.8,-0.2,-0.2,-0.1),(-1,-0.1,-0.1,0),(-0.5,0.5,0,0))}	{((0.2,0.5,0.5,0.9),(0.5,0.5,1,1),(0.1,0.1,0.1,0.1)),((-0.3,-0.3,-0.1,-0.1),(-0.5,-0.2,-0.2,0),(-1,-1,-0.4,-0.4))}	{((0.1,0.2,0.2,0.3),(0.8,0.8,1,1),(0,0,0.6,1)),((-1,-1,-1,-0.2),(-0.6,-0.6,0,0),(-1,-1,-0.5,-0.3))}	{((0.2,0.3,0.4,0.7),(0.4,0.4,0.5,1),(0.2,0.2,0.2,0.2)),((-0.5,-0.4,-0.3,0),(0,0,0,0),(-1,-0.6,-0.2,-0.1))}	{((0.1,0.5,0.6,1),(0.3,0.3,0.5,0.5),(0,0.2,0.2,0.7)),((-0.8,-0.2,-0.2,-0.1),(-1,-0.1,-0.1,0),(-0.5,0.5,0,0))}	{((0.1,0.5,0.6,1),(0.3,0.3,0.5,0.5),(0,0.2,0.2,0.7)),((-0.8,-0.2,-0.2,-0.1),(-1,-0.1,-0.1,0),(-0.5,0.5,0,0))}
A_4	{((0.1,0.5,0.6,1),(0.3,0.3,0.5,0.5),(0,0.2,0.2,0.7)),((-0.8,-0.2,-0.2,-0.1),(-1,-0.1,-0.1,0),(-0.5,-0.5,0,0))}	{((0.3,0.5,0.6,0.7),(0.2,0.3,0.8,0.8),(0.1,0.4,0.4,0.4)),((-0.5,-0.5,-0.1,-0.1),(-0.9,-0.1,-0.1,-0.1),(-0.9,-0.9,-0.2,-0.1))}	{((0.2,0.3,0.4,0.7),(0.4,0.4,0.5,1),(0.2,0.2,0.2,0.2)),((-0.5,-0.4,-0.3,0),(0,0,0,0),(-1,-0.6,-0.2,-0.1))}	{((0.2,0.3,0.4,0.7),(0.4,0.4,0.5,1),(0.2,0.2,0.2,0.2)),((-0.5,-0.4,-0.3,0),(0,0,0,0),(-1,-0.6,-0.2,-0.1))}	{((0.2,0.3,0.4,0.7),(0.4,0.4,0.5,1),(0.2,0.2,0.2,0.2)),((-0.5,-0.4,-0.3,0),(0,0,0,0),(-1,-0.6,-0.2,-0.1))}	{((0.2,0.3,0.4,0.7),(0.4,0.4,0.5,1),(0.2,0.2,0.2,0.2)),((-0.5,-0.4,-0.3,0),(0,0,0,0),(-1,-0.6,-0.2,-0.1))}	{((0.2,0.5,0.5,0.9),(0.5,0.5,1,1),(0.1,0.1,0.1,0.1)),((-0.3,-0.3,-0.1,-0.1),(-0.5,-0.2,-0.2,0),(-1,-1,-0.4,-0.4))}	{((0.2,0.5,0.5,0.9),(0.5,0.5,1,1),(0.1,0.1,0.1,0.1)),((-0.3,-0.3,-0.1,-0.1),(-0.5,-0.2,-0.2,0),(-1,-1,-0.4,-0.4))}	{((0.2,0.3,0.4,0.7),(0.4,0.4,0.5,1),(0.2,0.2,0.2,0.2)),((-0.5,-0.4,-0.3,0),(0,0,0,0),(-1,-0.6,-0.2,-0.1))}
A_5	{((0.4,0.5,0.5,0.6),(0.8,0.8,1,1),(0,0,0.6,1)),((-1,-1,-0.9,-0.9),(-0.2,-0.2,-0.1,-0.1),(-1,-1,-0.4,-0.4))}	{((0.2,0.5,0.5,0.9),(0.5,0.5,1,1),(0.1,0.1,0.1,0.1)),((-0.3,-0.3,-0.1,-0.1),(-0.5,-0.2,-0.2,0),(-1,-1,-0.4,-0.4))}	{((0.3,0.5,0.6,0.7),(0.2,0.3,0.8,0.8),(0.1,0.4,0.4,0.4)),((-0.5,-0.5,-0.1,-0.1),(-0.9,-0.1,-0.1,-0.1),(-0.9,-0.9,-0.2,-0.1))}	{((0.2,0.3,0.4,0.7),(0.4,0.4,0.5,1),(0.2,0.2,0.2,0.2)),((-0.5,-0.4,-0.3,0),(0,0,0,0),(-1,-0.6,-0.2,-0.1))}	{((0.3,0.5,0.6,0.7),(0.2,0.3,0.8,0.8),(0.1,0.4,0.4,0.4)),((-0.5,-0.5,-0.1,-0.1),(-0.9,-0.1,-0.1,-0.1),(-0.9,-0.9,-0.2,-0.1))}	{((0.2,0.3,0.4,0.7),(0.4,0.4,0.5,1),(0.2,0.2,0.2,0.2)),((-0.5,-0.4,-0.3,0),(0,0,0,0),(-1,-0.6,-0.2,-0.1))}	{((0.1,0.5,0.6,1),(0.3,0.3,0.5,0.5),(0,0.2,0.2,0.7)),((-0.8,-0.2,-0.2,-0.1),(-1,-0.1,-0.1,0),(-0.5,0.5,0,0))}	{((0.3,0.5,0.6,0.7),(0.2,0.3,0.8,0.8),(0.1,0.4,0.4,0.4)),((-0.5,-0.5,-0.1,-0.1),(-0.9,-0.1,-0.1,-0.1),(-0.9,-0.9,-0.2,-0.1))}	{((0.2,0.3,0.4,0.7),(0.4,0.4,0.5,1),(0.2,0.2,0.2,0.2)),((-0.5,-0.4,-0.3,0),(0,0,0,0),(-1,-0.6,-0.2,-0.1))}
A_6	{((0.1,0.2,0.2,0.3),(0.8,0.8,1,1),(0,0,0.6,1)),((-1,-1,-1,-0.2),(-0.6,-0.6,0,0),(-1,-1,-0.5,-0.3))}	{((0.1,0.5,0.6,1),(0.3,0.3,0.5,0.5),(0,0.2,0.2,0.7)),((-0.8,-0.2,-0.2,-0.1),(-1,-0.1,-0.1,0),(-0.5,0.5,0,0))}	{((0.2,0.5,0.5,0.9),(0.5,0.5,1,1),(0.1,0.1,0.1,0.1)),((-0.3,-0.3,-0.1,-0.1),(-0.5,-0.2,-0.2,0),(-1,-1,-0.4,-0.4))}	{((0.3,0.5,0.6,0.7),(0.2,0.3,0.8,0.8),(0.1,0.4,0.4,0.4)),((-0.5,-0.5,-0.1,-0.1),(-0.9,-0.1,-0.1,-0.1),(-0.9,-0.9,-0.2,-0.1))}	{((0.2,0.5,0.5,0.9),(0.5,0.5,1,1),(0.1,0.1,0.1,0.1)),((-0.3,-0.3,-0.1,-0.1),(-0.5,-0.2,-0.2,0),(-1,-1,-0.4,-0.4))}	{((0.3,0.5,0.6,0.7),(0.2,0.3,0.8,0.8),(0.1,0.4,0.4,0.4)),((-0.5,-0.5,-0.1,-0.1),(-0.9,-0.1,-0.1,-0.1),(-0.9,-0.9,-0.2,-0.1))}	{((0.4,0.5,0.5,0.6),(0.8,0.8,1,1),(0,0,0.6,1)),((-1,-1,-0.9,-0.9),(-0.2,-0.2,-0.1,-0.1),(-1,-1,-0.4,-0.4))}	{((0.2,0.5,0.5,0.9),(0.5,0.5,1,1),(0.1,0.1,0.1,0.1)),((-0.3,-0.3,-0.1,-0.1),(-0.5,-0.2,-0.2,0),(-1,-1,-0.4,-0.4))}	{((0.2,0.3,0.4,0.7),(0.4,0.4,0.5,1),(0.2,0.2,0.2,0.2)),((-0.5,-0.4,-0.3,0),(0,0,0,0),(-1,-0.6,-0.2,-0.1))}
A_7	{((0.1,0.3,0.5,0.9),(0.0,2,0.4,0.6),(1,1,1,1)),((-0.7,-0.5,-0.3,-0.1),(-1,-1,-0.8,-0.8),(-0.2,-0.1,0,0))}	{((0.4,0.5,0.5,0.6),(0.8,0.8,1,1),(0,0,0.6,1)),((-1,-1,-0.9,-0.9),(-0.2,-0.2,-0.1,-0.1),(-1,-1,-0.4,-0.4))}	{((0.1,0.5,0.6,1),(0.3,0.3,0.5,0.5),(0,0.2,0.2,0.7)),((-0.8,-0.2,-0.2,-0.1),(-1,-0.1,-0.1,0),(-0.5,0.5,0,0))}	{((0.2,0.5,0.5,0.9),(0.5,0.5,1,1),(0.1,0.1,0.1,0.1)),((-0.3,-0.3,-0.1,-0.1),(-0.5,-0.2,-0.2,0),(-1,-1,-0.4,-0.4))}	{((0.1,0.5,0.6,1),(0.3,0.3,0.5,0.5),(0,0.2,0.2,0.7)),((-0.8,-0.2,-0.2,-0.1),(-1,-0.1,-0.1,0),(-0.5,0.5,0,0))}	{((0.2,0.5,0.5,0.9),(0.5,0.5,1,1),(0.1,0.1,0.1,0.1)),((-0.3,-0.3,-0.1,-0.1),(-0.5,-0.2,-0.2,0),(-1,-1,-0.4,-0.4))}	{((0.1,0.2,0.2,0.3),(0.8,0.8,1,1),(0,0,0.6,1)),((-1,-1,-1,-0.2),(-0.6,-0.6,0,0),(-1,-1,-0.5,-0.3))}	{((0.1,0.5,0.6,1),(0.3,0.3,0.5,0.5),(0,0.2,0.2,0.7)),((-0.8,-0.2,-0.2,-0.1),(-1,-0.1,-0.1,0),(-0.5,0.5,0,0))}	{((0.4,0.5,0.5,0.6),(0.8,0.8,1,1),(0,0,0.6,1)),((-1,-1,-0.9,-0.9),(-0.2,-0.2,-0.1,-0.1),(-1,-1,-0.4,-0.4))}
A_8	{((0.2,0.3,0.4,0.7),(0.4,0.4,0.5,1),(0.2,0.2,0.2,0.2)),((-0.5,-0.4,-0.3,0),(0,0,0,0),(-1,-0.6,-0.2,-0.1))}	{((0.1,0.2,0.2,0.3),(0.8,0.8,1,1),(0,0,0.6,1)),((-1,-1,-1,-0.2),(-0.6,-0.6,0,0),(-1,-1,-0.5,-0.3))}	{((0.4,0.5,0.5,0.6),(0.8,0.8,1,1),(0,0,0.6,1)),((-1,-1,-0.9,-0.9),(-0.2,-0.2,-0.1,-0.1),(-1,-1,-0.4,-0.4))}	{((0.1,0.5,0.6,1),(0.3,0.3,0.5,0.5),(0,0.2,0.2,0.7)),((-0.8,-0.2,-0.2,-0.1),(-1,-0.1,-0.1,0),(-0.5,0.5,0,0))}	{((0.4,0.5,0.5,0.6),(0.8,0.8,1,1),(0,0,0.6,1)),((-1,-1,-0.9,-0.9),(-0.2,-0.2,-0.1,-0.1),(-1,-1,-0.4,-0.4))}	{((0.1,0.5,0.6,1),(0.3,0.3,0.5,0.5),(0,0.2,0.2,0.7)),((-0.8,-0.2,-0.2,-0.1),(-1,-0.1,-0.1,0),(-0.5,0.5,0,0))}	{((0.1,0.3,0.5,0.9),(0.0,2,0.4,0.6),(1,1,1,1)),((-0.7,-0.5,-0.3,-0.1),(-1,-1,-0.8,-0.8),(-0.2,-0.1,0,0))}	{((0.3,0.5,0.6,0.7),(0.2,0.3,0.8,0.8),(0.1,0.4,0.4,0.4)),((-0.5,-0.5,-0.1,-0.1),(-0.9,-0.1,-0.1,-0.1),(-0.9,-0.9,-0.2,-0.1))}	{((0.1,0.2,0.2,0.3),(0.8,0.8,1,1),(0,0,0.6,1)),((-1,-1,-1,-0.2),(-0.6,-0.6,0,0),(-1,-1,-0.5,-0.3))}	{((0.1,0.2,0.2,0.3),(0.8,0.8,1,1),(0,0,0.6,1)),((-0.7,-0.5,-0.3,-0.1),(-1,-1,-0.8,-0.8),(-0.2,-0.1,0,0))}
A_9	{((0.3,0.5,0.6,0.7),(0.2,0.3,0.8,0.8),(0.1,0.4,0.4,0.4)),((-0.5,-0.5,-0.1,-0.1),(-0.9,-0.1,-0.1,-0.1),(-0.9,-0.9,-0.2,-0.1))}	{((0.1,0.3,0.5,0.9),(0,0.2,0.4,0.6),(1,1,1,1)),((-0.7,-0.5,-0.3,-0.1),(-1,-1,-0.8,-0.8),(-0.2,-0.1,0,0))}	{((0.1,0.2,0.2,0.3),(0.8,0.8,1,1),(0,0,0.6,1)),((-1,-1,-1,-0.2),(-0.6,-0.6,0,0),(-1,-1,-0.5,-0.3))}	{((0.4,0.5,0.5,0.6),(0.8,0.8,1,1),(0,0,0.6,1)),((-1,-1,-0.9,-0.9),(-0.2,-0.2,-0.1,-0.1),(-1,-1,-0.4,-0.4))}	{((0.1,0.2,0.2,0.3),(0.8,0.8,1,1),(0,0,0.6,1)),((-1,-1,-1,-0.2),(-0.6,-0.6,0,0),(-1,-1,-0.5,-0.3))}	{((0.4,0.5,0.5,0.6),(0.8,0.8,1,1),(0,0,0.6,1)),((-1,-1,-0.9,-0.9),(-0.2,-0.2,-0.1,-0.1),(-1,-1,-0.4,-0.4))}	{((0.2,0.3,0.4,0.7),(0.4,0.4,0.5,1),(0.2,0.2,0.2,0.2)),((-0.5,-0.4,-0.3,0),(0,0,0,0),(-1,-0.6,-0.2,-0.1))}	{((0.2,0.5,0.5,0.9),(0.5,0.5,1,1),(0.1,0.1,0.1,0.1)),((-0.3,-0.3,-0.1,-0.1),(-0.5,-0.2,-0.2,0),(-1,-1,-0.4,-0.4))}	{((0.2,0.3,0.4,0.7),(0.4,0.4,0.5,1),(0.2,0.2,0.2,0.2)),((-0.5,-0.4,-0.3,0),(0,0,0,0),(-1,-0.6,-0.2,-0.1))}
A_{10}	{((0.3,0.5,0.6,0.7),(0.2,0.3,0.8,0.8),(0.1,0.4,0.4,0.4)),((-0.5,-0.5,-0.1,-0.1),(-0.9,-0.1,-0.1,-0.1),(-0.9,-0.9,-0.2,-0.1))}	{((0.2,0.3,0.4,0.7),(0.4,0.4,0.5,1),(0.2,0.2,0.2,0.2)),((-0.5,-0.4,-0.3,0),(0,0,0,0),(-1,-0.6,-0.2,-0.1))}	{((0.1,0.3,0.5,0.9),(0.0,2,0.4,0.6),(1,1,1,1)),((-0.7,-0.5,-0.3,-0.1),(-1,-1,-0.8,-0.8),(-0.2,-0.1,0,0))}	{((0.1,0.2,0.2,0.3),(0.8,0.8,1,1),(0,0,0.6,1)),((-1,-1,-1,-0.2),(-0.6,-0.6,0,0),(-1,-1,-0.5,-0.3))}	{((0.1,0.3,0.5,0.9),(0,0.2,0.4,0.6),(1,1,1,1)),((-0.7,-0.5,-0.3,-0.1),(-1,-1,-0.8,-0.8),(-0.2,-0.1,0,0))}	{((0.1,0.2,0.2,0.3),(0.8,0.8,1,1),(0,0,0.6,1)),((-1,-1,-1,-0.2),(-0.6,-0.6,0,0),(-1,-1,-0.5,-0.3))}	{((0.1,0.3,0.5,0.9),(0,0.2,0.4,0.6),(1,1,1,1)),((-0.7,-0.5,-0.3,-0.1),(-1,-1,-0.8,-0.8),(-0.2,-0.1,0,0))}	{((0.1,0.2,0.2,0.3),(0.8,0.8,1,1),(0,0,0.6,1)),((-1,-1,-1,-0.2),(-0.6,-0.6,0,0),(-1,-1,-0.5,-0.3))}	{((0.1,0.3,0.5,0.9),(0,0.2,0.4,0.6),(1,1,1,1)),((-0.7,-0.5,-0.3,-0.1),(-1,-1,-0.8,-0.8),(-0.2,-0.1,0,0))}
A_{11}	{((0.2,0.3,0.4,0.7),(0.4,0.4,0.5,1),(0.2,0.2,0.2,0.2)),((-0.5,-0.4,-0.3,0),(0,0,0,0),(-1,-0.6,-0.2,-0.1))}	{((0.1,0.2,0.2,0.3),(0.8,0.8,1,1),(0,0,0.6,1)),((-1,-1,-1,-0.2),(-0.6,-0.6,0,0),(-1,-1,-0.5,-0.3))}	{((0.2,0.3,0.4,0.7),(0.4,0.4,0.5,1),(0.2,0.2,0.2,0.2)),((-0.5,-0.4,-0.3,0),(0,0,0,0),(-1,-0.6,-0.2,-0.1))}	{((0.1,0.3,0.5,0.9),(0.0,2,0.4,0.6),(1,1,1,1)),((-0.7,-0.5,-0.3,-0.1),(-1,-1,-0.8,-0.8),(-0.2,-0.1,0,0))}	{((0.2,0.3,0.4,0.7),(0.4,0.4,0.5,1),(0.2,0.2,0.2,0.2)),((-0.5,-0.4,-0.3,0),(0,0,0,0),(-1,-0.6,-0.2,-0.1))}	{((0.1,0.3,0.5,0.9),(0.0,2,0.4,0.6),(1,1,1,1)),((-0.7,-0.5,-0.3,-0.1),(-1,-1,-0.8,-0.8),(-0.2,-0.1,0,0))}	{((0.1,0.2,0.2,0.3),(0.8,0.8,1,1),(0,0,0.6,1)),((-1,-1,-1,-0.2),(-0.6,-0.6,0,0),(-1,-1,-0.5,-0.3))}	{((0.1,0.2,0.2,0.3),(0.8,0.8,1,1),(0,0,0.6,1)),((-1,-1,-0.9,-0.9),(-0.2,-0.2,-0.1,-0.1),(-1,-1,-0.4,-0.4))}	{((0.1,0.2,0.2,0.3),(0.8,0.8,1,1),(0,0,0.6,1)),((-1,-1,-1,-0.2),(-0.6,-0.6,0,0),(-1,-1,-0.5,-0.3))}

Table 2. The second BTNN matrix

	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8	C_9
A_1	{((0.4,0.5,0.5,0.6),(0.8,0.8,1,1),(0,0,0.6,1)),((-1,-1,-0.9,-0.9),(-0.2,-0.2,-0.1,-0.1),(-1,-1,-0.4,-0.4))}	{((0.1,0.3,0.5,0.9),(0,0.2,0.4,0.6),(1,1,1,1)),((-0.7,-0.5,-0.3,-0.1),(-1,-1,-0.8,-0.8),(-0.2,-0.1,0,0))}	{((0.1,0.2,0.2,0.3),(0.8,0.8,1,1),(0,0,0.6,1)),((-1,-1,-1,-0.2),(-0.6,-0.6,0,0),(-1,-1,-0.5,-0.3))}	{((0.4,0.5,0.5,0.6),(0.8,0.8,1,1),(0,0,0.6,1)),((-1,-1,-0.9,-0.9),(-0.2,-0.2,-0.1,-0.1),(-1,-1,-0.4,-0.4))}	{((0.1,0.5,0.6,1),(0.3,0.3,0.5,0.5),(0,0.2,0.2,0.7)),((-0.8,-0.2,-0.2,-0.1),(-1,-0.1,0,0),(-0.5,0.5,0,0))}	{((0.2,0.5,0.5,0.9),(0.5,0.5,1,1),(0.1,0.1,0.1,0.1)),((-0.3,-0.3,-0.1,-0.1),(-0.5,-0.2,-0.2,0),(-1,-1,-0.4,-0.4))}	{((0.3,0.5,0.6,0.7),(0.2,0.3,0.8,0.8),(0.1,0.4,0.4,0.4)),((-0.5,-0.5,-0.1,-0.1),(-0.9,-0.1,-0.1,-0.1),(-0.9,-0.9,-0.2,-0.1))}	{((0.1,0.2,0.2,0.3),(0.8,0.8,1,1),(0,0,0.6,1)),((-1,-1,-1,-0.2),(-0.6,-0.6,0,0),(-1,-1,-0.5,-0.3))}	{((0.1,0.5,0.6,1),(0.3,0.3,0.5,0.5),(0,0.2,0.2,0.7)),((-0.8,-0.2,-0.2,-0.1),(-1,-0.1,-0.1,0),(-0.5,0.5,0,0))}
A_2	{((0.1,0.2,0.2,0.3),(0.8,0.8,1,1),(0,0,0.6,1)),((-1,-1,-1,-0.2),(-0.6,-0.6,0,0),(-1,-1,-0.5,-0.3))}	{((0.3,0.5,0.6,0.7),(0.2,0.3,0.8,0.8),(0.1,0.4,0.4,0.4)),((-0.5,-0.5,-0.1,-0.1),(-0.9,-0.1,-0.1,-0.1),(-0.9,-0.9,-0.2,-0.1))}	{((0.2,0.5,0.5,0.9),(0.5,0.5,1,1),(0.1,0.1,0.1,0.1)),((-0.3,-0.3,-0.1,-0.1),(-0.5,-0.2,-0.2,0),(-1,-1,-0.4,-0.4))}	{((0.4,0.5,0.5,0.6),(0.8,0.8,1,1),(0,0,0.6,1)),((-1,-1,-0.9,-0.9),(-0.2,-0.2,-0.1,-0.1),(-1,-1,-0.4,-0.4))}	{((0.4,0.5,0.5,0.6),(0.8,0.8,1,1),(0,0,0.6,1)),((-1,-1,-0.9,-0.9),(-0.2,-0.2,-0.1,-0.1),(-1,-1,-0.4,-0.4))}	{((0.4,0.5,0.5,0.6),(0.8,0.8,1,1),(0,0,0.6,1)),((-1,-1,-0.9,-0.9),(-0.2,-0.2,-0.1,-0.1),(-1,-1,-0.4,-0.4))}	{((0.1,0.2,0.2,0.3),(0.8,0.8,1,1),(0,0,0.6,1)),((-1,-1,-1,-0.2),(-0.6,-0.6,0,0),(-1,-1,-0.5,-0.3))}	{((0.4,0.5,0.5,0.6),(0.8,0.8,1,1),(0,0,0.6,1)),((-1,-1,-0.9,-0.9),(-0.2,-0.2,-0.1,-0.1),(-1,-1,-0.4,-0.4))}	{((0.4,0.5,0.5,0.6),(0.8,0.8,1,1),(0,0,0.6,1)),((-1,-1,-0.9,-0.9),(-0.2,-0.2,-0.1,-0.1),(-1,-1,-0.4,-0.4))}	{((0.1,0.5,0.6,1),(0.3,0.3,0.5,0.5),(0,0.2,0.2,0.7)),((-0.8,-0.2,-0.2,-0.1),(-1,-0.1,-0.1,0),(-0.5,0.5,0,0))}
A_3	{((0.1,0.3,0.5,0.9),(0.0,2,0.4,0.6),(1,1,1,1)),((-0.7,-0.5,-0.3,-0.1),(-1,-1,-0.8,-0.8),(-0.2,-0.1,0,0))}	{((0.1,0.2,0.2,0.3),(0.8,0.8,1,1),(0,0,0.6,1)),((-1,-1,-1,-0.2),(-0.6,-0.6,0,0),(-1,-1,-0.5,-0.3))}	{((0.4,0.5,0.5,0.6),(0.8,0.8,1,1),(0,0,0.6,1)),((-1,-1,-0.9,-0.9),(-0.2,-0.2,-0.1,-0.1),(-1,-1,-0.4,-0.4))}	{((0.1,0.2,0.2,0.3),(0.8,0.8,1,1),(0,0,0.6,1)),((-1,-1,-1,-0.2),(-0.6,-0.6,0,0),(-1,-1,-0.5,-0.3))}	{((0.4,0.5,0.5,0.6),(0.8,0.8,1,1),(0,0,0.6,1)),((-1,-1,-0.9,-0.9),(-0.2,-0.2,-0.1,-0.1),(-1,-1,-0.4,-0.4))}	{((0.1,0.2,0.2,0.3),(0.8,0.8,1,1),(0,0,0.6,1)),((-1,-1,-1,-0.2),(-0.6,-0.6,0,0),(-1,-1,-0.5,-0.3))}	{((0.1,0.3,0.5,0.9),(0.0,2,0.4,0.6),(1,1,1,1)),((-0.7,-0.5,-0.3,-0.1),(-1,-1,-0.8,-0.8),(-0.2,-0.1,0,0))}	{((0.1,0.2,0.2,0.3),(0.8,0.8,1,1),(0,0,0.6,1)),((-1,-1,-1,-0.2),(-0.6,-0.6,0,0),(-1,-1,-0.5,-0.3))}	{((0.1,0.5,0.6,1),(0.3,0.3,0.5,0.5),(0,0.2,0.2,0.7)),((-0.8,-0.2,-0.2,-0.1),(-1,-0.1,-0.1,0),(-0.5,0.5,0,0))}
A_4	{((0.2,0.3,0.4,0.7),(0.4,0.4,0.5,1),(0.2,0.2,0.2,0.2)),((-0.5,-0.4,-	{((0.1,0.2,0.2,0.3),(0.8,0.8,1,1),(0,0,0.6,1)),((-1,-1,-1,-0.2),(-0.6,-	{((0.1,0.3,0.5,0.9),(0.0,2,0.4,0.6),(1,1,1,1)),((-0.7,-0.5,-0.3,-0.1),(-1,-1,-	{((0.1,0.3,0.5,0.9),(0.0,2,0.4,0.6),(1,1,1,1)),((-0.7,-0.5,-0.3,-0.1),(-1,-1,-	{((0.1,0.2,0.2,0.3),(0.8,0.8,1,1),(0,0,0.6,1)),((-1,-1,-1,-0.2),(-0.6,-	{((0.2,0.3,0.4,0.7),(0.4,0.4,0.5,1),(0.2,0.2,0.2,0.2)),((-0.5,-0.4,-	{((0.1,0.3,0.5,0.9),(0.0,2,0.4,0.6),(1,1,1,1)),((-0.7,-0.5,-0.3,-0.1),(-1,-1,-	{((0.2,0.5,0.5,0.9),(0.5,0.5,1,1),(0.1,0.1,0.1,0.1)),((-0.3,-	{((0.4,0.5,0.5,0.6),(0.8,0.8,1,1),(0,0,0.6,1)),((-1,-1,-0.9,-0.9),(-0.2,-0.2,-

	0.3,0),(0,0,0,0),(-1,-0.6,-0.2,-0.1))}	0.6,0,0),(-1,-1,-0.5,-0.3))}	0.8,-0.8),(-0.2,-0.1,0,0))}	0.8,-0.8),(-0.2,-0.1,0,0))}	0.6,0,0),(-1,-1,-0.5,-0.3))}	0.3,0),(0,0,0,0),(-1,-0.6,-0.2,-0.1))}	0.8,-0.8),(-0.2,-0.1,0,0))}	0.5,-0.2,-0.2,0),(-1,-1,-0.4,-0.4))}	0.1,-0.1),(-1,-1,-0.4,-0.4))}
A5	{((0.3,0.5,0.6,0.7),(0.2,0.3,0.8,0.8),(0.1,0.4,0.4,0.4)),((-0.5,-0.5,-0.1,-0.1),(-0.9,-0.1,-0.1,-0.1),(-0.9,-0.9,-0.2,-0.1))}	{((0.4,0.5,0.5,0.6),(0.8,0.8,1,1),(0,0,0.6,1)),((-1,-1,-0.9,-0.9),(-0.2,-0.2,-0.1,-0.1),(-1,-1,-0.4,-0.4))}	{((0.1,0.2,0.2,0.3),(0.8,0.8,1,1),(0,0,0.6,1)),((-1,-1,-1,-0.2),(-0.6,-0.6,0,0),(-1,-1,-0.5,-0.3))}	{((0.2,0.3,0.4,0.7),(0.4,0.4,0.5,1),(0.2,0.2,0.2,0.2)),((-0.5,-0.4,-0.3,0),(0,0,0,0),(-1,-0.6,-0.2,-0.1))}	{((0.1,0.3,0.5,0.9),(0.0,2.0,4.0,6),(1,1,1,1)),((-0.7,-0.5,-0.3,-0.1),(-1,-1,-0.8,-0.8),(-0.2,-0.1,0,0))}	{((0.3,0.5,0.6,0.7),(0.2,0.3,0.8,0.8),(0.1,0.4,0.4,0.4)),((-0.5,-0.5,-0.1,-0.1),(-0.9,-0.1,-0.1,-0.1),(-0.9,-0.2,-0.1,-0.1))}	{((0.2,0.3,0.4,0.7),(0.4,0.4,0.5,1),(0.2,0.2,0.2,0.2)),((-0.5,-0.4,-0.3,0),(0,0,0,0),(-1,-0.6,-0.2,-0.1))}	{((0.1,0.5,0.6,1),(0.3,0.3,0.5,0.5),(0.0,2.0,2.0,7)),((-0.8,-0.2,-0.2,-0.1),(-1,-1,-0.1,-0),(-0.5,-0.5,0,0))}	{((0.2,0.5,0.5,0.9),(0.5,0.5,1,1),(0.1,0.1,0.1,0.1)),((-0.3,-0.3,-0.1,-0.1),(-0.5,-0.2,-0.2,0),(-1,-1,-0.4,-0.4))}
A6	{((0.4,0.5,0.5,0.6),(0.8,0.8,1,1),(0,0,0.6,1)),((-1,-1,-0.9,-0.9),(-0.2,-0.2,-0.1,-0.1),(-1,-1,-0.4,-0.4))}	{((0.4,0.5,0.5,0.6),(0.8,0.8,1,1),(0,0,0.6,1)),((-1,-1,-0.9,-0.9),(-0.2,-0.2,-0.1,-0.1),(-1,-1,-0.4,-0.4))}	{((0.4,0.5,0.5,0.6),(0.8,0.8,1,1),(0,0,0.6,1)),((-1,-1,-0.9,-0.9),(-0.2,-0.2,-0.1,-0.1),(-1,-1,-0.4,-0.4))}	{((0.3,0.5,0.6,0.7),(0.2,0.3,0.8,0.8),(0.1,0.4,0.4,0.4)),((-0.5,-0.5,-0.1,-0.1),(-0.9,-0.1,-0.1,-0.1),(-0.9,-0.2,-0.1,-0.1))}	{((0.2,0.3,0.4,0.7),(0.4,0.4,0.5,1),(0.2,0.2,0.2,0.2)),((-0.5,-0.4,-0.3,0),(0,0,0,0),(-1,-0.6,-0.2,-0.1))}	{((0.4,0.5,0.5,0.6),(0.8,0.8,1,1),(0,0,0.6,1)),((-1,-1,-0.9,-0.9),(-0.2,-0.2,-0.1,-0.1),(-1,-1,-0.4,-0.4))}	{((0.3,0.5,0.6,0.7),(0.2,0.3,0.8,0.8),(0.1,0.4,0.4,0.4)),((-0.5,-0.5,-0.1,-0.1),(-0.9,-0.1,-0.1,-0.1),(-0.9,-0.2,-0.1,-0.1))}	{((0.2,0.5,0.5,0.9),(0.5,0.5,1,1),(0.1,0.1,0.1,0.1)),((-0.3,-0.3,-0.1,-0.1),(-0.5,-0.2,-0.2,0),(-1,-1,-0.4,-0.4))}	{((0.1,0.5,0.6,1),(0.3,0.3,0.5,0.5),(0.0,2.0,2.0,7)),((-0.8,-0.2,-0.2,-0.1),(-1,-1,-0.1,-0),(-0.5,-0.5,0,0))}
A7	{((0.1,0.2,0.2,0.3),(0.8,0.8,1,1),(0,0,0.6,1)),((-1,-1,-1,-0.2),(-0.6,-0.6,0,0),(-1,-1,-0.5,-0.3))}	{((0.1,0.2,0.2,0.3),(0.8,0.8,1,1),(0,0,0.6,1)),((-1,-1,-1,-0.2),(-0.6,-0.6,0,0),(-1,-1,-0.5,-0.3))}	{((0.1,0.2,0.2,0.3),(0.8,0.8,1,1),(0,0,0.6,1)),((-1,-1,-1,-0.2),(-0.6,-0.6,0,0),(-1,-1,-0.5,-0.3))}	{((0.2,0.5,0.5,0.9),(0.5,0.5,1,1),(0.1,0.1,0.1,0.1)),((-0.3,-0.3,-0.1,-0.1),(-0.5,-0.2,-0.2,0),(-1,-1,-0.4,-0.4))}	{((0.4,0.5,0.5,0.6),(0.8,0.8,1,1),(0,0,0.6,1)),((-1,-1,-0.9,-0.9),(-0.2,-0.2,-0.1,-0.1),(-1,-1,-0.4,-0.4))}	{((0.1,0.2,0.2,0.3),(0.8,0.8,1,1),(0,0,0.6,1)),((-1,-1,-1,-0.2),(-0.6,-0.6,0,0),(-1,-1,-0.5,-0.3))}	{((0.2,0.5,0.5,0.9),(0.5,0.5,1,1),(0.1,0.1,0.1,0.1)),((-0.3,-0.3,-0.1,-0.1),(-0.5,-0.2,-0.2,0),(-1,-1,-0.4,-0.4))}	{((0.1,0.5,0.6,1),(0.3,0.3,0.5,0.5),(0.0,2.0,2.0,7)),((-0.8,-0.2,-0.2,-0.1),(-1,-1,-0.1,-0),(-0.5,-0.5,0,0))}	{((0.2,0.5,0.5,0.9),(0.5,0.5,1,1),(0.1,0.1,0.1,0.1)),((-0.3,-0.3,-0.1,-0.1),(-0.5,-0.2,-0.2,0),(-1,-1,-0.4,-0.4))}
A8	{((0.1,0.3,0.5,0.9),(0.0,2.0,4.0,6),(1,1,1,1)),((-0.7,-0.5,-0.3,-0.1),(-1,-1,-0.8,-0.8),(-0.2,-0.1,0,0))}	{((0.1,0.3,0.5,0.9),(0.0,2.0,4.0,6),(1,1,1,1)),((-0.7,-0.5,-0.3,-0.1),(-1,-1,-0.8,-0.8),(-0.2,-0.1,0,0))}	{((0.1,0.3,0.5,0.9),(0.0,2.0,4.0,6),(1,1,1,1)),((-0.7,-0.5,-0.3,-0.1),(-1,-1,-0.8,-0.8),(-0.2,-0.1,0,0))}	{((0.1,0.5,0.6,1),(0.3,0.3,0.5,0.5),(0.0,2.0,2.0,7)),((-0.8,-0.2,-0.2,-0.1),(-1,-1,-0.1,-0),(-0.5,-0.5,0,0))}	{((0.1,0.2,0.2,0.3),(0.8,0.8,1,1),(0,0,0.6,1)),((-1,-1,-1,-0.2),(-0.6,-0.6,0,0),(-1,-1,-0.5,-0.3))}	{((0.1,0.3,0.5,0.9),(0.0,2.0,4.0,6),(1,1,1,1)),((-0.7,-0.5,-0.3,-0.1),(-1,-1,-0.8,-0.8),(-0.2,-0.1,0,0))}	{((0.1,0.5,0.6,1),(0.3,0.3,0.5,0.5),(0.0,2.0,2.0,7)),((-0.8,-0.2,-0.2,-0.1),(-1,-1,-0.1,-0),(-0.5,-0.5,0,0))}	{((0.1,0.5,0.6,1),(0.3,0.3,0.5,0.5),(0.0,2.0,2.0,7)),((-0.8,-0.2,-0.2,-0.1),(-1,-1,-0.1,-0),(-0.5,-0.5,0,0))}	{((0.2,0.3,0.4,0.7),(0.4,0.4,0.5,1),(0.2,0.2,0.2,0.2)),((-0.5,-0.4,-0.3,0),(0,0,0,0),(-1,-0.6,-0.2,-0.1))}
A9	{((0.2,0.3,0.4,0.7),(0.4,0.4,0.5,1),(0.2,0.2,0.2,0.2)),((-0.5,-0.4,-0.3,0),(0,0,0,0),(-1,-0.6,-0.2,-0.1))}	{((0.2,0.3,0.4,0.7),(0.4,0.4,0.5,1),(0.2,0.2,0.2,0.2)),((-0.5,-0.4,-0.3,0),(0,0,0,0),(-1,-0.6,-0.2,-0.1))}	{((0.2,0.3,0.4,0.7),(0.4,0.4,0.5,1),(0.2,0.2,0.2,0.2)),((-0.5,-0.4,-0.3,0),(0,0,0,0),(-1,-0.6,-0.2,-0.1))}	{((0.2,0.3,0.4,0.7),(0.4,0.4,0.5,1),(0.2,0.2,0.2,0.2)),((-0.5,-0.4,-0.3,0),(0,0,0,0),(-1,-0.6,-0.2,-0.1))}	{((0.1,0.3,0.5,0.9),(0.0,2.0,4.0,6),(1,1,1,1)),((-0.7,-0.5,-0.3,-0.1),(-1,-1,-0.8,-0.8),(-0.2,-0.1,0,0))}	{((0.2,0.3,0.4,0.7),(0.4,0.4,0.5,1),(0.2,0.2,0.2,0.2)),((-0.5,-0.4,-0.3,0),(0,0,0,0),(-1,-0.6,-0.2,-0.1))}	{((0.1,0.3,0.5,0.9),(0.0,2.0,4.0,6),(1,1,1,1)),((-0.7,-0.5,-0.3,-0.1),(-1,-1,-0.8,-0.8),(-0.2,-0.1,0,0))}	{((0.2,0.3,0.4,0.7),(0.4,0.4,0.5,1),(0.2,0.2,0.2,0.2)),((-0.5,-0.4,-0.3,0),(0,0,0,0),(-1,-0.6,-0.2,-0.1))}	{((0.1,0.3,0.5,0.9),(0.0,2.0,4.0,6),(1,1,1,1)),((-0.7,-0.5,-0.3,-0.1),(-1,-1,-0.8,-0.8),(-0.2,-0.1,0,0))}
A10	{((0.3,0.5,0.6,0.7),(0.2,0.3,0.8,0.8),(0.1,0.4,0.4,0.4)),((-0.5,-0.5,-0.1,-0.1),(-0.9,-0.1,-0.1,-0.1),(-0.9,-0.9,-0.2,-0.1))}	{((0.3,0.5,0.6,0.7),(0.2,0.3,0.8,0.8),(0.1,0.4,0.4,0.4)),((-0.5,-0.5,-0.1,-0.1),(-0.9,-0.1,-0.1,-0.1),(-0.9,-0.9,-0.2,-0.1))}	{((0.3,0.5,0.6,0.7),(0.2,0.3,0.8,0.8),(0.1,0.4,0.4,0.4)),((-0.5,-0.5,-0.1,-0.1),(-0.9,-0.1,-0.1,-0.1),(-0.9,-0.9,-0.2,-0.1))}	{((0.1,0.3,0.5,0.9),(0.0,2.0,4.0,6),(1,1,1,1)),((-0.7,-0.5,-0.3,-0.1),(-1,-1,-0.8,-0.8),(-0.2,-0.1,0,0))}	{((0.2,0.3,0.4,0.7),(0.4,0.4,0.5,1),(0.2,0.2,0.2,0.2)),((-0.5,-0.4,-0.3,0),(0,0,0,0),(-1,-0.6,-0.2,-0.1))}	{((0.3,0.5,0.6,0.7),(0.2,0.3,0.8,0.8),(0.1,0.4,0.4,0.4)),((-0.5,-0.5,-0.1,-0.1),(-0.9,-0.1,-0.1,-0.1),(-0.9,-0.2,-0.2,0),(-1,-1,-0.4,-0.4))}	{((0.2,0.3,0.4,0.7),(0.4,0.4,0.5,1),(0.2,0.2,0.2,0.2)),((-0.5,-0.4,-0.3,0),(0,0,0,0),(-1,-0.6,-0.2,-0.1))}	{((0.1,0.3,0.5,0.9),(0.0,2.0,4.0,6),(1,1,1,1)),((-0.7,-0.5,-0.3,-0.1),(-1,-1,-0.8,-0.8),(-0.2,-0.1,0,0))}	{((0.1,0.2,0.2,0.3),(0.8,0.8,1,1),(0,0,0.6,1)),((-1,-1,-1,-0.2),(-0.6,-0.6,0,0),(-1,-1,-0.5,-0.3))}
A11	{((0.2,0.5,0.5,0.9),(0.5,0.5,1,1),(0.1,0.1,0.1,0.1)),((-0.3,-0.3,-0.1,-0.1),(-0.5,-0.2,-0.2,0),(-1,-1,-0.4,-0.4))}	{((0.2,0.5,0.5,0.9),(0.5,0.5,1,1),(0.1,0.1,0.1,0.1)),((-0.3,-0.3,-0.1,-0.1),(-0.5,-0.2,-0.2,0),(-1,-1,-0.4,-0.4))}	{((0.2,0.5,0.5,0.9),(0.5,0.5,1,1),(0.1,0.1,0.1,0.1)),((-0.3,-0.3,-0.1,-0.1),(-0.5,-0.2,-0.2,0),(-1,-1,-0.4,-0.4))}	{((0.1,0.2,0.2,0.3),(0.8,0.8,1,1),(0,0,0.6,1)),((-1,-1,-1,-0.2),(-0.6,-0.6,0,0),(-1,-1,-0.5,-0.3))}	{((0.2,0.3,0.5,0.6,0.7),(0.2,0.3,0.8,0.8),(0.1,0.4,0.4,0.4)),((-0.5,-0.5,-0.1,-0.1),(-0.9,-0.1,-0.1,-0.1),(-0.9,-0.2,-0.2,0),(-1,-1,-0.4,-0.4))}	{((0.2,0.5,0.5,0.9),(0.5,0.5,1,1),(0.1,0.1,0.1,0.1)),((-0.3,-0.3,-0.1,-0.1),(-0.5,-0.2,-0.2,0),(-1,-1,-0.4,-0.4))}	{((0.3,0.5,0.6,0.7),(0.2,0.3,0.8,0.8),(0.1,0.4,0.4,0.4)),((-0.5,-0.5,-0.1,-0.1),(-0.9,-0.1,-0.1,-0.1),(-0.9,-0.9,-0.2,-0.1))}	{((0.1,0.2,0.2,0.3),(0.8,0.8,1,1),(0,0,0.6,1)),((-1,-1,-1,-0.2),(-0.6,-0.6,0,0),(-1,-1,-0.5,-0.3))}	{((0.4,0.5,0.5,0.6),(0.8,0.8,1,1),(0,0,0.6,1)),((-1,-1,-0.9,-0.9),(-0.2,-0.2,-0.1,-0.1),(-1,-1,-0.4,-0.4))}

Table 3. The third BTNN matrix

	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8	C_9
A1	{((0.2,0.3,0.4,0.7),(0.4,0.4,0.5,1),(0.2,0.2,0.2,0.2)),((-0.5,-0.4,-0.3,0),(0,0,0,0),(-1,-0.6,-0.2,-0.1))}	{((0.1,0.3,0.5,0.9),(0.0,2.0,4.0,6),(1,1,1,1)),((-0.7,-0.5,-0.3,-0.1),(-1,-1,-0.8,-0.8),(-0.2,-0.1,0,0))}	{((0.1,0.2,0.2,0.3),(0.8,0.8,1,1),(0,0,0.6,1)),((-1,-1,-1,-0.2),(-0.6,-0.6,0,0),(-1,-1,-0.5,-0.3))}	{((0.4,0.5,0.5,0.6),(0.8,0.8,1,1),(0,0,0.6,1)),((-1,-1,-0.9,-0.9),(-0.2,-0.2,-0.1,-0.1),(-1,-1,-0.4,-0.4))}	{((0.1,0.5,0.6,1),(0.3,0.3,0.5,0.5),(0.0,2.0,2.0,7)),((-0.8,-0.2,-0.2,-0.1),(-1,-1,0.1,-0.1),(-0.5,-0.5,0,0))}	{((0.2,0.5,0.5,0.9),(0.5,0.5,1,1),(0.1,0.1,0.1,0.1)),((-0.3,-0.3,-0.1,-0.1),(-0.5,-0.2,-0.2,0),(-1,-1,-0.4,-0.4))}	{((0.3,0.5,0.6,0.7),(0.2,0.3,0.8,0.8),(0.1,0.4,0.4,0.4)),((-0.5,-0.5,-0.1,-0.1),(-0.9,-0.1,-0.1,-0.1),(-0.9,-0.9,-0.2,-0.1))}	{((0.1,0.2,0.2,0.3),(0.8,0.8,1,1),(0,0,0.6,1)),((-1,-1,-1,-0.2),(-0.6,-0.6,0,0),(-1,-1,-0.5,-0.3))}	{((0.1,0.3,0.5,0.9),(0.0,2.0,4.0,6),(1,1,1,1)),((-0.7,-0.5,-0.3,-0.1),(-1,-1,-0.8,-0.8),(-0.2,-0.1,0,0))}
A2	{((0.2,0.3,0.4,0.7),(0.4,0.4,0.5,1),(0.2,0.2,0.2,0.2)),((-0.5,-0.4,-0.3,0),(0,0,0,0),(-1,-0.6,-0.2,-0.1))}	{((0.3,0.5,0.6,0.7),(0.2,0.3,0.8,0.8),(0.1,0.4,0.4,0.4)),((-0.5,-0.5,-0.1,-0.1),(-0.9,-0.1,-0.1,-0.1),(-0.9,-0.9,-0.2,-0.1))}	{((0.2,0.5,0.5,0.9),(0.5,0.5,1,1),(0.1,0.1,0.1,0.1)),((-0.3,-0.3,-0.1,-0.1),(-0.5,-0.2,-0.2,0),(-1,-1,-0.4,-0.4))}	{((0.1,0.5,0.6,1),(0.3,0.3,0.5,0.5),(0.0,2.0,2.0,7)),((-0.8,-0.2,-0.2,-0.1),(-1,-1,0.1,-0.1),(-0.5,-0.5,0,0))}	{((0.4,0.5,0.5,0.6),(0.8,0.8,1,1),(0,0,0.6,1)),((-1,-1,-0.9,-0.9),(-0.2,-0.2,-0.1,-0.1),(-1,-1,-0.4,-0.4))}	{((0.1,0.2,0.2,0.3),(0.8,0.8,1,1),(0,0,0.6,1)),((-1,-1,-1,-0.2),(-0.6,-0.6,0,0),(-1,-1,-0.5,-0.3))}	{((0.2,0.3,0.4,0.7),(0.4,0.4,0.5,1),(0.2,0.2,0.2,0.2)),((-0.5,-0.4,-0.3,0),(0,0,0,0),(-1,-0.6,-0.2,-0.1))}	{((0.2,0.3,0.4,0.7),(0.4,0.4,0.5,1),(0.2,0.2,0.2,0.2)),((-0.5,-0.4,-0.3,0),(0,0,0,0),(-1,-0.6,-0.2,-0.1))}	{((0.4,0.5,0.5,0.6),(0.8,0.8,1,1),(0,0,0.6,1)),((-1,-1,-0.9,-0.9),(-0.2,-0.2,-0.1,-0.1),(-1,-1,-0.4,-0.4))}
A3	{((0.1,0.3,0.5,0.9),(0.0,2.0,4.0,6),(1,1,1,1)),((-0.7,-0.5,-0.3,-0.1),(-1,-1,-0.8,-0.8),(-0.2,-0.1,0,0))}	{((0.1,0.2,0.2,0.3),(0.8,0.8,1,1),(0,0,0.6,1)),((-1,-1,-1,-0.2),(-0.6,-0.6,0,0),(-1,-1,-0.5,-0.3))}	{((0.4,0.5,0.5,0.6),(0.8,0.8,1,1),(0,0,0.6,1)),((-1,-1,-0.9,-0.9),(-0.2,-0.2,-0.1,-0.1),(-1,-1,-0.4,-0.4))}	{((0.1,0.5,0.6,1),(0.3,0.3,0.5,0.5),(0.0,2.0,2.0,7)),((-0.8,-0.2,-0.2,-0.1),(-1,-1,0.1,-0.1),(-0.5,-0.5,0,0))}	{((0.2,0.5,0.5,0.9),(0.5,0.5,1,1),(0.1,0.1,0.1,0.1)),((-0.3,-0.3,-0.1,-0.1),(-0.5,-0.2,-0.2,0),(-1,-1,-0.4,-0.4))}	{((0.1,0.3,0.5,0.9),(0.0,2.0,4.0,6),(1,1,1,1)),((-0.7,-0.5,-0.3,-0.1),(-1,-1,-0.8,-0.8),(-0.2,-0.1,0,0))}	{((0.1,0.3,0.5,0.9),(0.0,2.0,4.0,6),(1,1,1,1)),((-0.7,-0.5,-0.3,-0.1),(-1,-1,-0.8,-0.8),(-0.2,-0.1,0,0))}	{((0.1,0.5,0.6,1),(0.3,0.3,0.5,0.5),(0.0,2.0,2.0,7)),((-0.8,-0.2,-0.2,-0.1),(-1,-1,0.1,-0.1),(-0.5,-0.5,0,0))}	{((0.1,0.5,0.6,1),(0.3,0.3,0.5,0.5),(0.0,2.0,2.0,7)),((-0.8,-0.2,-0.2,-0.1),(-1,-1,0.1,-0.1),(-0.5,-0.5,0,0))}
A4	{((0.4,0.5,0.5,0.6),(0.8,0.8,1,1),(0,0,0.6,1)),((-1,-1,-0.9,-0.9),(-0.2,-0.2,-0.1,-0.1),(-1,-1,-0.4,-0.4))}	{((0.1,0.2,0.2,0.3),(0.8,0.8,1,1),(0,0,0.6,1)),((-1,-1,-1,-0.2),(-0.6,-0.6,0,0),(-1,-1,-0.5,-0.3))}	{((0.1,0.3,0.5,0.9),(0.0,2.0,4.0,6),(1,1,1,1)),((-0.7,-0.5,-0.3,-0.1),(-1,-1,-0.8,-0.8),(-0.2,-0.1,0,0))}	{((0.2,0.3,0.4,0.7),(0.4,0.4,0.5,1),(0.2,0.2,0.2,0.2)),((-0.5,-0.4,-0.3,0),(0,0,0,0),(-1,-0.6,-0.2,-0.1))}	{((0.3,0.5,0.6,0.7),(0.2,0.3,0.8,0.8),(0.1,0.4,0.4,0.4)),((-0.5,-0.5,-0.1,-0.1),(-0.9,-0.1,-0.1,-0.1),(-0.9,-0.9,-0.2,-0.1))}	{((0.2,0.3,0.4,0.7),(0.4,0.4,0.5,1),(0.2,0.2,0.2,0.2)),((-0.5,-0.4,-0.3,0),(0,0,0,0),(-1,-0.6,-0.2,-0.1))}	{((0.1,0.2,0.2,0.3),(0.8,0.8,1,1),(0,0,0.6,1)),((-1,-1,-1,-0.2),(-0.6,-0.6,0,0),(-1,-1,-0.5,-0.3))}	{((0.2,0.5,0.5,0.9),(0.5,0.5,1,1),(0.1,0.1,0.1,0.1)),((-0.3,-0.3,-0.1,-0.1),(-0.5,-0.2,-0.2,0),(-1,-1,-0.4,-0.4))}	{((0.4,0.5,0.5,0.6),(0.8,0.8,1,1),(0,0,0.6,1)),((-1,-1,-0.9,-0.9),(-0.2,-0.2,-0.1,-0.1),(-1,-1,-0.4,-0.4))}
A5	{((0.1,0.5,0.6,1),(0.3,0.3,0.5,0.5),(0.0,2.0,2.0,7)),((-0.8,-0.2,-0.2,-0.1),(-1,-1,0.1,-0.1),(-0.5,-0.5,0,0))}	{((0.4,0.5,0.5,0.6),(0.8,0.8,1,1),(0,0,0.6,1)),((-1,-1,-0.9,-0.9),(-0.2,-0.2,-0.1,-0.1),(-1,-1,-0.4,-0.4))}	{((0.1,0.2,0.2,0.3),(0.8,0.8,1,1),(0,0,0.6,1)),((-1,-1,-1,-0.2),(-0.6,-0.6,0,0),(-1,-1,-0.5,-0.3))}	{((0.1,0.3,0.5,0.9),(0,0.2,0.4,0.6),(1,1,1,1)),((-0.7,-0.5,-0.3,-0.1),(-1,-1,-0.8,-0.8),(-0.2,-0.1,0,0))}	{((0.2,0.3,0.4,0.7),(0.4,0.4,0.5,1),(0.2,0.2,0.2,0.2)),((-0.5,-0.4,-0.3,0),(0,0,0,0),(-1,-0.6,-0.2,-0.1))}	{((0.3,0.5,0.6,0.7),(0.2,0.3,0.8,0.8),(0.1,0.4,0.4,0.4)),((-0.5,-0.5,-0.1,-0.1),(-0.9,-0.1,-0.1,-0.1),(-0.9,-0.9,-0.2,-0.1))}	{((0.4,0.5,0.5,0.6),(0.8,0.8,1,1),(0,0,0.6,1)),((-1,-1,-0.9,-0.9),(-0.2,-0.2,-0.1,-0.1),(-1,-1,-0.4,-0.4))}	{((0.3,0.5,0.6,0.7),(0.2,0.3,0.8,0.8),(0.1,0.4,0.4,0.4)),((-0.5,-0.5,-0.1,-0.1),(-0.9,-0.1,-0.1,-0.1),(-0.9,-0.9,-0.2,-0.1))}	{((0.2,0.3,0.4,0.7),(0.4,0.4,0.5,1),(0.2,0.2,0.2,0.2)),((-0.5,-0.4,-0.3,0),(0,0,0,0),(-1,-0.6,-0.2,-0.1))}
A6	{((0.1,0.5,0.6,1),(0.3,0.3,0.5,0.5),(0.0,2.0,2.0,7)),((-0.8,-0.2,-0.2,-0.1),(-1,-1,0.1,-0.1),(-0.5,-0.5,0,0))}	{((0.1,0.5,0.6,1),(0.3,0.3,0.5,0.5),(0.0,2.0,2.0,7)),((-0.8,-0.2,-0.2,-0.1),(-1,-1,0.1,-0.1),(-0.5,-0.5,0,0))}	{((0.4,0.5,0.5,0.6),(0.8,0.8,1,1),(0,0,0.6,1)),((-1,-1,-0.9,-0.9),(-0.2,-0.2,-0.1,-0.1),(-1,-1,-0.4,-0.4))}	{((0.1,0.5,0.6,1),(0.3,0.3,0.5,0.5),(0.0,2.0,2.0,7)),((-0.8,-0.2,-0.2,-0.1),(-1,-1,0.1,-0.1),(-0.5,-0.5,0,0))}	{((0.3,0.5,0.6,0.7),(0.2,0.3,0.8,0.8),(0.1,0.4,0.4,0.4)),((-0.5,-0.5,-0.1,-0.1),(-0.9,-0.1,-0.1,-0.1),(-0.9,-0.2,-0.2,0),(-1,-1,-0.4,-0.4))}	{((0.2,0.5,0.5,0.9),(0.5,0.5,1,1),(0.1,0.1,0.1,0.1)),((-0.3,-0.3,-0.1,-0.1),(-0.5,-0.2,-0.2,0),(-1,-1,-0.4,-0.4))}	{((0.1,0.5,0.6,1),(0.3,0.3,0.5,0.5),(0.0,2.0,2.0,7)),((-0.8,-0.2,-0.2,-0.1),(-1,-1,-0.1,-0),(-0.5,-0.5,0,0))}	{((0.2,0.5,0.5,0.9),(0.5,0.5,1,1),(0.1,0.1,0.1,0.1)),((-0.3,-0.3,-0.1,-0.1),(-0.5,-0.2,-0.2,0),(-1,-1,-0.4,-0.4))}	{((0.3,0.5,0.6,0.7),(0.2,0.3,0.8,0.8),(0.1,0.4,0.4,0.4)),((-0.5,-0.5,-0.1,-0.1),(-0.9,-0.1,-0.1,-0.1),(-0.9,-0.9,-0.2,-0.1))}
A7	{((0.2,0.5,0.5,0.9),(0.5,0.5,1,1),(0.1,0.1,0.1,0.1)),((-0.3,-0.3,-0.1,-0.1),(-0.5,-	{((0.2,0.5,0.5,0.9),(0.5,0.5,1,1),(0.1,0.1,0.1,0.1)),((-0.3,-0.3,-0.1,-0.1),(-0.5,-	{((0.1,0.2,0.2,0.3),(0.8,0.8,1,1),(0,0,0.6,1)),((-1,-1,-1,-0.2),(-0.6,-	{((0.2,0.5,0.5,0.9),(0.5,0.5,1,1),(0.1,0.1,0.1,0.1)),((-0.3,-0.3,-0.1,-0.1),(-0.5,-	{((0.2,0.5,0.5,0.9),(0.5,0.5,1,1),(0.1,0.1,0.1,0.1)),((-0.3,-0.3,-0.1,-0.1),(-0.5,-	{((0.1,0.5,0.6,1),(0.3,0.3,0.5,0.5),(0.0,2.0,2.0,7)),((-0.8,-0.2,-0.2,-0.1),(-1,-	{((0.2,0.5,0.5,0.9),(0.5,0.5,1,1),(0.1,0.1,0.1,0.1)),((-0.3,-0.3,-0.1,-0.1),(-0.5,-	{((0.1,0.5,0.6,1),(0.3,0.3,0.5,0.5),(0.0,2.0,2.0,7)),((-0.8,-0.2,-0.2,-0.1),(-1,-	{((0.4,0.5,0.5,0.6),(0.8,0.8,1,1),(0,0,0.6,1)),((-1,-1,-0.9,-0.9),(-0.2,-0.2,-0.1,-

	0.2,-0.2,0),(-1,-1,-0.4,-0.4)}}	0.2,-0.2,0),(-1,-1,-0.4,-0.4)}}	0.6,0,0),(-1,-1,-0.5,-0.3)}}	0.2,-0.2,0),(-1,-1,-0.4,-0.4)}}	0.2,-0.2,0),(-1,-1,-0.4,-0.4)}}	0.1,-0.1,0),(-0.5,-0.5,0,0)}}	0.2,-0.2,0),(-1,-1,-0.4,-0.4)}}	0.1,-0.1,0),(-0.5,-0.5,0,0)}}	0.1),(-1,-1,-0.4,-0.4)}}
A8	{((0.2,0.5,0.5,0.9),(0.5,0.5,1,1),(0.1,0.1,0.1,0.1)),((-0.3,-0.3,-0.1,-0.1),(-0.5,-0.2,-0.2,0),(-1,-1,-0.4,-0.4))}	{((0.3,0.5,0.6,0.7),(0.2,0.3,0.8,0.8),(0.1,0.4,0.4,0.4)),((-0.5,-0.5,-0.1,-0.1),(-0.9,-0.1,-0.1,-0.1),(-0.9,-0.9,-0.2,-0.1))}	{((0.1,0.3,0.5,0.9),(0,0.2,0.4,0.6),(1,1,1,1)),((-0.7,-0.5,-0.3,-0.1),(-1,-1,-0.8,-0.8),(-0.2,-0.1,0,0))}	{((0.3,0.5,0.6,0.7),(0.2,0.3,0.8,0.8),(0.1,0.4,0.4,0.4)),((-0.5,-0.5,-0.1,-0.1),(-0.9,-0.1,-0.1,-0.1),(-0.9,-0.9,-0.2,-0.1))}	{((0.1,0.5,0.6,1),(0.3,0.3,0.5,0.5),(0,0.2,0.2,0.7)),((-0.8,-0.2,-0.2,-0.1),(-1,-0.1,-0.1,0),(-0.5,-0.5,0,0))}	{((0.4,0.5,0.5,0.6),(0.8,0.8,1,1),(0,0,0.6,1)),((-1,-1,-0.9,-0.9),(-0.2,-0.2,-0.1,-0.1),(-1,-1,-0.4,-0.4))}	{((0.3,0.5,0.6,0.7),(0.2,0.3,0.8,0.8),(0.1,0.4,0.4,0.4)),((-0.5,-0.5,-0.1,-0.1),(-0.9,-0.1,-0.1,-0.1),(-0.9,-0.9,-0.2,-0.1))}	{((0.1,0.3,0.5,0.9),(0,0.2,0.4,0.6),(1,1,1,1)),((-0.7,-0.5,-0.3,-0.1),(-1,-1,-0.8,-0.8),(-0.2,-0.1,0,0))}	{((0.4,0.5,0.5,0.6),(0.8,0.8,1,1),(0,0,0.6,1)),((-1,-1,-0.9,-0.9),(-0.2,-0.2,-0.1,-0.1),(-1,-1,-0.4,-0.4))}
A9	{((0.3,0.5,0.6,0.7),(0.2,0.3,0.8,0.8),(0.1,0.4,0.4,0.4)),((-0.5,-0.5,-0.1,-0.1),(-0.9,-0.1,-0.1,-0.1),(-0.9,-0.9,-0.2,-0.1))}	{((0.2,0.3,0.4,0.7),(0.4,0.4,0.5,1),(0.2,0.2,0.2,0.2)),((-0.5,-0.4,-0.3,0),(0,0,0,0),(-1,-0.6,-0.2,-0.1))}	{((0.2,0.3,0.4,0.7),(0.4,0.4,0.5,1),(0.2,0.2,0.2,0.2)),((-0.5,-0.4,-0.3,0),(0,0,0,0),(-1,-0.6,-0.2,-0.1))}	{((0.2,0.3,0.4,0.7),(0.4,0.4,0.5,1),(0.2,0.2,0.2,0.2)),((-0.5,-0.4,-0.3,0),(0,0,0,0),(-1,-0.6,-0.2,-0.1))}	{((0.4,0.5,0.5,0.6),(0.8,0.8,1,1),(0,0,0.6,1)),((-1,-1,-0.9,-0.9),(-0.2,-0.2,-0.1,-0.1),(-1,-1,-0.4,-0.4))}	{((0.1,0.2,0.2,0.3),(0.8,0.8,1,1),(0,0,0.6,1)),((-1,-1,-1,-0.2),(-0.6,-0.6,0,0),(-1,-1,-0.5,-0.3))}	{((0.2,0.3,0.4,0.7),(0.4,0.4,0.5,1),(0.2,0.2,0.2,0.2)),((-0.5,-0.4,-0.3,0),(0,0,0,0),(-1,-0.6,-0.2,-0.1))}	{((0.3,0.5,0.6,0.7),(0.2,0.3,0.8,0.8),(0.1,0.4,0.4,0.4)),((-0.5,-0.5,-0.1,-0.1),(-0.9,-0.1,-0.1,-0.1),(-0.9,-0.9,-0.2,-0.1))}	{((0.2,0.3,0.4,0.7),(0.4,0.4,0.5,1),(0.2,0.2,0.2,0.2)),((-0.5,-0.4,-0.3,0),(0,0,0,0),(-1,-0.6,-0.2,-0.1))}
A10	{((0.3,0.5,0.6,0.7),(0.2,0.3,0.8,0.8),(0.1,0.4,0.4,0.4)),((-0.5,-0.5,-0.1,-0.1),(-0.9,-0.1,-0.1,-0.1),(-0.9,-0.9,-0.2,-0.1))}	{((0.1,0.3,0.5,0.9),(0,0.2,0.4,0.6),(1,1,1,1)),((-0.7,-0.5,-0.3,-0.1),(-1,-1,-0.8,-0.8),(-0.2,-0.1,0,0))}	{((0.3,0.5,0.6,0.7),(0.2,0.3,0.8,0.8),(0.1,0.4,0.4,0.4)),((-0.5,-0.5,-0.1,-0.1),(-0.9,-0.1,-0.1,-0.1),(-0.9,-0.9,-0.2,-0.1))}	{((0.1,0.3,0.5,0.9),(0.0,0.2,0.4,0.6),(1,1,1,1)),((-0.7,-0.5,-0.3,-0.1),(-1,-1,-0.8,-0.8),(-0.2,-0.1,0,0))}	{((0.1,0.2,0.2,0.3),(0.8,0.8,1,1),(0,0,0.6,1)),((-1,-1,-1,-0.2),(-0.6,-0.6,0,0),(-1,-1,-0.5,-0.3))}	{((0.1,0.3,0.5,0.9),(0.0,0.2,0.4,0.6),(1,1,1,1)),((-0.7,-0.5,-0.3,-0.1),(-1,-1,-0.8,-0.8),(-0.2,-0.1,0,0))}	{((0.1,0.3,0.5,0.9),(0,0.2,0.4,0.6),(1,1,1,1)),((-0.7,-0.5,-0.3,-0.1),(-1,-1,-0.8,-0.8),(-0.2,-0.1,0,0))}	{((0.1,0.2,0.2,0.3),(0.8,0.8,1,1),(0,0,0.6,1)),((-1,-1,-1,-0.2),(-0.6,-0.6,0,0),(-1,-1,-0.5,-0.3))}	{((0.1,0.3,0.5,0.9),(0,0.2,0.4,0.6),(1,1,1,1)),((-0.7,-0.5,-0.3,-0.1),(-1,-1,-0.8,-0.8),(-0.2,-0.1,0,0))}
A11	{((0.2,0.3,0.4,0.7),(0.4,0.4,0.5,1),(0.2,0.2,0.2,0.2)),((-0.5,-0.4,-0.3,0),(0,0,0,0),(-1,-0.6,-0.2,-0.1))}	{((0.1,0.2,0.2,0.3),(0.8,0.8,1,1),(0,0,0.6,1)),((-1,-1,-1,-0.2),(-0.6,-0.6,0,0),(-1,-1,-0.5,-0.3))}	{((0.2,0.5,0.5,0.9),(0.5,0.5,1,1),(0.1,0.1,0.1,0.1)),((-0.3,-0.3,-0.1,-0.1),(-0.5,-0.2,-0.2,0),(-1,-1,-0.4,-0.4))}	{((0.1,0.2,0.2,0.3),(0.8,0.8,1,1),(0,0,0.6,1)),((-1,-1,-1,-0.2),(-0.6,-0.6,0,0),(-1,-1,-0.5,-0.3))}	{((0.1,0.3,0.5,0.9),(0,0.2,0.4,0.6),(1,1,1,1)),((-0.7,-0.5,-0.3,-0.1),(-1,-1,-0.8,-0.8),(-0.2,-0.1,0,0))}	{((0.2,0.3,0.4,0.7),(0.4,0.4,0.5,1),(0.2,0.2,0.2,0.2)),((-0.5,-0.4,-0.3,0),(0,0,0,0),(-1,-0.6,-0.2,-0.1))}	{((0.1,0.2,0.2,0.3),(0.8,0.8,1,1),(0,0,0.6,1)),((-1,-1,-1,-0.2),(-0.6,-0.6,0,0),(-1,-1,-0.5,-0.3))}	{((0.1,0.2,0.2,0.3),(0.8,0.8,1,1),(0,0,0.6,1)),((-1,-1,-1,-0.2),(-0.6,-0.6,0,0),(-1,-1,-0.5,-0.3))}	{((0.1,0.2,0.2,0.3),(0.8,0.8,1,1),(0,0,0.6,1)),((-1,-1,-1,-0.2),(-0.6,-0.6,0,0),(-1,-1,-0.5,-0.3))}

Table 4. The normalization matrix.

	C1	C2	C3	C4	C5	C6	C7	C8	C9
A1	0.833333333	0.966495	0.916667	1	0.583333	0.757576	0.824176	0.916667	0.818627
A2	0.861111111	0.773196	0.694444	0.722222	1	1	0.824176	1	0.911765
A3	0.972222222	0.881443	1	0.694444	0.796296	0.94697	0.980769	0.583333	0.617647
A4	0.85	0.938144	0.759259	0.650463	0.717593	0.590909	0.980769	0.694444	0.897059
A5	0.911111111	1	0.842593	0.650463	0.701389	0.70202	0.840659	0.657407	0.627451
A6	1	0.804124	0.898148	0.657407	0.643519	0.868687	0.901099	0.694444	0.642157
A7	0.991666667	0.969072	0.805556	0.694444	0.759259	0.79798	0.912088	0.583333	0.95098
A8	0.841666667	0.920103	0.912037	0.62037	0.833333	0.891414	0.848901	0.715278	0.867647
A9	0.772222223	0.724227	0.666667	0.694444	0.928241	0.893939	0.771978	0.643519	0.688725
A10	0.833333334	0.780928	0.752315	0.884259	0.775463	0.901515	0.901099	0.789352	0.936275
A11	0.711111111	0.938144	0.643519	0.900463	0.701389	0.765152	1	0.944444	1

Table 5. The weighted matrix.

	C1	C2	C3	C4	C5	C6	C7	C8	C9
A1	0.084642639	0.115376	0.113413	0.130666	0.051135	0.076948	0.083713	0.113413	0.090162
A2	0.087464061	0.092301	0.085919	0.09437	0.087659	0.101571	0.083713	0.123724	0.100419
A3	0.098749746	0.105223	0.123724	0.090741	0.069803	0.096185	0.099618	0.072172	0.068026
A4	0.086335492	0.111992	0.093938	0.084994	0.062904	0.060019	0.099618	0.085919	0.0988
A5	0.092542619	0.119376	0.104249	0.084994	0.061483	0.071305	0.085387	0.081337	0.069106
A6	0.101571167	0.095993	0.111122	0.085901	0.05641	0.088234	0.091526	0.085919	0.070725
A7	0.100724741	0.115684	0.099666	0.090741	0.066556	0.081052	0.092642	0.072172	0.104739
A8	0.085489066	0.109838	0.112841	0.081062	0.073049	0.090542	0.086224	0.088497	0.09556
A9	0.078435513	0.086455	0.082482	0.090741	0.081369	0.090798	0.078411	0.079618	0.075854
A10	0.084642639	0.093224	0.093079	0.115543	0.067977	0.091568	0.091526	0.097662	0.103119
A11	0.072228386	0.111992	0.079618	0.11766	0.061483	0.077717	0.101571	0.11685	0.110137

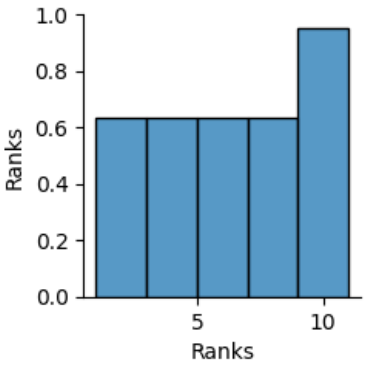

Fig 2. The rank of the alternatives.

Sensitivity analysis

Sensitivity analysis is covered in this area to examine how different criterion weights affect the outcome. The necessary sensitivity analysis findings are shown in Fig 3 and Fig. 4,5. We take into consideration the following scenarios for this purpose:

Situation I When evaluating alternatives, we consider the LMAW -based technique for determining the objective weight of criterion, which is = 1.0 in in this instance. Each option's overall assessment degree is calculated using this model. Alternative 2 is prioritized in the best and alternative 9 is ranked as the worst.

Situation 2. We increased the criteria weights by 14% and other criteria have the same weights. Then we applied the MARCOS method to rank the alternatives. We show the alternative 2 is the best and alternative 9 is the worst.

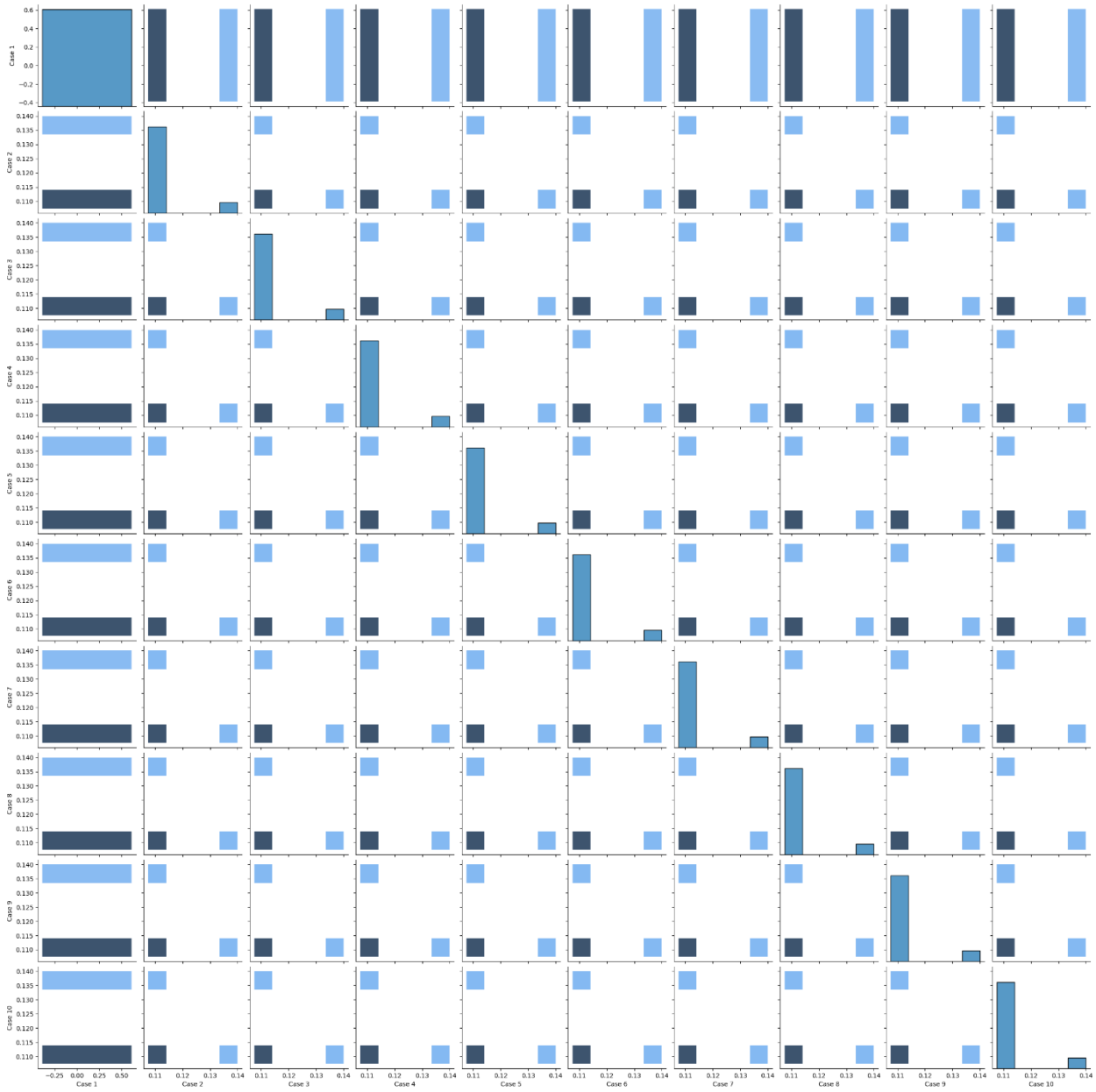

Fig 3. The criteria weights with different cases.

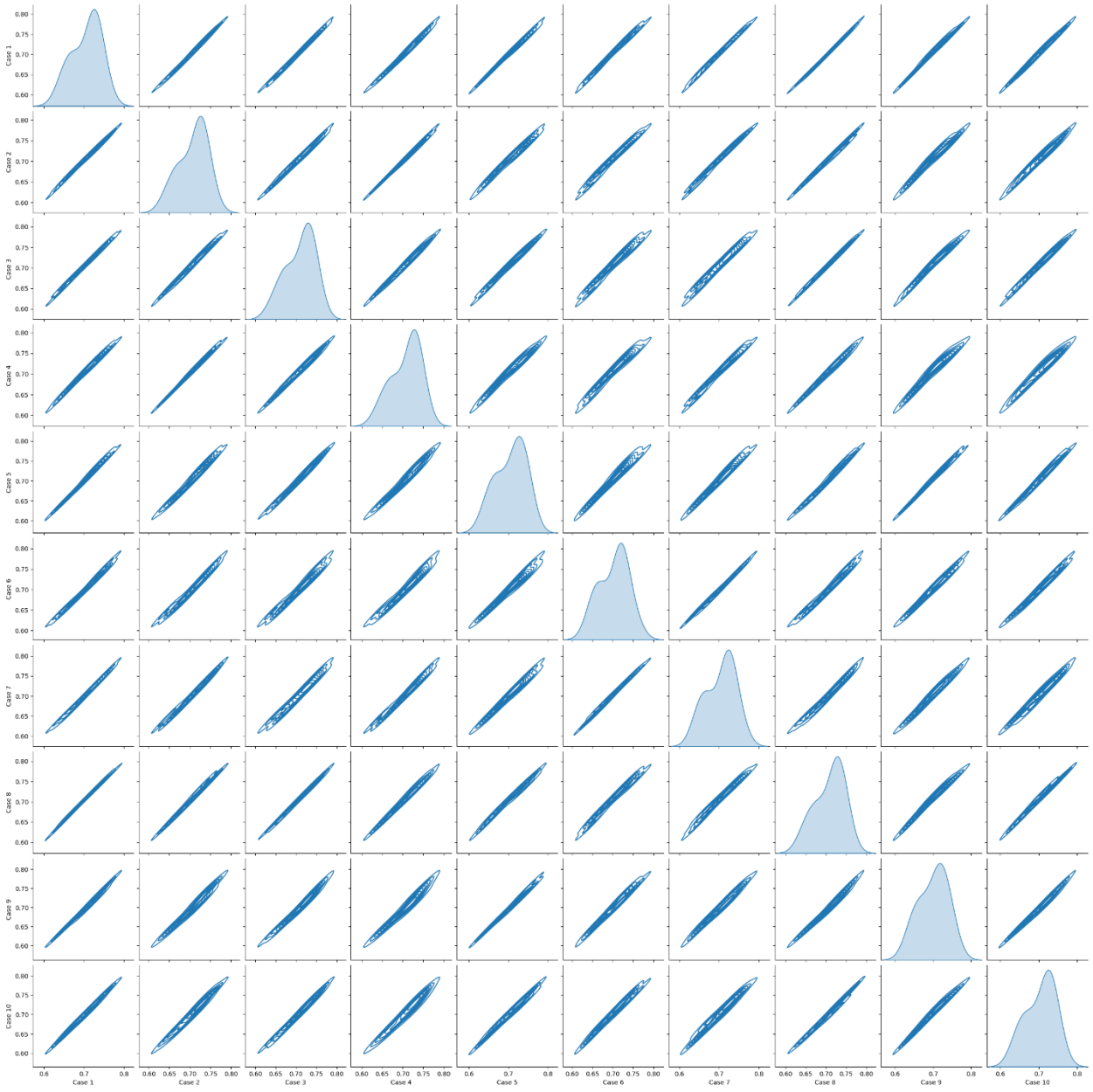

Fig 4. The results of the MARCOS method.

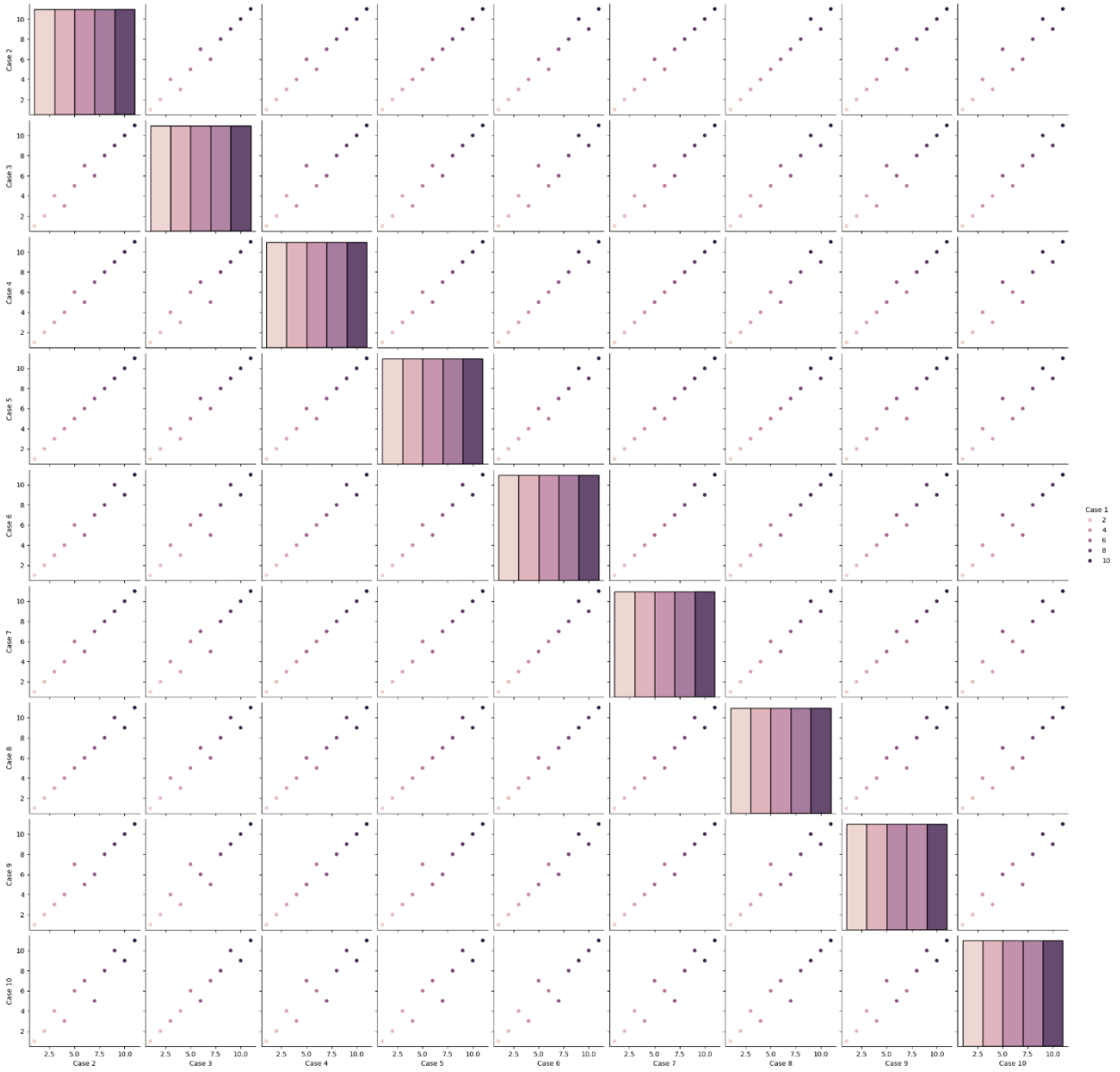

Fig 5. The rank of alternatives with different cases.

5. Conclusions

When deciding on several criteria, the neutrosophic sets and their information aggregation operators have a crucial rule. The generalized form of the neutrosophic set, the bipolar trapezoidal neutrosophic set, and the information aggregation operators under the bipolar trapezoidal neutrosophic were all covered in this study. Furthermore, we suggested two techniques to address various scenarios in the bipolar trapezoidal neutrosophic environment during MCDM operations. The benefits of these comparisons were examined when the outcomes of these algorithms were contrasted with corresponding experimental cases. The steps of the suggested algorithms were. The LMAW method is used under the neutrosophic set to compute

the criteria weights. The MARCOS method is used to rank the alternatives. The results of sensitivity analysis show alternative 2 is the best and alternative 9 is the worst.

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Solving a Global-Mixed Integer Signomial Geometric Fractional Programming Problem

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Abstract. This article addresses mixed integer fractional signomial geometric programming (MIFSGP) problems, which have been widely used in industrial design. In this paper, first, we convert fractional signomial programming into a nonfractional problem so that it maintains its geometric structure. Then, convex relaxation is used to reach a mixed integer global solution. Although, in many cases, we obtain a better objective function value with this process, designers may still be dissatisfied with the rupture between the original objective function value and the relaxed value. Therefore, we apply a spatial branch and bound algorithm to decrease that distance to an acceptable extent and maintain the global solution. Finally, a real design problem is considered to evaluate the efficiency and accuracy of the proposed technique.

Keywords: geometric programming, fractional programming, mixed integer programming, non-convex functions, spatial branch and bound algorithm.

1. **Introduction**

Fractional geometric programming (FGP) is applied to solve a class of geometric programming problems to minimize the fractional objective function under definite constraints. A few methods have been utilized in recent decades to convert a fractional signomial objective function into a nonfractional signomial objective function to reach the optimal solution via

common mathematical programming techniques. In a mixed integer fractional signomial geometric programming problem (MIFSGP), the objective function is a quotient of two signomial functions subject to certain constraints with integer and continuous decision variables; this approach is an essential part of geometric programming problems in the wide scope of engineering design, management and finance. For instance, Ray and Saini (2001) [10], Arora (1989) [3], Tsai (2005) [12], and Shirinnejad et al [11] addressed a few methods to solve real (FGP) problems in design engineering.

In a real nonfractional signomial case, the number of iterations of the solver is also significantly reduced, and an integer solution is obtained by reformulating the problem, changing negative power variables to positive power variables and applying our proposed approach. In this work, first, we define a new variable for the fractional objective function. This technique formulates an MIFGP problem by adding new constraints to a nonfractional mixed integer geometric programming (MIGP) problem.

Since the formulated problem still contains some concave terms in the objective functions or constraints and is sometimes more than before, this feature still results in local solutions to the problem. In addition, the existence of integer variables generally makes the feasible region non-convex. Therefore, applying convex relaxation in most cases will yield the lowest possible value for the objective function of the original (MIFGP) problem. Therefore, by using a spatial branch and bound algorithm (SBB), we find a feasible solution for the MIGP problems and the tightest global lower bound to the local original lower bound for the objective function simultaneously.

2. Mixed integer signomial geometric fractional programming (MISGFP) problems

A signomial function consists of a sum of positive or negative terms that are products of power functions, i.e.,

$$p(x) = f(x_1, x_2, \ldots, x_n) = \sum_{t=1}^{T} \sigma_t c_t \prod_{j=1}^{n} x_j^{\alpha_{tj}}, \tag{1}$$

Where x is a vector that contains positive variables of real or integer types. In each signomial function, T and n represent the number of terms and variables, respectively. C_t is the absolute value of the coefficients, and σ_t is the sign of the coefficient ($+1$ or -1). If $\sigma_t = +1$ in all terms of a signomial function, the function is called a posynomial. A (MISGFP) programming problem is defined in its typical form as follows:

Obtain $x = (x_1, x_2, x_3, \ldots, x_n)^T$ to

$$\begin{aligned} \text{Minimize} \qquad & f(x) = \frac{p(x)}{q(x)}, \qquad q(x) > 0, \\ \text{Subject to} \qquad & g_i(x) \leq \xi_i, \qquad \xi_i = \pm 1, \quad i = 1, 2, \ldots, m. \end{aligned} \tag{2}$$

where $p(x)$, $q(x)$ and $g_i(x)$ are signomial functions and

$$x = (x_1, x_2, x_3, \ldots, x_n)^T \in \boldsymbol{X},$$

$\boldsymbol{X}$ is a vector of real or integer positive variables x,

$$0 \leq \underline{x}_j \leq x_j \leq \bar{x}_j, \qquad j = 1, 2, \ldots, n.$$

It is assumed that the mentioned problem is feasible and has an optimal solution.

3. **Strategy of reformulation**

This article presents a convenient technique for converting a non-convex problem (MISGFP) into a convex nonfractional mixed integer signomial geometric programming problem (MISGP). Consider the following signomial geometric fractional programming problem:

$$\begin{aligned} \text{Minimize} \qquad & f(x) = \frac{p(x)}{q(x)} = \frac{\sum_{t=1}^{T_0} \sigma_{0t} c_{0t} \prod_{j=1}^{n} x_j^{\alpha_{0tj}}}{\sum_{t=1}^{T_0'} \sigma'_{0t} c'_{0t} \prod_{j=1}^{n} x_j^{\alpha'_{0tj}}}, \\ \text{Subject to} \qquad & g_i(x) \leq \xi_i, \qquad \xi_i = \mp 1, \qquad i = 1, 2, \ldots, m. \end{aligned} \tag{3}$$

where

$$g_i(x) = \sum_{t=1}^{T_i} \sigma_{it} c_{it} \prod_{j=1}^{n} x_j^{\alpha_{it}},$$

$q(x) > 0$,

for $x \in \boldsymbol{X}$ and for all $x = (x_1, x_2, x_3, \ldots, x_n)^T \in \boldsymbol{X} \subseteq \mathbb{R}, \qquad 0 \leq \underline{x}_j \leq x_j \leq \bar{x}_j$,

$j = 1, 2, \ldots, n$,

$T_0 =$ Number of terms in $p(x)$,

$T_0' =$ Number of terms in $q(x)$,

$c_{0t}, c'_{0t} \in \mathbb{R}^+$,

let $q(x) = x_{n+1}$, $x_{n+1} > 0$.
We have:

$$f(x) = x_{n+1}^{-1} * p(x),$$

To find the lower and upper bounds of x_{n+1}, we should solve the following two sub problems:

$$\begin{aligned} & L_{n+1} := \min q(x), \\ \text{Subject to} \quad & g_i(x) \leq \xi_i, \\ & \xi = \mp 1, \qquad i = 1, 2, \ldots, m, \quad \text{for } x \in \boldsymbol{X}. \end{aligned}$$

and

$$\begin{aligned} & U_{n+1} := \max q(x), \\ \text{Subject to} \quad & g_i(x) \leq \xi_i, \\ & \xi = \mp 1, \qquad i = 1, 2, \ldots, m, \quad \text{for } x \in \boldsymbol{X}. \end{aligned}$$

Therefore, problem (3) leads to the following signomial geometric programming problem, which contains a new equality constraint, $x_{n+1}^{-1} * q(x) = 1$.

$$\begin{aligned} \text{Minimize} \quad & x_{n+1}^{-1} * p(x) && (4) \\ \text{Subject to} \quad & x_{n+1}^{-1} * q(x) = 1, && (5) \\ & g_i(x) \leq \xi_i, \\ & \xi_i = \mp 1, \qquad i = 1, 2, \ldots m, \\ & L_{n+1} \leq x_{n+1} \leq U_{n+1}, \\ & x > 0, \end{aligned}$$

We can replace equation (4) with two inequalities as follows:

$$x_{n+1}^{-1} * q(x) \leq 1, \tag{6}$$

$$-x_{n+1}^{-1} * q(x) \leq -1. \tag{7}$$

Or equivalent of(7)

$$x_{n+1} * q^{-1}(x) \leq 1. \tag{8}$$

Therefore, the original MISGFP problem is reformulated to:

$$\begin{aligned} \text{Minimize} \quad & x_{n+1}^{-1} * p(x), \\ \text{Subject to} \quad & x_{n+1}^{-1} * q(x) \leq 1, \\ & x_{n+1} * q^{-1}(x) \leq 1, \\ & g_i(x) \leq \xi_i, && (9) \\ & \xi_i = \mp 1, \qquad i = 1, 2, \ldots, m, \\ & L_{n+1} \leq x_{n+1} \leq U_{n+1}, \\ & x > 0. \end{aligned}$$

Where

$$p\left(x\right)=\sum_{t=1}^{T_0}\sigma_{0t}c_{0t}\prod_{j=1}^{n}x_j^{\alpha_{0tj}},$$
$$q\left(x\right)=\sum_{t=1}^{T_0'}\sigma'_{0t}c'_{0t}\prod_{j=1}^{n}x_j^{\alpha'_{0tj}},$$
$$g_i(x)=\sum_{t=1}^{T_i}\sigma_{it}c_{it}\prod_{j=1}^{n}x_j^{\alpha_{itj}},$$
$$q(x)>0,\qquad \forall\, x=(x_1,x_2,\ldots,x_n)^T\in \boldsymbol{X}\subseteq\mathbb{R},\qquad 0\leq \underline{x}_j\leq x_j\leq \bar{x}_j,\qquad j=1,2,\ldots,n.$$

Proposition 3.1. *If* $0<q\left(x^*\right)=x_{n+1}^*$*, where* x^* *is the optimal solution of* (4)*,* x_n^* *is the optimal solution of* (3)*,*

$$\forall x\neq x^*,\qquad \left(x_{n+1}^*\right)^{-1}*p\left(x^*\right)\leq\left(x_{n+1}^*\right)^{-1}*p(x). \tag{10}$$

Proof. Suppose $x^*=(x_1^*,x_2^*,\ldots x_n^*)$ is an optimal solution of (4); consequently,

$$\left(x_{n+1}^*\right)^{-1}*q\left(x^*\right)=1,$$
$$g_i(x^*)\leq\xi_i,$$
$$\xi=\mp 1,\qquad i=1,2,\ldots,m.$$

Thus, x^*, is satisfied in the constraint of (3). Additionally, by replacing

$$q\left(x\right)=x_{n+1},$$

and

$$q\left(x^*\right)=x_{n+1}^*,$$

in the objective function (4), we have:

$$\begin{gathered}x_{n+1}^{*-1}*\sum_{t=1}^{T_0}\sigma_{0t}c_{0t}\prod_{j=1}^{n}x_j^{*\alpha_{0tj}}\leq x_{n+1}^{-1}*\sum_{t=1}^{T_0}\sigma_{0t}c_{0t}\prod_{j=1}^{n}x_j^{\alpha_{0t}}\\ \frac{\sum_{t=1}^{T_0}\sigma_{0t}c_{0t}\prod_{j=1}^{n}x_j^{*\alpha_{0tj}}}{\sum_{t=1}^{T_0'}\sigma'_{0t}c'_{0t}\prod_{j=1}^{n}x_j^{*\alpha'_{0tj}}}\leqslant\frac{\sum_{t=1}^{T_0}\sigma_{0t}c_{0t}\prod_{j=1}^{n}x_j^{\alpha_{0tj}}}{\sum_{t=1}^{T_0'}\sigma'_{0t}c'_{0t}\prod_{j=1}^{n}x_j^{\alpha'_{0tj}}}.\end{gathered}\tag{11}$$

Hence, x^* is an optimal solution of (3). □

4. Convexification strategies

Converting to convex relaxation is an efficient tool for obtaining a global solution in non-convex (MIGP) problems. It is applied to expand the feasible set of (4) and achieve a lower bound on the optimal solution of the (MIGP) problems. Since the nonlinear functions in the constraints and objective of (4) are signomial functions, each signomial function is convex if all the terms are convex [7]. The requirements of convexification for every signomial term are provided with the following theorem. See [8].

Theorem 4.1 (Maranas and Floudas (1995)). *A positive signomial term*

$$f(x) = c\prod_{i=1}^{n} x_i^{\alpha_i},$$

is convex if one of the following requirements holds:

(1) $\alpha_i \leq 0, \qquad (i = 1, \ldots, n).$

(2) $\exists\, k \neq i, \qquad \alpha_k > 0, \qquad \alpha_i \leq 0, \qquad \sum_{i=1}^{n} \alpha_i \geq 1.$

A negative signomial term

$$f\,(x) = c\prod_{i=1}^{n} x_i^{\alpha_i} (c < 0),$$

is convex if $\alpha_i > 0$ *(for* $i = 1, \ldots, n$*) and*

$$\sum_{i}^{n} \alpha_i \leq 1.$$

Theorem 4.1 states that it is possible to convexify every positive or negative signomial term by using power transformations; therefore, this fact is applied in this paper to convexify non-convex signomial functions.

After utilizing power transformations to convexify a signomial term, the power transformation functions need to be approximated by piecewise linear functions. This study used a standard piecewise linear approximation, special order set type 2 (SOS2) (Beale and Tomlin) [5].

5. Bound assessment algorithm

The spatial branch-and-bound algorithm is one of the best-known techniques for obtaining a precise or at least an ε-approximate solution to a non-convex mixed integer geometric programming problem. The lower bound of the objective function is computed for the entire feasible region. The SBB algorithm works in a common application of tight lower bounds estimated through convex relaxation. To improve the quality of our method used to determine the lower bounds of the MIFGP (2), we applied (using AMPL [6]), a streamlined "partial SBB" algorithm [1, 9], to find ε-approximate solutions for every small positive ε. This algorithm works recursively by dividing the search area along the coordinate direction that contributes most to the difference between the lower and upper bounds on the optimal objective function value determined in every subproblem. For a nonlinear minimization problem, the lower bound

is usually computed by constructing and solving a convex relaxation. The upper bound can simply be a local minimum found by an NLP solver.

At each branching step, the most reassuring node is registered, and the others are discarded. Since our purpose is to solve an MIGFP problem with integer variables, branching may be necessary for both continuous and integer decision variables.

6. **Proposed Algorithm: The partial depth-first SBB algorithm**

Input P as a non-convex MIGP problem.

Devote count while converting the non-convex problem to a convex one.

(1) Allocate o→ count and

Find x^*, through an objective function with the value S^*, which can solve P in local form.

(2) A convex relaxation R is used for the primal problem P.

(3) Find the minimum value of S to obtain the optimum solution S^* using relaxation R.

(4) Choose branching point $\grave{x}_i$ for the ith variable.

(5) We add a new constraint as follows to define P_0 as P

$$x_i^L \leq x_i \leq \grave{x}_i.$$

(6) P_1 is defined as P by adding the constraint $\grave{x}_i \leq x_i \leq x_i^U$.

(7) P_k is used as a convex relaxation for R_k, $k \in \{0, 1\}$.

(8) Similarly, we used $\grave{x}_k$ as the optimum of P_k by the S_k value, $k \in \{0, 1\}$.

(9) Let $l = (S_0, S_1)$, (the best lower bound to S^*).

(10) For $S_l > S^*$, the algorithm is stopped since the node cannot be further improved.

(11) For $S_l < S^*$, devote $S_l \to S$ and $\grave{x}_l \to \grave{x}$.

(12) In the case of the feasibility of x^* in P_l, solve local P_l to find x_i^* with S_i^*.

(13) ?For $S_i^* < S^*$, let $S_i^* \to S^*$ and $x_i^* \to x^*$, to improve the non-convex problem solution.

(14) The global optimum is achieved if $|S^* - S| < \varepsilon$. Here ε is an acceptable error.

(15) Dedicates $P_l \to P$ and $\grave{x}_l \to \grave{x}$ to update the ith branching variable and branching point$\grave{x}_i$.

(16) Finally, the count is increased, and the algorithm is repeated.

7. **Numerical example**

Example 7.1. [2] This case pertains to a design problem of a journal bearing. Its structure is an inverse problem, where the eccentricity and attitude angle are achieved for a certain load and speed. The volume of steel, the thickness of the intermediate layer and nickel barrier, and the dimensions of the plated overlay of the journal bearing are assumed to be unknown. Hence, some of these parameters of the model are known and estimated by engineers. Suppose that x_1 is the radial clearance, x_2 is the fluid force, x_3 is the diameter, x_4 the rotation speed

and x_5 is the length-to-diameter ratio. The variables of these parameters are concentrated between some specified lower and upper bounds.

The following mathematical programming formulation depicts the above mentioned problem as a signomial geometric programming problem:

$$\begin{aligned} \text{Minimize} \quad & 0.5x_1^2x_2x_4x_5 + 1.1x_1^{-1}x_2^{-1}x_3^{-1}, \\ \text{Subject to} \quad & 8.4x_1x_2^{-1}x_3^{-1}x_4^{-1}x_5 \leq 4.2, \\ & 0.5x_2x_3 + x_1 + x_4^{-1}x_5^{-1} + 1.6x_3x_4 \leq 1. \end{aligned}$$

The above problem is a concave signomial programming problem. Upon solving the problem, the local objective value was found to be $F = 3.561$. The objective function of the above problem can be easily converted into a fractional function as follows:

$$\frac{0.5x_1^3x_2^2x_3x_4x_5 + 1.1}{x_1x_2x_3},$$

Therefore, the original problem is modified by defining x_6 as a new variable where $x_6 = x_1x_2x_3$,

$$\begin{aligned} \text{Minimize} \quad & 0.5x_1^3x_2^2x_3x_4x_5x_6^{-1} + 1.1x_6^{-1}, \\ \text{Subject to} \quad & x_1x_2x_3x_6^{-1} \leq 1, \\ & x_1^{-1}x_2^{-1}x_3^{-1}x_6 \leq 1, \\ & 8.4x_1x_2^{-1}x_3^{-1}x_4^{-1}x_5 \leq 4.2, \\ & 0.5x_2x_3 + x_1 + x_4^{-1}x_5^{-1} + 1.6x_3x_4 \leq 1. \end{aligned}$$

The mentioned problem has seven non-convex terms for obtaining a global solution. Several power transformations and piecewise linear approximations are applied to underestimate convexified terms. Therefore, the non-convex primary programming problem is now converted to

the folowing convex programming problem:

$$\begin{aligned}
\text{Minimize} \quad & 0.5Z_1^6 y_2^{-2} y_3^{-1} y_4^{-1} y_5^{-1} x_6^{-1} + 1.1 x_6^{-1}, \\
& y_1^{-1} y_2^{-1} y_3^{-1} x_6^{-1} \leq 1, \\
& 8.4 k_1^5 x_2^{-1} x_3^{-1} x_4^{-1} y_5^{-1} \leq 4.2, \\
& 0.5 y_2^{-1} z_3^2 + k_1^5 x_4^{-1} x_5^{-1} + 1.6 z_3^2 y_4^{-1} \leq 1, \\
& x_1^{-1} x_2^{-1} x_3^{-1} y_6^{-1} \leq 1.
\end{aligned}$$

where

$$\begin{aligned}
& z_1 = x_1^{0.5}, Z_1 = L\left(x_1^{0.5}\right) \\
& k_1 = x_1^{0.2}, \ K_1 = L\left(x_1^{0.2}\right) \\
& z_3 = x_3^{0.5}, \ Z_3 = L\left(x_3^{0.5}\right) \\
& y_i = x_i^{-1}, \qquad Y_i = L\left(x_i^{-1}\right), \qquad i = 1,2,3,4,5,6, \\
& x_2 \in Z, \qquad x_1, x_2, x_3, x_4, x_5, x_6 \in \mathbb{R} \\
& 0.1 \leqslant x_1 \leqslant 0.5, \\
& 9 \leqslant x_2 \leqslant 17, \\
& 0.01 \leqslant x_3 \leqslant 0.21, \\
& 0.1 \leqslant x_4 \leqslant 1.7, \\
& 0.1 \leqslant x_5 \leqslant 1.8, \\
& 0.1 \leqslant x_6 \leqslant 0.5.
\end{aligned}$$

$L\left(x_1^{0.5}\right), L(x_1^{0.2}), L\left(x_3^{0.5}\right), L\left(x_i^{-1}\right)$ are the piecewise linearized expressions of the non-convex terms $x_1^{0.5}, x_1^{0.2}, x_3^{0.5}$ and convex terms x_i^{-1} for $i = 1, 2, \ldots, 6$, respectively. Since $x_2 \in Z$, we choose integer break points to have an integer solution directly. By solving this program with LINGO 18.0 software, where the tolerable error was specified as 0.001, the following solution is obtained. A comparison of the original non-convex problem and the convex problem is shown in Table 1. Although the optimum value of the convex problem is much better than that of the non-convex problem and the number of iterations is much less than that of the original non-convex problem, this inequality may not be desirable for decision makers.

TABLE 1. The optimal conciliation obtained solutions.

problem	F	x_1	x_2	x_3	x_4	x_5	iteration
Non-convex(P)	$F^* = 3.55257$	0.3317	9.7297	0.1199	0.8687	1.5275	130
Convex(R)	$\hat{F} = 2.82846$	0.5	9	0.1275	0.9	0.9	43

Therefore, the difference between the objectives of two problems is:

$$\left|F^* - \hat{F}\right| = 0.72511.$$

Suppose the maximum difference between objectives F^* and $\hat{F}$ is accepted for $\varepsilon = 10^{-2}$. This means that we should find a tighter relaxation when implementing the proposed Algorithm.

Applying the suggested Algorithm, the following steps are implemented to reach the minimum value of $\varepsilon \leq 10^{-2}$. Therefore, the global solution of the convex problem is obtained with an acceptable difference from that of the primary non-convex problem. The remaining steps are presented in Table 2 as follows:

Hint: The branching variable, objective value and punching node are abbreviated as B.V., O.V., and P.N., respectively.

TABLE 2. Table 2 shows the global solution obtained after 16 iterations with an acceptable tolerance of $\varepsilon = 0.0086$ as follows.

steps	B.V	V.v in R_0	O. $V\grave{\hat{F}}_0$	V. V in R_1	O. V$\hat{F}_1$	$\left\lvert\grave{\hat{F}}_0 - F_0^*\right\rvert$	$\left\lvert\grave{\hat{F}}_1 - F_0^*\right\rvert$	pruned node
1	x_5	0.85	1.97	1.5	2.85b	b1.58257	b0.7	R_0
2	x_5	1.5	2.875	1.52	2.895	0.67757	0.65757	R_0
3	x_2	8	2.895	10	2.82125	0.67757	0.73132	R_1
4	x_2	8	2.895	9	2.921	0.67757	0.63157	R_1
5	x_3	0.120	3.752271	0.2	2.41922	0.1997	1.1335	R_0
6	x_3	0.11	3.41	0.12	3.75227	0.14257	0.1997	R_0
7	x_3	0.105	3.85211	0.11	3.41	0.29954	0.14257	R_0
8	x_3	0.108	3.81242	0.110	3.41	0.25985	0.14257	R_0
9	x_3	0.109	3.43	0.110	3.41	0.12257	0.14257	R_1
10	x_3	0.1085	3.85	0.109	3.43	0.24743	0.12257	R_0
11	x_3	0.1088	3.44	0.109	3.43	0.11257	0.12257	R_1
12	x_4	0.7	3.44372	0.85	3.6887	0.10885	0.13613	R_1
13	x_4	0.76	3.4801	0.85	3.6887	0.07247	0.13613	R_1
14	x_4	0.756	3.54006	0.8	3.8799	0.01251	0.32733	R_1
15	x_1	0.45	3.5422	0.5	3.54006	0.01037	0.0125	R_1
16	x_1	0.43	3.54398	0.45	3.5422	0.0086	0.01037	R_0

TABLE 3. Comparison of the results obtained from the initial problem and the relaxed problem.

Problem	**O.V**	x_1	x_2	x_3	x_4	x_5
Original (P)	$F^* = 3.55257$	0.3317	9.729	0.1199	0.8687	1.5275
Relaxed (R)	$\hat{F} = 3.54398$	0.43	9	0.1088	0.706	1.52

The results of the present algorithm indicate the effectiveness of the spatial branch and bound technique. According to Table 3, we conclude that with the help of the proposed algorithm, we reach a global optimum, which is very close to the solution of the original problem and is sometimes even an integer solution to the problem of interest.

8. **Conclusion**

In this paper, a mixed-integer fractional geometric programming problem (MIFGP) as an NP-hard problem is discussed. To reach a global solution, we first converted the non-convex geometric programming problem to a convex problem to underestimate the original non-convex problem. We used the spatial branch and bound technique to increase the consistency and efficiency of the relaxation strategies. The results showed that implementing this algorithm on the convexified signomial programming problem can make the lower bound of the relaxed problem much closer to the lower bound of its non-convex programming problem.

Conflict of interest: the authors are hereby declare that there is no any conflict of interest regarding the present article.

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The potential of alliances for regional productive development from the Tutorial Centers of the University of Cartagena measured with the support of Plithogenic SWOT Analysis

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Abstract. The University of Cartagena in Colombia, with its 104 recognized research groups in various fields such as health, social sciences, engineering, basic sciences, environment, and sustainable development, expands its academic reach by establishing Tutorial Centers in Bolívar and Córdoba. The objective of these centers is to offer distance learning opportunities, thus increasing access to education for the local population. This study explores the potential of forming strategic alliances with the productive sector, taking advantage of the resources and strategic benefits of these centers. In this paper, we measure the potentialities, based on the Plithogenic SWOT Analysis. The SWOT analysis is used to measure the possible strengths, weaknesses, opportunities, and threats to the development of an organization or a project. In particular, the potentialities are calculated as the combination of strengths and opportunities. On the other hand, the variables to be measured are multiple and of different natures, which is why we use the Plithogeny theory. For the study, we used the opinion of a group of 5 experts.

Keywords: Plothogeny, Plithogenic Set, SWOT Analysis, Potentials, Plithogenic SWOT Analysis.

1 Introduction

The University of Cartagena, located in Colombia, is an academic institution that has 104 research groups recognized in the Scienti platform of the Ministry of Science, Technology, and Innovation. The research lines of these groups focus on the areas of health, social sciences, engineering, basic sciences, the environment, and sustainable development, among others.

In addition to its activities developed in the face-to-face academic programs, the university has established Tutorial Centers in the departments of Bolívar and Córdoba, to expand educational opportunities through distance education to the local population.

Given the strategic location of these tutorial centers and their potential to influence the socio-economic landscape, it is necessary to explore the possibilities of forming alliances with the productive sector. These alliances are not only beneficial for academic growth but also strategic for regional development.

Institutional guidelines and network transformation are necessary to embed applied research in academic settings, which highlights the justification for having well-defined objectives and being able to highlight the impact on the community. These centers have the potential to serve as resources for practical solutions to local problems, thus promoting regional development.

This study aims to analyze the potential of generating alliances with the productive sector from the perspective of the resources of these tutorial centers, focusing on the available resources and the strategic advantages they offer. This research is justified by the need to take advantage of the academic capacities of these centers to impact economic and social growth in the regions of departments such as Bolívar and Córdoba. This is because education and research are important axes for regional transformation.

Collaboration between academia and the productive sector is a symbiotic relationship that can enhance innovation and technological development. Survey and interview data indicate that there are efforts to strengthen these links, albeit with limitations in resources and coordination. The triple helix theory, which describes the in-

teraction between university, industry, and government, can be a useful framework for understanding and strengthening these relationships.

This collaboration benefits not only students, who gain practical experience and applied knowledge but also the productive sector, which can access new technologies and methodologies developed in academia. However, a concerted effort is needed to overcome challenges such as lack of funding and shortage of adequate infrastructure for research.

This article is dedicated to quantitatively determining the potential of the alliance between the University of Cartagena and the productive centers of the region and the country. The variables that constitute the study are of different dimensions, there are pedagogical, social, and economic dimensions, among others. In addition, these variables interact with each other in a dynamic way, where there may be indeterminations. That is why we chose the theory of Plithogeny to carry out the measurement.

Plithogeny is the theory that generalizes Neutrosophy, where the interaction between different concepts, their opposites, and their neutralities is taken into account, to form a unique dynamism more general than classical dialectics [1-4].

On the other hand, the SWOT technique is used to measure the strengths, weaknesses, opportunities, and threats that an organization or project faces [5-7]. Weaknesses and strengths are part of what affects the organization internally, while opportunities and threats affect it from the outside. The combination of Strengths + Opportunities makes up the Potentials. This method will be combined with Plithogeny to enrich and gain accuracy in the results.

The paper presents the following structure, is followed by a Materials and Methods section where the concepts of Plithogeny, SWOT and the Plithogenic SWOT analysis presented in [8] are recalled. Later, the Results section appears, where the study is carried out. We finish with the conclusions.

2 Materials and Methods

The section is dedicated to recalling the concepts of plithogenic sets, SWOT analysis, and the method proposed in [8] on Plithogenic SWOT analysis.

2.1 Plithogenic sets

Let U be a universe of discourse, and P a non-empty set of elements, $P \subseteq U$. Let A be a non-empty set of *uni-dimensional* attributes $A = \{\alpha_1, \alpha_2, \dots, \alpha_m\}$, $m \geq 1$, and $\alpha \in A$ is a given attribute whose spectrum of all the possible values (or states) is the non-empty set S, where S can be a set of finite discrete, $S = \{s_1, s_2, \dots, s_l\}$, $1 \leq l < \infty$, or infinitely denumerable set $S = \{s_1, s_2, \dots, s_\infty\}$, or an infinitely uncountable set (continuous), $S =]a, b[$, $a < b$, where] ... [is an open, semi-open or a closed interval set of real numbers or another set, [1-4, 9-13].

Let V be a non-empty subset of S, where V is the range of all attribute values needed by experts for the application. Each element $x \in P$ is characterized by the values of all attributes in $V = \{v_1, v_2, \dots, v_n\}$, for $n \geq 1$.

In the set of values of the attribute V, in general, there is a dominant attribute value, which is determined by experts in its application. The dominant attribute value means the most important attribute value that experts are interested in.

Each attribute value $v \in V$ has a corresponding degree of appurtenance $d(x, v)$ of the element x, to the set P, concerning some given criteria.

The degree of appurtenance can be either a fuzzy degree of appurtenance, an intuitionistic fuzzy degree of appurtenance, or a neutrosophic degree of appurtenance to the plithogenic set.

Thus, the *attribute value appurtenance degree function* is:

$$\forall x \in P, d: P \times V \to P\ ([0,1]^z) \qquad (1)$$

Such that $d(x, v)$ is a subset of $[0, 1]^z$, where $\mathcal{P}([0, 1]^z)$ is the power set of $[0, 1]^z$, where z = 1 (fuzzy degree of appurtenance), z = 2 (intuitionistic fuzzy degree of appurtenance), or z = 3 (neutrosophic degree of appurtenance).

Let $|V| \geq 1$ be the cardinal. Let $c: V \times V \to [0, 1]$ be the *attribute value contradiction degree function* between any two attribute values v_1 and v_2, denoted by $c(v_1, v_2)$, and satisfying the following axioms:

1. $c(v_1, v_1) = 0$, the degree of contradiction between the same attribute values is zero;
2. $c(v_1, v_2) = c(v_2, v_1)$, commutativity.

We can define a fuzzy attribute value contradiction degree function (c as before, we denote by c_F to distinguish it from the following two), an intuitive fuzzy attribute value contradiction degree function ($c_{IF}: V \times V \to$

$[0,1]^2$), or more generally, a neutrosophic attribute value contradiction degree function ($c_N: V \times V \to [0,1]^3$), the latter one can be used to increase the complexity of the calculation, but also to increase the accuracy.

We mainly calculate the degree of contradiction between the values of uni-dimensional attributes. For multi-dimensional attribute values, we divide them into their corresponding one-dimensional attribute values.

The attribute value contradiction degree function helps the plithogenic aggregation operators and the plithogenic inclusion (partial order) to obtain a more accurate result.

The attribute value contradiction degree function is designed in each field where a plithogenic set is used according to the application to be solved. If ignored, aggregations still work, but the result may lose accuracy.

So (P, a, V, d, c) is called a *plithogenic set* [1-4]:

1. Where "P" is a set, "a" is an attribute (multi-dimensional in general), "V" is the range of values of the attribute, "d" is the degree of appurtenance of the attribute value of each element x to the set P for some given criteria ($x \in P$), and "d" means "d_F" or "d_{IF}" or "d_N", when it is a degree of fuzzy membership, an intuitionistic fuzzy membership, or a degree of neutrosophic membership, respectively, of an element x to the plithogenic set P;
2. "c" means "c_F" or "c_{IF}" or "c_N", when it is a fuzzy attribute value contradiction degree function, intuitionistic fuzzy attribute value contradiction degree function, or neutrosophic attribute value contradiction degree function, respectively.

Functions $d(\cdot,\cdot)$ and $c(\cdot,\cdot)$ are defined according to the applications that experts need to solve.

Then, the following notation is used:

$x(d(x,V))$, where $d(x,V) = \{d(x,v), \text{for all } v \in V\}, \forall x \in P$.

The attribute value contradiction degree function is calculated between each attribute value with respect to the dominant attribute value (denoted by v_D) in particular, and with respect to other attribute values as well.

The attribute value contradiction degree function c evaluated between the values of two attributes is used in the definition of plithogenic aggregation operators (intersection (AND), union (OR), implication ($\Longrightarrow$), equivalence ($\Longleftrightarrow$), inclusion (partial order), and other plithogenic aggregation operators that combine two or more degrees of values of the attribute based on a t-norm and a t-conorm.

Most plithogenic aggregation operators are linear combinations of one fuzzy t-norm (denoted by $\wedge_F$) with one fuzzy t-conorm (denoted by $\vee_F$), but nonlinear combinations can also be constructed.

If the t-norm is applied over the dominant attribute value denoted by v_D, and the contradiction between v_D and v_2 is $c(v_D, v_2)$, then v_2 is applied over the attribute value as follows:

$$[1 - c(v_D, v_2)] \cdot t_{norm}(v_D, v_2) + c(v_D, v_2) \cdot t_{conorm}(v_D, v_2) \quad (2),$$

Or, by using symbols:

$$[1 - c(v_D, v_2)] \cdot (v_D \wedge_F v_2) + c(v_D, v_2) \cdot (v_D \vee_F v_2) \quad (3).$$

Similarly, if the t-conorm is applied on the dominant attribute value denoted by v_D, and the contradiction between v_D and v_2 is $c(v_D, v_2)$, then on the attribute value v_2 it is applied:

$$[1 - c(v_D, v_2)] \cdot t_{conorm}(v_D, v_2) + c(v_D, v_2) \cdot t_{norm}(v_D, v_2) \quad (4).$$

Or, by using symbols:

$$[1 - c(v_D, v_2)] \cdot (v_D \vee_F v_2) + c(v_D, v_2) \cdot (v_D \wedge_F v_2) \quad (5).$$

The *Plithogenic Neutrosophic Intersection* is defined as:

$$(a_1, a_2, a_3) \wedge_P (b_1, b_2, b_3) = \left(a_1 \wedge_F b_1, \frac{1}{2}[(a_2 \wedge_F b_2) + (a_2 \vee_F b_2)], a_3 \vee_F b_3\right) \quad (6),$$

The *Plithogenic Neutrosophic Union* is defined as:

$$(a_1, a_2, a_3) \vee_P (b_1, b_2, b_3) = \left(a_1 \vee_F b_1, \frac{1}{2}[(a_2 \wedge_F b_2) + (a_2 \vee_F b_2)], a_3 \wedge_F b_3\right) \quad (7),$$

In other words, with regard to what applies to membership, the opposite applies to non-membership, while in indeterminacy the average between them applies.

The *Plithogenic Neutrosophic Inclusion* is defined as follows:

Since the degrees of contradiction are $c(a_1, a_2) = c(a_2, a_3) = c(b_1, b_2) = c(b_2, b_3) = 0.5$, applies: $a_2 \geq [1 - c(a_1, a_2)]b_2$ or $a_2 \geq (1 - 0.5)b_2$ or $a_2 \geq 0.5b_2$ while $c(a_1, a_3) = c(b_1, b_3) = 1$.

Having $a_1 \leq b_1$ the opposite is done for $a_3 \geq b_3$, hence $(a_1, a_2, a_3) \leq_P (b_1, b_2, b_3)$ if and only if $a_1 \leq b_1$, $a_2 \geq 0.5b_2$, and $a_3 \geq b_3$.

2.2 SWOT Analysis

The SWOT analysis is a methodology for studying the situation of a company or a project, analyzing its internal characteristics (weaknesses and strengths) and its external situation (threats and opportunities) in a squared matrix, [5-7].

The analysis consists of four steps:

- External analysis,
- Internal analysis,
- Preparation of the SWOT matrix,
- Determination of the strategy to be used.

The organization does not exist and cannot exist outside an environment, the environment that surrounds it; So external analysis allows us to identify the opportunities and threats that the context may present to an organization. These are the components of external analysis.

Opportunities are those positive factors that are generated in the environment and that, once identified, can be taken advantage of. Threats are negative situations, external to the program or project, that can attack it, so when appropriate, it may be necessary to design the right strategy to be able to circumvent them.

The internal elements to be analyzed during the SWOT analysis correspond to the strengths and weaknesses that have to do with the availability of capital resources, personnel, assets, product quality, internal and market structure, and consumer perception, among others.

The internal analysis allows for fixing the strengths and weaknesses of the organization, carrying out a study that allows knowing the quantity and quality of the resources and processes with which the organization has.

Strengths are all those internal and positive elements that differentiate the program or project from others of the same kind. Weaknesses relate to all those elements, energy resources, skills, and assets that the organization already has and that constitute barriers to the good performance of the organization. Weaknesses are internal problems that once they are identified and after developing an appropriate strategy, can and should be eliminated.

The four elements of the analysis are located in a matrix and evaluated by experts. These results are aggregated using the percentages of the assessments.

From the combination of strengths and opportunities arise the potentialities, which indicate the most promising lines of action for the organization. Limitations, determined by a combination of weaknesses and threats, pose a serious warning. Risks (combination of strengths and threats) and challenges (combination of weaknesses and opportunities), determined by their corresponding combination of factors, will require careful consideration when setting the course that the organization must take toward the desirable future.

2.3 Plithogenic SWOT Analysis

The method proposed in [8] uses the following:

To calculate the results of the aggregation by the 4 quadrants, viz. Strength-Opportunity (SO), Strength-Threat (ST), Weakness-Opportunity (WO), and Weakness-Threat (WT), formulas 8, 9, 10 and 11 are used, respectively. This article does not differentiate any dominant aspect of these formulas [14-16].

$$\mathrm{SO} = \bigwedge_{p_i} \bigwedge_{p_j} \left(\left(\omega_{o_i} \bigwedge_p \omega_{s_j} \right) \bigwedge_p E_{o_i s_j} \right) \qquad (8),$$

Where $\bigwedge_p$ is the Plithogenic Neutrosophic Intersection based on the minimum t-norm and the maximum t-conorm (see Equation 6), ω_{o_i} is the weight given to the ith opportunity, ω_{s_j} is the weight given to the jth strength, $E_{o_i s_j}$ is the evaluation assigned to the intersection of the ith opportunity with the jth strength.

$$\mathrm{ST} = \bigwedge_{p_i} \bigwedge_{p_j} \left(\left(\omega_{T_i} \bigwedge_p \omega_{s_j} \right) \bigwedge_p E_{T_i s_j} \right) \qquad (9),$$

ω_{T_i} is the weight given to the ith threat, and $E_{T_i s_j}$ is the assessment assigned to the intersection between the ith threat with the jth strength.

$$\mathrm{WO} = \bigwedge_{p_i} \bigwedge_{p_j} \left(\left(\omega_{w_i} \bigwedge_p \omega_{o_j} \right) \bigwedge_p E_{w_i o_j} \right) \qquad (10),$$

ω_{w_i} is the weight given to the ith weakness, and $E_{w_i o_j}$ is the evaluation assigned to the intersection of the ith weakness with the jth opportunity.

$$\mathrm{WT} = \bigwedge_{p_i} \bigwedge_{p_j} \left(\left(\omega_{w_i} \bigwedge_p \omega_{T_j} \right) \bigwedge_p E_{w_i T_j} \right) \qquad (11).$$

See also [17-20].

3 Results

The method proposed in [8] uses some measurement scales, which are shown in Tables 1, 2.

Language expression	Plithogenic number (T, I, F)
Very poor (VP)	(0.10, 0.75, 0.85)
Poor (P)	(0.25, 0.60, 0.80)
Medium poor (MP)	(0.40, 0.70, 0.50)
Medium (M)	(0.50, 0.40, 0.60)
Medium good (MG)	(0.65, 0.30, 0.45)
Good (G)	(0.80, 0.10, 0.30)
Very good (VG)	(0.95, 0.05, 0.05)

Table 1: Linguistic values associated with plithogenic numbers for the evaluation according to the criteria. Source: [8].

Language expression	Plithogenic number (T, I, F)
Low significance (LS)	(0.10, 0.70, 0.80)
Equal significance (ES)	(0.30, 0.40, 0.80)
Robust significance (RS)	(0.50, 0.40, 0.60)
Very robust significance (VRS)	(0.70, 0.30, 0.10)
Absolute significance (AS)	(0.90, 0.10, 0.10)

Table 2: Linguistic values associated with plithogenic numbers for the evaluation of the weight of the criteria. Source: [8].

Five specialists were hired, who gave their opinions based on the linguistic measurement scales shown in Tables 1 and 2. The Strengths, Weaknesses, Opportunities, and Threats shown below were identified:

The opportunities are as follows:

O1- The time is right for the alliance to be established, as society in general and the State will welcome it. There is a regional need for development, so any positive experience is welcome.

O2- There are experiences in other regions of the world and the country that can serve as a basis for a local integration program.

The threats identified were:

T1- The tangible results of any program implemented may not be visible in less than a year, which may discourage local government and society in general from collaborating with them.

T2- There may be a lack of acceptance, indifference, or ignorance on the part of the general population to realize the advantages they are receiving through this means and this can lead to apathy and little recognition. Especially when popular language is not used for social dissemination or, what is worse, when there is no such dissemination at all.

Weaknesses were:

W1. The situation of teacher-researchers is complex in terms of their ability to transfer knowledge and interact with other actors.

W2. The tutorial centers must not only be involved in education and training but must also connect with the community and production centers. This is something they are not used to.

W3. For their part, production centers may not be accustomed to the language and methods of academia and therefore there may be a certain degree of incomprehension in the integration of academia and production.

W4. There may be a lack of financial and technological resources, insufficient motivation among students and teachers, and a lack of a clear definition of lines of research.

W5. An institutional policy aimed at strengthening the research infrastructure is required.

W6. It is crucial to foster a more robust research culture among students and teachers so that academic activities transcend beyond pedagogical exercises and become rigorous research that generates new knowledge.

The strengths are:

S1. Academia can support production in the incorporation of new technologies that help them gain greater efficiency and effectiveness in products and services.

S2. The productive system can help universities as a provider of material and financial resources in supporting teaching work.
S3. The productive system can practically complement the knowledge that students receive in the academy.
S4. There is the possibility of creating a solid academic community and participation in national and international research networks.

In summary, each of the specialists gave an opinion on the aspects of the scales shown in Table 1, comparing the internal criteria (S and W) with the external ones (O and T). They evaluated the weight or importance of each criterion on the scale shown in Table 2. The median of the results for all the specialists was then found, obtaining a single value of weight and evaluation for all the criteria, aggregated as the collective result of all the specialists.

The results are shown in Tables 3 and 4.

	Opportunities			Threats	
	No.	O1	O2	T1	T2
Strengths	S1	(0.95, 0.05, 0.05)	(0.80, 0.10, 0.30)	(0.50, 0.40, 0.60)	(0.40, 0.70, 0.50)
	S2	(0.95, 0.05, 0.05)	(0.95, 0.05, 0.05)	(0.50, 0.40, 0.60)	(0.40, 0.70, 0.50)
	S3	(0.80, 0.10, 0.30)	(0.95, 0.05, 0.05)	(0.25, 0.60, 0.80)	(0.40, 0.70, 0.50)
	S4	(0.95, 0.05, 0.05)	(0.80, 0.10, 0.30)	(0.25, 0.60, 0.80)	(0.25, 0.60, 0.80)
Weaknesses	W1	(0.80, 0.10, 0.30)	(0.80, 0.10, 0.30)	(0.50, 0.40, 0.60)	(0.40, 0.70, 0.50)
	W2	(0.95, 0.05, 0.05)	(0.80, 0.10, 0.30)	(0.80, 0.10, 0.30)	(0.95, 0.05, 0.05)
	W3	(0.80, 0.10, 0.30)	(0.95, 0.05, 0.05)	(0.50, 0.40, 0.60)	(0.40, 0.70, 0.50)
	W4	(0.95, 0.05, 0.05)	(0.80, 0.10, 0.30)	(0.25, 0.60, 0.80)	(0.25, 0.60, 0.80)
	W5	(0.95, 0.05, 0.05)	(0.80, 0.10, 0.30)	(0.50, 0.40, 0.60)	(0.40, 0.70, 0.50)
	W6	(0.95, 0.05, 0.05)	(0.95, 0.05, 0.05)	(0.25, 0.60, 0.80)	(0.25, 0.60, 0.80)

Table 3: SWOT matrix of the aspects measured in the form of plithogenic numbers. Source: the authors.

Aspect to measure	Aggregated weight by the specialists
O1	(0.90, 0.10, 0.10)
O2	(0.90, 0.10, 0.10)
T1	(0.90, 0.10, 0.10)
T2	(0.90, 0.10, 0.10)
S1	(0.90, 0.10, 0.10)
S2	(0.90, 0.10, 0.10)
S3	(0.90, 0.10, 0.10)
S4	(0.90, 0.10, 0.10)
W1	(0.90, 0.10, 0.10)
W2	(0.90, 0.10, 0.10)
W3	(0.90, 0.10, 0.10)
W4	(0.90, 0.10, 0.10)
W5	(0.90, 0.10, 0.10)
W6	(0.90, 0.10, 0.10)

Table 4: Neutrosophic numbers obtained from calculating the median of the weights evaluated by the specialists. Source: the authors.

The results achieved were the following, using Equations 8-11:

$\mathrm{SO} = (0.80, 0.08928, 0.30)$, $\mathrm{ST} = (0.25, 0.35586, 0.80)$, $\mathrm{WO} = (0.80, 0.07908, 0.30)$, $\mathrm{WT} = (0.25, 0.34919, 0.80)$.

The qualitative evaluation of the results obtained is obtained using the distance formula 12, to compare them with each of the elements in Table 1 and assign the linguistic value corresponding to the smallest distance:

$$d(Q, TV_v) = \sqrt{(Q - TV_v) \cdot (Q - TV_v)'} \qquad (12)$$

Where Q denotes the plithogenic number evaluating the quadrant, TV_v is the plithogenic number of the vth member of Table 1, $T(\cdot)$, $I(\cdot)$, and $F(\cdot)$ are their truth, indeterminate, and falseness values, respectively, and $'$ is the matrix transpose operator.

Thus, the SO quadrant (potentials) is rated as “Good” with a distance of 0.29418; the ST quadrant (risks) is rated as “Poor” with a distance of 0.24413, the WO quadrant (challenges) is rated as “Good” with a distance equal to 0.02092, and the WT quadrant (limitations) is rated as “Poor” with a distance equal to 0.25081.

Conclusion

This paper was dedicated to studying the potential of using the Tutorial Centers of the University of Cartagena de Indias in Colombia to create cooperation between productive centers and academia to contribute to the economic and social development of the region. To achieve this, we relied on the opinion of five experts, who evaluated the situation of the Strengths, Weaknesses, Opportunities, and Threats to carry out any integration program in this sense. The evaluations were carried out with the help of a Plithogenic SWOT analysis. This allowed us to take into account the plurality of variables involved, the indeterminacy, and uncertainty to measure the main aspects that influence the success of the projects. The main result is that the potential is good, and the challenges have good prospects of being overcome. As for the risks and limitations, they have low ratings; therefore we must work on them, looking for tactics and strategies to avoid failure due to these aspects.

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Enhancing Medical X-Ray Image Classification with Neutrosophic Set Theory and Advanced Deep Learning Models

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Abstract

The classification of medical images presents significant challenges due to the presence of noise, uncertainty, and indeterminate information. Traditional deep learning models often struggle to manage this, leading to reduced diagnostic accuracy, especially when dealing with low-quality or ambiguous conditions. This paper proposes a hybrid approach that integrates Neutrosophic Set (NS) theory with deep learning models to enhance X-ray image classification under uncertain conditions. NS theory introduces three domains: True (T), Indeterminate (I), and False (F) to manage image uncertainty and noise, allowing deep learning models to better interpret complex, ambiguous visual information. To evaluate the approach, five state-of-the-art deep learning models—MobileNet, ResNet50, VGG16, DenseNet121, and InceptionV3 are utilized, and their performance was evaluated on two different medical image datasets: Cervical spine injuries detection and chest disease classification. The results indicate that models trained on NS-transformed data, particularly DenseNet and MobileNet, yield superior outcomes compared to those trained on the original data, achieving significantly higher accuracy, precision, and recall. This demonstrates that incorporating NS theory into deep learning models significantly enhances their ability to classify uncertain and noisy X-ray images, providing a robust solution for improving diagnostic accuracy in medical imaging.

Keywords: Neutrosophic Set (NS), Image Classification, X-ray Imaging, Deep Learning, Cervical spine, Chest Diseases.

1. Introduction

Medical imaging plays a pivotal role in diagnosing and managing diseases, especially in critical areas such as chest diseases and Cervical spinal injuries. X-rays are the most widely used due to their cost-effectiveness and rapid results, particularly in the detection of chest diseases and bone fractures [1]. However,

despite their ubiquity, the interpretation of X-ray images presents significant challenges for both clinicians and automated systems. Artificial Intelligence, particularly deep learning models have shown remarkable success in image classification tasks, offering state-of-the-art performance in a range of medical applications [2]. Despite the power of AI, medical images, especially X-rays, often exhibit ambiguity and incomplete information due to varying factors such as exposure settings, image noise, and patient positioning inconsistencies can hide critical details and introduce uncertainty [3]. Addressing these uncertainties is essential for building more robust and reliable classification models Which lead to more accurate diagnosis process [4].

Deep learning has revolutionized medical image analysis, with convolutional neural networks (CNNs) being the most prominent technique used [5]. CNNs such as MobileNet, ResNet, VGG16, DenseNet121, and Inception offer powerful feature extraction capabilities, enabling them to capture intricate patterns in medical images [6, 7]. These models have been widely adopted for tasks ranging from disease detection to segmentation. However, despite their impressive performance in image classification, most existing methods lack mechanisms to effectively manage the uncertainty and noise inherent in medical imaging data, which can lead to suboptimal diagnostic accuracy [8]. Deep learning models such as CNNs, while powerful in many contexts, struggle when dealing with noisy or ambiguous data. Additionally, the inherent complexity of human anatomy makes it difficult to distinguish between subtle abnormalities in cases such as spinal fractures or chest diseases [9]. These gaps underscore the need for models capable of managing uncertainty while maintaining high classification accuracy.

Neutrosophic Set (NS) theory offers a promising approach to handle uncertainty in image classification. NS theory was introduced by Florentin Smarandache in 1995 and unified 1999 [10]. It extends classical binary logic by introducing three domains: truth (T), falsity (F), and indeterminacy (I) [11]. These domains allow for the representation of incomplete, inconsistent, and uncertain information, which is often present in medical X-rays. By converting pixel values into these three domains, NS theory helps to emphasize the certainty of certain image features while minimizing the impact of uncertain or noisy areas. Neutrosophic Sets operate by assigning three membership values (T, I, F) to each element within a set. Each value lies in the interval [0, 1], representing the degrees to which the element belongs to the truth, indeterminacy, and falsity domains [12]. This tripartite model is uniquely advantageous in scenarios where information is incomplete or conflicting, such as medical imaging [13, 14]. For instance, in an X-ray of a fracture, some regions may be clearly indicative of a fracture (high truth), some may not indicate a fracture (high falsity), while other regions, perhaps due to poor image quality or ambiguous features may fall into an indeterminate zone (high indeterminacy).

By explicitly modeling the areas of uncertainty, NS provides a more refined approach to interpreting medical images. This framework has been applied in various image processing tasks, such as segmentation and classification, particularly in the medical field [14, 15]. In the context of X-ray image classification, few studies have integrated uncertainty management techniques, such as NS theory, into the preprocessing or training pipeline of deep learning models, leaving a significant gap in leveraging uncertainty-aware methods for medical X-ray analysis. In our approach, the NS framework is applied to X-ray images, aiding in the classification task by enhancing images feature representation and improving robustness in uncertain scenarios. Thus, integrating uncertainty management methods, such as NS theory, into deep learning pipelines could potentially improve performance in challenging medical scenarios.

In this paper, we propose a hybrid approach that integrates NS theory with deep learning models to enhance the classification of medical X-ray images. Our methodology leverages NS theory during the preprocessing phase to address image uncertainty, such as noise and low contrast, and transform the images into three NS images domains (T, I, F), which may improve the interpretability and robustness of deep learning models. To demonstrate the efficacy of this approach, we conduct case studies on two distinct datasets cervical spine fracture and dislocation, and chest diseases, which show the framework's ability to generalize across diverse medical imaging tasks. To evaluate the approach, five state-of-the-art deep learning models were used including MobileNet, ResNet50, VGG16, DenseNet121, and InceptionV3. The Experimental results indicate that models trained on NS-transformed data, particularly DenseNet121 and MobileNet, demonstrate superior performance compared to those trained on the original data in terms of accuracy, precision, and recall. The key contributions of this work include:

a. Introducing an NS-based pre-processing step to manage uncertainty in X-ray medical images, making it easier for deep learning models to extract meaningful patterns.
b. Demonstrating the impact of NS theory in improving classification performance across five state-of-the-art models: MobileNet, ResNet50, VGG16, DenseNet121, and InceptionV3.
c. Evaluating the effectiveness of our approach on two distinct medical datasets (spinal fracture X-rays and chest disease images)

This hybrid methodology addresses limitations in existing deep learning approaches by offering a more reliable solution for medical image classification in uncertain environments. Our results underscore the potential of NS theory as a transformative tool for handling uncertainty in x-rays images, to be more understanded for DL models, which make this a interesting area for further investigation.

The remainder of this paper is structured as follows: Section 2 provides a detailed overview of the related work in the areas of Neutrosophic Sets and deep learning for medical image classification. Section 3 describes the methodology used in this study, including the NS-based preprocessing technique, and the architecture of the deep learning models (MobileNet, ResNet, VGG, DenseNet, Inception). In Section 4, we present the experimental setup, the datasets, and evaluation metrics. Section 5 includes the results, the comparisons of the models, and a detailed discussion of the results. Finally, Section 6 concludes the paper with a summary of findings and suggestions for future work.

2. Related Work

In this section, we will review the application of Neutrosophic Set Theory and deep learning techniques in the classification of medical images, particularly focusing on X-rays images. The review will cover various approaches used to enhance accuracy, highlight key methodologies and utilize models that have been employed in recent studies, and discuss their strengths, limitations, and the gaps that remain in achieving more accurate and reliable approaches.

Cei et al. [16] introduced a novel NS-based deep learning approach for analyzing digital mammograms, specialized X-ray images of the breast used to detect and diagnose breast cancer, specifically for the detection and classification of microcalcifications (MCs), which are critical early indicators of breast cancer. The technique utilized the membership sets of NS to map digital mammograms into three distinct domains: T, I, and F. These domains were then employed to train a convolutional neural network (CNN) model, facilitating tasks such as lesion detection and regional clustering. The proposed method achieved a sensitivity of 92.5% for detecting MC clusters, with an area under the curve (AUC) of 0.908 and 0.872 on the validation and test sets, respectively. These results underscore the effectiveness of combining NS and DL techniques in automating the detection and classification of MC clusters in digital mammography images.

Khalifa [17], produced a study to explore the impact of neutrosophic sets on deep learning models using a limited dataset of COVID-19 X-ray images. The images were transformed into the NS domain, which consists of three categories: true (T), indeterminacy (I), and false (F) images. These transformed images were then used to train various DL models includes AlexNet, GoogLeNet, and ResNet18. The performance of the models was evaluated across four domain-original images and the three NS domains—comparing results based on accuracy, precision, recall, and F1 score. The findings demonstrated that incorporating NS into DL models could significantly enhance testing accuracy, particularly in the context of limited COVID-19 datasets, highlighting its potential for improving diagnostic accuracy. The authors in [18] explored the use of chest CT scans for early detection of COVID-19, they proposed a hybrid method which integrates binary cross-entropy, transfer

learning, and deep convolutional neural networks techniques for enhanced classification accuracy. Pre-trained models like ResNet (50), VGG (19), VGG (16), and Inception V3 were applied to the DCNNs to improve the classification process. The results showed that the pre-trained models achieved accuracies of 99.07%, 98.70%, 98.55%, and 96.23%, respectively, using the Adam optimizer. Another study addressed the feature selection problem in COVID-19 detection using chest X-ray images by integrating metaheuristic algorithms with a deep learning model [19]. In this approach, the VGG19 deep network was employed for feature extraction, and a feature selection method was applied to identify the most significant features, enhancing the classification performance. The selected features were then fed into an optimized neural network, with optimization driven by a hybrid metaheuristic optimizer. The results demonstrated that the proposed method achieved an accuracy of 99.88%, showcasing its effectiveness in improving COVID-19 detection.

The study presented in [20] introduces the BoneNet-NS technique for classifying fractures in X-ray images. This approach integrates deep learning with neutrosophic set methodologies to effectively manage aleatoric uncertainty in medical imaging. The research proposes two frameworks for combining NS with DL models, named BoneNet-NS1 and BoneNet-NS2. Utilizing a dataset of 4,924 X-ray images, various DL models, including Xception, ResNet52V2, DenseNet121, and a customized CNN, were evaluated to distinguish between fractured and non-fractured classes. Statistical analyses demonstrated that BoneNet-NS2 achieved remarkable performance metrics across most DL models when working with NS image domain instead of the original images, Specifically, with the ResNet52V2 model, BoneNet-NS2 achieved an accuracy of 99.7%, a log loss of 0.006, F1-score of 99.7.

Jennifer et al. [21] employed a neutrosophic approach to differentiate between lung infection types. By classifying chest x-ray images into True (T), False (F), and Indeterminacy (I) set memberships, this approach effectively reduces fuzziness while retaining critical information for feature extraction of lung opacity. The preprocessing stage utilizes alpha-mean and beta-enhancement operations to decrease indeterminacy and enhance relevant image components. Subsequently, the enhanced neutrosophic images are classified using several deep learning models, including ResNet-50, VGG-16, and XGBoost. Experimental evaluations conducted on the ActualMed COVID-19 Chest X-ray and COVID-19 radiography datasets reveal that the enhanced neutrosophic images achieve a notable accuracy of 97.33%, surpassing the performance of other domain sets.

In [22], a deep neural network model utilizing neutrosophic features for skin cancer diagnosis was introduced. The model applied neutrosophic-based lesion segmentation to reduce noise and improve classification accuracy on PH2, ISIC 2017, ISIC 2018, and ISIC 2019 datasets. The network, built with Inception and residual blocks, achieved high accuracy rates of 99.50%, 99.33%, 98.56%, and 98.04% on the respective datasets, outperforming many existing classifiers. These results highlight

the effectiveness of combining neutrosophic techniques with deep learning for skin cancer detection.

Authors in [23] introduced a hybrid to address the need for rapid and accurate COVID-19 detection using chest X-ray images. The study utilized Neutrosophic techniques (NTs) combined with machine learning (ML) methods to create an automated tool for classifying X-rays into COVID-19 positive or negative cases. Morphological features (MFs) and principal component analysis (PCA) were employed to extract key features from the images. Compared to RT-PCR tests, which are costly and require expert personnel, chest X-rays provide a more accessible and efficient alternative for COVID-19 diagnosis. The model achieved high performance metrics, including 98.46% accuracy, 98.19% precision, 98.18% sensitivity, demonstrating its effectiveness and superiority over other diagnostic methods.

In [24], an encoder-decoder deep neural network incorporating neutrosophic set theory and indeterminacy fusion is proposed for segmenting White Blood Cells (WBCs). The model leverages indeterminacy within the NS domain to enhance the segmentation of WBCs into distinct nucleus and cytoplasm regions. This approach addresses the limitations of prior methods, which often overlook internal structures. The model surpasses three original encoder-decoder networks, achieving high precision rates and the greatest mean segmentation accuracy of 0.95301, demonstrating the effectiveness of integrating NS-based indeterminacy in improving medical image segmentation.

Another hybrid method combining neutrosophic and convolutional neural networks (NS-CNN) is proposed in [25] for the classification of brain tumors as benign or malignant. The method first segments MRI images using the neutrosophic set – expert maximum fuzzy-sure entropy (NS-EMFSE) approach. In the classification stage, features from the segmented brain images are extracted using a CNN and then classified using SVM and KNN classifiers. Based on a 5-fold cross-validation on 80 benign and 80 malignant tumor samples, the experimental results demonstrated that the CNN features performed best with the SVM classifier, achieving an average accuracy of 95.62%.

Despite notable advancements in the application of NS within DL methodologies for addressing uncertainty and noise in medical image analysis, significant gaps remain regarding robustness and accuracy. Many studies report limitations in classification performance, often relying on small datasets, which may yield unpredictable results over time. Furthermore, there is a pressing need for more comprehensive evaluations of NS combined with DL techniques in the medical field. In this context, testing the proposed approach across multiple datasets not only enhances the robustness and generalizability of the model but also provides a more holistic understanding of its performance across diverse medical imaging scenarios.

3. Methodology

The methodology is designed to combine the Neutrosophic Set (NS) theory and deep learning models for effective classification of medical X-ray images. It consists of three core components: first, the theoretical foundation and equations governing the NS domain and its application to image uncertainty handling are introduced. This is followed by an explanation of the deep learning models utilized, including MobileNet, ResNet50, VGG16, DenseNet121, and inceptionV3, highlighting their architecture and suitability for the task at hand. Finally, the proposed approach, which integrates the NS-based pre-processing with the deep learning models to address the challenges of image noise, ambiguity, and the need for accurate diagnosis through automated classification, is detailed.

3.1. Neutrosophic Set (NS) Definitions and Preliminaries

Neutrosophic Set (NS) theory extends traditional fuzzy set theory by introducing the concepts of indeterminacy, allowing for a more comprehensive representation of uncertainty in data. NS decomposes information into three distinct components: Truth (T), Indeterminacy (I), and Falsity (F). NS is particularly useful for handling uncertain, incomplete, and inconsistent data, which are common in medical imaging due to noise and low contrast in X-rays. The Key Definitions are:

- Truth (T): Represents the degree of truth or certainty in the image's pixel values.
- Indeterminacy (I): Captures the uncertainty or ambiguity present in the image.
- Falsity (F): Denotes the degree of falsehood or error in the pixel values.

- **Neutrosophic Sets (NS) Domains**

In NS, each element x is characterized by three independent degrees: truth, indeterminacy, and falsehood. Mathematically, this can be expressed as follows:

$$NS(x) = \{T(x),\ I(x),\ F(x)\} \tag{1}$$

$$0 \leq T(x), I(x), F(x) \leq 1 \tag{2}$$

$$0 \leq T(x) + I(x) + F(x) \leq 3 \tag{3}$$

where: $T(x)$ represents the degree of truth, $I(x)$ denotes the degree of indeterminacy, $F(x)$ indicates the degree of falsehood. In image transformation process to Neutrosophic Domains, each pixel of an image is preprocessed by Neutrosophic logic to calculate its T, F, I components to determine its domain. Given an X-ray image G with pixel intensities, Pixel in image is represent as $P(x, y)$ the image can be transformed into the domains using the following:

- First, to handle image noise and uncertainty, the local mean $g(x,y)$ of an image G is calculated using a convolution operation. Then Calculate absolute difference between the original image and its mean $O(x,y)$.

$$g(x,y) = \frac{1}{25}\sum_{i=-2}^{2}\sum_{j=-2}^{2} G(x+i, y+j) \tag{4}$$

$$O(x,y) = abs(G(x,y) - g(x,y)) \tag{5}$$

- Second the NS domains for the pixel $P(x,y)$ can be represented as follows:

$$P_{NS}(x,y) = \{T(x,y), I(x,y), T(x,y)\} \tag{6}$$

$$T(x,y) = \frac{g(x,y) - g_{min}}{g_{max} - g_{min}} \tag{7}$$

$$I(x,y) = \frac{O(x,y) - O_{min}}{O_{max} - O_{min}} \tag{8}$$

$$F(x,y) = 1 - T(x,y) \tag{9}$$

where $\boldsymbol{x}$ and $\boldsymbol{y}$ variables typically represent the pixel coordinates in a 2D image, $G(x,y)$ refers to the pixel value at the coordinate (x,y) in the image G, $\boldsymbol{I}$ and $\boldsymbol{j}$ are indices used to denote the neighboring pixels around a given pixel (x,y). g_{max} is the highest average pixel value within a defined neighborhood. g_{min} is the lowest average pixel value within that same neighborhood. O_{max} is the highest absolute difference observed across the neighborhood. O_{min} is the lowest absolute difference observed [25, 26]. By applying these transformations to each pixel of the image, the X-ray is decomposed into its NS components. This allows for better feature extraction in images with uncertainty, aiding in medical diagnoses where details are often unclear.

- **Entropy in Neutrosophic Sets**

Entropy is typically used to quantify the uncertainty or randomness in the image data [27]. It can be used to assess the information content of the image. Higher entropy indicates more complexity or uncertainty in the pixel values, which could influence how to define T, I, and F. and to adjust the parameters in those equations, essentially influencing how you interpret the pixel values around each G(x, y). The Entropy can be employed to evaluate the uncertainty associated with the image data. The entropy H of a neutrosophic set can be defined as:

$$Entropy\ (E) = -\sum_{k=1}^{N} P(K)\log(P(K)) \tag{10}$$

$$E_{Ns} = E_T + E_I + E_F \tag{11}$$

This equation provides a measure of the uncertainty within the neutrosophic representation of the image, where $p(k)$ is the probability of occurrence of pixel value k in the image, and E_T , E_I , and E_F are entropies for True, Indeterminacy, and False, respectively.

To adjusts the influence of the entropy of the image on the degree of truth and balance the response of the NS approach under different image conditions, additional parameters can be used such as $Alpha\,(\alpha)$ and $Beta\,(\beta)$, $Alpha\,(\alpha)$ parameter controls the weight of entropy in the classification. A higher value indicates a greater reliance on the image's information content, enhancing the representation of areas of interest in the image. While $\boldsymbol{Beta\,(\beta)}$ Acts as a complementary measure that reflects the degree of uncertainty in the classification process. It allows for a balance between certainty and uncertainty, making the model robust against noise and variations in the input data.

$$\alpha = \alpha_{min} + \frac{(\alpha_{min} - \alpha_{min})\,(E_n - E_{min})}{(E_{max} - E_{min})} \tag{12}$$

$$\beta = 1 - \alpha \tag{13}$$

where E_{max} is the maximum possible entropy for an image of size $m \times n$, E_{min}represents the minimum entropy, which occurs when the image is completely uniform (complete certainty), while α_{min} and α_{min} provide bounds for the influence of entropy on the NS framework, allowing you to control how the model reacts to uncertainty in different images.

3.2. Deep Learning Models

Deep learning has emerged as a powerful approach for solving complex problems in various fields, including medical image analysis. At the core of deep learning is the use of neural networks, particularly Convolutional Neural Networks (CNNs) [28], which have revolutionized image classification tasks by automatically learning hierarchical feature representations directly from data. CNNs are specifically designed to process grid-like data, such as images, by utilizing convolutional layers that apply filters to detect important features like edges, textures, and patterns. These features are learned through multiple layers of abstraction, allowing CNNs to capture both low-level and high-level details within the images. In the medical imaging, CNNs can automatically identify complex and subtle patterns that may not be easily visible to the human eye, making them invaluable for tasks such as disease diagnosis and fracture detection. However, training CNNs from scratch requires vast amounts of labeled data and significant computational resources. To overcome this limitation, we leverage transfer learning [29], a technique that allows pretrained models to be adapted to new domains. Transfer learning uses models that have already been trained on large datasets, such

as ImageNet, and fine-tunes them on the target medical datasets [30]. This approach not only reduces the need for large amounts of training data but also significantly accelerates the learning process.

In this study, deep learning models are trained to work within the **neutrosophic domain**, where medical images are transformed to capture true, false, and indeterminate information based on neutrosophic sets. By extracting these features, the models are equipped to handle the inherent uncertainty present in medical imaging data. CNNs are well-suited for this task as they can capture important structural features in medical images and can be adapted to work with the nuanced information provided by the neutrosophic transformation. This allows the models to focus on extracting the most relevant and informative features from noisy or ambiguous data, leading to more robust classification results.

To further enhance performance, we employed transfer learning techniques to adapt pretrained CNN models to the new medical image domains. This process allows the models to retain useful general features from their original training (on natural images) while adapting to the specific task of medical image classification. By fine-tuning the pretrained models on the medical X-ray datasets, we can extract more meaningful features specific to fractures and chest diseases. This hybrid approach of combining neutrosophic sets with deep learning models provides a powerful framework for handling complex medical images. In this study, we applied a set of well-established deep learning models with advantages in terms of computational efficiency and feature extraction capabilities to classify medical X-ray images:

- MobileNet: Is a streamlined CNN architecture designed for computational efficiency by utilizing depthwise separable convolutions [31].
- ResNet50: Introduces residual connections, allowing deeper networks by bypassing the vanishing gradient problem. These shortcut connections enable better gradient flow, making it easier to train very deep models [32].
- VGG16: Is a deep CNN model with a uniform layer structure, known for its simplicity and effectiveness in image classification [33].
- DenseNet121: provides enhanced feature propagation and reduced vanishing gradient issues, which makes it possible to retrieve complex image features more reliably [34].
- InceptionV3: This model incorporates techniques like factorized convolutions to enhance efficiency and reduce computational costs without sacrificing performance [35].

These models have been widely used for image classification tasks and have demonstrated strong performance in various medical imaging applications. They were selected for their proven ability to learn complex features from data and their potential to address the challenges of classification tasks with x-ray images. By

employing these models, we harnessed the complementary strengths of these models to classify medical X-ray images, each offering unique benefits in terms of feature extraction and computational efficiency.

3.3. The Proposed NS-DL Approach

The integration of NS with Deep Learning DL techniques presents a promising avenue for enhancing the classification of medical X-ray images. This approach aims to leverage the strengths of both methodologies, addressing uncertainties inherent in medical imaging while utilizing the powerful feature extraction capabilities of deep learning models. By combining NS with DL, it can achieve more robust classification outcomes that account for ambiguous, incomplete, or inconsistent information present in medical images. Neutrosophic Logic offers a framework for representing and managing the uncertainty associated with medical images, allowing for more comprehensive information representation. It enables deep learning to extract relevant features from images while NS addresses the vagueness and ambiguity in pixel information.

The proposed approach consists of a multi-step architecture that seamlessly integrates NS and DL for enhanced medical image classification:

1. Image Acquisition and Preprocessing: Obtain X-ray images and preprocess them to ensure uniformity in size and format.
2. NS Domains Conversion: Apply neutrosophic transformation to the images to convert them into a neutrosophic representation. This involves producing the NS domains: truth, indeterminacy, and falsity, providing a richer representation of the image data.
3. Feature Extraction Using Deep Learning: using pre-trained deep learning models (MobileNet, ResNet50, VGG16, DenseNet121, and inceptionV3) to extract essential features from the neutrosophic images.
4. Classification and Fine-Tuning: Adding a classifier that integrates the features obtained from the deep learning model with the neutrosophic representations, enhancing the model's ability to classify the images.
5. Evaluation and Comparison: Assess the performance of the integrated model against traditional deep learning models that do not utilize NS.

In our case. NS theory can be applied to represent the uncertainty and noise present in the X-rays image data by converting the image into the neutrosophic domain, where each pixel is assigned to three memberships: truth (T), indeterminacy (I), and falsity (F). following the formula:

$$T(x,y) = \mu T(x,y), I(x,y) = \mu I(x,y), F(x,y) = \mu F(x,y) \quad (14)$$

where $\mu T(x,y), \mu I(x,y)$ and $\mu F(x,y)$ are the actual calculated membership functions that map the pixel's values into the truth, indeterminacy, and falsity components respectively. These values are typically normalized between 0 and 1. These three

components represent different aspects of the pixel's characteristics. (T) represents the degree to which a pixel belongs to a region of interest (e.g., a bone or fracture). (I) captures the uncertainty or vagueness in the pixel, which may be due to noise, low contrast, or blurring in the X-ray image. (F) represents the degree to which a pixel does not belong to the region of interest. Fig.1. shows an sample of x-ray images and their corresponding T, I, and F neutrosophic image domains.

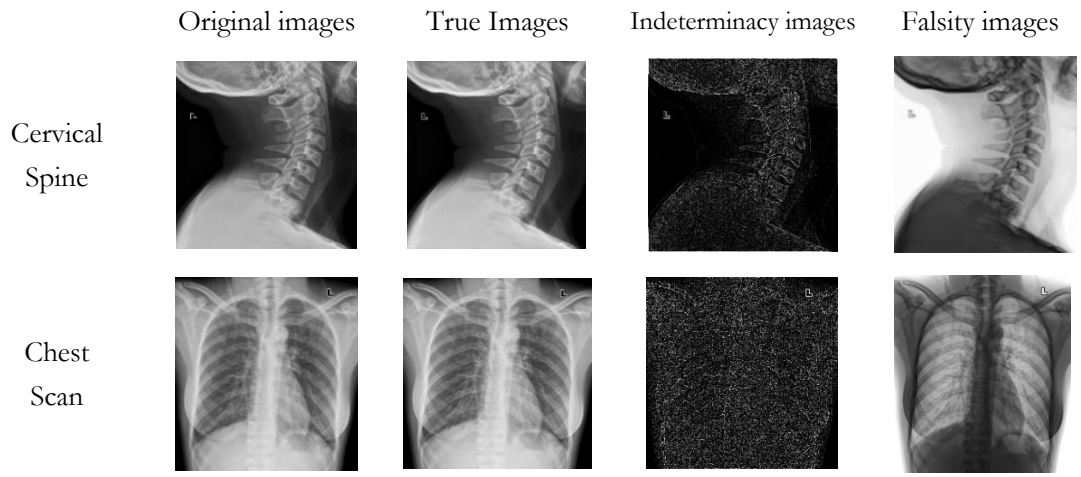


Figure 1: Sample X-ray Images with Corresponding True (T), Indeterminacy (I), and False (F) Neutrosophic Image Domains

Transfer learning technique [29] is used to optimize the performance of these models, where the pretrained versions MobileNet, ResNet50, VGG16, DenseNet121, and InceptionV3, originally trained on the ImageNet dataset, were fine-tuned for our specific medical X-ray datasets. Additionally custom dense layers were added on top of the pretrained models for classification into the target classes of the spine fracture and chest disease datasets. The models were trained using the Adam optimizer, with a categorical crossentropy loss function [36].

$$Minimize: loss(CCE) = -\sum_{i=1}^{M} y_i \cdot \log \breve{y}_i \tag{15}$$

where y_i is true valur $\breve{y}_i$ is shorthand for a vector that contains all the outputs that were predicted based on the training samples. Fine-tuning allowed the models to adapt to the specific features of the medical images while leveraging the powerful feature extraction capabilities they had already learned from ImageNet. Fig.2 show the architecture for the proposed NS-DL approach, While Algorithm.1 showing the main its main steps.

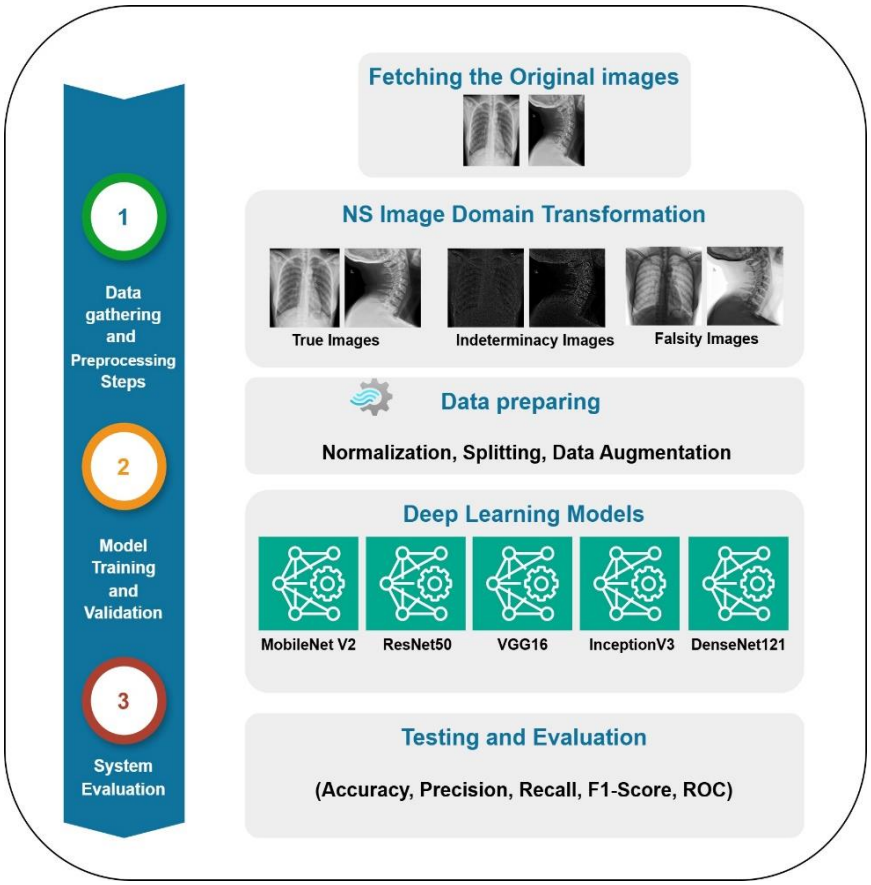


Figure 2: Architecture of the Proposed NS-DL approach

Algorithm 1: The Proposed NS-DL Approach

Input: Input_dataset(G), number of epochs (T).
Output: Model Score (S).

1. Preprocessing the Input Dataset:
   - Convert *G* to grayscale to simplify pixel intensity calculations and reduce computational complexity.
2. Pixel-Level Calculations:
   - Calculate pixel intensity *G (x, y),* which represents the value of each pixel at coordinates (x, y)
   - Calculate local mean *O min,* which helps identify localized intensity variations in the image.
3. Generate Neutrosophic Domains: Compute Truth, Indeterminacy, Falsity domains.
   - T (x, y) = (G(x, y) - G_min) / (G_max - G_min)
   - I (x, y) = abs(G(x, y) - O_min) / (O_max - O_min)
   - F (x, y) = 1 - T(x, y)
   - Normalize the resulting image output for consistent input to the DL models.
5. Model Implementation:
   - Utilize five state-of-the-art deep learning models: MobileNet, ResNet, VGG, DenseNet, and Inception

- Fine-tune each model to adapt to the specific features of the NS images.

6. Train each model separately on the three NS domains images.
   - t = 0 // for the current epoch
   - While t < T // number of epochs
     - Computing the score function
     - Check Early stopping conditions (if satisfied -> end training)
     - Updates the model's weights.

   t = t +1

7. After training, Calculate the final score *S* of the model and evaluate it.

4. Experimental Setup and Analysis

In this section, we describe the datasets, environment setup, and evaluation metrics used to assess the performance of the proposed approach. The experiments are designed to evaluate the effectiveness of integrating Neutrosophic Sets (NS) with deep learning models (MobileNet, ResNet, VGG16, DenseNet121, and Inception) for classifying X-ray images. We focus on the classification of medical X-rays with the goal of improving diagnostic accuracy through feature extraction and uncertainty management using NS.

4.1. Utilized Dataset

In this study, the proposed models were evaluated using two distinct X-ray image datasets. The first dataset focuses on Cervical Spine Fracture and Dislocation, while the second is utilized for Chest Disease classification [37]. Each dataset comprises images that have been meticulously labeled by medical experts to ensure accuracy. The Chest Disease X-ray Dataset contains images categorized into four labels: Hemothorax, Pneumothorax, Flail Chest, and Normal. In contrast, the Cervical Spine X-ray Dataset includes three labels: Fracture, Dislocation, and Normal. Both datasets consist of images in JPEG and JPG formats, each with a resolution of 256 x 256 pixels. Table 1 presents the distribution of images for each class in both datasets, while Fig.3 shows the proportional distribution of Image Classes, providing insight into the balance and representation of each category used for training and evaluation.

Table 1: Image Class Distribution for utilized X-ray Datasets

Chest Disease Dataset			Cervical Spine dataset		
Class ID	Class Name	Number of images	Class ID	Class Name	Number of images
0	Flail	525	0	Dislocation	530
1	Hemothorax	422	1	Fracture	772
2	Normal	481	2	Normal	707
3	Pneumothorax	522			

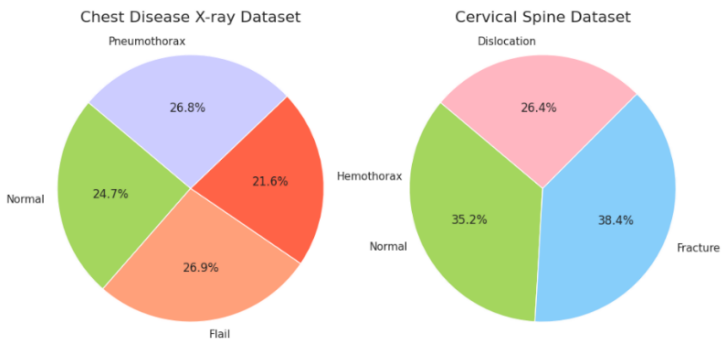


Fig 3: Percentage distributions of Image Classes in X-ray Datasets

In the image preprocessing steps. Each image is converted to grayscale to simplify the data and focus on essential features without color distractions. Following this, neutrosophic transformation is applied, generating three NS domains: True, Indeterminate, and False. This transformation aids in capturing the nuanced details of the X-ray images, facilitating better feature extraction. Pixel normalization is then performed on the transformed images. This process standardizes the pixel values, ensuring they fall within a specific range, it helps accelerate the training process and improves the model's convergence by preventing issues related to varying scales of pixel values. Furthermore, Data augmentation techniques are applied next to enhance the diversity of the training dataset and mitigate the risk of overfitting [38]. Common augmentation strategies include rotation, flipping, zooming, and shifting of images. These techniques allow the model to learn more robust features and improve its generalization to unseen data. Finally, the datasets are split into training, validation, and test sets with a ratio of 70%, 15%, and 15%, respectively. The validation set plays a critical role in monitoring the model's performance during training, allowing for early stopping and hyperparameter tuning without overfitting to the training data. This systematic approach ensures that the model is well-prepared to achieve optimal performance on unseen test data.

4.2. Environment Setup and Hyperparameters Tuning

All experiments are conducted in a computational environment with the following specifications:

- Hardware: NVIDIA Tesla P100 GPU with 16GB of memory, 30GB RAM, and an Intel Xeon processor.
- Software: The experiments are implemented using Python 3.10, TensorFlow 2.15, Karas API 3.3. The system runs on a Kaggle environment.

For deep learning models, the hyperparameters are carefully tuned to optimize performance, all utilized models (MobileNet, ResNet, VGG16, DenseNet121, and Inception) were Initialized with pretrained ImageNet weights, a learning rate of

0.001, batch size of 32, and an Adam optimizer [39]. The model is fine-tuned for 100 epochs with early stopping technique [40]. To fine-tune the pretrained models, we incorporated custom dense layers on top to classify the target categories in both the spine fracture and chest disease datasets. This fine-tuning process enabled the models to adjust to the unique characteristics of the medical images while still benefiting from the robust feature extraction capabilities previously acquired from ImageNet [28]. The specific configurations of the new layers and their hyperparameters for are presented in Table2.

Tabel 2: The configurations of the new layers and hyperparameters settings

Layer	Parameter	Value
Dense layer_1	Number of units	128
	Activation function	Rule
Dense layer_2	Number of units	64
	Activation function	Rule
Dense layer_3	Number of units	N_classes
	Activation function	softmax
	Optimizer	Adam
	Learning rate	0.001
	Epochs	100
	Early stopping	Monitor=loss, Patience=10

4.3. Evaluation Metrices

To evaluate the performance of the proposed approach across the cervical spine fracture and chest disease classification tasks, a comprehensive set of evaluation metrics was employed: (Accuracy, Precision, Recall, F1-Score) These metrics provide insight into the model's effectiveness in identifying and categorizing X-ray images, The mathematical formulas for the utilization of evaluation metrices can be defined as follows:

$$\text{Accuercy} = (TP + TN)/\, total\_predictions \tag{16}$$

$$\text{Precision} = TP/\,(TP + FP) \tag{17}$$

$$\text{Recall} = TP/\,(TP + FN) \tag{18}$$

$$\text{F1} - \text{score} = 2 * (precision * recall)/\,(precision + recall) \tag{19}$$

where TP is the correctly classified positive cases, TN is the correctly classified negative cases, FP is the incorrectly classified positive cases, and FN is the incorrectly classified negative cases. These four metrices allow us to assess the performance of different models from various perspectives, $Accuracy$ measures the overall correctness of the model by calculating the ratio of correctly classified images to the total number of images, $Precision$ evaluates the model's ability to correctly

identify positive instances, providing a focus on the relevance of the true positive predictions. High precision indicates that the model makes fewer false positive errors, which is crucial in medical diagnosis to avoid misidentifying normal cases as diseased, $Recall$, also known as sensitivity or true positive rate, reflects the model's capability to retrieve all relevant instances. A higher recall demonstrates the model's ability to correctly detect diseased cases, making it an essential metric in the context of medical imaging, where missed diagnoses could have severe consequences, and $F1\ score$ combines precision and recall into a single metric, acting as their harmonic mean. This score is particularly useful when there is an imbalance in the dataset, as it provides a more nuanced evaluation by considering both false positives and false negatives.

Receiver Operating Characteristic (ROC) curves and the associated Area Under the Curve (AUC) are also used to offer a graphical representation of the model's ability to differentiate between the positive and negative classes [41]. The ROC curve plots the true positive rate against the false positive rate, providing insight into the trade-off between sensitivity and specificity at various classification thresholds. A higher AUC value suggests that the model has a strong ability to distinguish between classes, which is critical for clinical decision-making.

$$AUC = \int_0^1 TPR\ (FPR) \quad d(FPR) \tag{20}$$

where TPR is the true positive Rate, and FPR is the false positive rate. The Area Under the ROC Curve (AUC-ROC) gives a single scalar value summarizing the performance of the classifier, where the perfect classifier has an AUC-ROC of 1. This collection of metrics provides a comprehensive evaluation of the model's performance, covering both overall accuracy and the balance between false positives and false negatives, and ensures that the evaluation process thoroughly captures the strengths and limitations of the models, it is particularly important in the medical domain, where misclassification can have significant consequences.

5. Results and Discussion

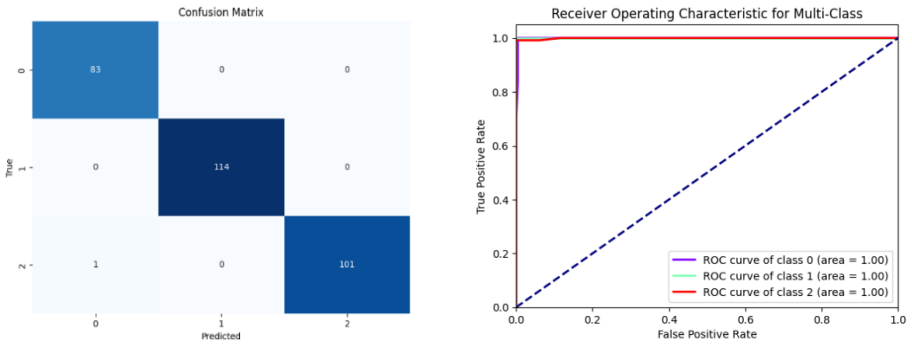
In this section, we present the comparative performance of deep learning models trained on both the original and NS transformed data. The models were evaluated on two medical imaging datasets: spinal injury detection and chest disease classification. The results comparison focuses on the impact of NS image transformation, the classification performance and the. The primary evaluation metrics used include accuracy, precision, recall, and F1-score. Results are broken down by the original dataset and the NS-transformed data, specifically the True (T), Indeterminacy (I), and Falsity (F) domains of the NS theory. We highlight the models' improvements in uncertainty handling, leading to better classification accuracy and overall performance.

5.1. Results in Cervical Spine dataset

The results for the spinal injury detection dataset demonstrate the effectiveness of applying NS transformations to the image data. As shown in Table 3, on the original dataset, MobileNet demonstrated the highest overall accuracy of 99.00%, with F1-score value of 98.96%. Upon applying NS transformations, significant enhancements were observed. In the True (T) domain, both InceptionV3 and DenseNet121 attained notable accuracies of 99.67%, accompanied by F1-scores of 99.64% and 99.69%, respectively. The confusion matrices and ROC curves for InceptionV3 and DenseNet121 are presented in Fig 4 and Fig 5, respectively. Furthermore, in the Indeterminacy (I) and Falsity (F) domains, both DenseNet121 and MobileNet exhibited remarkable improvements, each achieving an accuracy of 99.67% with balanced performance metrics. These results indicate that NS transformations significantly bolster the performance of deep learning models in spinal injury detection, improving their accuracy and robustness across various conditions.

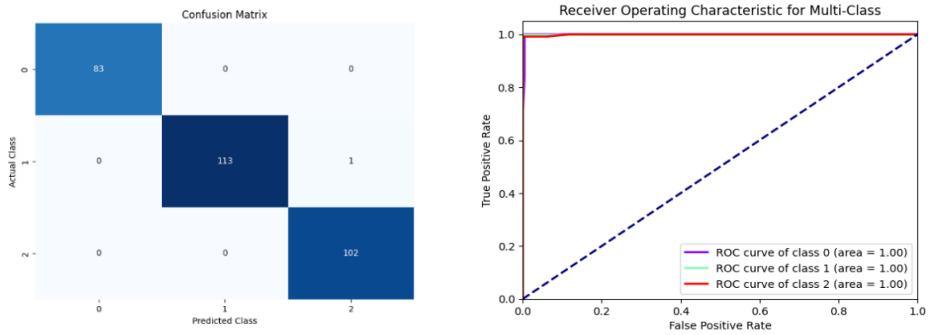
Table 3: Performance Metrics of DL Models on Original and NS Domains C-Spinal Injury Detection Datasets

Data	Model	Accuracy	Precision	Recall	F1-Score
Original Data	MobileNet	**99.00**	**98.81**	**99.15**	**98.96**
	ResNet50	95.99	95.72	96.15	96.91
	VGG16	98.33	98.21	98.32	98.26
	InceptionV3	96.99	97.05	97.21	97.11
	DenseNet121	**99.00**	**99.81**	**99.10**	**99.94**
True (T) NS domain Data	MobileNet	99.33	99.18	99.31	99.23
	ResNet50	97.32	97.05	87.31	97.17
	VGG16	98.66	98.63	98.25	98.43
	InceptionV3	**99.67**	**99.60**	**99.67**	**99.64**
	DenseNet121	**99.67**	**99.68**	**99.71**	**99.69**
Indeterminacy (I) NS domain Data	MobileNet	**99.33**	**99.17**	**99.38**	**99.26**
	ResNet50	96.99	96.71	96.89	96.69
	VGG16	99.33	99.05	99.41	99.22
	InceptionV3	98.33	97.81	98.56	98.13
	DenseNet121	**99.67**	**99.60**	**99.67**	**99.64**
Falsity (F) NS domain data	MobileNet	**99.67**	**99.72**	**99.57**	**99.64**
	ResNet50	96.99	96.65	97.04	96.83
	VGG16	98.33	97.98	98.13	98.05
	InceptionV3	98.33	98.26	97.75	97.98
	DenseNet121	**99.33**	**99.33**	**99.31**	**99.31**



(A) (B)

Figure 4: Performance Analysis of the Inception Model on True (T) Domain Images: (a) Confusion Matrix; (b) ROC Curve.



(A) (B)

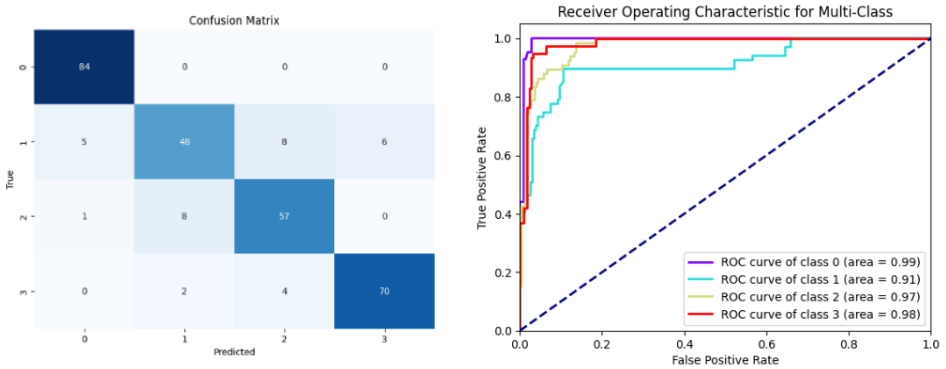
Figure 5: Performance Analysis of the DenseNet Model on True (T) Domain Images: (a) Confusion Matrix; (b) ROC Curve.

5.2. Results in Chest Disease Classification

For the chest disease classification dataset, similar trends were observed, with NS-transformed data providing a notable boost in performance. As shown in Table 4, the DenseNet model achieved the highest performance on the original dataset, with an accuracy of 86.69%. In the NS-transformed images, DenseNet excelled in the True domain, achieving an accuracy of 88.40%, followed closely by the MobileNet model in the Indeterminacy domain, which significantly improved from its original performance, attaining an accuracy of 88.05%. The confusion matrices and ROC curves for these two best models are presented in Figures 6 and 7. These results underscore the effectiveness of NS transformations in enhancing the performance of deep learning models for chest disease classification, particularly highlighting DenseNet121's capabilities in accurately detecting conditions within this dataset.

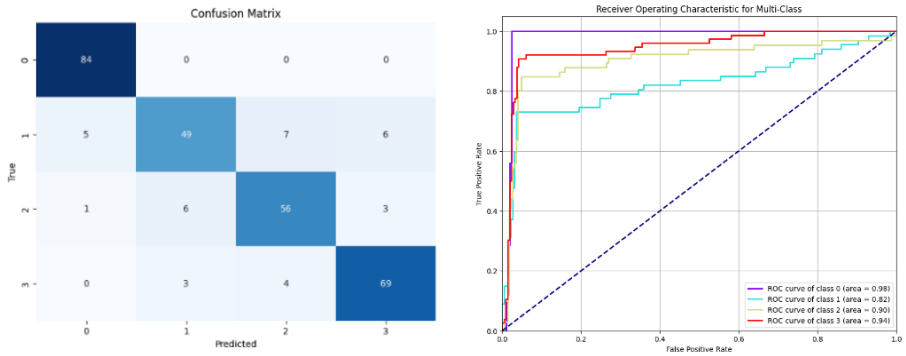
Table 4: Performance Metrics of DL Models on Original and NS Domains Chest Diseases Classification Datasets

Data	Model	Accuracy	Precision	Recall	F1-Score
Original Data	MobileNet	85.32	86.27	85.25	85.20
	ResNet50	75.43	75.68	75.37	75.43
	VGG16	83.96	84.39	83.79	83.34
	InceptionV3	83.28	82.49	82.44	82.37
	DenseNet121	**86.69**	**86.43**	**86.21**	**86.06**
True (T) NS domain Data	MobileNet	**87.37**	**86.97**	**86.73**	**86.64**
	ResNet50	76.79	76.07	75.78	75.50
	VGG16	84.30	84.62	84.10	84.28
	InceptionV3	86.35	86.24	85.95	85.62
	DenseNet121	**88.40**	**87.70**	**87.53**	**87.48**
Indeterminacy (I) NS domain Data	MobileNet	**88.05**	**87.46**	**87.19**	**87.19**
	ResNet50	79.86	79.65	79.98	79.39
	VGG16	81.91	81.69	81.40	81.40
	InceptionV3	83.62	83.24	82.68	82.78
	DenseNet121	**87.37**	**87.07**	**86.63**	**86.65**
Falsity (F) NS domain data	MobileNet	86.69	86.20	86.58	86.32
	ResNet50	76.45	76.28	75.93	75.21
	VGG16	83.96	83.41	83.52	82.94
	InceptionV3	82.94	82.84	83.85	82.74
	DenseNet121	**87.71**	**87.47**	**88.26**	**87.63**


Confusion Matrix
True
Predicted
Receiver Operating Characteristic for Multi-Class
True Positive Rate
False Positive Rate
ROC curve of class 0 (area = 0.99)
ROC curve of class 1 (area = 0.91)
ROC curve of class 2 (area = 0.97)
ROC curve of class 3 (area = 0.98)

(A) (B)

Figure 6: Performance Analysis of the DenseNet Model on True (T) Domain Images: (a) Confusion Matrix; (b) ROC Curve.



(A) (B)

***Figure 7**: Performance Analysis of the MobileNet Model on Indeterminacy (I) Domain Images: (a) Confusion Matrix; (b) ROC Curve.*

5.3.Summary of Experiments

The results of the experiments clearly demonstrate the positive impact of Neutrosophic Set (NS) transformations on the performance of deep learning models for medical image classification across both the cervical spine and chest disease classification datasets. For the C-spinal injury detection dataset, NS-transformed models exhibited significant improvements in performance metrics. DenseNet121 and Inceprtion emerged as the top performer in the True (T) domain, achieving an accuracy of **99.67%**, while the original dataset accuracy was **99.00%**, 96.99%. respectively. Incepton also showed notable enhancement, achieving an accuracy of **99.33%** in the Indeterminacy domain compared to its original accuracy of **99.00%**. These improvements underscore the effectiveness of NS transformations in reducing uncertainty and enhancing the models' ability to extract relevant features. In the chest disease classification dataset, DenseNet121 again demonstrated the highest accuracy among the models, achieving 88.40% on the True domain, a significant improvement from 86.69% on the original dataset. MobileNet performed well too, with an accuracy of 87.37% on the True domain, showing a marked increase from 85.32% on the original data. Other models, such as InceptionV3 and VGG16, also benefited from NS transformations, indicating that these methods enhance overall classification performance across various conditions. These results highlight the potential of integrating NS theory into medical image classification workflows, particularly in scenarios characterized by uncertainty. enabling more accurate capture of key features associated with spinal injuries and chest diseases, which could lead to improved diagnostic accuracy in real-world clinical applications.

6. Conclusion and future Work

This study presents a hybrid approach that integrates Neutrosophic Set (NS) theory with deep learning models to enhance the classification of medical X-ray images across two challenging datasets: cervical spine injuries and chest disease classification. Traditional deep learning models often struggle with the inherent

noise and uncertainty present in medical images, leading to reduced diagnostic accuracy. By incorporating NS theory, which categorizes information into True (T), Indeterminate (I), and False (F) domains, we effectively addressed these challenges. Our results demonstrate that models trained on NS-transformed data consistently outperformed those trained on original datasets. Specifically, DenseNet121 achieved the highest accuracy in both datasets, with 88.40% for chest disease classification and 99.97% for cervical spine injury detection. MobileNet and Inception modes also exhibited significant improvements in the different NS domain. These findings highlight the effectiveness of combining NS theory with state-of-the-art deep learning models, as the transformed data allowed for better feature extraction and more accurate predictions across various medical imaging scenarios. The incorporation of NS theory not only enhances the ability to manage uncertainty in medical images but also provides a robust solution for improving diagnostic accuracy. However, this study has certain limitations. One limitation is the reliance on only X-ray medical datasets (cervical spine and chest diseases). While these datasets are valuable, they may not fully represent the diversity and complexity of medical images encountered in real-world clinical environments. The generalizability of our approach to other medical imaging domains requires further investigation. Additionally, while NS theory provides an improved mechanism for handling uncertainty, it may introduce extra computational overhead, which could be a challenge for resource-constrained environments. Future work will focus on expanding the application of NS theory to additional medical imaging modalities, such as MRI and CT scans, to further validate its effectiveness. Additionally Future studies will also aim to optimize the computational efficiency of the NS-DL approach, particularly in terms of processing time and resource utilization. This will help ensure that our approach is not only accurate but also feasible for deployment in real-time clinical settings.

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A Concise Formalization of Hyperrealism-Antirealism in Physics Using Neutrosophic Logic

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Abstract. In [6], the relationship between hyperrealism and antirealism in physics is proposed. Hyperrealism-Antirealism describes a spectrum where theories transition from being empirically grounded (realist) to speculative frameworks dominated by untestable assumptions, thereby detaching from observable reality.

This paper presents a concise mathematical formalization of the spectrum of *hyperrealism-antirealism* in theoretical physics and the philosophy of science. We propose a method to quantify the extent to which a theory is "hyperreal" or "antirealist" using a ratio of testable to untestable statements. Additionally, we provide several theorems that outline the conditions under which theories shift into hyperreal or antirealist domains. Finally, we explore how Neutrosophic Logic [38] can capture intermediate degrees between realism and antirealism, offering a nuanced perspective on the spectrum.

Keywords: Hyperrealism, Antirealism, Realism, Neutrosophic Logic

1. Preliminaries and Definitions

This section provides an introduction to the foundational concepts and definitions required for the discussions in this paper.

1.1. *Physical Theory and Realism*

This subsection provides an explanation of Physical Theory and Realism. Relevant definitions and simple examples are presented below.

Definition 1.1 (Set)**.** [22] A *set* is a collection of distinct objects, known as elements, that are clearly defined, allowing any object to be identified as either belonging to or not belonging

to the set. If A is a set and x is an element of A, this membership is denoted by $x \in A$. Sets are typically represented using curly brackets.

Definition 1.2 (Physical Theory)**.** [6] Let $\mathcal{T}$ be a physical theory expressed as a formal system with:

- A set of axioms or postulates $A = \{A_1, A_2, \ldots, A_n\}$.
- A set of theorems or derived statements $S = \{s_1, s_2, \ldots\}$ obtainable from A by rules of inference.

We assume that $\mathcal{T}$ is sufficiently well-defined so that each $s \in S$ has a clear meaning in principle.

Definition 1.3 (Testable vs. Untestable Statements)**.** [6] Partition S into two subsets:

$$S_{\text{test}} = \{\, s \in S : s \text{ is falsifiable or empirically verifiable in principle}\},$$

$$S_{\text{untest}} = \{\, s \in S : s \text{ is not empirically testable in principle or is operationally meaningless}\}.$$

We do not distinguish here whether S_{untest} statements are *in principle* untestable or just untestable given current technology; we treat them uniformly as "not testable."

Example 1.4 (Quantum Field Theory (QFT))**.** (cf. [20, 28, 28]) A *Quantum Field Theory* $\mathcal{T}_{\text{QFT}}$ typically includes:

- Axioms/postulates: Field operators, canonical commutation relations, etc.
- Derived statements: Predictions about particle scattering cross-sections, decay rates, and vacuum fluctuations.

$S = \{$scattering amplitude relations, vacuum state expansions, renormalization group equations, etc.$\}$.

Many statements here are *testable* via high-energy experiments (e.g. collider data for scattering cross-sections). However, certain assumptions about non-perturbative vacuum structure or extreme energy behaviors could remain *untestable* with current technology.

Example 1.5 (Cosmological Dark Energy Models)**.** Consider a physical theory of *Dark Energy* that posits a particular scalar field or exotic fluid as the driver of accelerated cosmic expansion(cf. [7, 31, 33]). The domain of statements:

$$S = \{\text{Hubble parameter evolution, equation of state parameter,}$$

$$\text{cosmic microwave background constraints, etc.}\}.$$

A subset of these can be tested with astrophysical data (e.g. supernova observations, cosmological distance measurements), thus belonging to S_{test}. More speculative claims (e.g. about the ultimate fate of the universe beyond observable horizons or about higher-dimensional modifications) might be effectively *untestable* with present or near-future technology, placing them in S_{untest}.

Example 1.6 (In the Context of Quantum Gravity)**.** A *Quantum Gravity* framework $\mathcal{T}_{\mathrm{QG}}$ (cf. [15, 18]) might hypothesize:

- S_{test}: Low-energy quantum gravitational effects potentially observable in precise measurements of graviton scattering at extremely high energies (though technologically challenging).
- S_{untest}: Specific features of spacetime foam or Planck-scale discretization that may be inaccessible to direct experimentation.

Thus, while some approximate predictions about gravitational corrections might be tested indirectly, the deeper structure of spacetime at 10^{-35} meters could remain untestable in practice.

1.2. *Hyperrealism-Antirealism Measure*

This subsection presents the definition of the Hyperrealism-Antirealism Measure. Additionally, we provide several concrete examples of Realist, Hyperrealist, and Antirealist concepts, as discussed in [6].

Definition 1.7 (Hyperrealism-Antirealism Measure)**.** [6] Define the *Hyperrealism-Antirealism Ratio* (HAR) of the theory $\mathcal{T}$ as

$$\mathrm{HAR}(\mathcal{T}) \;=\; \frac{\left|S_{\mathrm{untest}}\right|}{\left|S_{\mathrm{test}}\right|}.$$

We say:

- $\mathcal{T}$ is *realist* or *empirically grounded* if $\mathrm{HAR}(\mathcal{T}) < 1$.
- $\mathcal{T}$ is *hyperrealist* if $\mathrm{HAR}(\mathcal{T}) \geq 1$.
- $\mathcal{T}$ is *antirealist* in the extreme if $\mathrm{HAR}(\mathcal{T}) \to \infty$, i.e., if $\left|S_{\mathrm{test}}\right|$ is finite (or negligible) while $\left|S_{\mathrm{untest}}\right|$ grows arbitrarily large.

Example 1.8 (Solid-State Physics)**.** (cf. [5, 17]) A typical *solid-state* model for semiconductors or superconductors includes statements about electron band structures, doping effects, and transport phenomena. Much of this is directly testable via laboratory measurements of conductivity, critical temperatures, tunneling spectra, and so on. Thus,

$$\mathrm{HAR}(\mathcal{T}_{\mathrm{solid}}) = \frac{|S_{\mathrm{untest}}|}{|S_{\mathrm{test}}|} < 1,$$

since the majority of statements (e.g. doping concentration vs. conductivity) can be empirically verified. This places standard solid-state physics squarely in the *realist* category.

Example 1.9 (Stoichiometric Chemical Bonding)**.** (cf. [34]) Consider a simple *chemical theory* focusing on stoichiometry and bond energy approximations. Most statements, such as

predicted yields of reactions under certain conditions, are testable in a basic chemistry lab. Hence

$$\mathrm{HAR}(\mathcal{T}_{\mathrm{chem}}) = \frac{|S_{\mathrm{untest}}|}{|S_{\mathrm{test}}|} \ll 1,$$

reflecting a highly *realist* approach with strong empirical grounding.

Example 1.10 (Electromagnetism as Realist or Mildly Hyperrealist)**.** *Classical electromagnetism* (Maxwell's equations) [16, 23, 36] is also highly testable. Most statements, such as the inverse-square law for electric fields, can be verified. However, some advanced claims (e.g. certain theoretical boundary conditions or idealized continuous distributions) might not be directly testable, so

$$S_{\mathrm{test}} \ > \ S_{\mathrm{untest}},$$

and typically $\mathrm{HAR}(\mathcal{T}_{\mathrm{EM}}) < 1$. Thus it remains primarily *realist*. In specialized theoretical corners (say, extremely idealized boundary conditions or vacuum states), there may be a mild drift toward *hyperreal* statements, but overall the testable component is dominant.

Example 1.11 (String Theory: Hyperrealist Domain)**.** *String Theory* [41] posits that fundamental particles are tiny vibrating strings living in higher-dimensional spaces. Many core aspects, such as the shape of extra dimensions or the existence of a vast "landscape" of possible vacua, remain experimentally unverified or difficult to test.

Thus, while some statements might be in S_{test} (e.g., certain low-energy effective predictions), a large fraction of the theory's claims lie in

$$S_{\mathrm{untest}} = \{\text{existence of extra dimensions, specific compactifications, multi-brane setups, etc.}\}.$$

If the untestable claims outnumber the testable predictions,

$$\mathrm{HAR}(\mathcal{T}_{\mathrm{string}}) = \frac{|S_{\mathrm{untest}}|}{|S_{\mathrm{test}}|} \ \geq 1,$$

placing String Theory in a *hyperrealist* regime. Whether it remains so depends on future experimental possibilities.

Example 1.12 (Quantum Cosmology)**.** *Quantum cosmology* [4] often deals with the wavefunction of the universe, decoherence at Planck epochs, and speculative boundary conditions (e.g. Hartle–Hawking no-boundary proposal). While some broad consequences might be tested indirectly (through cosmic microwave background patterns), many core statements (like the initial wavefunction of the entire universe) may remain effectively untestable. Consequently,

$$\mathrm{HAR}(\mathcal{T}_{\mathrm{qcosmo}}) \geq 1$$

if untestable statements dominate. This places quantum cosmology in a *hyperrealist* regime, at least until novel observations or conceptual breakthroughs reduce S_{untest}.

Example 1.13 (Multiverse Hypotheses)**.** *Multiverse scenarios* (cf. [26]) propose that our universe is one among many, each with potentially different physical constants. While certain anthropic arguments might be testable in a limited sense, many claims (e.g. the existence of infinitely many parallel universes) are *untestable* by definition. Thus

$$|S_{\text{untest}}| \gg |S_{\text{test}}|,$$

and the theory's HAR is typically ≥ 1, indicating a *hyperrealist* orientation.

Example 1.14 (A Fully Untestable Speculative Theory: Antirealist Limit)**.** Consider a hypothetical framework full of *metaphysical claims* (cf. [27]) with no empirical handle:

$$S_{\text{untest}} = \{\text{“Invisible pink unicorns cause time fluctuations,” etc.}\}, \quad S_{\text{test}} = \varnothing.$$

Hence,

$$\text{HAR}(\mathcal{T}_{\text{extreme}}) = \frac{|S_{\text{untest}}|}{|S_{\text{test}}|} = \frac{\text{finite or infinite}}{0} \to \infty.$$

Such a theory is *antirealist in the extreme*, offering no measurable or falsifiable predictions.

Example 1.15 (“Simulation Hypothesis” without Empirical Anchors)**.** (cf. [21,35]) A version of the *simulation hypothesis* posits that our entire universe is a computer simulation run by higher beings. If framed in a way that absolutely no test could ever falsify or confirm it (i.e. no predicted anomalies), then

$$S_{\text{test}} = \varnothing, \quad S_{\text{untest}} \neq \varnothing, \quad \text{HAR} \to \infty.$$

This is *antirealist in the extreme*, since no empirical method can, in principle, verify or refute the scenario.

Example 1.16 (Mythical Ether Theory (Unrevisable))**.** Historically, the *luminiferous ether* [29, 30] was once hypothesized as the medium for light propagation. After experiments (e.g. Michelson–Morley) failed to detect it, a purely ad hoc “ether” with undetectable properties became *unfalsifiable*. If one persists in an ether theory that automatically adjusts postulates to evade every new test, then

$$S_{\text{test}} \to 0, \quad S_{\text{untest}} \text{ grows.}$$

Hence HAR $\to \infty$. This effectively renders the theory *antirealist*, as it ceases to produce meaningful testable claims.

2. **Results of This Paper**

This section outlines the main results of this paper.

2.1. *Properties of Hyperrealism*

In this subsection, we explore the properties of Hyperrealism and related concepts through several theorems. The theorems are presented below.

Theorem 2.1 (Threshold for Hyperrealism)**.** *Let $\mathcal{T}$ be a physical theory with two disjoint sets of statements: S_{test} (testable) and S_{untest} (untestable). Suppose there exists a constant $\alpha \geq 1$ such that*

$$|S_{\text{untest}}| \;\geq\; \alpha \left|S_{\text{test}}\right|.$$

Then the Hyperrealism–Antirealism Ratio, defined as

$$\text{HAR}(\mathcal{T}) \;=\; \frac{|S_{\text{untest}}|}{|S_{\text{test}}|},$$

satisfies $\text{HAR}(\mathcal{T}) \geq 1$*. In particular, $\mathcal{T}$ lies in the hyperrealist regime.*

Proof. By definition of $\text{HAR}(\mathcal{T})$,

$$\text{HAR}(\mathcal{T}) = \frac{\left|S_{\text{untest}}\right|}{\left|S_{\text{test}}\right|} \geq \frac{\alpha \left|S_{\text{test}}\right|}{\left|S_{\text{test}}\right|} = \alpha \;\geq\; 1.$$

Hence $\text{HAR}(\mathcal{T}) \geq 1$, placing $\mathcal{T}$ in at least a *hyperrealist* domain. □

Remark 2.2. This result formalizes the intuition that if untestable statements dominate (or at least match) the testable ones by a constant factor $\alpha \geq 1$, the theory departs from a strictly empirical grounding and enters the hyperrealist regime.

Theorem 2.3 (Antirealist Limit)**.** *Let $\mathcal{T}$ be a physical theory whose testable statements remain bounded in number, i.e. $\left|S_{\text{test}}\right| = T_0 < \infty$, while its untestable statements $\left|S_{\text{untest}}\right|$ grow unboundedly over successive reformulations or expansions. Then*

$$\lim_{\textit{expansions}} \text{HAR}(\mathcal{T}) \;=\; \lim_{n\to\infty} \frac{|S_{\text{untest}}|(n)}{T_0} \;=\; \infty.$$

Hence, in the limit, $\mathcal{T}$ becomes antirealist.

Proof. Label each expansion or new version of the theory by an index $n \in \mathbb{N}$, and let $U(n) = |S_{\text{untest}}|(n)$ denote the number of untestable statements at stage n. Assume $U(n) \to \infty$ as $n \to \infty$, while $|S_{\text{test}}| = T_0$ remains fixed (or does not grow comparably). Then

$$\text{HAR}(\mathcal{T})(n) \;=\; \frac{U(n)}{T_0} \;\xrightarrow[n\to\infty]{}\; \infty.$$

Thus, as n increases, the Hyperrealism–Antirealism Ratio diverges, implying $\mathcal{T}$ enters the *antirealist* regime. □

Corollary 2.4 (Surfeit of Speculation Implies Antirealism)**.** *If* $|S_{\text{untest}}|$ *grows without bound, while* $|S_{\text{test}}|$ *stays finite or increases more slowly, then eventually*

$$\text{HAR}(\mathcal{T}) \to \infty,$$

driving the theory into an antirealist *limit.*

Remark 2.5 (Scope and Limitations)**.** Although these measures (ratio definitions and thresholds) offer a convenient way to quantify how far a theory drifts from empirical verifiability, the distinction between "testable" and "untestable" can be somewhat fluid. Future technological or conceptual breakthroughs may convert currently untestable statements into testable ones. Nevertheless, these theorems underscore a key principle: once untestable content becomes overwhelmingly large, the theory effectively detaches from strict empirical foundations and moves toward hyperrealism or even antirealism.

2.2. *Neutrosophic Realism Index*

The Neutrosophic Realism Index is a concept derived from the application of Neutrosophic Logic to the framework of realism. Neutrosophic Logic [8, 11, 37–40] is widely recognized as a generalization of Fuzzy Logic [42] and Intuitionistic Fuzzy Logic [2, 3]. The principles of Neutrosophic Logic have been applied across various fields, including graph theory [9, 12, 13], topology [19, 32], algebra [24, 25], automata [14], and linguistics [10], demonstrating its versatility and broad utility. Below, we present the definition of the Neutrosophic Realism Index, along with relevant theorems and properties.

Definition 2.6 (Neutrosophic Logic)**.** [38] Neutrosophic Logic extends classical logic by assigning to each proposition a truth value comprising three components:

$$v(A) = (T, I, F),$$

where $T, I, F \in [0, 1]$ represent the degrees of truth, indeterminacy, and falsity, respectively.

Example 2.7 (Real-Life Application of Neutrosophic Logic: Weather Forecast)**.** (cf. [1]) Consider a weather forecast predicting that it will rain tomorrow. Using Neutrosophic Logic, this prediction can be expressed as:

$$v(\text{"It will rain tomorrow"}) = (0.7, 0.2, 0.1),$$

where:

- $T = 0.7$: There is a 70% confidence that the statement is true based on meteorological data.

- $I = 0.2$: There is a 20% level of indeterminacy due to factors such as changing weather conditions or measurement uncertainties.
- $F = 0.1$: There is a 10% chance that the statement is false, possibly due to model inaccuracies or unforeseen events.

This representation highlights the nuanced uncertainties involved in real-world predictions, making Neutrosophic Logic suitable for domains like weather forecasting, decision-making, and risk assessment.

Definition 2.8 (Neutrosophic Realism Index)**.** [6] Let each statement $s \in S$ have a triple (T_s, I_s, F_s), where $T_s, I_s, F_s \in [0,1]$ with $T_s + I_s + F_s \leq 3$. Define the *Neutrosophic Realism Index* (NRI) for the theory as an average:

$$\mathrm{NRI}(\mathcal{T}) \;=\; \frac{1}{|S|}\sum_{s\in S} T_s.$$

Example 2.9 (A Highly Realist Theory)**.** Let $\mathcal{T}_1$ have $|S| = 4$ statements: $S = \{s_1, s_2, s_3, s_4\}$. Suppose each statement has (T_s, I_s, F_s) as follows:

$$s_1 : (0.9, 0.05, 0.05), \quad s_2 : (1.0, 0.0, 0.0), \quad s_3 : (0.85, 0.10, 0.05), \quad s_4 : (0.95, 0.02, 0.03).$$

Then the sum of T_s is $(0.9 + 1.0 + 0.85 + 0.95) = 3.70$. Dividing by $|S| = 4$, we get

$$\mathrm{NRI}(\mathcal{T}_1) = \frac{3.70}{4} = 0.925.$$

Since 0.925 is quite high (close to 1), this theory is *strongly realist* by Theorems 2.12 and 2.14.

Example 2.10 (A Partially Realist Theory)**.** Let $\mathcal{T}_2$ have $|S| = 3$ statements with:

$$s_1 : (0.5, 0.4, 0.1), \quad s_2 : (0.6, 0.2, 0.2), \quad s_3 : (0.4, 0.35, 0.25).$$

The sum of T_s is $(0.5 + 0.6 + 0.4) = 1.5$. Hence

$$\mathrm{NRI}(\mathcal{T}_2) = \frac{1.5}{3} = 0.5.$$

If we define $\alpha = 0.3$ and $\beta = 0.8$ as the thresholds, we see 0.5 falls in the intermediate range $\alpha \leq 0.5 < \beta$. So $\mathcal{T}_2$ is *partially realist*, reflecting moderate truth membership but also significant indeterminacy or falsehood.

Example 2.11 (A Low-NRI (Hyperrealist/Antirealist) Theory)**.** Consider $\mathcal{T}_3$ with $|S| = 3$ statements:

$$s_1 : (0.1, 0.5, 0.4), \quad s_2 : (0.0, 0.4, 0.6), \quad s_3 : (0.05, 0.3, 0.65).$$

The sum of T_s is $(0.1 + 0.0 + 0.05) = 0.15$, so

$$\mathrm{NRI}(\mathcal{T}_3) = \frac{0.15}{3} = 0.05.$$

That is very low, indicating most statements are closer to indeterminate or false than to true. By Theorem 2.12, this places $\mathcal{T}_3$ in the strongly antirealist/hyperrealist domain.

Theorem 2.12 (Realism vs. Indeterminacy)**.** *If NRI($\mathcal{T}$) is very low (close to 0), it indicates minimal average truth membership, which implies strong antireal or hyperreal tendencies. If NRI($\mathcal{T}$) is high (close to 1 or beyond), the theory is strongly realist. Intermediate values of NRI reflect partial realism, acknowledging significant uncertainty (large I_s) or partial falsehood (larger F_s).*

Proof. By Definition,

$$\mathrm{NRI}(\mathcal{T}) = \frac{1}{|S|}\sum_{s\in S} T_s.$$

If NRI($\mathcal{T}$) is near 0, that sum of T_s over all statements must be near 0, indicating that most statements have $T_s \approx 0$. Because $T_s + I_s + F_s \leq 3$, such statements must have either I_s or F_s significant. In neutrosophic logic, large I_s or F_s aligns with hyperrealism/antirealism (the statements are either indeterminate or false from a realist standpoint). Conversely, if NRI($\mathcal{T}$) ≈ 1 or larger, then most statements have T_s close to 1, indicating strong agreement with a realist or "true" stance. Intermediate values of T_s imply partial truth or increased indeterminacy. This direct link to the sum of truth degrees proves the theorem. □

Theorem 2.13 (Bounds on NRI)**.** *For any theory $\mathcal{T}$ with statements $\{s_1, \ldots, s_n\}$ assigned triples (T_s, I_s, F_s) in $[0, 1]$, we have:*

$$0 \;\leq\; \mathit{NRI}(\mathcal{T}) \;\leq\; 3,$$

where 3 is an absolute maximum only if $T_s = 3$ is allowed; in practice, typical implementations take $T_s + I_s + F_s = 1$, so $0 \leq \mathrm{NRI}(\mathcal{T}) \leq 1$.

Proof. By definition, $T_s \in [0, 1]$ for each statement. If we allow the extended sum $T_s+I_s+F_s \leq 3$, then the maximum T_s for a single statement can be 3. However, in many neutrosophic logic formulations, one confines $T_s + I_s + F_s = 1$. Under that typical constraint, $T_s \leq 1$. Summing n such values and dividing by n yields a maximum of 1 for NRI($\mathcal{T}$). The minimum is clearly 0, achieved if $T_s = 0$ for all s. □

Theorem 2.14 (Thresholds for Classification)**.** *Fix two constants $0 < \alpha < \beta \leq 1$. We may define realism categories based on NRI($\mathcal{T}$):*

- *If NRI($\mathcal{T}$) $< \alpha$, the theory is* antirealist-leaning *or* hyperrealist-leaning.
- *If $\alpha \leq$ NRI($\mathcal{T}$) $< \beta$, the theory is* partially realist *(significant uncertainty or partial falsehood).*
- *If NRI($\mathcal{T}$) $\geq \beta$, the theory is* strongly realist.

Proof. This follows from Theorem 2.12 and the observation that NRI($\mathcal{T}$) can be partitioned into different ranges. Such threshold-based classification is common in fuzzy or neutrosophic logic frameworks, where intervals represent distinct qualitative categories. □

Theorem 2.15 (NRI and Global Consistency)**.** *Assume each statement in S must satisfy $T_s + I_s + F_s = 1$ (a common simplification in neutrosophic frameworks). Then:*

$$NRI(\mathcal{T}) + \frac{1}{|S|}\sum_{s\in S}(I_s + F_s) = 1.$$

Hence

$$NRI(\mathcal{T}) = 1 - \frac{1}{|S|}\sum_{s\in S}(I_s + F_s).$$

Proof. If $T_s + I_s + F_s = 1$ for each s, summing this over all $s \in S$ gives:

$$\sum_{s\in S} T_s + \sum_{s\in S} I_s + \sum_{s\in S} F_s = |S|.$$

Divide both sides by $|S|$:

$$\frac{1}{|S|}\sum_{s\in S} T_s + \frac{1}{|S|}\sum_{s\in S} I_s + \frac{1}{|S|}\sum_{s\in S} F_s = 1.$$

The left term is precisely NRI($\mathcal{T}$), and the other two terms sum to the average $(I_s + F_s)$. So the statement follows. □

Corollary 2.16 (Trade-off of Realism, Indeterminacy, and Falsehood)**.** *Under the same assumption, if NRI($\mathcal{T}$) is high, then $\frac{1}{|S|}\sum_{s\in S}(I_s + F_s)$ must be low, meaning the theory does not rely heavily on indeterminacy or falsehood. Conversely, if NRI($\mathcal{T}$) is low, then $I_s + F_s$ must be high on average, implying an antirealist or hyperrealist stance.*

Proof. Immediate from Theorem 2.15: NRI($\mathcal{T}$) $= 1 - \frac{1}{|S|}\sum_{s\in S}(I_s + F_s)$. If one term is large, the other is correspondingly small. □

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Data Availability

This research is purely theoretical, involving no data collection or analysis. We encourage future researchers to pursue empirical investigations to further develop and validate the concepts introduced here.

Ethical Approval

As this research is entirely theoretical in nature and does not involve human participants or animal subjects, no ethical approval is required.

Conflicts of Interest

The authors confirm that there are no conflicts of interest related to the research or its publication.

Disclaimer

This work presents theoretical concepts that have not yet undergone practical testing or validation. Future researchers are encouraged to apply and assess these ideas in empirical contexts. While every effort has been made to ensure accuracy and appropriate referencing, unintentional errors or omissions may still exist. Readers are advised to verify referenced materials on their own. The views and conclusions expressed here are the authors' own and do not necessarily reflect those of their affiliated organizations.

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Risk Assessment of Municipal Water Supply and Drainage Project Costs Using Neutrosophic Numbers

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Abstract: Risk assessment of municipal water supply and drainage project costs is a multi-criteria decision making (MCDM) problem. Water supply and drainage have various conflict criteria. Two MCDM methods are used in this study such as SIWEC and MACONT. The SIWEC method is used to compute the criteria weights and the MACONT method is used to rank the alternatives. These methods are used under the trapezoidal fuzzy neutrosophic sets (TFNSs) to deal with uncertainty and vague information. Eight criteria and eleven alternatives ate used to evaluate the MCDM problem. The sensitivity analysis is conducted to show the stability of the ranks. The comparative analysis is performed to show the effectiveness of the proposed approach.

Keywords: Trapezoidal Fuzzy Neutrosophic Numbers; Water Supply; Risk Assessment; Drainage Project Costs.

1. Introduction and Literature Review

The ties between people, society, and nature have fallen out of balance due to the rapid advancement of social and economic growth. Resource depletion, the energy crisis, ozone depletion, global warming, acid rain, eutrophication, smog, and other environmental issues have all attracted a lot of attention in many nations across the world. Despite creating enormous economic value and solving the housing crisis, the construction sector has polluted the environment[1], [2]. As a result, it is impossible to overlook how greenhouse gas emissions from the usage of fossil fuels contribute to global warming. Fossil fuels are the major energy source. Therefore, to lessen the influence on the environment and achieve sustainable growth, it is crucial to select materials with lower energy consumption and employ low-carbon energy instead of fossil fuels. Building water supply and drainage systems are essential components, thus it's important to optimize system design to conserve energy and enhance management to lessen their effect on the surroundings and save money[3], [4]. This study proposes a multi-criteria decision

making (MCDM) approach for risk assessment of municipal water supply and drainage project costs. But this MCDM approach has uncertainty information. To address the ambiguity and unpredictability of numerous real-world practical issues, Zadeh [5]introduced fuzzy sets (FS), which are defined by a membership degree that ranges within the unit interval [0, 1]. Atanassov proposed the idea of intuitionistic fuzzy sets (IFS), which is a generalization of the concept of fuzzy sets (FS) and added a new component to the definition of fuzzy sets that establishes the degree of non-membership. Over the past few decades, intuitionistic fuzzy sets have been extensively researched and used due to their advantages over fuzzy sets. Cuong and Kreinovichs [6] added the degree of neutral membership based on the two memberships of IFS to create the idea of picture fuzzy sets (PFS). Nevertheless, there are several issues that require attention that FS, IFS, and PFS cannot handle. The neutrosophic set (NS), however, can manage this challenging circumstance. The vote outcome can be represented by a neutrosophic number (NN), which is the NN's truth-membership degree (T), falsity-membership degree (F), and indeterminacy-membership degree (I)[7], [8]. A truth-membership degree, an indeterminacy-membership degree, and a falsity-membership degree define the neutrosophic set, which was developed by Smarandache [9]. It is a generalization of sets, including intuitionistic fuzzy sets, fuzzy sets, and crisp sets. Numerous researchers have recently focused on this innovative idea, leading to the development, enhancement, and expansion of the neutrosophic. Under the trapezoidal fuzzy neutrosophic numbers environment, multi-attribute decision-making problems are challenging, especially when the attribute weight is entirely unknown, and the attribute value data is lacking. TAN et al. [10] suggests a gray theory and information entropy-based decision-making approach as a remedy. First, a simpler and more logical approach than the conventional complements method is put forth to define the missing information using trapezoidal fuzzy neutrosophic numbers to address the issue of attribute value incompleteness in the decision matrix. The definition of new trapezoidal fuzzy neutrosophic entropy is put forth and used to calculate the attribute weight to address the issue of totally unknown attribute weights. The best option is then chosen by ranking the possibilities using the gray relation analysis method of grey theory. Tan et al. [11] proposed two MAGDM issue, where the attribute weights are unknown and the alternative attribute value is in the form of a TFNSs. One method is based on the TOPSIS method, while the other is based on the TFNNs operator. The TFNSs information is then aggregated using a suggested TFNNs ordered weighted arithmetic averaging operator. The best option in a TFNSs environment is then determined by ranking the options using the score and accuracy functions of a TFNNs. Furthermore, an expanded TOPSIS approach is suggested to handle TFNSs data. The effectiveness and suitability of the suggested group decision making techniques are illustrated by a sensitivity analysis and an illustrative example. To create a more practical and effective decision support system, numerous esteemed scholars have been examining the problems of imprecision, inconsistency, and ambiguous information in decision making. To address those problems, Irvanizam [12] proposed novel MCGDM approach. This approach combines TFNNs with the original MABAC method. Initially, the suggested approach aggregates expert independent judgments using a few aggregation operators after converting them into TFNNs in the form of

linguistic variables. The examined options are then ranked using the score and accuracy functions. The suggested MABAC chooses the optimal option based on the distance measure between the alternatives and the boundary approximation area. The requirements for information representations and believability levels in contemporary decision-making (DM) systems are growing increasingly intricate and sophisticated. According to the literature currently in publication, truth, indeterminacy, and falsity membership functions provide a better way to represent TFNNs. Unfortunately, because TFNNs lacks a measure of trust, it communicates its flaw. TFNNs should be connected to its degree of trustworthiness in order to increase its creditability. Ye et al. [13]introduced the idea of credibility TFNNs, a novel framework of TFNV connected to the measure of credibility, to explain why TFNNs and its credibility level should be combined. By combining the content of a TFNNs with a corresponding credibility level TFNNs have the advantage of being able to depict human knowledge and judgments that are uncertain and contradictory in a more credible manner. They then suggest the score function of TFNNs as well as the operational laws. Lastly, a real-world DM example of slope decision schemes is presented to demonstrate the effectiveness and suitability of the developed DM model in the instance of TFNNs.

A more modern MCDM approach that is distinct from most others, particularly in terms of data normalization, is Mixed Aggregation by Comprehensive Normalization Technique (MACONT). Unlike most other MCDM techniques, the MACONT employs three data normalization processes at the same time: linear max-min normalization (N3), linear ratio-based normalization (N2), and linear sum-based normalization (N1)[14], [15]. By making this distinction, the MACONT can minimize errors when compared to other MCDM systems that only use one data normalization technique.

2. Preliminaries

We provide some concepts about the trapezoidal fuzzy neutrosophic sets (TFNSs) and trapezoidal fuzzy neutrosophic numbers (TFNNs) with their operations rules.

Definition 1

Trapezoidal fuzzy neutrosophic sets can be defined such as:

$$N = \{(x, T_N(x), I_N(x), F_N(x))|x \in X\} \quad (1)$$

$$T_N(x) \subset [0,1], I_N(x) \subset [0,1], F_N(x) \subset [0,1] \quad (2)$$

$$T_N(x) = \left(t_N^1(x), t_N^2(x), t_N^3(x), t_N^4(x)\right) : X \to [0,1] \quad (3)$$

$$I_N(x) = \left(i_N^1(x), i_N^2(x), i_N^3(x), i_N^4(x)\right) : X \to [0,1] \quad (4)$$

$$F_N(x) = \left(f_N^1(x), f_N^2(x), f_N^3(x), f_N^4(x)\right) : X \to [0,1] \quad (5)$$

$$0 \le t_N^4(x) + i_N^4(x) + f_N^4(x) \le 3 \quad (6)$$

Definition 2

The TFNNs can be defined as:

$$n = \big((a_1, a_2, a_3, a_4), (b_1, b_2, b_3, b_4), (c_1, c_2, c_3, c_4)\big)$$

$$T_N(x) = \begin{pmatrix} \frac{x-a_1}{a_2-a_1} & a_1 \le x \le a_2 \\ 1 & a_2 \le x \le a_3 \\ \frac{a_4-x}{a_4-a_3} & a_3 \le x \le a_4 \\ 0 & otherwise \end{pmatrix} \tag{7}$$

$$I_N(x) = \begin{pmatrix} \frac{b_2-x}{b_2-b_1} & b_1 \le x \le b_2 \\ 0 & b_2 \le x \le b_3 \\ \frac{x-b_3}{b_4-b_3} & b_3 \le x \le b_4 \\ 1 & otherwise \end{pmatrix} \tag{8}$$

$$F_N(x) = \begin{pmatrix} \frac{c_2-x}{c_2-c_1} & c_1 \le x \le c_2 \\ 0 & c_2 \le x \le c_3 \\ \frac{x-c_3}{c_4-c_3} & c_3 \le x \le c_4 \\ 1 & otherwise \end{pmatrix} \tag{9}$$

Definition 3

Let $n_1 = \big((a_1, a_2, a_3, a_4), (b_1, b_2, b_3, b_4), (c_1, c_2, c_3, c_4)\big)$ and $n_2 = \big((e_1, e_2, e_3, e_4), (f_1, f_2, f_3, f_4), (g_1, g_2, g_3, g_4)\big)$ be two TFNNs. Then the following operations such as:

$$n_1 \oplus n_2 = \begin{pmatrix} \begin{pmatrix} a_1 + e_1 - a_1e_1, a_2 + e_2 - a_2e_2, \\ a_3 + e_3 - a_3e_3 - a_4 + e_4 - a_4e_4 \end{pmatrix}, \\ (b_1f_1, b_2f_2, b_3f_3, b_4f_4\ ), \\ (c_1g_1, c_2g_2, c_3g_3, c_4g_4) \end{pmatrix} \tag{10}$$

$$n_1 \otimes n_2 = \begin{pmatrix} (a_1e_1, a_2e_2, a_3e_3, a_4e_4), \\ \begin{pmatrix} b_1 + f_1 - b_1f_1, b_2 + f_2 - b_2f_2, \\ b_3 + f_3 - b_3f_3 - b_4 + f_4 - b_4f_4 \end{pmatrix}, \\ \begin{pmatrix} c_1 + g_1 - c_1g_1, c_2 + g_2 - c_2g_2, \\ c_3 + g_3 - c_3g_3 - c_4 + g_4 - c_4g_4 \end{pmatrix} \end{pmatrix} \tag{11}$$

$$\Upsilon n_1 = \begin{pmatrix} \begin{pmatrix} 1-(1-a_1)^\Upsilon, 1-(1-a_2)^\Upsilon, \\ 1-(1-a_3)^\Upsilon, 1-(1-a_4)^\Upsilon \end{pmatrix}, \\ \big(b_1^\Upsilon, b_2^\Upsilon, b_3^\Upsilon, b_4^\Upsilon\big), \\ \big(c_1^\Upsilon, c_2^\Upsilon, c_3^\Upsilon, c_4^\Upsilon\big) \end{pmatrix} \tag{12}$$

$$n_1^\Upsilon = \begin{pmatrix} \big(a_1^\Upsilon, a, a_3^\Upsilon, a_4^\Upsilon\big), \\ \begin{pmatrix} 1-(1-b_1)^\Upsilon, 1-(1-b_2)^\Upsilon, \\ 1-(1-b_3)^\Upsilon, 1-(1-b_4)^\Upsilon \end{pmatrix}, \\ \begin{pmatrix} 1-(1-c_1)^\Upsilon, 1-(1-c_2)^\Upsilon, \\ 1-(1-c_3)^\Upsilon, 1-(1-c_4)^\Upsilon \end{pmatrix} \end{pmatrix} \tag{13}$$

Definition 4.

The score function can be defined as:

$$s(n) = \frac{1}{3}\left(2 + \frac{\sum_{i=1}^{4} a_i}{4} - \frac{\sum_{i=1}^{4} b_i}{4} - \frac{\sum_{i=1}^{4} c_i}{4}\right) \quad (14)$$

3. Proposed Approach under Trapezoidal Fuzzy Neutrosophic Sets

This section provides two methods under the TFNSs to compute the criteria weights and rank the alternatives. This section is divided into two stages. In the first stage, we discussed the steps of the SIWEC to compute the criteria weights. In the second stage, we discuss the steps of the MACONT method to rank the alternatives. The SIWEC approach was developed to make things easier for the decision makers (DM) and those who compute the criteria weights. Because DM's ratings are employed, this approach falls under the category of subjective approaches for judging the significance of criteria. When using this method, two versions will be created: the TFNSs-SIWEC method, which uses the DM ratings in the form of linguistic values, and the simple SIWEC method, which uses ratings in the form of crisp numbers[16]. Guidelines for the future development of this method in various forms will be established in this way.

1. Build the decision matrix.

This matrix is built based on the opinions of experts. Experts use the TFNNs to evaluate the criteria and alternatives.

$$K = \begin{bmatrix} k_{11} & \cdots & k_{1n} \\ \vdots & \ddots & \vdots \\ k_{m1} & \cdots & k_{mn} \end{bmatrix} \quad (15)$$

2. Apply the score function

We applied the score function to obtain crisp values.

3. Combine the experts opinions.

We combine the opinions of experts into a single matrix.

4. Normalize the decision matrix.

We normalize the combined decision matrix such as:

$$n_{ij} = \frac{k_{ij}}{\max k_{ij}} \quad (16)$$

5. Compute the standard deviation

We compute the standard deviation for each criterion.

6. Multiply the values of standard deviation by the normalization values such as:

$$v_{ij} = n_{ij} * stand \quad (17)$$

7. Compute the sum of the weights of certain criteria.

$$s_j = \sum_{j=1}^{n} v_j \tag{18}$$

8. Compute the final score of criteria weights.

$$w_j = \frac{s_j}{\sum_{j=1}^{n} s_j} \tag{19}$$

Then we apply the steps of the MACONT method to rank the alternatives. We used this method under the TFNNs to deal with uncertainty information[17], [18].

1. We start with the combined decision matrix.

2. Normalize the decision matrix for positive and negative criteria such as:

$$y_{ij}^{1} = \frac{k_{ij}}{\sum_{i=1}^{m} k_{ij}} \tag{20}$$

$$y_{ij}^{1} = \frac{1/k_{ij}}{\sum_{i=1}^{m} 1/k_{ij}} \tag{21}$$

$$y_{ij}^{2} = \frac{k_{ij}}{\max k_{ij}} \tag{22}$$

$$y_{ij}^{2} = \frac{\min k_{ij}}{k_{ij}} \tag{23}$$

$$y_{ij}^{3} = \frac{(k_{ij} - \min k_{ij})}{(\max k_{ij} - \min k_{ij})} \tag{24}$$

$$y_{ij}^{3} = \frac{(k_{ij} - \max k_{ij})}{(\min k_{ij} - \max k_{ij})} \tag{25}$$

3. Combined the normalization matrix

$$d_{ij} = \alpha y_{ij}^{1} + \beta y_{ij}^{2} + (1 - \alpha - \beta) y_{ij}^{3} \tag{26}$$

Where $\alpha = \beta = 1/3$

4. Compute the distance between each alternative and reference alternative such as:

$$E_1(a_i) = \gamma \frac{p_i}{\sqrt{\sum_{i=1}^{m} (p_i)^2}} + (1 - \gamma) \frac{Q_i}{\sqrt{\sum_{i=1}^{m} (q_i)^2}} \tag{27}$$

$$E_2(a_i) = \delta \max\left(w_j\left(d_{ij} - d_j^{-}\right)\right) + (1 - \delta) \min\left(w_j\left(d_{ij} - d_j^{-}\right)\right) \tag{28}$$

Where d_j^{-} refers to the average value and the value of δ and γ are between 0 and 1.

6. Compute the final comprehensive score

$$E(a_i) = \frac{1}{2}\left(E_1(a_i) + \frac{E_2(a_i)}{\sqrt{\sum_{i=1}^{m} \left(E_2(a_i)\right)^2}}\right) \tag{29}$$

7. Rank the alternatives.

Table 1. Criteria of Municipal Water Supply and Drainage Project Costs.

Criteria	Alternatives
Social Risks	Monitoring Systems
Stakeholder Risks	Use of Cost Buffer
Environmental Risks	Supplier Agreements
Investment Risk	Climate-Resilient Infrastructure Investments
Budget Overrun Risk	Advanced Project Management
Compliance Risks	Comprehensive Feasibility Study
Climate Risks	Cost Estimation Models
Technology Risk	Contingency Planning
	Community Engagement
	Sustainable Material Selection
	Smart Water Management Systems

4. Results of the Application

We introduce the results of risk assessment of municipal water supply and drainage project costs with different criteria and alternatives. We used 8 criteria and 11 alternatives to evaluate this case study as shown in Table 1. Three experts have evaluated the criteria and alternatives as shown in Table 2. We used the TFNNs to evaluate the criteria and alternatives.

1. We built the decision matrix using the TFNNs as shown in Table 2.

2. We applied the score function to obtain crisp values.

3. We combined the experts opinions to obtain one decision matrix.

4. We normalize the decision matrix using Eq. (16) as shown in Table 3.

5. We compute the standard deviation

6. We multiply the values of standard deviation by the normalization values using Eq. (17) as shown in Table 4.

7. We compute the sum of the weights of certain criteria using Eq. (18).

8. We compute the final score of criteria weights using Eq. (19) as shown in Fig 1. Then we rank the criteria weights as shown in Fig 2.

Table 2. The TFNNs of municipal water supply and drainage project costs.

	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8
A_1	((0,0,0,0),(1,1,1,1),(1,1,1,1))	((0,0.1,0.2,0.3),(0.7,0.8,0.9,1),(0.7,0.8,0.9,1))	((0.1,0.2,0.3,0.4),(0.4,0.6,0.7,0.9),(0.4,0.6,0.7,0.9))	((0.3,0.4,0.5,0.6),(0.2,0.4,0.5,0.7),(0.2,0.4,0.5,0.7))	((0.5,0.6,0.7,0.8),(0,0.2,0.3,0.4),(0,0.2,0.3,0.4))	((0.7,0.8,0.9,1),(0,0.1,0.2,0.3),(0,0.1,0.2,0.3))	((1,1,1,1),(0,0,0,0),(0,0,0,0))	((0.1,0.2,0.3,0.4),(0.4,0.6,0.7,0.9),(0.4,0.6,0.7,0.9))
A_2	((0,0,0,0),(1,1,1,1),(1,1,1,1))	((1,1,1,1),(0,0,0,0),(0,0,0,0))	((0.7,0.8,0.9,1),(0,0.1,0.2,0.3),(0,0.1,0.2,0.3))	((0.5,0.6,0.7,0.8),(0,0.2,0.3,0.4),(0,0.2,0.3,0.4))	((0.3,0.4,0.5,0.6),(0.2,0.4,0.5,0.7),(0.2,0.4,0.5,0.7))	((0.1,0.2,0.3,0.4),(0.4,0.6,0.7,0.9),(0.4,0.6,0.7,0.9))	((0,0,0,0),(1,1,1,1),(1,1,1,1))	((0.3,0.4,0.5,0.6),(0.2,0.4,0.5,0.7),(0.2,0.4,0.5,0.7))
A_3	((0,0.1,0.2,0.3),(0.7,0.8,0.9,1),(0.7,0.8,0.9,1))	((0.1,0.2,0.3,0.4),(0.4,0.6,0.7,0.9),(0.4,0.6,0.7,0.9))	((0.3,0.4,0.5,0.6),(0.2,0.4,0.5,0.7),(0.2,0.4,0.5,0.7))	((0.5,0.6,0.7,0.8),(0,0.2,0.3,0.4),(0,0.2,0.3,0.4))	((0.7,0.8,0.9,1),(0,0.1,0.2,0.3),(0,0.1,0.2,0.3))	((0,0.1,0.2,0.3),(0.7,0.8,0.9,1),(0.7,0.8,0.9,1))	((0,0.1,0.2,0.3),(0.7,0.8,0.9,1),(0.7,0.8,0.9,1))	((0.5,0.6,0.7,0.8),(0,0.2,0.3,0.4),(0,0.2,0.3,0.4))
A_4	((0.3,0.4,0.5,0.6),(0.2,0.4,0.5,0.7),(0.2,0.4,0.5,0.7))	((0.1,0.2,0.3,0.4),(0.4,0.6,0.7,0.9),(0.4,0.6,0.7,0.9))	((0,0.1,0.2,0.3),(0.7,0.8,0.9,1),(0.7,0.8,0.9,1))	((0,0,0,0),(1,1,1,1),(1,1,1,1))	((1,1,1,1),(0,0,0,0),(0,0,0,0))	((0,0,0,0),(1,1,1,1),(1,1,1,1))	((0.1,0.2,0.3,0.4),(0.4,0.6,0.7,0.9),(0.4,0.6,0.7,0.9))	((0.7,0.8,0.9,1),(0,0.1,0.2,0.3),(0,0.1,0.2,0.3))
A_5	((0.5,0.6,0.7,0.8),(0,0.2,0.3,0.4),(0,0.2,0.3,0.4))	((0.3,0.4,0.5,0.6),(0.2,0.4,0.5,0.7),(0.2,0.4,0.5,0.7))	((0.1,0.2,0.3,0.4),(0.4,0.6,0.7,0.9),(0.4,0.6,0.7,0.9))	((0,0.1,0.2,0.3),(0.7,0.8,0.9,1),(0.7,0.8,0.9,1))	((0,0,0,0),(1,1,1,1),(1,1,1,1))	((1,1,1,1),(0,0,0,0),(0,0,0,0))	((0.3,0.4,0.5,0.6),(0.2,0.4,0.5,0.7),(0.2,0.4,0.5,0.7))	((1,1,1,1),(0,0,0,0),(0,0,0,0))
A_6	((0.5,0.6,0.7,0.8),(0,0.2,0.3,0.4),(0,0.2,0.3,0.4))	((0.5,0.6,0.7,0.8),(0,0.2,0.3,0.4),(0,0.2,0.3,0.4))	((0.3,0.4,0.5,0.6),(0.2,0.4,0.5,0.7),(0.2,0.4,0.5,0.7))	((0.5,0.6,0.7,0.8),(0,0.2,0.3,0.4),(0,0.2,0.3,0.4))	((1,1,1,1),(0,0,0,0),(0,0,0,0))	((0.7,0.8,0.9,1),(0,0.1,0.2,0.3),(0,0.1,0.2,0.3))	((0.5,0.6,0.7,0.8),(0,0.2,0.3,0.4),(0,0.2,0.3,0.4))	((0.7,0.8,0.9,1),(0,0.1,0.2,0.3),(0,0.1,0.2,0.3))

A7	((0.7,0.8,0.9,1),(0,0.1,0.2,0.3),(0,0.1,0.2,0.3))	((0.7,0.8,0.9,1),(0,0.1,0.2,0.3),(0,0.1,0.2,0.3))	((0.1,0.2,0.3,0.4),(0.4,0.6,0.7,0.9),(0.4,0.6,0.7,0.9))	((0.7,0.8,0.9,1),(0,0.1,0.2,0.3),(0,0.1,0.2,0.3))	((0.7,0.8,0.9,1),(0,0.1,0.2,0.3),(0,0.1,0.2,0.3))	((0.5,0.6,0.7,0.8),(0,0.2,0.3,0.4),(0,0.2,0.3,0.4))	((0.7,0.8,0.9,1),(0,0.1,0.2,0.3),(0,0.1,0.2,0.3))	((0.5,0.6,0.7,0.8),(0,0.2,0.3,0.4),(0,0.2,0.3,0.4))
A8	((0.7,0.8,0.9,1),(0,0.1,0.2,0.3),(0,0.1,0.2,0.3))	((1,1,1,1),(0,0,0,0),(0,0,0,0))	((0,0.1,0.2,0.3),(0.7,0.8,0.9,1),(0.7,0.8,0.9,1))	((1,1,1,1),(0,0,0,0),(0,0,0,0))	((0.5,0.6,0.7,0.8),(0,0.2,0.3,0.4),(0,0.2,0.3,0.4))	((0.3,0.4,0.5,0.6),(0.2,0.4,0.5,0.7),(0.2,0.4,0.5,0.7))	((1,1,1,1),(0,0,0,0),(0,0,0,0))	((0,0.1,0.2,0.3),(0.7,0.8,0.9,1),(0.7,0.8,0.9,1))
A9	((1,1,1,1),(0,0,0,0),(0,0,0,0))	((0,0,0,0),(1,1,1,1),(1,1,1,1))	((0,0,0,0),(1,1,1,1),(1,1,1,1))	((0,0,0,0),(1,1,1,1),(1,1,1,1))	((0.3,0.4,0.5,0.6),(0.2,0.4,0.5,0.7),(0.2,0.4,0.5,0.7))	((0.1,0.2,0.3,0.4),(0.4,0.6,0.7,0.9),(0.4,0.6,0.7,0.9))	((0,0,0,0),(1,1,1,1),(1,1,1,1))	((1,1,1,1),(0,0,0,0),(0,0,0,0))
A10	((1,1,1,1),(0,0,0,0),(0,0,0,0))	((0,0.1,0.2,0.3),(0.7,0.8,0.9,1),(0.7,0.8,0.9,1))	((1,1,1,1),(0,0,0,0),(0,0,0,0))	((0,0.1,0.2,0.3),(0.7,0.8,0.9,1),(0.7,0.8,0.9,1))	((0.1,0.2,0.3,0.4),(0.4,0.6,0.7,0.9),(0.4,0.6,0.7,0.9))	((0,0.1,0.2,0.3),(0.7,0.8,0.9,1),(0.7,0.8,0.9,1))	((0,0.1,0.2,0.3),(0.7,0.8,0.9,1),(0.7,0.8,0.9,1))	((0.1,0.2,0.3,0.4),(0.4,0.6,0.7,0.9),(0.4,0.6,0.7,0.9))
A11	((0,0,0,0),(1,1,1,1),(1,1,1,1))	((0.1,0.2,0.3,0.4),(0.4,0.6,0.7,0.9),(0.4,0.6,0.7,0.9))	((0.7,0.8,0.9,1),(0,0.1,0.2,0.3),(0,0.1,0.2,0.3))	((0.1,0.2,0.3,0.4),(0.4,0.6,0.7,0.9),(0.4,0.6,0.7,0.9))	((0,0.1,0.2,0.3),(0.7,0.8,0.9,1),(0.7,0.8,0.9,1))	((0,0,0,0),(1,1,1,1),(1,1,1,1))	((0.1,0.2,0.3,0.4),(0.4,0.6,0.7,0.9),(0.4,0.6,0.7,0.9))	((0.1,0.2,0.3,0.4),(0.4,0.6,0.7,0.9),(0.4,0.6,0.7,0.9))
	C1	C2	C3	C4	C5	C6	C7	C8
A1	((0.3,0.4,0.5,0.6),(0.2,0.4,0.5,0.7),(0.2,0.4,0.5,0.7))	((0,0.1,0.2,0.3),(0.7,0.8,0.9,1),(0.7,0.8,0.9,1))	((0.1,0.2,0.3,0.4),(0.4,0.6,0.7,0.9),(0.4,0.6,0.7,0.9))	((0.3,0.4,0.5,0.6),(0.2,0.4,0.5,0.7),(0.2,0.4,0.5,0.7))	((0.5,0.6,0.7,0.8),(0,0.2,0.3,0.4),(0,0.2,0.3,0.4))	((0.7,0.8,0.9,1),(0,0.1,0.2,0.3),(0,0.1,0.2,0.3))	((1,1,1,1),(0,0,0,0),(0,0,0,0))	((0.1,0.2,0.3,0.4),(0.4,0.6,0.7,0.9),(0.4,0.6,0.7,0.9))
A2	((0.1,0.2,0.3,0.4),(0.4,0.6,0.7,0.9),(0.4,0.6,0.7,0.9))	((1,1,1,1),(0,0,0,0),(0,0,0,0))	((0.7,0.8,0.9,1),(0,0.1,0.2,0.3),(0,0.1,0.2,0.3))	((0.3,0.4,0.5,0.6),(0.2,0.4,0.5,0.7),(0.2,0.4,0.5,0.7))	((0.3,0.4,0.5,0.6),(0.2,0.4,0.5,0.7),(0.2,0.4,0.5,0.7))	((0.1,0.2,0.3,0.4),(0.4,0.6,0.7,0.9),(0.4,0.6,0.7,0.9))	((0.3,0.4,0.5,0.6),(0.2,0.4,0.5,0.7),(0.2,0.4,0.5,0.7))	((0.3,0.4,0.5,0.6),(0.2,0.4,0.5,0.7),(0.2,0.4,0.5,0.7))
A3	((0,0.1,0.2,0.3),(0.7,0.8,0.9,1),(0.7,0.8,0.9,1))	((0.1,0.2,0.3,0.4),(0.4,0.6,0.7,0.9),(0.4,0.6,0.7,0.9))	((0.3,0.4,0.5,0.6),(0.2,0.4,0.5,0.7),(0.2,0.4,0.5,0.7))	((0.1,0.2,0.3,0.4),(0.4,0.6,0.7,0.9),(0.4,0.6,0.7,0.9))	((0.3,0.4,0.5,0.6),(0.2,0.4,0.5,0.7),(0.2,0.4,0.5,0.7))	((0,0.1,0.2,0.3),(0.7,0.8,0.9,1),(0.7,0.8,0.9,1))	((0.1,0.2,0.3,0.4),(0.4,0.6,0.7,0.9),(0.4,0.6,0.7,0.9))	((0.5,0.6,0.7,0.8),(0,0.2,0.3,0.4),(0,0.2,0.3,0.4))
A4	((0,0,0,0),(1,1,1,1),(1,1,1,1))	((0.1,0.2,0.3,0.4),(0.4,0.6,0.7,0.9),(0.4,0.6,0.7,0.9))	((0,0.1,0.2,0.3),(0.7,0.8,0.9,1),(0.7,0.8,0.9,1))	((0,0.1,0.2,0.3),(0.7,0.8,0.9,1),(0.7,0.8,0.9,1))	((0.1,0.2,0.3,0.4),(0.4,0.6,0.7,0.9),(0.4,0.6,0.7,0.9))	((0,0,0,0),(1,1,1,1),(1,1,1,1))	((0,0.1,0.2,0.3),(0.7,0.8,0.9,1),(0.7,0.8,0.9,1))	((0.7,0.8,0.9,1),(0,0.1,0.2,0.3),(0,0.1,0.2,0.3))
A5	((1,1,1,1),(0,0,0,0),(0,0,0,0))	((0.3,0.4,0.5,0.6),(0.2,0.4,0.5,0.7),(0.2,0.4,0.5,0.7))	((0.1,0.2,0.3,0.4),(0.4,0.6,0.7,0.9),(0.4,0.6,0.7,0.9))	((0,0,0,0),(1,1,1,1),(1,1,1,1))	((0,0.1,0.2,0.3),(0.7,0.8,0.9,1),(0.7,0.8,0.9,1))	((1,1,1,1),(0,0,0,0),(0,0,0,0))	((0,0,0,0),(1,1,1,1),(1,1,1,1))	((0.5,0.6,0.7,0.8),(0,0.2,0.3,0.4),(0,0.2,0.3,0.4))
A6	((0.3,0.4,0.5,0.6),(0.2,0.4,0.5,0.7),(0.2,0.4,0.5,0.7))	((0.3,0.4,0.5,0.6),(0.2,0.4,0.5,0.7),(0.2,0.4,0.5,0.7))	((0.3,0.4,0.5,0.6),(0.2,0.4,0.5,0.7),(0.2,0.4,0.5,0.7))	((1,1,1,1),(0,0,0,0),(0,0,0,0))	((0,0,0,0),(1,1,1,1),(1,1,1,1))	((0.3,0.4,0.5,0.6),(0.2,0.4,0.5,0.7),(0.2,0.4,0.5,0.7))	((1,1,1,1),(0,0,0,0),(0,0,0,0))	((0.7,0.8,0.9,1),(0,0.1,0.2,0.3),(0,0.1,0.2,0.3))
A7	((0.1,0.2,0.3,0.4),(0.4,0.6,0.7,0.9),(0.4,0.6,0.6,0.7,0.9))	((0.1,0.2,0.3,0.4),(0.4,0.6,0.7,0.9),(0.4,0.6,0.6,0.7,0.9))	((0.1,0.2,0.3,0.4),(0.4,0.6,0.7,0.9),(0.4,0.6,0.6,0.7,0.9))	((0.7,0.8,0.9,1),(0,0.1,0.2,0.3),(0.7,0.8,0.9,1),(0.7,0.8,0.9,1))	((0.3,0.4,0.5,0.6),(0.2,0.4,0.5,0.7),(0.2,0.4,0.5,0.7))	((0.1,0.2,0.3,0.4),(0.4,0.6,0.7,0.9),(0.4,0.6,0.7,0.9))	((0.7,0.8,0.9,1),(0,0.1,0.2,0.3),(0.7,0.8,0.9,1),(0.7,0.8,0.9,1))	((0.5,0.6,0.7,0.8),(0,0.2,0.3,0.4),(0,0.2,0.3,0.4))
A8	((0,0.1,0.2,0.3),(0.7,0.8,0.9,1),(0.7,0.8,0.9,1))	((0,0.1,0.2,0.3),(0.7,0.8,0.9,1),(0.7,0.8,0.9,1))	((0,0.1,0.2,0.3),(0.7,0.8,0.9,1),(0.7,0.8,0.9,1))	((0.5,0.6,0.7,0.8),(0,0.2,0.3,0.4),(0,0.2,0.3,0.4))	((0.1,0.2,0.3,0.4),(0.4,0.6,0.7,0.9),(0.4,0.6,0.7,0.9))	((0,0.1,0.2,0.3),(0.7,0.8,0.9,1),(0.7,0.8,0.9,1))	((0.5,0.6,0.7,0.8),(0,0.2,0.3,0.4),(0,0.2,0.3,0.4))	((0.5,0.6,0.7,0.8),(0,0.2,0.3,0.4),(0,0.2,0.3,0.4))
A9	((0,0,0,0),(1,1,1,1),(1,1,1,1))	((0,0,0,0),(1,1,1,1),(1,1,1,1))	((0,0,0,0),(1,1,1,1),(1,1,1,1))	((0,0,0,0),(1,1,1,1),(1,1,1,1))	((0,0.1,0.2,0.3),(0.7,0.8,0.9,1),(0.7,0.8,0.9,1))	((0,0,0,0),(1,1,1,1),(1,1,1,1))	((0,0.1,0.2,0.3),(0.7,0.8,0.9,1),(0.7,0.8,0.9,1))	((0,0,0,0),(1,1,1,1),(1,1,1,1))
A10	((1,1,1,1),(0,0,0,0),(0,0,0,0))	((1,1,1,1),(0,0,0,0),(0,0,0,0))	((1,1,1,1),(0,0,0,0),(0,0,0,0))	((0,0.1,0.2,0.3),(0.7,0.8,0.9,1),(0.7,0.8,0.9,1))	((0,0,0,0),(1,1,1,1),(1,1,1,1))	((1,1,1,1),(0,0,0,0),(0,0,0,0))	((0,0,0,0),(1,1,1,1),(1,1,1,1))	((0,0.1,0.2,0.3),(0.7,0.8,0.9,1),(0.7,0.8,0.9,1))
A11	((0.7,0.8,0.9,1),(0,0.1,0.2,0.3),(0,0.1,0.2,0.3))	((0.7,0.8,0.9,1),(0,0.1,0.2,0.3),(0,0.1,0.2,0.3))	((0.7,0.8,0.9,1),(0,0.1,0.2,0.3),(0,0.1,0.2,0.3))	((0.1,0.2,0.3,0.4),(0.4,0.6,0.7,0.9),(0.4,0.6,0.7,0.9))	((1,1,1,1),(0,0,0,0),(0,0,0,0))	((0.7,0.8,0.9,1),(0,0.1,0.2,0.3),(0,0.1,0.2,0.3))	((1,1,1,1),(0,0,0,0),(0,0,0,0))	((0.1,0.2,0.3,0.4),(0.4,0.6,0.7,0.9),(0.4,0.6,0.7,0.9))
	C1	C2	C3	C4	C5	C6	C7	C8
A1	((0,0,0,0),(1,1,1,1),(1,1,1,1))	((0,0.1,0.2,0.3),(0.7,0.8,0.9,1),(0.7,0.8,0.9,1))	((0.1,0.2,0.3,0.4),(0.4,0.6,0.7,0.9),(0.4,0.6,0.7,0.9))	((0.3,0.4,0.5,0.6),(0.2,0.4,0.5,0.7),(0.2,0.4,0.5,0.7))	((0.5,0.6,0.7,0.8),(0,0.2,0.3,0.4),(0,0.2,0.3,0.4))	((0.7,0.8,0.9,1),(0,0.1,0.2,0.3),(0,0.1,0.2,0.3))	((1,1,1,1),(0,0,0,0),(0,0,0,0))	((0.1,0.2,0.3,0.4),(0.4,0.6,0.7,0.9),(0.4,0.6,0.7,0.9))
A2	((1,1,1,1),(0,0,0,0),(0,0,0,0))	((1,1,1,1),(0,0,0,0),(0,0,0,0))	((0.7,0.8,0.9,1),(0,0.1,0.2,0.3),(0,0.1,0.2,0.3))	((0.5,0.6,0.7,0.8),(0,0.2,0.3,0.4),(0,0.2,0.3,0.4))	((0.3,0.4,0.5,0.6),(0.2,0.4,0.5,0.7),(0.2,0.4,0.5,0.7))	((0.1,0.2,0.3,0.4),(0.4,0.6,0.7,0.9),(0.4,0.6,0.7,0.9))	((0,0,0,0),(1,1,1,1),(1,1,1,1))	((0.3,0.4,0.5,0.6),(0.2,0.4,0.5,0.7),(0.2,0.4,0.5,0.7))
A3	((0.7,0.8,0.9,1),(0,0.1,0.2,0.3),(0,0.1,0.2,0.3))	((0,0,0,0),(1,1,1,1),(1,1,1,1))	((0.3,0.4,0.5,0.6),(0.2,0.4,0.5,0.7),(0.2,0.4,0.5,0.7))	((0.5,0.6,0.7,0.8),(0,0.2,0.3,0.4),(0,0.2,0.3,0.4))	((0.7,0.8,0.9,1),(0,0.1,0.2,0.3),(0,0.1,0.2,0.3))	((0,0.1,0.2,0.3),(0.7,0.8,0.9,1),(0.7,0.8,0.9,1))	((1,1,1,1),(0,0,0,0),(0,0,0,0))	((0.5,0.6,0.7,0.8),(0,0.2,0.3,0.4),(0,0.2,0.3,0.4))
A4	((0.5,0.6,0.7,0.8),(0,0.2,0.3,0.4),(0,0.2,0.3,0.4))	((1,1,1,1),(0,0,0,0),(0,0,0,0))	((0,0,0,0),(1,1,1,1),(1,1,1,1))	((0,0,0,0),(1,1,1,1),(1,1,1,1))	((0,0,0,0),(1,1,1,1),(1,1,1,1))	((0,0,0,0),(1,1,1,1),(1,1,1,1))	((0.7,0.8,0.9,1),(0,0.1,0.2,0.3),(0,0.1,0.2,0.3))	((0.7,0.8,0.9,1),(0,0.1,0.2,0.3),(0,0.1,0.2,0.3))
A5	((0.3,0.4,0.5,0.6),(0.2,0.4,0.5,0.7),(0.2,0.4,0.5,0.7))	((0.7,0.8,0.9,1),(0,0.1,0.2,0.3),(0,0.1,0.2,0.3))	((1,1,1,1),(0,0,0,0),(0,0,0,0))	((0,0,0,0),(1,1,1,1),(1,1,1,1))	((1,1,1,1),(0,0,0,0),(0,0,0,0))	((0,0,0,0),(1,1,1,1),(1,1,1,1))	((0.5,0.6,0.7,0.8),(0,0.2,0.3,0.4),(0,0.2,0.3,0.4))	((1,1,1,1),(0,0,0,0),(0,0,0,0))
A6	((0.1,0.2,0.3,0.4),(0.4,0.6,0.7,0.9),(0.4,0.6,0.7,0.9))	((0.5,0.6,0.7,0.8),(0,0.2,0.3,0.4),(0,0.2,0.3,0.4))	((0.7,0.8,0.9,1),(0,0.1,0.2,0.3),(0,0.1,0.2,0.3))	((1,1,1,1),(0,0,0,0),(0,0,0,0))	((0,0,0,0),(1,1,1,1),(1,1,1,1))	((0,0,0,0),(1,1,1,1),(1,1,1,1))	((0.3,0.4,0.5,0.6),(0.2,0.4,0.5,0.7),(0.2,0.4,0.5,0.7))	((0.7,0.8,0.9,1),(0,0.1,0.2,0.3),(0,0.1,0.2,0.3))
A7	((0,0.1,0.2,0.3),(0.7,0.8,0.9,1),(0.7,0.8,0.9,1))	((0,0.1,0.2,0.3),(0.7,0.8,0.9,1),(0.7,0.8,0.9,1))	((0,0.1,0.2,0.3),(0.7,0.8,0.9,1),(0.7,0.8,0.9,1))	((0.3,0.4,0.5,0.6),(0.2,0.4,0.5,0.7),(0.2,0.4,0.5,0.7))	((0.1,0.2,0.3,0.4),(0.4,0.6,0.7,0.9),(0.4,0.6,0.7,0.9))	((0.3,0.4,0.5,0.6),(0.2,0.4,0.5,0.7),(0.2,0.4,0.5,0.7))	((0,0,0,0),(1,1,1,1),(1,1,1,1))	((0.5,0.6,0.7,0.8),(0,0.2,0.3,0.4),(0,0.2,0.3,0.4))
A8	((0,0,0,0),(1,1,1,1),(1,1,1,1))	((0.1,0.2,0.3,0.4),(0.4,0.6,0.7,0.9),(0.4,0.6,0.7,0.9))	((0.3,0.4,0.5,0.6),(0.2,0.4,0.5,0.7),(0.2,0.4,0.5,0.7))	((0.5,0.6,0.7,0.8),(0,0.2,0.3,0.4),(0,0.2,0.3,0.4))	((0.3,0.4,0.5,0.6),(0.2,0.4,0.5,0.7),(0.2,0.4,0.5,0.7))	((0.5,0.6,0.7,0.8),(0,0.2,0.3,0.4),(0,0.2,0.3,0.4))	((0,0.1,0.2,0.3),(0.7,0.8,0.9,1),(0.7,0.8,0.9,1))	((1,1,1,1),(0,0,0,0),(0,0,0,0))
A9	((1,1,1,1),(0,0,0,0),(0,0,0,0))	((0,0.1,0.2,0.3),(0.7,0.8,0.9,1),(0.7,0.8,0.9,1))	((0.1,0.2,0.3,0.4),(0.4,0.6,0.7,0.9),(0.4,0.6,0.7,0.9))	((0.3,0.4,0.5,0.6),(0.2,0.4,0.5,0.7),(0.2,0.4,0.5,0.7))	((0.1,0.2,0.3,0.4),(0.4,0.6,0.7,0.9),(0.4,0.6,0.7,0.9))	((0.3,0.4,0.5,0.6),(0.2,0.4,0.5,0.7),(0.2,0.4,0.5,0.7))	((0,0,0,0),(1,1,1,1),(1,1,1,1))	((0.7,0.8,0.9,1),(0,0.1,0.2,0.3),(0,0.1,0.2,0.3))
A10	((1,1,1,1),(0,0,0,0),(0,0,0,0))	((0,0,0,0),(1,1,1,1),(1,1,1,1))	((0,0.1,0.2,0.3),(0.7,0.8,0.9,1),(0.7,0.8,0.9,1))	((0,0.1,0.2,0.3),(0.7,0.8,0.9,1),(0.7,0.8,0.9,1))	((0,0.1,0.2,0.3),(0.7,0.8,0.9,1),(0.7,0.8,0.9,1))	((0,0.1,0.2,0.3),(0.7,0.8,0.9,1),(0.7,0.8,0.9,1))	((0,0.1,0.2,0.3),(0.7,0.8,0.9,1),(0.7,0.8,0.9,1))	((0,0.1,0.2,0.3),(0.7,0.8,0.9,1),(0.7,0.8,0.9,1))
A11	((0,0,0,0),(1,1,1,1),(1,1,1,1))	((0.1,0.2,0.3,0.4),(0.4,0.6,0.7,0.9),(0.4,0.6,0.7,0.9))	((0,0,0,0),(1,1,1,1),(1,1,1,1))	((0,0.1,0.2,0.3),(0.7,0.8,0.9,1),(0.7,0.8,0.9,1))	((0,0,0,0),(1,1,1,1),(1,1,1,1))	((0,0.1,0.2,0.3),(0.7,0.8,0.9,1),(0.7,0.8,0.9,1))	((0.1,0.2,0.3,0.4),(0.4,0.6,0.7,0.9),(0.4,0.6,0.7,0.9))	((0.3,0.4,0.5,0.6),(0.2,0.4,0.5,0.7),(0.2,0.4,0.5,0.7))

Table 3. Normalized matrix by SIWEC method.

	C1	C2	C3	C4	C5	C6	C7	C8
A1	0.172222	0.15	0.372549	0.567073	0.992481	1	1	0.347561
A2	0.438889	1	1	0.72561	0.699248	0.372549	0.172222	0.567073
A3	0.383333	0.211111	0.607843	0.652439	1	0.176471	0.488889	0.804878
A4	0.416667	0.544444	0.117647	0.054878	0.593985	0	0.438889	0.932927

A_5	0.75	0.627778	0.640523	0.054878	0.518797	0.784314	0.416667	1
A_6	0.522222	0.661111	0.738562	1	0.834586	0.928105	0.75	0.932927
A_7	0.438889	0.561111	0.535948	0.932927	0.947368	0.745098	0.672222	0.804878
A_8	0.333333	0.488889	0.320261	0.902439	0.706767	0.54902	0.627778	0.689024
A_9	0.666667	0.05	0.124183	0.189024	0.443609	0.326797	0.05	0.676829
A_{10}	1	0.383333	0.843137	0.22561	0.210526	0.575163	0.1	0.439024
A_{11}	0.283333	0.494444	0.666667	0.286585	0.518797	0.392157	0.544444	0.420732

Table 4. The multiplication values between standard deviation and normalization values.

	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8
A_1	0.040395	0.039981	0.105454	0.203757	0.249536	0.313634	0.289139	0.078375
A_2	0.102942	0.266542	0.28306	0.260722	0.17581	0.116844	0.049796	0.127875
A_3	0.089912	0.05627	0.172056	0.234431	0.251427	0.055347	0.141357	0.1815
A_4	0.09773	0.145118	0.033301	0.019718	0.149344	0	0.1269	0.210375
A_5	0.175914	0.167329	0.181306	0.019718	0.130439	0.245987	0.120475	0.2255
A_6	0.122488	0.176214	0.209057	0.359314	0.209837	0.291085	0.216855	0.210375
A_7	0.102942	0.14956	0.151705	0.335214	0.238194	0.233688	0.194366	0.1815
A_8	0.078184	0.13031	0.090653	0.324259	0.1777	0.172191	0.181515	0.155375
A_9	0.156368	0.013327	0.035151	0.067919	0.111535	0.102495	0.014457	0.152625
A_{10}	0.234552	0.102175	0.238659	0.081065	0.052932	0.180391	0.028914	0.099
A_{11}	0.066456	0.13179	0.188707	0.102974	0.130439	0.122994	0.15742	0.094875

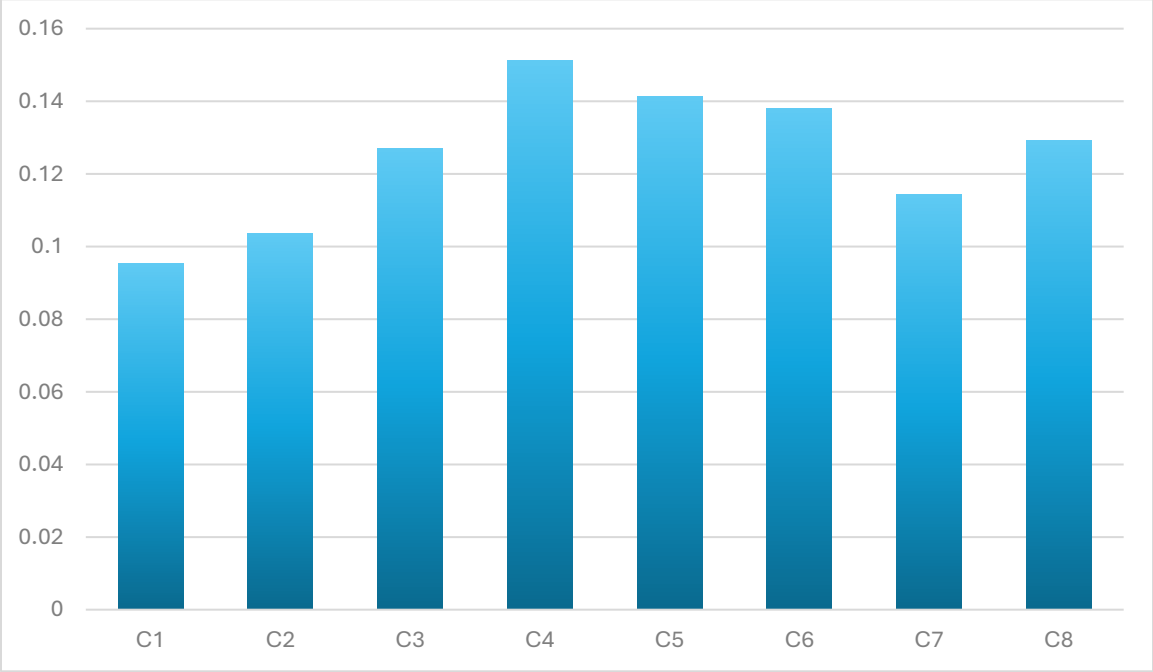

Fig 1. The criteria weights.

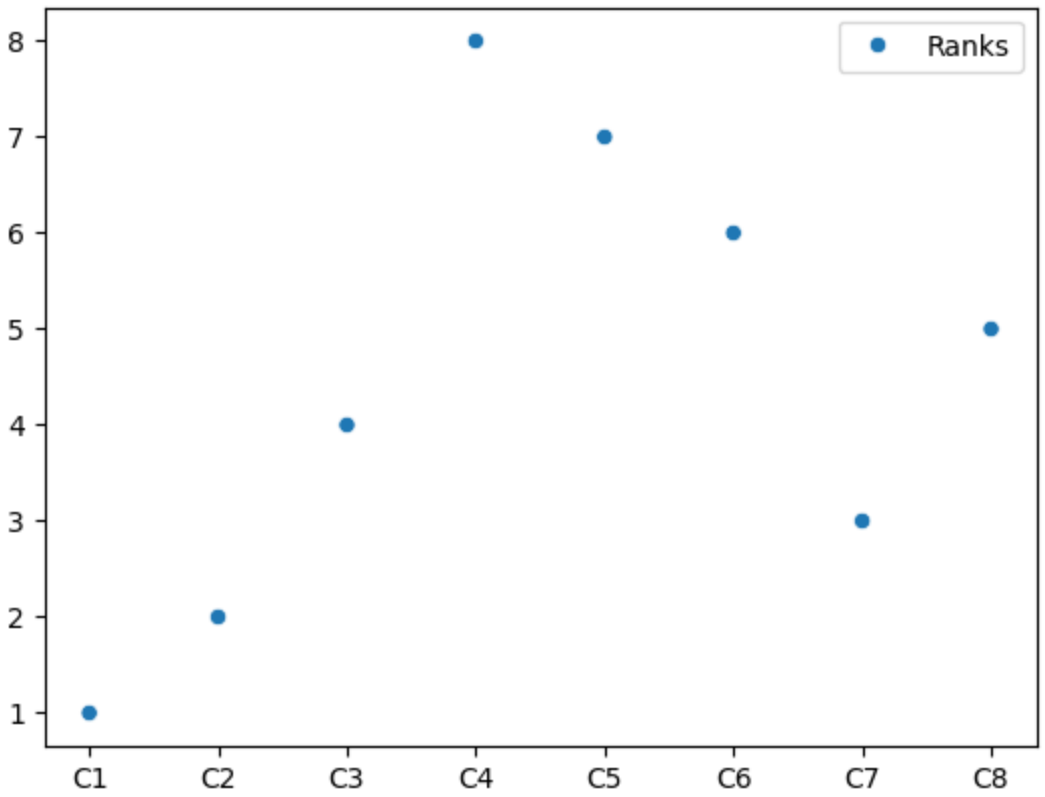

Fig 2. The ranks of criteria.

We show the results of the ranking method.

2. We normalize the decision matrix for positive and negative criteria by three different methods using Eqs. (21-25) as shown in Table 5.

3. We combined the normalization matrix using Eq. (26). We put the value of $\alpha = \beta = 1/3$ as shown in Table 6.

4. We Compute the distance between each alternative and reference alternative using Eqs. (27-28).

6. We compute the final comprehensive score using Eq. (29).

7. Rank the alternatives as shown in Fig 3.

Table 5. The three normalization matrices.

	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8
A_1	0.031860226	0.029001	0.062432	0.101418	0.132931	0.17095	0.190074	0.045637
A_2	0.081192189	0.19334	0.167579	0.129771	0.093656	0.063687	0.032735	0.07446
A_3	0.070914697	0.040816	0.101862	0.116685	0.133938	0.030168	0.092925	0.105685
A_4	0.077081192	0.105263	0.019715	0.009815	0.079557	0	0.083421	0.122498
A_5	0.138746146	0.121375	0.107338	0.009815	0.069486	0.134078	0.079197	0.131305
A_6	0.096608428	0.12782	0.123768	0.178844	0.111782	0.158659	0.142555	0.122498
A_7	0.081192189	0.108485	0.089814	0.166848	0.126888	0.127374	0.127772	0.105685

A_8	0.061664954	0.094522	0.053669	0.161396	0.094663	0.093855	0.119324	0.090472
A_9	0.123329907	0.009667	0.020811	0.033806	0.059416	0.055866	0.009504	0.088871
A_{10}	0.184994861	0.074114	0.141292	0.040349	0.028197	0.098324	0.019007	0.057646
A_{11}	0.052415211	0.095596	0.11172	0.051254	0.069486	0.067039	0.103485	0.055244
	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8
A_1	0.172222	0.15	0.372549	0.567073	0.992481	1	1	0.347561
A_2	0.438889	1	1	0.72561	0.699248	0.372549	0.172222	0.567073
A_3	0.383333	0.211111	0.607843	0.652439	1	0.176471	0.488889	0.804878
A_4	0.416667	0.544444	0.117647	0.054878	0.593985	0	0.438889	0.932927
A_5	0.75	0.627778	0.640523	0.054878	0.518797	0.784314	0.416667	1
A_6	0.522222	0.661111	0.738562	1	0.834586	0.928105	0.75	0.932927
A_7	0.438889	0.561111	0.535948	0.932927	0.947368	0.745098	0.672222	0.804878
A_8	0.333333	0.488889	0.320261	0.902439	0.706767	0.54902	0.627778	0.689024
A_9	0.666667	0.05	0.124183	0.189024	0.443609	0.326797	0.05	0.676829
A_{10}	1	0.383333	0.843137	0.22561	0.210526	0.575163	0.1	0.439024
A_{11}	0.283333	0.494444	0.666667	0.286585	0.518797	0.392157	0.544444	0.420732
	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8
A_1	0	0.105263	0.288889	0.541935	0.990476	1	1	0
A_2	0.322148	1	1	0.709677	0.619048	0.372549	0.128655	0.336449
A_3	0.255034	0.169591	0.555556	0.632258	1	0.176471	0.461988	0.700935
A_4	0.295302	0.520468	0	0	0.485714	0	0.409357	0.897196
A_5	0.697987	0.608187	0.592593	0	0.390476	0.784314	0.385965	1
A_6	0.422819	0.643275	0.703704	1	0.790476	0.928105	0.736842	0.897196
A_7	0.322148	0.538012	0.474074	0.929032	0.933333	0.745098	0.654971	0.700935
A_8	0.194631	0.461988	0.22963	0.896774	0.628571	0.54902	0.608187	0.523364
A_9	0.597315	0	0.007407	0.141935	0.295238	0.326797	0	0.504673
A_{10}	1	0.350877	0.822222	0.180645	0	0.575163	0.052632	0.140187
A_{11}	0.134228	0.467836	0.622222	0.245161	0.390476	0.392157	0.520468	0.11215

Table 6. The combined normalization matrix.

	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8
A_1	0.061224735	0.095806	0.24605	0.417321	0.733814	0.751285	0.757022	0.117959
A_2	0.284883384	0.758002	0.750274	0.540485	0.48549	0.27989	0.112949	0.327039
A_3	0.238287832	0.143414	0.435134	0.48364	0.740181	0.13258	0.359339	0.553543
A_4	0.266245163	0.403099	0.041209	0.019408	0.396348	0	0.320436	0.675506
A_5	0.545818475	0.468021	0.461395	0.019408	0.332675	0.589243	0.303145	0.739392
A_6	0.354776712	0.493989	0.54018	0.753653	0.600101	0.697271	0.562503	0.675506
A_7	0.284883384	0.416084	0.377358	0.701545	0.69561	0.559781	0.501987	0.553543
A_8	0.196351835	0.359819	0.204031	0.67786	0.491857	0.41247	0.467405	0.443195
A_9	0.475925147	0.0179	0.046461	0.123623	0.269003	0.245518	0.017851	0.431579
A_{10}	0.755498458	0.277585	0.624218	0.152046	0.071617	0.432112	0.056755	0.205076
A_{11}	0.154415838	0.364147	0.482405	0.199416	0.332675	0.294622	0.402566	0.187653

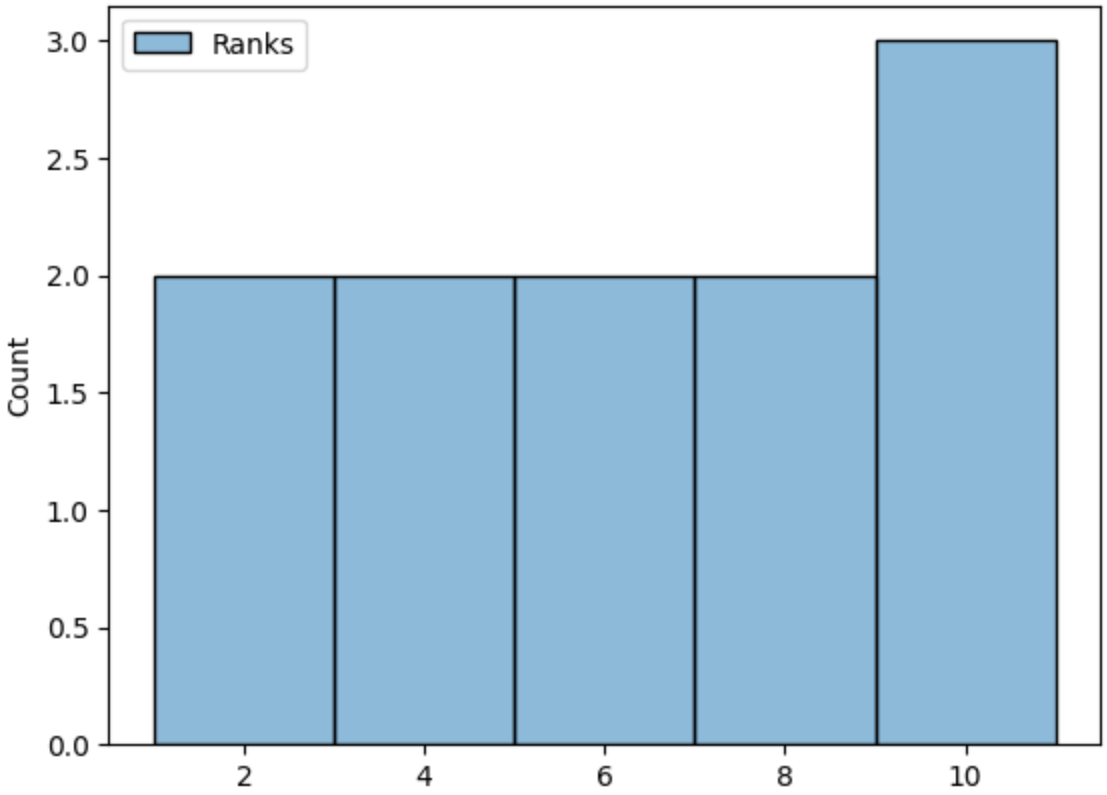

Fig 3. Ranks of alternatives.

Then we applied the sensitivity analysis to show different ranks of alternatives under different values. The sensitivity analysis was conducted with three cases.

In the first case we change the value of δ then we applied the alternative method to show the rank of alternatives. The rank of alternatives is shown in Fig 4 the results show alternative 6 is the best, followed by alternative 7, alternative 2, and alternative 8. We show alternative 9 is the worst. We show alternative 6 is the best in all cases and alternative 9 is the worst. So, the rank of alternatives is stable under different cases.

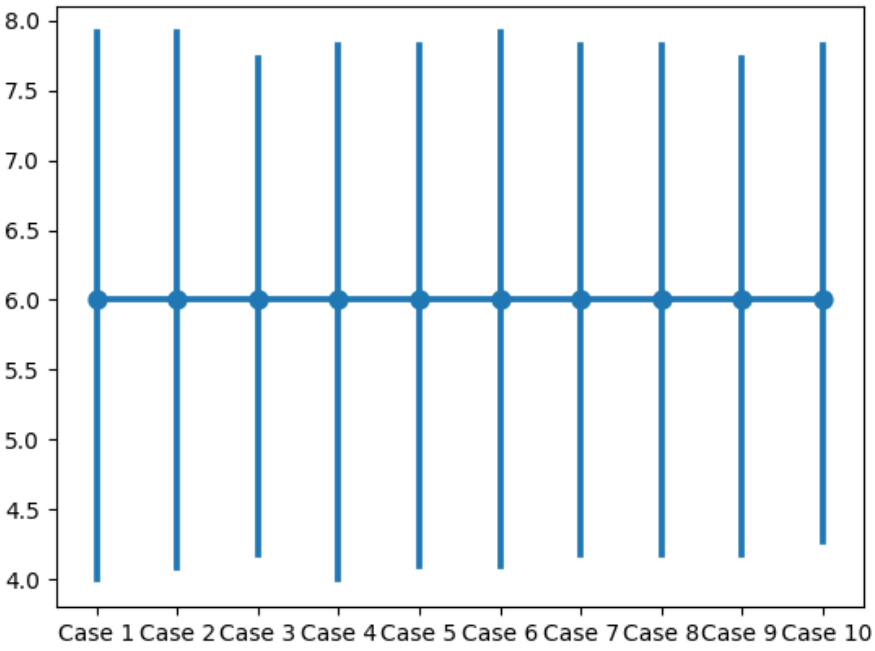

Fig 4. The rank of alternatives under δ values.

In the second case we change the value of γ then we applied the alternative method to show the rank of alternatives. The rank of alternatives is shown in Fig 5. The results show alternative 6 is the best, followed by alternative 7, alternative 2, and alternative 5. We show alternative 9 is the worst. We show alternative 6 is the best in all cases and alternative 9 is the worst. So, the rank of alternatives is stable under different cases.

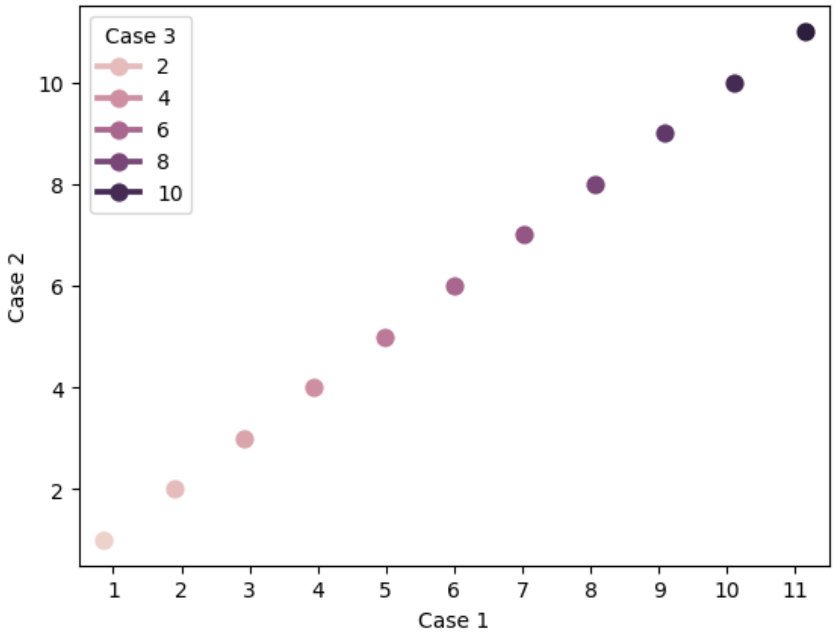

Fig 5. The rank of alternatives under γ values.

In the third case we change the value of α and β then we applied the alternative method to show the rank of alternatives. The rank of alternatives is shown in Fig 6. the results show alternative 6 is the best, followed by alternative 7, alternative 2, and alternative 5. We show alternative 9 is the worst. We show alternative 6 is the best in all cases and alternative 9 is the worst. So, the rank of alternatives is stable under different cases.

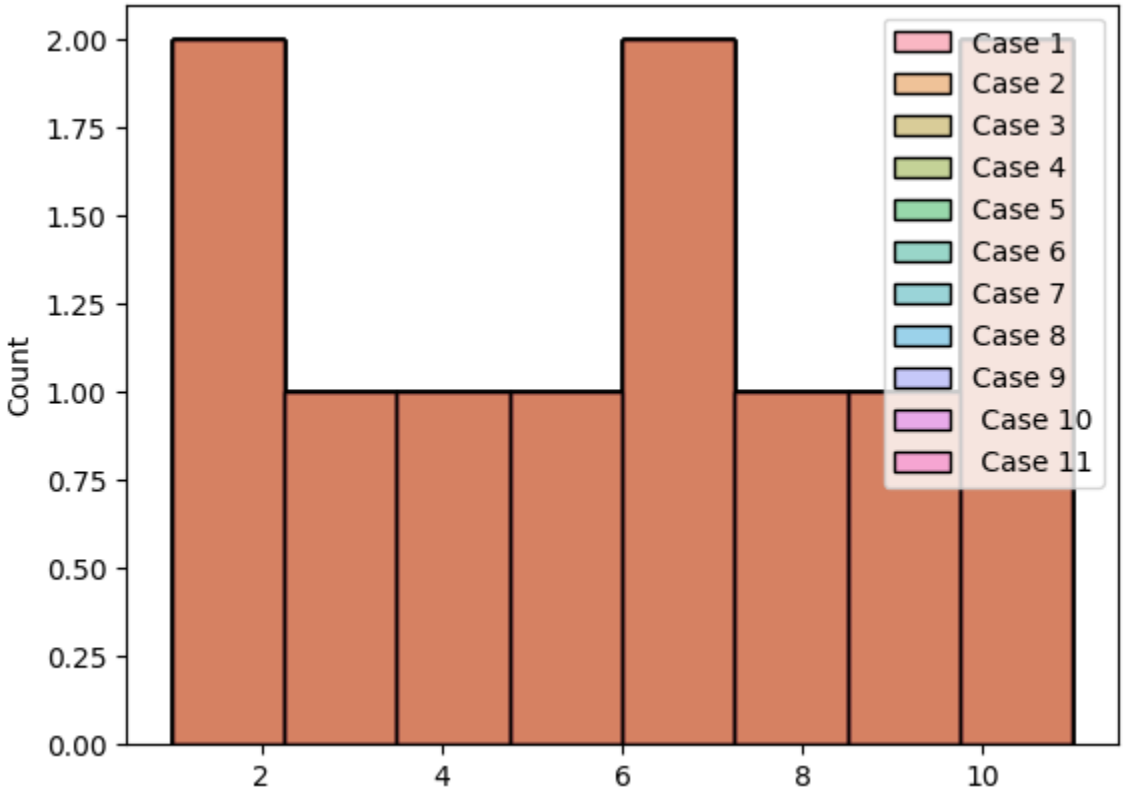

Fig 6. The rank of alternatives under α and β values.

Comparative Study

In this study, we compare the proposed approach by different methods to show the ranks of alternatives. Our study is compared with different methods such as VIKOR, MABAC, TOPSIS, and EDAS methods. The aim of this study is to show the effectiveness of the proposed approach. Fig 7 shows the ranks of relations of each method. Fig 8 shows the ranks of each method. Th results show the proposed approach is effective compared to other different methods.

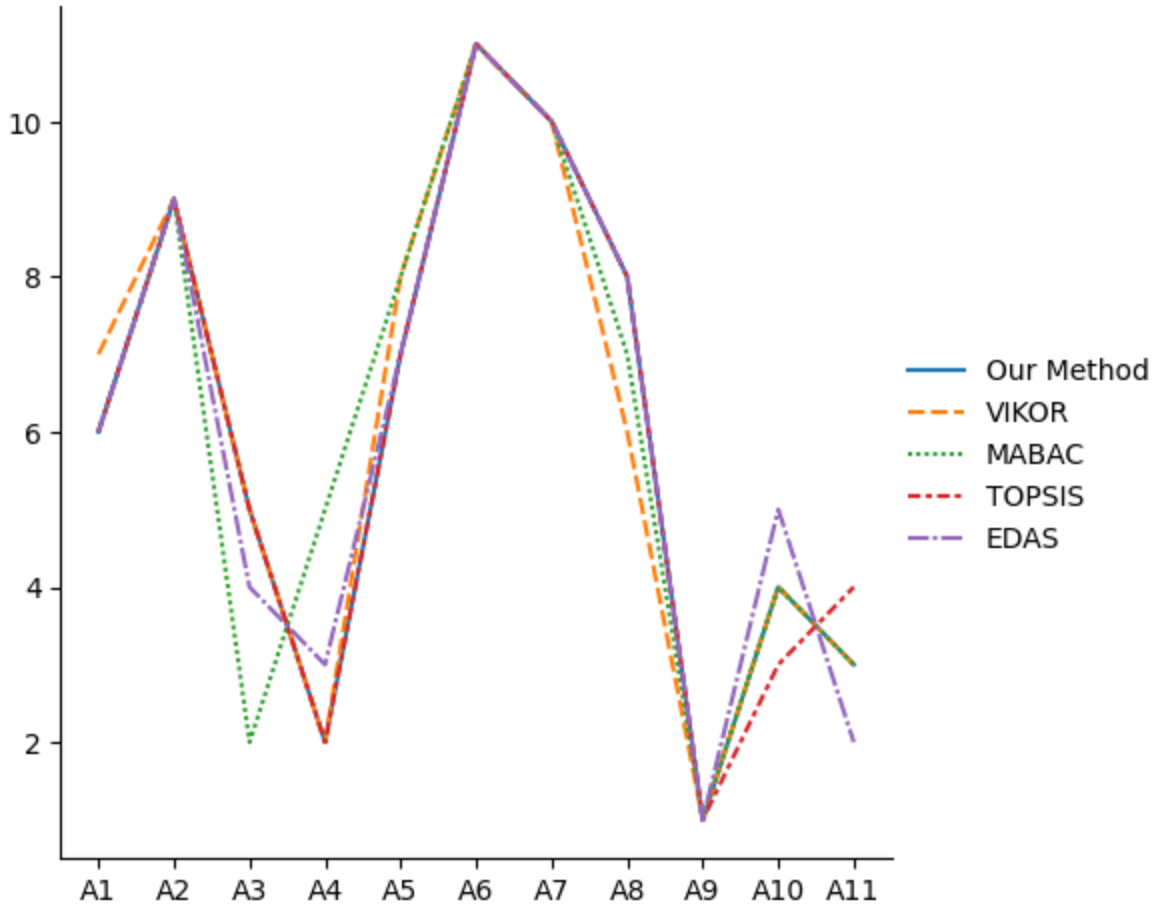

Fig 7. Relations between comparative methods.

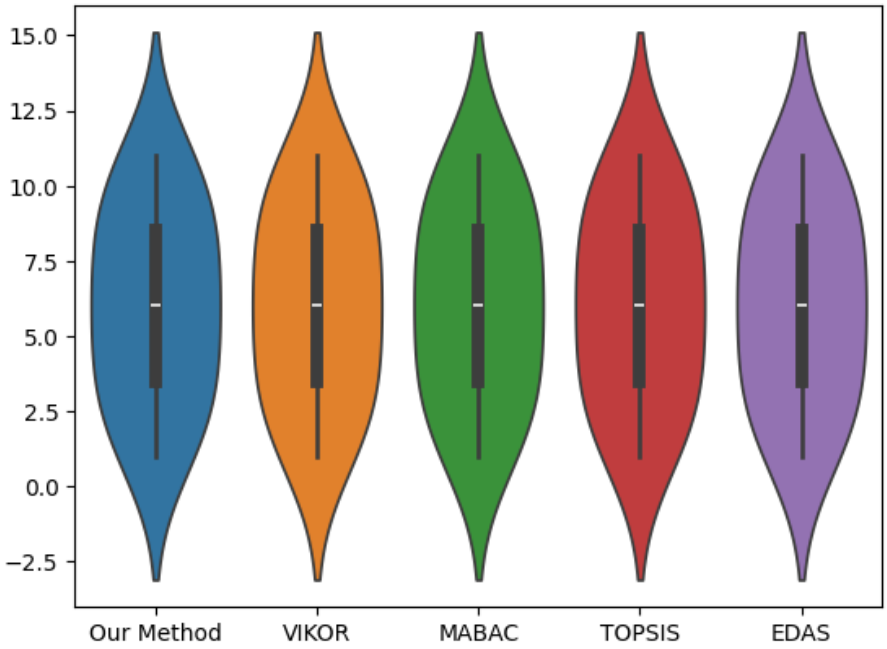

Fig 8. The ranks of each method.

In our method, we show alternative 6 is the best, followed by alternative 7, alternative 2, and alternative 8.

In VIKOR method, we show alternative 6 is the best, followed by alternative 7, alternative 2, and alternative 5. In MABAC method, we show alternative 6 is the best, followed by alternative 7, alternative 2, and alternative 5. In the TOPSIS method, we show alternative 6 is the best, followed by alternative 7, alternative 2, and alternative 8. In EDAS method, we show alternative 6 is the best, followed by alternative 7, alternative 2, and alternative 8.

5. Conclusions

We proposed a MCDM approach for Risk assessment of municipal water supply and drainage project costs. Two MCDM methods are used to evaluate this problem. One method is used to compute the criteria weights and another method is used to rank the alternatives. 8 criteria and 11 alternatives are used to evaluate this problem. The results show alternative 6 is the best in all cases. The sensitivity analysis was conducted to show the stability of the ranks of alternatives. We performed it with three cases and ten scenarios in each case. The results show the ranks of alternatives are stable in different cases. Then we performed comparative analysis. We compared our model with other MCDM methods. The results showed that our model is effective compared with other MCDM methods.

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Edge connectivity of a neutrosophic graph

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Abstract

Neutosophic graphs are an extension of fuzzy and intuitionistic fuzzy graphs by including the uncertainty, vagueness, and indeterminacy that are normal in the real world. This paper looks into the edge connectivity of a neutrosophic graph, which is a basic parameter that shows how strong and fault-tolerant networks are that are modelled by these graphs. Edge connectivity, which is the smallest number of edges that need to be taken away from a graph to make it trivial or disconnected, is a key concept in figuring out how strong and resilient networks are in many situations. This paper also addresses computational challenges related to determining edge connectivity in neutrosophic graphs. We develop efficient algorithms that minimize computational overhead and ensure accuracy in identifying the critical edge sets. We analyze the performance of these algorithms through both theoretical complexity assessments and empirical evaluations on benchmark datasets. Some of the most important things that the study found were critical edges that, when removed, have a big effect on how connected the graph is and how indeterminacy affects the strength of networks. The research underscores the importance of incorporating neutrosophic parameters into graph connectivity studies to better model and analyse systems characterized by uncertainty and partial knowledge.

Keywords: Cut-edge; Edge connectivity; Graph algorithms; Neutrosophic graph

1. Introduction

Zadeh [1] introduced the theory of fuzzy sets, revolutionizing the handling of uncertainty and imprecision in mathematical models. Fuzzy set theory added vagueness to classical set theory by giving membership grades from 0 to 1. This enabled its application in various domains, including decision-making [2, 3]. Over the years, researchers have built on this base to create fuzzy graphs (FGs), which model relationships when there is uncertainty. FGs have been used in communication networks, social systems, and optimization problems [4, 5]. An important generalization of fuzzy sets, known as intuitionistic fuzzy sets (IFSs), was introduced by Atanassov [6]. IFSs incorporate an additional parameter, the degree of non-membership, to address uncertainty with greater precision. This addition made it possible for intuitionistic fuzzy graphs (IFGs), which have been studied a lot for their ability to model systems with two levels of uncertainty, like social networks, decision support systems, and cell phone networks [7, 8, 9]. Researchers have extensively studied the connectivity properties of fuzzy and intuitionistic fuzzy graphs. They have focused on measures like cut vertices, bridges, and strong paths, which are important for understanding how stable and resilient networks are.

In spite of these improvements, both FGs and IFGs don't deal with systems that are inherently uncertain, like relationships that can't be clearly put into membership or nonmembership categories. To get around this problem, Smarandache came up with neutrosophic sets [10], a structure with three parts: truth-membership (TT), indeterminacy-membership (I), and falsity-membership (F). Then Neutrosophic graphs (NGs) take this idea and apply it to graph theory, giving us a better way to model networks that are uncertain, vague, and uncertain all at the same time. Then Raut et al. wrote a lot of papers about neutrosophic theory, Fermatean neutrosophic theory, and how they could be used [11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23]. Connectivity is a fundamental parameter for any graph structure, as it reflects the robustness and reliability of the network it represents. The edge connectivity of a graph, defined as the minimum number of edges whose removal disconnects the graph, plays a critical role in evaluating network stability. A lot of research has been done on edge connectivity for crisp, fuzzy, and intuitionistic fuzzy graphs, but not as much on how it applies to neutrosophic graphs. The fact that NGs contain uncertainty adds to their complexity, calling for new ways to measure and analyze connectivity.

In this paper, we look into how edges connect in neutrosophic graphs and come up with new definitions and methods that work with this type of graph. We include the truth, indeterminacy, and falsity parameters of edges in the analysis. This gives us a

more complete picture of how robust a network is. Section-2 provides the necessary preliminaries, including fundamental concepts of neutrosophic graphs. Section-3 presents the mathematical formulation of neutrosophic edge removal and the neutrosophic edge measure. Section-4 presents a theorem, which presents neutrosophic edge connectivity is the classical edge connectivity of the graph. Section-5 presents an algorithm to compute edge connectivity in a neutrosophic graph. Section-6 presents an example, and section-7 defines results and Finally, Section-8 concludes the study with key findings and outlines potential directions for future research. This study adds to the basic ideas of neutrosophic graph theory and gives us useful tools for looking at how strong complex systems are when there is uncertainty, vagueness, and indeterminacy.

2. Preliminaries

2.1 Neutrosophic Graph

A neutrosophic graph $G = (V, E, \tau, \iota, \phi)$ consists of:

- V: Set of vertices.
- E: Set of edges.
- $\tau: E \to [0,1]$ represent Truth membership function.
- $\iota: E \to [0,1]$ represent Indeterminacy membership function.
- $\phi: E \to [0,1]$ represent Falsity membership function.

Each edge $e \in E$ is represented as $e = (u, v, \tau, \iota, \phi)$, where $\tau, \iota \; and \; \phi$ satisfy $0 \leq \tau_e + \iota_e + \phi_e \leq 1$.

2.2 Edge Connectivity

For a graph G:

- Edge connectivity $\lambda(G)$: Minimum number of edges that need to be removed to disconnect G.

- For neutrosophic graphs, edge connectivity considers the neutrosophic weights (τ, ι, ϕ)

3. Mathematical Formulation

3.1 Neutrosophic Edge Removal

Let $E' \subseteq E$ be a subset of edges. Removing E' results in a subgraph $G' = (V, E \setminus E')$. The neutrosophic impact of removing E' is defined as:

$d(G, G') = \sum_{e \in E'}((1 - \tau_e) + \iota_e + \phi_e)$.

3.2 Edge Connectivity Measure

The neutrosophic edge connectivity $\lambda_{\mathrm{N}}(\mathrm{G})$ is:

- $\lambda_N(G) = min(\lambda_N(G), d(G, G'))$is disconnected $d(G, G') = \sum_{e \in E'}((1 - \tau_e) + \iota_e + \phi_e)$.

4. Theorem:

For any neutrosophic graph $G = (V, E, \tau, \iota, \phi)$ the neutrosophic edge connectivity $\lambda_N(G)$ satisfies:$\lambda_N(G) \geq \lambda(\mathrm{G})$, where $\lambda(G)$, is the classical edge connectivity of the graph.

Proof:

From Definitions of Classical Edge Connectivity λ(G), The minimum number of edges that need to be removed to disconnect the graph G.

Neutrosophic Edge Connectivity $\lambda_N(G)$ Defined as:

$$\lambda_N(G) = min \sum_{e \in E'} \big((1 - \tau_e) + \iota_e + \phi_e\big)$$

where (τ, ι, ϕ) are the truth, indeterminacy, and falsity membership values of the edges in E'.

Steps-1: Classical Case as a Special Case:

- If we assign $\tau_e = 1, \iota_e = 0$ and $\phi_e = 0$ for all $e \in E$, the neutrosophic graph G reduces to a classical graph.
- Under this assignment: $\sum_{e\in E'}((1-\tau_e)+\iota_e+\phi_e) = \sum_{e\in E'}\big((1-1)+0+0\big) = \mathrm{E}'$

  Hence, $\lambda_N(G) = \lambda(\mathrm{G})$ in the classical case.

Steps-2: General Neutrosophic Graph:

For a neutrosophic graph, edge weights include τ_e, ι_e, ϕ_e The neutrosophic impact of removing an edge e is given by: $w(e) = 1 - \tau_e + \iota_e + \phi_e$.. This weight $w(e) \geq 0$ since $0 \leq \tau_e + \iota_e + \phi_e \leq 1$ and $\tau_e + \iota_e + \phi_e \leq 1$.

Steps-3 : Minimum Edge Set to Disconnect G:

- Let E' be the edge set with the minimum neutrosophic impact that disconnects G.
- The neutrosophic impact is:

$$d(G, G') = \sum w(e) = \sum_{e\in E'}((1-\tau_e+\iota_e+\phi_e)$$

- Since $w(e) \geq 1 - \tau_e$, the neutrosophic impact satisfies: $d(G, G') \geq |E'|$, where $|E'|$ is the number of edges in E'.

Steps-4: Relation to Classical Edge Connectivity:

The classical edge connectivity $\lambda(G)$ corresponds to the minimum $|E'|$ for which G' becomes disconnected.

- Hence, the neutrosophic edge connectivity, which considers $w(e) \geq 1$ for all edges in the classical sense, satisfies: $\lambda_N(G) \geq \lambda(G)$.

The neutrosophic edge connectivity $\lambda_N(G)$ is always greater than or equal to the classical edge connectivity $\lambda(G)$ because neutrosophic weights assign additional contributions from indeterminacy and falsity measures, making $\lambda_N(G)$at least as large as $\lambda(G)$.

5. Algorithm: Compute Edge Connectivity in a Neutrosophic Graph

Input:

- A neutrosophic graph $G = (V, E, \tau, \iota, \phi)$

  Where, V is Set of vertices and E is Set of edges with neutrosophic weights (τ, ι, ϕ).

Output:

- The neutrosophic edge connectivity $\lambda_N(G)$, the minimum neutrosophic impact required to disconnect the graph.

Steps:

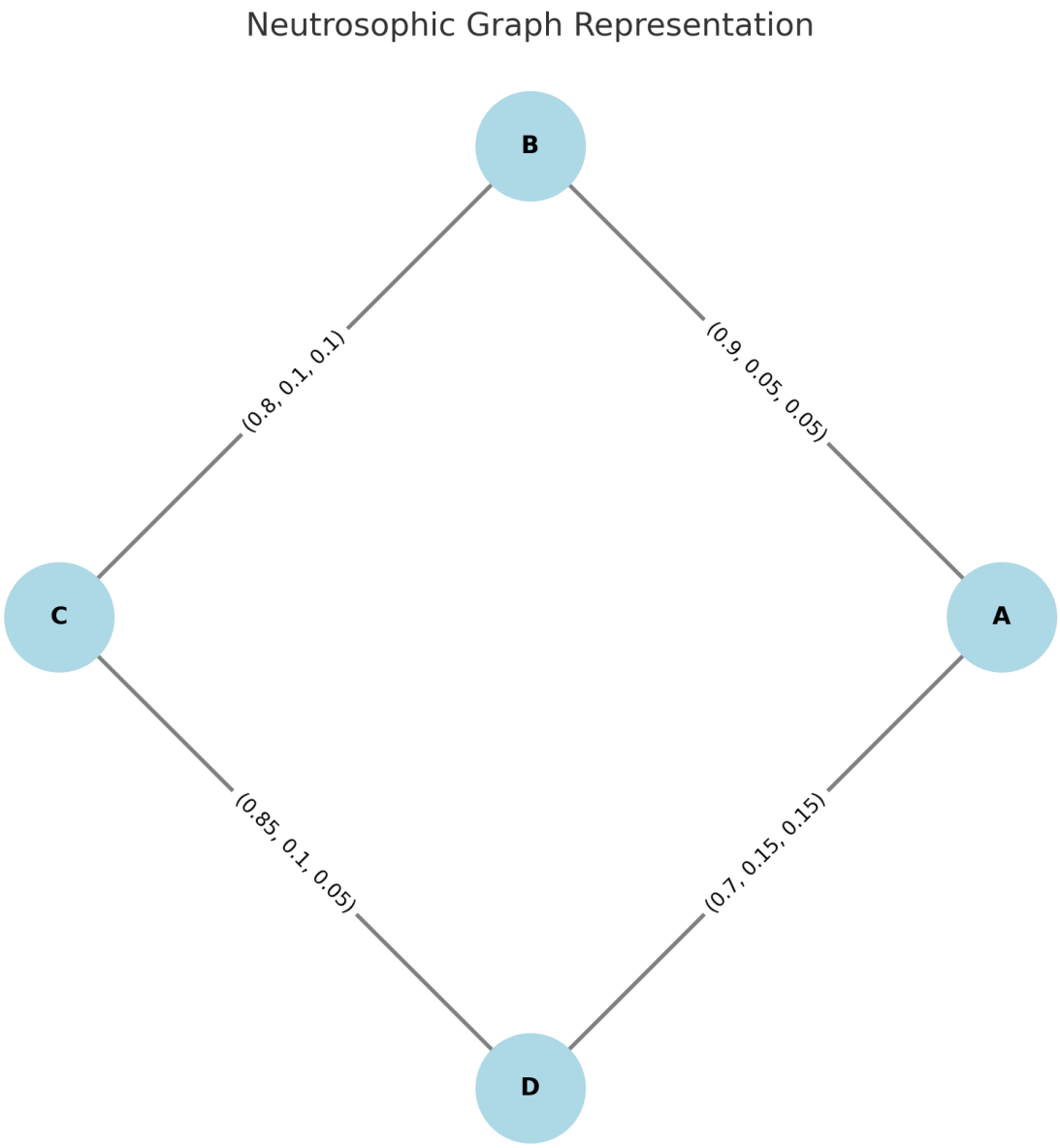
2. Initialization:
   - Set $\lambda_N(G) \to \infty$.
3. Identify Edge Subsets:
   - List all subsets of edges $E' \subseteq E$.
4. Iterate Over Subsets:
   - For each E', perform the following:

- Remove edges in E' from G, forming a subgraph $G' = (V, E \setminus E')$.
- Check if G' is disconnected (using graph traversal techniques like DFS or BFS).

5. Calculate Neutrosophic Impact:
   - For each subset E' that disconnects G, compute the neutrosophic impact: $d(G, G') = \sum_{e \in E'}((1 - \tau_e) + \iota_e + \phi_e)$.
6. Update Edge Connectivity:
   - Update $\lambda_N(G) = min(\lambda_N(G), d(G, G'))$
7. Return Result:
   - Return the computed $\lambda_N(G)$.

6. Example: Edge Connectivity Computation

Here is the graphical representation of the neutrosophic graph from the provided example. Each edge is labelled with its corresponding neutrosophic weights (τ, ι, ϕ).



Graph Description:

- Vertices: $V = \{A, B, C, D\}$.
- Edges: $E = \{e_1, e_2, e_3, e_4\}$
  - $e_1 = (\mathrm{A}, \mathrm{B}) = (0.9,0.05,0.05)$
  - $e_2 = (\mathrm{B}, \mathrm{C}) = (0.8,0.1,0.1)$
  - $e_3 = (\mathrm{C}, \mathrm{D}) = (0.85,0.1,0.05)$

- $e_4 = (\mathrm{D}, \mathrm{A}) = 0.7,0.15,0.15)$

Steps:

1. Edge Subsets:
   - Identify subsets of edges that disconnect the graph:
     - $\{e_1, e_3\}$: Disconnects A and C from B and D.
     - $\{e_2, e_4\}$: Disconnects B and D from A and C.
2. Compute Neutrosophic Impact:
   - For$\{e_1, e_3\}$

$$d(G, G') = (1 - 0.9 + 0.05 + 0.05) + (1 - 0.85 + 0.1 + 0.05) = 0.2 + 0.3 = 0.5$$

   - For$\{e_2, e_4\}$:

$$d(G, G') = (1 - 0.8 + 0.1 + 0.1) + (1 - 0.7 + 0.15 + 0.15) = 0.4 + 0.6 = 1.0$$

3. Minimum Neutrosophic Impact:

$$\lambda_N(G) = 0.5$$

7. Result

The neutrosophic edge connectivity

$$\lambda_N(G) = 0.5$$

8. Conclusion

In this paper, we advance our theoretical understanding of edge connectivity in neutrosophic graphs and provide practical tools for its application in diverse fields. There are suggestions for future research, such as looking into how vertex connectivity changes in neutrosophic graphs and how it might be used in new areas like artificial intelligence, cybersecurity, and complex system analysis. This work lays the groundwork for further exploration of neutrosophic graph theory and its implications for network science and decision-making under uncertainty.

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Neutrosophic augmented Lagrange multipliers method Nonlinear Programming Problems Constrained by Inequalities

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Abstract: Operations research uses scientific methods that take the language of mathematics as a basis and uses the computer, without which it would not be possible to achieve numerical solutions to the problems raised. It is concerned with applying scientific methods to complex issues in the management and direction of large systems in various fields and private and governmental businesses in peace and war in politics, management, economics, planning and implementation and in all aspects of life. In issues that require sound solutions, when solutions are numerous and options are multiple, we need a decision based on sound scientific foundations that take into account all the circumstances and changes that the decision maker may face during the course of work, and help him make a decision that leaves nothing to chance or luck. For this reason, operations research has provided many methods that help us transform life issues into mathematical models whose optimal solution is the appropriate decision. The nonlinear programming method is one of the most important methods presented by operations research because most problems, when modeled, result in a nonlinear model, which prompted many students and researchers to search for appropriate scientific methods to solve these models. One of the most important and most widely used of these methods is the Lagrange multipliers method.

Keywords: Operations research; nonlinear programming; Lagrange multipliers; neutrosophic science; neutrosophic nonlinear programming; neutrosophic augmented Lagrange multipliers method.

1. Introduction

Operations research is the applied aspect of mathematics and is one of the most important modern sciences that deals with practical issues and meets the desire and demand of decision-makers to obtain ideal decisions through the methods it provides that are appropriate for all issues. We know that operations research methods depend on the data attached to each issue and this data is values that were set through observation and its accuracy depends on who collects it and is appropriate for working conditions similar to the conditions in which this data was collected. Before the emergence of neutrosophic science, we relied on this data to study real issues and accept the results as they are, but after the emergence of neutrosophic science, the science that caused a major revolution in all fields and proved its ability to give results that enjoy a margin of freedom and suit all conditions, many researchers interested in scientific development have turned to reformulating many scientific concepts using the concepts of this science. We now have neutrosophic numbers - neutrosophic groups - neutrosophic probabilities - neutrosophic statistics - neutrosophic differentiation - neutrosophic

integration - neutrosophic linear programming - neutrosophic dynamic programming - neutrosophic simulation - neutrosophic transport models… [1-11, 16-18].

Nonlinear programming is one of the most important methods presented by operations research because most practical problems lead to nonlinear models. Therefore, in a previous study [12], we formulated some basic concepts of nonlinear programming using the concepts of neutrosophic science. In the two studies [13,14], we presented a neutrosophic study of the graphical method and the Lagrange multipliers method in the case of constraints of the type of equality methods used to find the optimal solution for nonlinear models. In this study, we present a formulation of the augmented Lagrange multipliers method that is used to find the optimal solution for nonlinear models restricted by inequality constraints using the concepts of neutrosophic science [15]. That is, we will take the data of the problem under study as neutrosophic numbers with the following standard formula $N = a + bI$ where $a\ and\ b$ are real or complex coefficients, a represents the determinant part and bI represents the undetermined part of the undetermined number N and it can be $[\lambda_1\ ,\ \lambda_2\ ]$or $\{\lambda_1\ ,\ \lambda_1\ \}$ or … else is any set close to the true value a.

2. Discussion:

A mathematical model is a neutrosophic model if the variables in the objective function, constraints, or both are neutrosophic values. Suppose we want to determine the optimal values of a neutrosophic nonlinear function $\boldsymbol{f_N(x)}$ subject to a set of constraints in the form of equalities or inequalities. This type of problem is addressed through the Lagrange factorial method, a method that has received great attention from students and researchers in the field of operations research, and appropriate developments have been introduced to it that have enabled it to deal with most situations with high efficiency.

- **Previous study:**

1- Neutrosophic nonlinear programming problems constrained by equality constraints: [12]

Constrained nonlinear programming problem text:

For nonlinear programming problems constrained by the equation constraints defined as follows:

Find:

$$\boldsymbol{Z_N = f_N(X) \longrightarrow (Max\ or\ Min)}$$

Subject to the constraints:

$$\boldsymbol{g_{Ni}(X) = b_{Ni}\quad ; i = 1, 2, \dots, m}$$

$$\boldsymbol{X = (x_1, x_2, \dots, x_n) \in R^n}$$

Since the functions $\boldsymbol{f_N(x)\ and\ g_{Ni}(X)}$ are continuous functions whose partial derivatives of the first degree can be computed, the Lagrange factorial method requires us to form a function of the type:

$$\boldsymbol{L_N(X, \lambda_i) = f_N(X) - \lambda_i(g_{Ni}(X) - b_{Ni})}$$

Where we call $\boldsymbol{L_N(X, \lambda_i)}$ the Lagrange neutrosophic function and the coefficients $\boldsymbol{\lambda_i}$ refer to the Lagrange factorials and these factorials measure the rate of change in the optimal value of $\boldsymbol{f_N(X)}$ as a result of Making small modifications to $\boldsymbol{g_{Ni}(X)}$, by solving the set of equations:

$$\boldsymbol{\frac{\partial L_N}{\partial X} = 0\quad ,\quad \frac{\partial L_N}{\partial \lambda_i} = 0}$$

We get the optimal solution vector, which is a maximum neutrosophic value of the function, which can be a minimum or a maximum. To determine its type, we apply the following test:

Maximum and minimum values of a constrained nonlinear function:

We can determine whether the value of an objective function in a nonlinear model is a minimum or a maximum as follows:

The value is minimum if the objective function is convex and the set of constraints forms a convex region

The value is maximum if the objective function is concave and the set of constraints forms a convex region

To determine the type of function $\boldsymbol{f_N(x_1\ , x_2, \dots, x_n)}$ (convex or concave), we use the Hessian matrix for this function:

Hessian matrix of the neutrosophic function $f_N(x_1, x_2, \dots, x_n)$:

It is a square and symmetric matrix of order $n \times n$, denoted by $H_N(x_1, x_2, \dots, x_n)$ and is defined by the following relation:

$$H_N(x_1, x_2, \dots, x_n) = \left[\frac{\partial^2 f_N}{\partial x_i \partial x_j}\right]$$

The matrix is symmetric and must satisfy the following conditions:

- The elements of the main diagonal are positive.
- The main minor determinants are positive.

2- Nonlinear programming problems with inequalities: [15]

The problem of nonlinear programming of constrained by inequalities is defined as follows:

$$MaxZ = \mathbf{f}(x_1, x_2, \dots, x_n)$$

Subject to constraints in the form of inequalities:

$$g_i(x_1, x_2, \dots, x_n) \leq b_i$$

$$(x_1, x_2, \dots, x_n) \geq \mathbf{0}$$

In such problems, there is a direct relationship between the size of the problem under study and the size of the calculations required to determine the maximum point (maximum or minimum) of this problem, so the Lagrange method mentioned above is considered an impractical method to solve it. In order to solve these problems, researchers in the field of operations research have presented the augmented Lagrange multipliers method., which can be summarized in the following steps:

Step 1: We solve the problem without constraints, i.e.:

$$MaxZ = \mathbf{f}(x_1, x_2, \dots, x_n)$$

- ❖ If the resulting optimal value satisfies all constraints, then this solution is an optimal solution to the constrained problem. We conclude from this that the constraints of the problem are unnecessary.
- ❖ If the resulting optimal value does not satisfy all constraints, we move to the second step.

Step 2: We take $k = 1$ (the number of constraints used is k, we take one of the existing constraints) and convert this constraint to an equality constraint and search for the optimal solution for $\mathbf{f}(x_1, x_2, \dots, x_n)$ that is subject to $k = 1$ constraint in an equal manner using the Lagrange multipliers method mentioned above,

- ❖ If the resulting solution satisfies all the constraints of the problem, then the solution determines a local optimal point
- ❖ If the resulting value does not satisfy all the constraints, we delete this solution because it is not possible

We repeat this step for all possible sets of constraints, each of which consists of $k = 1$ equality constraint and record all the local optimal points that we obtain, then move to the third step.

Step 3: We take $k = 2$ (the number of constraints used is k. We take two constraints from the existing constraints) and convert these two constraints into equality constraints and search for the optimal solution for $\mathbf{f}(x_1, x_2, \dots, x_n)$ that is subject to $k = 2$ constraints in an equal manner using the Lagrange multipliers method mentioned above.

- ❖ If the resulting solution satisfies all the constraints of the problem, then the solution determines a local optimal point.
- ❖ If the resulting value does not satisfy all the constraints, we delete this solution because it is not possible.

We repeat this step for all possible sets of constraints each consisting of $k = 2$ equality constraints and record all the local optimal points we obtain.

We repeat the work for $k = 3$, $k = 4$ …, $k = m$

We calculate the values of the objective function for all local optimal points and choose the optimal value that satisfies the objective of the problem under study. The corresponding point is an absolute optimal point.

Note: If we do not encounter a local optimal point in the resulting solutions, we decide that the problem has no possible solution.

We illustrate the above through the following example:

Example1:

Find the optimal solution to the following nonlinear programming problem:

$$\boldsymbol{MaxZ = f(x_1, x_2) = -(2x_1 - 5)^2 - (2x_2 - 1)^2}$$

Subject to constraints:

$$\boldsymbol{x_1 + 2x_2 \leq 2}$$
$$\boldsymbol{x_1, x_2 \geq 0}$$

Solution:

Step 1: We solve the problem without restrictions:

To find the optimal value of $f(x)$ without constraints, we find the solutions to the following equations:

$$\frac{\partial f}{\partial x_1} = -4(2x_1 - 5) = 0$$
$$\frac{\partial f}{\partial x_2} = -4(2x_2 - 1) = 0$$

The solution to the set of equations is $\bar{\boldsymbol{x}} = (\boldsymbol{x_1, x_2}) = \left(\frac{5}{2}, \frac{1}{2}\right)$ This solution does not satisfy the first constraint in the problem, so we resort to applying steps (2) and (3) in the solution algorithm. Applying these two steps requires solving seven problems using the Lagrange multipliers method.

We summarize these problems and their solutions as follows:

For $k = 1$ for all constraints we get Problems 1, 2, 3:

Problem 1: For the constraint $x_1 = 0$:

The Lagrange function is:

$$L_1(x, \lambda) = -(2x_1 - 5)^2 - (2x_2 - 1)^2 - \lambda x_1$$

We calculate the partial derivatives of this function with respect to $\boldsymbol{x_1}$ ، $\boldsymbol{x_2}$ ، $\boldsymbol{\lambda}$:

$$\frac{\partial L}{\partial x_1} = -4(2x_1 - 5) - \lambda = 0$$
$$\frac{\partial L}{\partial x_2} = -4(2x_2 - 1) = 0$$
$$\frac{\partial L}{\partial \lambda} = x_1 = 0$$

The solution to the system of equations is the point $\bar{x} = (x_1, x_2, \lambda) = \left(0, \frac{1}{2}, -20\right)$ The point satisfies all the constraints and is a local maximum.

Problem 2: For the constraint $x_2 = 0$:

The Lagrange function is:

$$L_2(x, \lambda) = -(2x_1 - 5)^2 - (2x_2 - 1)^2 - \lambda x_2$$

We calculate the partial derivatives of this function with respect to x_1 ، x_2 ، λ:

$$\frac{\partial L}{\partial x_1} = -4(2x_1 - 5) = 0$$
$$\frac{\partial L}{\partial x_2} = -4(2x_2 - 1) - \lambda = 0$$
$$\frac{\partial L}{\partial \lambda} = 2x_2 = 0$$

The solution to the previous set of equations is the point $\bar{x} = (x_1, x_2, \lambda) = \left(\frac{5}{2}, 0, 0\right)$.

The point $\bar{x} = (x_1, x_2) = \left(\frac{5}{2}, 0\right)$ does not satisfy all the constraints since the first constraint is not satisfied Solution Rejected.

Problem 3: For the constraint $x_1 + 2x_2 - 2 = 0$:

The Lagrange function is:

$$L_3(x, \lambda) = -(2x_1 - 5)^2 - (2x_2 - 1)^2 - \lambda(x_1 + 2x_2 - 2)$$

We calculate the partial derivatives of this function with respect to x_1 ، x_2 ، λ:

$$\frac{\partial L_3}{\partial x_1} = -4(2x_1 - 5) - \lambda = 0$$
$$\frac{\partial L_3}{\partial x_2} = -4(2x_2 - 1) - 2\lambda = 0$$
$$\frac{\partial L_3}{\partial \lambda} = -(x_1 + 2x_2 - 2) = 0$$

The solution to the previous set of equations is $\bar{x} = (x_1, x_2, \lambda) = \left(\frac{22}{10}, \frac{-1}{10}, \frac{12}{5}\right)$The point $(x_1, x_2) = \left(\frac{22}{10}, \frac{-1}{10}\right)$does not satisfy the third constraint. The solution is rejected.

For $k = 2$ for all constraints we get Problems 4,5,6:

Problem 4: Taking the constraints $x_1 + 2x_2 - 2 = 0$, $x_1 = 0$:

The Lagrange function is

$$L_4(x, \lambda) = -(2x_1 - 5)^2 - (2x_2 - 1)^2 - \lambda_1(x_1 + 2x_2 - 2) - \lambda_2 x_1$$

We calculate the partial derivatives of this function with respect to x_1 ، x_2 ، λ_1، λ_2:

$$\frac{\partial L_4}{\partial x_1} = -4(2x_1 - 5) - \lambda_1 - \lambda_2 = 0$$
$$\frac{\partial L_4}{\partial x_2} = -4(2x_2 - 1) - 2\lambda_1 = 0$$
$$\frac{\partial L_4}{\partial \lambda_1} = -(x_1 + 2x_2 - 2) = 0$$
$$\frac{\partial L_4}{\partial \lambda_2} = -x_1 = 0$$

The solution to the previous set of equations is $\bar{x} = (x_1, x_2, \lambda) = (0{,}1, -2{,}22)$ The point $(x_1, x_2) = (0{,}1)$ satisfies all the constraints, so it is a local maximum.

Problem 5: We take the two constraints $x_1 + 2x_2 - 2 = 0$, $x_2 = 0$

The Lagrange function is:

$$L_5(x, \lambda) = -(2x_1 - 5)^2 - (2x_2 - 1)^2 - \lambda_1(x_1 + 2x_2 - 2) - \lambda_2 x_2$$

We calculate the partial derivatives of this function with respect to x_1 ، x_2 ، λ_1، λ_2:

$$\frac{\partial L_5}{\partial x_1} = -4(2x_1 - 5) - \lambda_1 = 0$$
$$\frac{\partial L_5}{\partial x_2} = -4(2x_2 - 1) - 2\lambda_1 - \lambda_2 = 0$$
$$\frac{\partial L_5}{\partial \lambda_1} = -(x_1 + 2x_2 - 2) = 0$$
$$\frac{\partial L_5}{\partial \lambda_2} = -x_2 = 0$$

The solution to the previous set of equations is $\bar{x} = (x_1, x_2, \lambda) = (2{,}0{,}4, -4)$ The point $(x_1, x_2) = (2{,}0)$ satisfies all the constraints, so it is a local maximum.

Problem 6: We take the two constraints $x_2 = 0$, $x_1 = 0$

The Lagrange function is:

$$L_6(x, \lambda) = -(2x_1 - 5)^2 - (2x_2 - 1)^2 - \lambda_1 x_1 - \lambda_2 x_2$$

We calculate the partial derivatives of this function with respect tox_1 ، x_2 ، λ_1، λ_2:

$$\frac{\partial L_6}{\partial x_1} = -4(2x_1 - 5) - \lambda_1 = 0$$
$$\frac{\partial L_6}{\partial x_2} = -4(2x_2 - 1) - \lambda_2 = 0$$
$$\frac{\partial L_6}{\partial \lambda_1} = -x_1 = 0$$
$$\frac{\partial L_6}{\partial \lambda_2} = -x_2 = 0$$

The solution to the previous set of equations is $\bar{x} = (x_1, x_2, \lambda) = (0,0,-20,-4)$The point $(x_1, x_2) = (0,0)$satisfies all the constraints and is a local maximum.

For $k = 3$ for all constraints we get Problem 7:

Problem 7: We take the constraints $x_1 + 2x_2 - 2 = 0$، $x_2 = 0$,$x_1 = 0$

The Lagrange function is:

$$L_7(x, \lambda) = -(2x_1 - 5)^2 - (2x_2 - 1)^2 - \lambda_1 x_1 - \lambda_2 x_2 - \lambda_3 (x_1 + 2x_2 - 2)$$

We calculate the partial derivatives of this function with respect to x_1 ، x_2 ، λ_1 ، λ_2 ، λ_3:

$$\frac{\partial L_7}{\partial x_1} = -4(2x_1 - 5) - \lambda_1 - \lambda_3 = 0$$

$$\frac{\partial L_7}{\partial x_2} = -4(2x_2 - 1) - \lambda_2 - 2\lambda_3 = 0$$

$$\frac{\partial L_7}{\partial \lambda_1} = -x_1 = 0$$

$$\frac{\partial L_7}{\partial \lambda_2} = -x_2 = 0$$

$$\frac{\partial L_7}{\partial \lambda_3} = -(x_1 + 2x_2 - 2) = 0$$

We note that there is no solution to the previous set of equations due to the contradiction of the last three equations. To determine the absolute optimal solution, i.e., the absolute maximum value, we take all possible points (local maximum points) and calculate the value of the objective function at each point and choose the largest value. The corresponding point is the absolute maximum solution point:

Problem 1: Point $\bar{x} = (x_1, x_2) = \left(0, \frac{1}{2}\right)$ is a local maximum point, value of the objective function:

$$f(x) = -25$$

Problem 4: Point $(x_1, x_2) = (0,1)$ This point is a local maximum point, value of the objective function:

$$f(x) = -26$$

Problem 5: Point $(x_1, x_2) = (2,0)$ is a local maximum point, value of the objective function:

$$f(x) = -2$$

Problem 6: Point $(x_1, x_2) = (0,0)$ is a local maximum point, value of the objective function:

$$f(x) = -26$$

By comparing the values of the objective function in the previous problems, we find that the best possible point is point $(x_1^*, x_2^*) = (2,0)$ and the absolute maximum value of the function is:

$$f^*(x) = -2$$

- **Current study:**

Neutrosophic Nonlinear programming problems constrained by inequalities:

The nonlinear programming problem constrained by inequalities is defined as follows:

$$(Max\ or\ Min) Z_N = f_N(x_1, x_2, \dots, x_n)$$

And subject to constraints in the form of inequalities:

$$g_{Ni}(x_1, x_2, \dots, x_n) \leq b_{Ni}$$

$$(x_1, x_2, \dots, x_n) \geq 0$$

The solution steps according to the expanded Lagrange multipliers method are as follows:

Step 1: We solve the problem without constraints, i.e.:

$$(Max\ or\ Min) Z_N = f_N(x_1, x_2, \dots, x_n)$$

If the resulting optimal neutrosophic value satisfies all constraints, then this solution is an optimal solution to the constrained neutrosophic problem, and we conclude Hence, the constraints of the problem are unnecessary if the resulting optimal value does not satisfy all the constraints.

Step 2: We take $k = 1$ (we take one of the constraints):

We convert this constraint to an equality constraint and search for the optimal solution for $f_N(x_1, x_2, \dots, x_n)$ that is subject to $k = 1$ constraint in an equal manner using the Lagrange neutrosophic multipliers method.

If the resulting solution satisfies all the constraints of the problem, then the solution determines a local neutrosophic optimal point. If the resulting neutrosophic optimal value does not satisfy all the constraints, we delete it because this solution is not possible.

We repeat the work for all possible sets of constraints, each of which consists of $k = 1$ equality constraint, and we record all the local neutrosophic optimal points that we obtain, then we move to step 3.

Step 3: We take $k = 2$ constraint (we take two of the constraints):

We convert this constraint to an equality constraint and search for the optimal solution for $f_N(x_1, x_2, \dots, x_n)$ that is subject to $k = 2$ constraint in an equal manner.

If the resulting neutrosophic optimum does not satisfy all the constraints, we delete it because this solution is not possible.

We repeat the work for all possible sets of constraints, each of which consists of $k = 2$ equality constraint, and we record all the local neutrosophic optimum points that we obtain.

We continue in this way until we reach $k = m$ constraint.

We calculate the value of the objective function for all the local optimal points we have obtained and choose the largest value if the goal is to maximize and the smallest value if the goal is to minimize. These values are the absolute neutrosophic optimal value.

We illustrate the above through the following example:

We take the mathematical model given in example (1):

The neutrosophic formula for the mathematical model given in example (1) is:

$$MaxZ_N = f_N(x_1, x_2) = (c_{1N}x_1 - d_{1N})^2 - (c_{2N}x_2 - d_{2N})^2$$

Subject to constraints:

$$a_{1N}x_1 + a_{2N}x_2 \leq b_N$$
$$x_1, x_2 \geq 0$$

Where $c_{1N}, c_{2N}, d_{1N}, d_{2N}, a_{1N}, a_{2N}, b_N$ are neutrosophic values.

It is worth noting that the existence of one neutrosophic value is sufficient for the nonlinear mathematical model to be a neutrosophic model. Accordingly, since the purpose of this research is to provide a neutrosophic formulation of the extended Lagrange multipliers method, we will take the neutrosophic mathematical model as follows:

Example 2:

Find the optimal solution to the following neutrosophic nonlinear programming problem:

$$MaxZ = \mathrm{f}(x_1, x_2) = -(2x_1 - 5)^2 - (2x_2 - 1)^2$$

Subject to the constraints:

$$x_1 + 2x_2 \leq [1,3]$$
$$x_1, x_2 \geq 0$$

In any of the constraints of the mathematical models, the second party expresses the available capabilities (The number of raw materials - possible working hours ...) We took it as a neutrosophic value in the form of a field whose minimum limit represents the available capabilities in the worst conditions and the maximum limit represents the available capabilities in the best conditions.

Step 1: We solve the problem without restrictions:

To find the optimal value of $f(x)$ without constraints, we find the solutions to the following equations:

$$\frac{\partial f}{\partial x_1} = -4(2x_1 - 5) = 0$$
$$\frac{\partial f}{\partial x_2} = -4(2x_2 - 1) = 0$$

The solution to the set of equations is $\bar{x} = (x_1, x_2) = \left(\frac{5}{2}, \frac{1}{2}\right)$ This solution does not satisfy the first constraint in the problem, so we resort to applying steps (2) and (3) in the solution algorithm. Applying these two steps requires solving seven problems using the Lagrange multipliers method.

We summarize these problems and their solutions as follows:

For $k = 1$ for all constraints we get Problems 1, 2, 3:

Problem 1: For the constraint $x_1 = 0$:

The Lagrange function is:

$$L(x, \lambda) = -(2x_1 - 5)^2 - (2x_2 - 1)^2 - \lambda x_1$$

We calculate the partial derivatives of this function with respect to x_1 ، x_2 ، λ.

$$\frac{\partial L}{\partial x_1} = -4(2x_1 - 5) - \lambda = 0$$
$$\frac{\partial L}{\partial x_2} = -4(2x_2 - 1) = 0$$
$$\frac{\partial L}{\partial \lambda} = x_1 = 0$$

The solution to the system of equations is the point $\bar{x} = (x_1, x_2, \lambda) = \left(0, \frac{1}{2}, -20\right)$ The point satisfies all the constraints and is a local maximum.

Problem 2: For the constraint $x_2 = 0$:

The Lagrange function is:

$$L(x, \lambda) = -(2x_1 - 5)^2 - (2x_2 - 1)^2 - \lambda x_2$$

We calculate the partial derivatives of this function with respect to x_1 ، x_2 ، λ.

$$\frac{\partial L}{\partial x_1} = -4(2x_1 - 5) = 0$$
$$\frac{\partial L}{\partial x_2} = -4(2x_2 - 1) - \lambda = 0$$
$$\frac{\partial L}{\partial \lambda} = 2x_2 = 0$$

The solution to the previous set of equations is the point $\bar{x} = (x_1, x_2, \lambda) = \left(\frac{5}{2}, 0,0\right)$.

The point $\bar{\boldsymbol{x}} = (\boldsymbol{x_1}, \boldsymbol{x_2}) = \left(\frac{\mathbf{5}}{\mathbf{2}}, \mathbf{0}\right)$ does not satisfy all the constraints since the first constraint is not satisfied Solution Rejected.

Problem 3: For the constraint $x_1 + 2x_2 - [1, 3] = 0$:

The Lagrange function is:

$$L(x, \lambda) = -(2x_1 - 5)^2 - (2x_2 - 1)^2 - \lambda(x_1 + 2x_2 - [1,3])$$

We calculate the partial derivatives of this function with respect to x_1 ، x_2 ، λ.

$$\frac{\partial L}{\partial x_1} = -4(2x_1 - 5) - \lambda = 0$$
$$\frac{\partial L}{\partial x_2} = -4(2x_2 - 1) - 2\lambda = 0$$
$$\frac{\partial L}{\partial \lambda} = -(x_1 + 2x_2 - [1,3]) = 0$$

The solution to the previous set of equations is:

$$\bar{x}_N = (x_{1N}, x_{2N}, \lambda_N) = \left(\left[1, \frac{6}{5}\right], \left[0, \frac{2}{5}\right], \left[\frac{52}{5}, 12\right]\right)$$

The point $\bar{x_N} = (x_{1N}, x_{2N}) = \left(\left[1, \frac{6}{5}\right], \left[0, \frac{2}{5}\right]\right)$does not satisfy the third constraint. The solution is rejected.

For $k = 2$ all constraints we get Problems 4,5,6:

Problem 4: Taking the constraints $x_1 + 2x_2 - [1, 3] = 0$, $x_1 = 0$:

The Lagrange function is:

$$L(x, \lambda) = -(2x_1 - 5)^2 - (2x_2 - 1)^2 - \lambda_1(x_1 + 2x_2 - [1,3]) - \lambda_2 x_1$$

We calculate the partial derivatives of this function with respect to x_1 ، x_2 ، λ_1، λ_2.

$$\frac{\partial L}{\partial x_1} = -4(2x_1 - 5) - \lambda_1 - \lambda_2 = 0$$
$$\frac{\partial L}{\partial x_2} = -4(2x_2 - 1) - 2\lambda_1 = 0$$
$$\frac{\partial L}{\partial \lambda_1} = -(x_1 + 2x_2 - [1,3]) = 0$$
$$\frac{\partial L}{\partial \lambda_2} = -x_1 = 0$$

The solution to the previous set of equations is:

$$\overline{x_N} = (x_{1N}, x_{2N}, \lambda_{1N}, \lambda_{2N}) = \left(0, \left[\frac{1}{2}, \frac{3}{2}\right], [0,4], [16,20]\right)$$

The point $\overline{x_N} = (x_{1N}, x_{2N}) = \left(0, \left[\frac{1}{2}, \frac{3}{2}\right]\right)$, not all constraints are satisfied since the first constraint is not satisfied, the solution is rejected.

Problem 5: We take the two constraints $x_1 + 2x_2 - [1,3] = 0$, $x_2 = 0$.

The Lagrange function is:

$$\mathrm{L}(x, \lambda) = -(2x_1 - 5)^2 - (2x_2 - 1)^2 - \lambda_1(x_1 + 2x_2 - [1,3]) - \lambda_2 x_2$$

We calculate the partial derivatives of this function with respect to x_1 ، x_2 ، λ_1، λ_2.

$$\frac{\partial L}{\partial x_1} = -4(2x_1 - 5) - \lambda_1 = 0$$
$$\frac{\partial L}{\partial x_2} = -4(2x_2 - 1) - 2\lambda_1 - \lambda_2 = 0$$
$$\frac{\partial L}{\partial \lambda_1} = -(x_1 + 2x_2 - [1,3]) = 0$$
$$\frac{\partial L}{\partial \lambda_2} = -x_2 = 0$$

The solution to the previous set of equations is:

$$\overline{x_N} = (x_{1N}, x_{2N}, \lambda_{1N}, \lambda_{2N}) = ([1,3], 0, [-4,12], [-20,12])$$

The point $\overline{x_N} = (x_{1N}, x_{2N}) = ([1,3], 0)$ satisfies all the constraints, so it is a local maximum.

Problem 6: We take the two constraints $x_2 = 0$, $x_1 = 0$:

The Lagrange function is:

$$L(x, \lambda) = -(2x_1 - 5)^2 - (2x_2 - 1)^2 - \lambda_1 x_1 - \lambda_2 x_2$$

We calculate the partial derivatives of this function with respect tox_1 ، x_2 ، λ_1، λ_2.

$$\frac{\partial L_6}{\partial x_1} = -4(2x_1 - 5) - \lambda_1 = 0$$
$$\frac{\partial L_6}{\partial x_2} = -4(2x_2 - 1) - \lambda_2 = 0$$
$$\frac{\partial L_6}{\partial \lambda_1} = -x_1 = 0$$
$$\frac{\partial L_6}{\partial \lambda_2} = -x_2 = 0$$

The solution to the previous set of equations is $\bar{x} = (x_1, x_2, \lambda) = (0,0, -20, -4)$The point $\bar{x} = (x_1, x_2) = (0,0)$satisfies all the constraints and is a local maximum.

For $k = 3$ for all constraints we get Problem 7:

Problem 7: We take the constraints $x_1 + 2x_2 - [1,3] = 0$، $x_2 = 0$, $x_1 = 0$:

The Lagrange function is:

$$L(x, \lambda) = -(2x_1 - 5)^2 - (2x_2 - 1)^2 - \lambda_1 x_1 - \lambda_2 x_2 - \lambda_3(x_1 + 2x_2 - [1,3])$$

We calculate the partial derivatives of this function with respect to x_1 ، x_2 ، λ_1 ، λ_2 ، λ_3.

$$\frac{\partial L}{\partial x_1} = -4(2x_1 - 5) - \lambda_1 - \lambda_3 = 0$$
$$\frac{\partial L}{\partial x_2} = -4(2x_2 - 1) - \lambda_2 - 2\lambda_3 = 0$$
$$\frac{\partial L}{\partial \lambda_1} = -x_1 = 0$$

$$\frac{\partial L}{\partial \lambda_2} = -x_2 = 0$$

$$\frac{\partial L}{\partial \lambda_3} = -(x_1 + 2x_2 - [1,3]) = 0$$

To determine the absolute optimal neutrosophic solution, i.e., the absolute neutrosophic maximum, we take all possible points (local maxima) and calculate the value of the objective function at each point and choose the largest values. The corresponding point is the absolute neutrosophic maximum solution point:

Problem 1: The point $\bar{\boldsymbol{x}} = (\boldsymbol{x_1}, \boldsymbol{x_2}) = \left(\mathbf{0}, \frac{\mathbf{1}}{\mathbf{2}}\right)$ is a local maximum point, the value of the objective function:

$$\boldsymbol{f(x) = -25}$$

Problem 3: The point $\overline{\boldsymbol{x_N}} = (\boldsymbol{x_{1N}}, \boldsymbol{x_{2N}}) = \left(\left[\mathbf{1}, \frac{\mathbf{6}}{\mathbf{5}}\right], \left[\mathbf{0}, \frac{\mathbf{2}}{\mathbf{5}}\right]\right)$ is a local maximum point, the value of the objective function:

$$\boldsymbol{f_N(x) \in [-33.8, -13]}$$

Problem 5: The point $\bar{\boldsymbol{x}} = (\boldsymbol{x_1}, \boldsymbol{x_2}) = ([\mathbf{1}, \mathbf{3}], \mathbf{0})$ is a local maximum point, the value of the objective function:

$$\boldsymbol{f_N(x) \in [-9, -1]}$$

Problem 6: The point $\bar{\boldsymbol{x}} = (\boldsymbol{x_1}, \boldsymbol{x_2}) = (\mathbf{0}, \mathbf{0})$ is a local maximum point, the value of the objective function:

$$\boldsymbol{f(x) = -26}$$

By comparing the values of the objective function in the previous questions, we find that the best possible point is the point $(\boldsymbol{x_1^*}, \boldsymbol{x_2^*}) = ([\mathbf{1}, \mathbf{3}], \mathbf{0})$ and the absolute maximum value of the function is:

$$\boldsymbol{f_N^*(x) \in [-9, -1]}$$

Note: This method does not guarantee an absolute optimal value for the problem, however the best possible point can be considered an absolute optimal point (general).

3. Conclusion and results:

In the previous study, we presented a neutrosophic formulation of the extended Lagrange multipliers method, and we summarize what we have reached through the following table:

	Classical Problem	**Neutrosophic Problem**
The Problem without restrictions	Optimal value of $f(x)$ without constraints Solution point $\bar{x} = (x_1, x_2) = \left(\frac{5}{2}, \frac{1}{2}\right)$ Does not satisfy all constraints as the first constraint	Optimal value of $f(x)$ without constraints Solution point $\bar{x} = (x_1, x_2) = \left(\frac{5}{2}, \frac{1}{2}\right)$ Does not satisfy all constraints as the first constraint
Problem1	$k = 1$ Constraint $x_1 = 0$ Solution point $\bar{x} = (x_1, x_2) = \left(0, \frac{1}{2}\right)$ It satisfies all constraints, so it is a local maximum Value of objective function $f(x) = -25$	$k = 1$ Constraint $x_1 = 0$ Solution point $\bar{x} = (x_1, x_2) = \left(0, \frac{1}{2}\right)$ It satisfies all constraints, so it is a local maximum Value of objective function $f(x) = -25$
Problem2	$k = 1$ Constraint $x_2 = 0$ Solution point $\bar{x} = (x_1, x_2) = \left(\frac{5}{2}, 0\right)$ Not all constraints are satisfied as the first constraint is not satisfied Solution rejected	$k = 1$ Constraint $x_2 = 0$ Solution point $\bar{x} = (x_1, x_2) = \left(\frac{5}{2}, 0\right)$ Not all constraints are satisfied as the first constraint is not satisfied Solution rejected
Problem3	$k = 1$ Constraint $x_1 + 2x_2 - 2 = 0$ Solution point $\bar{x} = (x_1, x_2) = \left(\frac{22}{10}, \frac{-1}{10}\right)$ Does not satisfy all constraints as the third constraint is not satisfied Solution rejected	$k = 1$ Constraint $x_1 + 2x_2 - [1,3] = 0$ Solution Point $\overline{x_N} = (x_{1N}, x_{2N}) = \left(\left[1, \frac{6}{5}\right], \left[0, \frac{2}{5}\right]\right)$ It satisfies all constraints, so it is a local maximum Value of objective function $f_N(x) \in [-33.8, -13]$
Problem4	$k = 2$ Constraints $x_1 + 2x_2 - 2 = 0 \quad , x_1 = 0$ The solution is the point $\bar{x} = (x_1, x_2) = (0,1)$ that satisfies all the constraints It is a local maximum	$k = 2$ Constraints $x_1 + 2x_2 - [1,3] = 0 \quad , x_1 = 0$ Solution Point $\overline{x_N} = (x_{1N}, x_{2N}) = \left(0, \left[\frac{1}{2}, \frac{3}{2}\right]\right)$ Does not satisfy all constraints as the first constraint is not satisfied

	The value of the objective function $f(x) = -26$	Solution rejected
Problem5	$k = 2$ Constraints $x_1 + 2x_2 - 2 = 0 \quad , x_2 = 0$ The solution is the point $\bar{x} = (x_1, x_2) = (2,0)$ It satisfies all the constraints, so it is a local maximum The value of the objective function $f(x) = -2$	$k = 2$ Constraints $x_1 + 2x_2 - [1,3] = 0 \quad , x_2 = 0$ Solution Point $\overline{x_N} = (x_{1N}, x_{2N}) = ([1,3], 0)$ It satisfies all constraints, so it is a local maximum Value of the objective function $f_N(x) \in [-9, -1]$
Problem6	$k = 2$ Constraints $x_2 = 0 \ , x_1 = 0$ The solution is the point $\bar{x} = (x_1, x_2) = (0,0)$ That satisfies all the constraints, so it is a local maximum The value of the objective function $f(x) = -26$	$k = 2$ Constraints $x_2 = 0 \ , x_1 = 0$ The solution is the point $\bar{x} = (x_1, x_2) = (0,0)$ That satisfies all the constraints, so it is a local maximum The value of the objective function $f(x) = -26$
Problem7	$k = 3$ Three constraints $x_1 + 2x_2 - 2 = 0 \quad , x_1 = 0 \, , x_2 = 0$ There is no solution to the problem	$k = 3$ Three constraints $x_2 = 0 \ , x_1 = 0 \, , x_1 + 2x_2 - [1,3] = 0$ There is no solution to the problem

Comparing the results of the solution of the classical and neutrosophic problems, we find:
Problem 3:
In the classical problem:
$k = 1$ and $x_1 + 2x_2 - 2 = 0$
The problem has a solution at the point $\bar{x} = (x_1, x_2) = \left(\frac{22}{10}, \frac{-1}{10}\right)$ but it does not satisfy all the constraints since the third constraint is not satisfied
A rejected solution
In the neutrosophic problem:
$k = 1$ with the neutrosophic constraint $x_1 + 2x_2 - [1,3] = 0$
The problem has a solution at the point $\overline{x_N} = (x_{1N}, x_{2N}) = \left(\left[1, \frac{6}{5}\right], \left[0, \frac{2}{5}\right]\right)$ which satisfies all the constraints and is a local maximum.
The value of the objective function $\boldsymbol{f_N(x) = [-33.8, -13]}$
Problem4:
In the classical problem:
$\boldsymbol{k = 2}$ and the constraints $\boldsymbol{x_1 + 2x_2 - 2 = 0 \quad , x_1 = 0}$
The problem has a solution which is the point $\boldsymbol{\bar{x} = (x_1, x_2) = (0, 1)}$ that satisfies all the constraints, so it is a local maximum.
The value of the objective function $\boldsymbol{f(x) = -26}$
In the neutrosophic problem:
$\boldsymbol{k = 2}$ and the constraints $\boldsymbol{x_1 + 2x_2 - [1, 3] = 0 \quad , x_1 = 0}$
The problem has a solution which is the point $\boldsymbol{\overline{x_N} = (x_{1N}, x_{2N}) = \left(0, \left[\frac{1}{2}, \frac{3}{2}\right]\right)}$ that does not satisfy all the constraints, since the first constraint is not satisfied
Rejected solution
In addition to the absolute optimal value:
In the classical problem:
The point $\boldsymbol{(x_1^*, x_2^*) = (2, 0)}$ and the absolute maximum value of the function is:

$$\boldsymbol{f^*(x) = -2}$$

In the neutrosophic problem:
The point $\boldsymbol{(x_1^*, x_2^*) = ([1, 3], 0)}$ and the absolute maximum value of the function is:

$$\boldsymbol{f_N^*(x) \in [-9, -1]}$$

The absolute maximum value in the classical problem belongs to the solution domain in the neutrosophic problem, which confirms what we mentioned in the text of the research that the neutrosophic values give results that have a margin of freedom and are suitable for all conditions.

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Hybrid SuperHyperSoft Model for Evaluation of Comprehensive Transportation Efficiency: Analysis and Results

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Abstract: In this research, we investigate the features of comprehensive transportation efficiency assessment. Optimizing the energy-saving and efficient comprehensive transportation structure is essential to further lowering logistics costs. We evaluate the transportation efficiency using the decision making model based on a set of criteria and alternatives. We used two methods. named BWM and MABAC methods. The BWM is used to compute the criteria weights and the MABAC method is used to rank the alternatives. These methods are used under the neutrosophic set to solve the uncertainty information. These methods are used with SuperHyperSoft to treat various criteria and sub criteria by using a set of sets. Eight criteria and nine alternatives are used in this study. The sensitivity analysis shows the proposed approach rank in stable under different cases. The comparative analysis shows the proposed approach is effective compared to other methods.

Keywords: SuperHyperSoft; Transportation Efficiency; Neutrosophic Set; Transportation Costs.

1. Introduction and Literature Review

A thorough and scientific transport structure may raise the general level of transportation development, lower transportation costs, and increase transportation efficiency. At the same time, protecting the environment through energy conservation and pollution reduction is very important[1], [2]. The cost, profit, and energy consumption of various means of transportation vary greatly; for example, long-distance road transportation is expensive and inefficient[3], [4]. In the transportation system, developing an efficient network assessment technique is crucial. Jin et al. [5]proposed the integration of transportation as a means of building a comprehensive transportation system; A thorough analysis of the features, operations, and makeup of the transportation system was conducted by Lu et al.[6]. Additionally, the source originated from the optimization dynamic of the transportation system, and the external and internal factors of structural dynamic optimization of the system are investigated, with further depiction discussed

from four aspects: acquisition, technology, fund, quality, and environment, respectively; Franco [7] examined a spatial general equilibrium model of a closed monocentric city with two means of transportation and suggested that parking improvements at the central business district might improve welfare and encourage more compact urban space. The concept of a fuzzy set (FS) was introduced to the world by Zadeh. According to FS theory, a real integer from the closed interval [0, 1] designates the membership grade of each element in a set. The concept of an intuitionistic FS (IFS) was later established by Atanassov as an expansion of FS. It is assumed that the components in IFS theory have both membership and non-membership grades, provided that their aggregate is less than or equal to unity[8], [9]. Atanassov also established several IFS properties. In addressing decision-making issues, both the FS and IFS theories play important roles. However, choosing the best alternative based on several exact or vague criteria is the main goal of decision-makers (DMs) in today's decision-making scenarios[10]. Experts may struggle to choose the appropriate object because they lack a sufficient level of cognitive understanding of the issue. The concept of a neutrosophic set (NS), which is defined by the membership grades of truth, indeterminacy, and falsehood for each element of the set, is used to get around this problem[11], [12]. Smarandache introduced the idea of NS. To demonstrate the significance of truth, indeterminacy, and falsity information on which people make judgments, DMs frequently use this notion[13]. To circumvent the limitations of neutrosophic theory and enable real-world applications, Wang et al. established the idea of single-valued NS (SVNS) with limited restrictions for the membership grades. Chinnadurai et al [14] proposed a study with three pieces. They started by talking about the idea of an intuitionistic neutrosophic soft set with interval values. To ensure that the supremum sum of the truth and falsity membership grades does not surpass unity, they applied an intuitionistic condition between them. Likewise, the membership grade for indeterminacy falls outside of the closed interval [0, 1].

Therefore, in their instance, the total of the truth, indeterminacy, and falsity membership grades does not surpass two. They defined some of its characteristics and introduced the concepts of necessity, possibility, concentration, and dilation operators. Secondly, we establish the similarity metric between two intuitionistic neutrosophic soft sets with interval values. Additionally, they compared it to current approaches to demonstrate its superiority.

Lastly, they created an algorithm and used the diagnosis of mental illnesses as an example. Even though similarity measures are essential for identifying mental illnesses, current techniques are rarely used to do so. Naturally, Ambivalence characterizes most psychiatric disease behaviors. Therefore, it is essential to use an interval-valued intuitionistic neutrosophic soft set to record the membership grades. They offer a method for identifying mental illnesses in this publication, and the suggested similarity metric is useful and appropriate for diagnosing mental illnesses in any neutrophilic setting.

Aliya et al. [15] presented two new types of operational rules for pairs of linguistic IVIN fuzzy numbers, which are called neutrality addition and scalar multiplication. These operations' primary concept is to incorporate the decision-maker's and scoring function's linguistic IVIN

fuzzy number. They establish the operational rules and IVIN fuzzy number. Lastly, using a variety of numerical examples, an MCDM technique based on the suggested operators is introduced and examined.

Fahmi et al. [16] introduced the idea of a IVIN fuzzy number. After a quick review of relevant features, they defined some score and accuracy functions for LIVINFNs. Additionally, as part of the geometric operators, they provided the geometric forms of the LIVINDFWG, LIVINDFOWG, and LIVINDFHWG operators.

They then go over its features and a few unique situations. Additionally, they proposed two novel MCDM techniques based on the LIVINDFWG and LIVINDFOWG operators that were created. Lastly, a typical example is given to compare the suggested approach with several other representative MCDM methods that are currently in use to confirm its superiority and efficacy.

The rest of this part is organized as follows: Section 2 shows some definitions of IVINNs. Section 3 shows the steps of the proposed approach. Section 4 shows the results of the proposed approach with sensitivity analysis and comparative analysis. Section 5 shows the conclusions of this study.

2. Preliminaries

This section shows some definitions of the Interval Valued Intuitionistic Neutrosophic Set (IVINS).

Definition 1

An IVINS in V can be defined as: $A = \{(V, T_A(V), I_A(V), F_A(V))\}$, where $T_A(V), I_A(V), F_A(V)$ are sub intervals of [0,1] and shows the functions of truth, indeterminacy, and falsity. The lower and upper of $T_A(V), I_A(V), F_A(V)$ can be defined as: $T_A^U(V), T_A^L(V), I_A^U(V), I_A^L(V), F_A^U(V), F_A^L(V)$.

$$0 \leq T_A^U(V) + F_A^U(V) \leq 1 \tag{1}$$

$$T_A^U(V), I_A^U(V), F_A^U(V) \geq 0 \tag{2}$$

$$0 \leq T_A^U(V) + I_A^U(V) + F_A^U(V) \leq 2 \tag{3}$$

Example 1

We can define the Interval Valued Intuitionistic Neutrosophic number (IVINN) as:

$$A = \begin{Bmatrix} ([0.3,0.4],[0.7,0.8],[0.1,0.2]), \\ ([0.4,0.5],[0.8,0.9],[0.2,0.3]), \\ ([0.6,0.7],[0.2,0.3],[0.2,0.3]) \end{Bmatrix} \tag{4}$$

Definition 2

Let two IVINNs as: $(A_1, S_1) and\ (A_2, S_2)$, then

$(A_1, S_1)\ Or\ (A_2, S_2)\ is\ an\ IVINN$ shown as $(A_1, S_1) \vee (A_2, S_2) = A_\vee, S_1 \times S_2$ where

$$A_{\vee}(q_1 \times q_2) = A_1(q_1) \cup A_2(q_2) \forall (q_1, q_2) \in S_1 \times S_2 \tag{5}$$

$$A_{\vee}(q_1, q_2) = \begin{pmatrix} \left[\vee\left(T^U_{A_1(q_1)}, T^U_{A_1(q_2)}\right), \vee\left(T^L_{A_1(q_1)}, T^L_{A_1(q_2)}\right)\right], \\ \left[\vee\left(I^U_{A_1(q_1)}, I^U_{A_1(q_2)}\right), \vee\left(I^L_{A_1(q_1)}, I^L_{A_1(q_2)}\right)\right], \\ \left[\vee\left(F^U_{A_1(q_1)}, F^U_{A_1(q_2)}\right), \vee\left(F^L_{A_1(q_1)}, F^L_{A_1(q_2)}\right)\right] \end{pmatrix} \tag{6}$$

$(A_1, S_1)\ and\ (A_2, S_2)\ is\ an\ IVINN$ shown as $(A_1, S_1) \wedge (A_2, S_2) = A_{\wedge}, S_1 \times S_2$ where

$$A_{\wedge}(q_1 \times q_2) = A_1(q_1) \cap A_2(q_2) \forall (q_1, q_2) \in S_1 \times S_2 \tag{7}$$

$$A_{\wedge}(q_1, q_2) = \begin{pmatrix} \left[\wedge\left(T^U_{A_1(q_1)}, T^U_{A_1(q_2)}\right), \wedge\left(T^L_{A_1(q_1)}, T^L_{A_1(q_2)}\right)\right], \\ \left[\wedge\left(I^U_{A_1(q_1)}, I^U_{A_1(q_2)}\right), \wedge\left(I^L_{A_1(q_1)}, I^L_{A_1(q_2)}\right)\right], \\ \left[\wedge\left(F^U_{A_1(q_1)}, F^U_{A_1(q_2)}\right), \wedge\left(F^L_{A_1(q_1)}, F^L_{A_1(q_2)}\right)\right] \end{pmatrix} \tag{8}$$

Definition 3

Let two IVINNs as: $(A_1, S_1) and\ (A_2, S_2)$, then

$(A_1, S_1)\ union\ (A_2, S_2)\ is\ an\ IVINN$ shown as $(A_1, S_1) \cup (A_2, S_2) = A_{\cup}, S_{\cup}$ where

$$S_{\cup}(S_1 \cup S_2) = \forall q \in A_{\cup} \tag{9}$$

$$A_{\cup}(q) = \begin{Bmatrix} \left(v, \left(T_{A_{1(q)}}, I_{A_{1(q)}}, F_{A_{1(q)}}\right)\right); \quad if\ q \in S_1 - S_2, \\ \left(v, \left(T_{A_{2(q)}}, I_{A_{2(q)}}, F_{A_{2(q)}}\right)\right); \quad if\ q \in S_2 - S_1, \\ \begin{Bmatrix} \left[\vee\left(T^U_{A_1(q_1)}, T^U_{A_1(q_2)}\right), \vee\left(T^L_{A_1(q_1)}, T^L_{A_1(q_2)}\right)\right], \\ \left[\vee\left(I^U_{A_1(q_1)}, I^U_{A_1(q_2)}\right), \vee\left(I^L_{A_1(q_1)}, I^L_{A_1(q_2)}\right)\right], \\ \left[\wedge\left(F^U_{A_1(q_1)}, F^U_{A_1(q_2)}\right), \wedge\left(F^L_{A_1(q_1)}, F^L_{A_1(q_2)}\right)\right] \end{Bmatrix} \quad if q \in S_2 \cup S_1 \end{Bmatrix} \tag{10}$$

$(A_1, S_1)\ intersection\ (A_2, S_2)\ is\ an\ IVINN$ shown as $(A_1, S_1) \cap (A_2, S_2) = A_{\cap}, S_{\cap}$ where

$$S_{\cap}(S_1 \cap S_2) = \forall q \in A_{\cap} \tag{11}$$

$$A_{\cup}(q) = \begin{Bmatrix} \left(v, \left(T_{A_{1(q)}}, I_{A_{1(q)}}, F_{A_{1(q)}}\right)\right); \quad if\ q \in S_1 - S_2, \\ \left(v, \left(T_{A_{2(q)}}, I_{A_{2(q)}}, F_{A_{2(q)}}\right)\right); \quad if\ q \in S_2 - S_1, \\ \begin{Bmatrix} \left[\wedge\left(T^U_{A_1(q_1)}, T^U_{A_1(q_2)}\right), \wedge\left(T^L_{A_1(q_1)}, T^L_{A_1(q_2)}\right)\right], \\ \left[\wedge\left(I^U_{A_1(q_1)}, I^U_{A_1(q_2)}\right), \wedge\left(I^L_{A_1(q_1)}, I^L_{A_1(q_2)}\right)\right], \\ \left[\vee\left(F^U_{A_1(q_1)}, F^U_{A_1(q_2)}\right), \vee\left(F^L_{A_1(q_1)}, F^L_{A_1(q_2)}\right)\right] \end{Bmatrix} \quad if q \in S_2 \cap S_1 \end{Bmatrix} \tag{12}$$

Definition 4 (SuperHyperSoft Set (SHSS))

The SHSS is an extension of HyperSoft set and has several HyperSoft set. The SHSS is used in this study to compute the criteria and sub criteria for selecting the best alternative based on the set of criteria.

Let the universe set $U = \{C_1, C_2, \ldots . C_n\}$. The power set of U is a $P(U)$ and Q_1, Q_2, Q_3 are select as a criteria. $P(Q_1) \times P(Q_2)\ and\ P(Q_3)\ are\ powersets\ of\ Q_1, Q_2, Q_3$

Let $F: P(Q_1) \times P(Q_2) \times P(Q_3) \to P(Q)\ where\ \times refers\ to\ cartesian\ product$, and this called SHSS over Q.

$$P(Q_1) \times P(Q_2) \times P(Q_3) = \begin{Bmatrix} \{Q_{11}\}, \{Q_{12}\}, \{Q_{11}, Q_{12}\} \times \\ \{Q_{21}\}, \{Q_{22}\}, \{Q_{21}, Q_{22}\} \times \\ \{Q_{31}\}, \{Q_{32}\}, \{Q_{33}\}, \{Q_{31}, Q_{32}\}, \{Q_{31}, Q_{33}\}, \\ \{Q_{32}, Q_{33}\}, \{Q_{31}, Q_{32}, Q_{33}\} \end{Bmatrix} \quad (13)$$

3. IVIN-BWM-MABAC Approach

This section shows the steps of the proposed approach. This section includes three parts, in the first part, we define the criteria and alternatives with a set of experts. In the second part, we compute the criteria weights by the BWM approach. In the third part, we rank the alternatives by the MABAC approach. These methods are used under the IVINS to dela with vague information.

IVIN-BWM

BWM, one of the newest MADM techniques, effectively addresses the inconsistency resulting from pairwise comparisons. Compared to the other methods, this approach is more reliable. BWM has been used in several research projects.

The BWM method's basic phases and structure are as follows:

Step 1: Common methods for choosing and selecting criteria, such as literature reviews, expert opinions, and other likely approaches.

Step 2: Using the thoughts and views of experts, determine which criteria are the best and worst.
Step 3: Using IVINNs, create the preferences matrix by comparing the best criterion to all others.

Step 4: Using IVINNs, create the preferences matrix by comparing the worst criterion to all others.

Step 5: Determine the relative relevance of the criteria by solving the next optimization model and calculating the final value and best weights (w1*, w2*, w3*, ….wn*).

$$minmax_j \left\{ \left| \left(W_b/W_j\right) - a_{B_j} \right|, \left| \left(W_j/W_w\right) - a_{j_w} \right| \right\} \quad (14)$$

The optimal value of reliability level

$$\sum_j W_j = 1 \quad (15)$$

$\min d$

$$\left|\left(W_B/W_j\right) - a_{B_j}\right| \leq d \; for \; all \; j \tag{16}$$

$$\left|\left(W_j/W_w\right) - a_{j_w}\right| \leq d \; for \; all \; j \tag{17}$$

$$\sum_j W_j = 1$$

$$W_j \geq 0 \; for \; all \; j \tag{18}$$

IVIN-MABAC

This part shows the steps of the MABAC methods.

Step 1. Create the assessment matrix.

We used the IVINNs to create the assessment matrix. Then we apply the score function to obtain crisp values. Then we combine the opinions of experts.

Step 2. Normalize the decision matrix

We can normalize the decision matrix based on the positive and cost criteria such as:

$$n_{ij} = \frac{x_{ij} - \min x_{ij}}{\max x_{ij} - \min x_{ij}} \tag{19}$$

$$n_{ij} = \frac{x_{ij} - \max x_{ij}}{\min x_{ij} - \max x_{ij}} \tag{20}$$

Where x_{ij} refers to the value in the decision matrix.

Step 3. Compute the weighted decision matrix.

$$f_{ij} = w_j + w_j n_{ij} \tag{21}$$

Step 4. Compute the border approximation method.

$$t_j = \left(\prod_{i=1}^{m} f_{ij}\right)^{\left(\frac{1}{m}\right)} \tag{22}$$

Step 5. Compute the distance from t_j

$$g_{ij} = f_{ij} - t_j \tag{23}$$

Step 6. Obtain the total distance

$$E_i = \sum_{j=1}^{n} g_{ij} \tag{24}$$

Step 7. Rank the alternatives.

4. Case Study

This section shows the results of the proposed approach to computing the criteria weights and rank alternatives. This study invited three experts to assess the criteria and alternatives. We collected eight criteria and nine alternatives. The criteria and suitable values: Economic Efficiency:

(Efficiency index less than 0.5, more than 0.5), Infrastructure Development: (index score less than 0.5, more than 0.5), Traffic Flow: (optimal or not optimal), Environmental Sustainability: (Sustainability index less than 0.5, more than 0.5), Public Transport Accessibility: (Efficiency index less than 0.5, more than 0.5), Smart Transportation Integration: (fully integrated, not integrated), Safety and Reliability: (low, middle, high), Government Policies: (low, middle, high)

The alternatives are: Ride-Sharing Services, Bicycle & Pedestrian Networks, Autonomous Vehicle Systems, High-Speed Rail Systems, Air Transport, Traditional Road Transport, Urban Metro Systems, Electric Bus Transit, Water Transport Systems

Each expert can select the best and worst criterion.

We use the IVINNs to create the preferences matrix by comparing the best criterion to all others.

We use the IVINNs to create the preferences matrix by comparing the worst criterion to all others.

We determine the relative relevance of the criteria using Eqs. (14-18). Then we compute the criteria weights as shown in Fig 1.

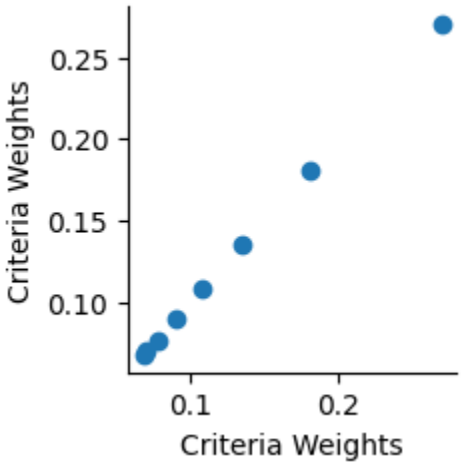

Fig 1. The importance of each criterion.

4.1. IVIN-MABAC

Based on the SHSS, we can suggest several sets for sub-criteria to rank the alternatives such as:

(Efficiency index more than 0.5) , (index score less than 0.5, more than 0.5), (optimal), (Sustainability more than 0.5), (Efficiency index more than 0.5), (fully integrated), (middle, high), (middle)

We proposed two sets such as:

Set 1: (Efficiency index more than 0.5) , (index score less than 0.5, more than 0.5), (optimal), (Sustainability more than 0.5), (Efficiency index more than 0.5), (fully integrated), (middle), (middle)

Set 2: (Efficiency index more than 0.5) , (index score less than 0.5, more than 0.5), (optimal), (Sustainability more than 0.5), (Efficiency index more than 0.5), (fully integrated), (high), (middle)

We applied the MABAC method based on the two sets. Based on Set 1:

Step 1. We create three assessment matrices between the criteria and alternatives using the IVINN as shown in Tables 1-3. Then we obtain crisp values. Then we combine the decision matrix.

Step 2. Eq. (19) is used to normalize the decision matrix

Step 3. Eq. (21) is used to compute the weighted decision matrix as shown in Fig 2.

Step 4. Then we compute the border approximation method using Eq. (22) as shown in Fig 3.
Step 5. Eq. (23) is used to compute the distance from t_j as shown in Fig 4.

Step 6. Eq. (24) is used to obtain the total distance

Step 7. We rank the alternatives as shown in Fig 5.

Table 1. The first IVINNs.

	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8
A_1	([0.1, 0.3], [0.4, 0.5], [0.1, 0.2])	([0.6, 0.7], [0.8, 0.9], [0.2, 0.3])	([0.2, 0.3], [0.4, 0.5], [0.1, 0.2])	([0.1, 0.2], [0.3, 0.4], [0.2, 0.3])	([0.1, 0.2], [0.3, 0.4], [0.2, 0.3])	([0.7, 0.8], [0.6, 0.7], [0.1, 0.2])	([0.3, 0.4], [0.5, 0.6], [0.3, 0.4])	([0.3, 0.4], [0.5, 0.6], [0.3, 0.4])
A_2	([0.1, 0.3], [0.4, 0.5], [0.1, 0.2])	([0.1, 0.2], [0.3, 0.4], [0.2, 0.3])	([0.7, 0.8], [0.6, 0.7], [0.1, 0.2])	([0.6, 0.7], [0.8, 0.9], [0.2, 0.3])	([0.2, 0.3], [0.4, 0.5], [0.1, 0.2])	([0.1, 0.2], [0.3, 0.4], [0.2, 0.3])	([0.1, 0.2], [0.3, 0.4], [0.2, 0.3])	([0.7, 0.8], [0.6, 0.7], [0.1, 0.2])
A_3	([0.8, 0.9], [0.7, 0.8], [0.0, 0.1])	([0.2, 0.3], [0.4, 0.5], [0.1, 0.2])	([0.2, 0.3], [0.4, 0.5], [0.1, 0.2])	([0.1, 0.2], [0.3, 0.4], [0.2, 0.3])	([0.6, 0.7], [0.8, 0.9], [0.2, 0.3])	([0.2, 0.3], [0.4, 0.5], [0.1, 0.2])	([0.2, 0.3], [0.4, 0.5], [0.1, 0.2])	([0.3, 0.4], [0.5, 0.6], [0.3, 0.4])
A_4	([0.1, 0.2], [0.3, 0.4], [0.2, 0.3])	([0.6, 0.7], [0.8, 0.9], [0.2, 0.3])	([0.1, 0.2], [0.3, 0.4], [0.2, 0.3])	([0.7, 0.8], [0.6, 0.7], [0.1, 0.2])	([0.7, 0.8], [0.6, 0.7], [0.1, 0.2])	([0.6, 0.7], [0.8, 0.9], [0.2, 0.3])	([0.6, 0.7], [0.8, 0.9], [0.2, 0.3])	([0.1, 0.2], [0.3, 0.4], [0.2, 0.3])
A_5	([0.2, 0.3], [0.4, 0.5], [0.1, 0.2])	([0.1, 0.3], [0.4, 0.5], [0.1, 0.2])	([0.2, 0.3], [0.4, 0.5], [0.1, 0.2])	([0.1, 0.2], [0.3, 0.4], [0.2, 0.3])	([0.1, 0.2], [0.3, 0.4], [0.2, 0.3])	([0.7, 0.8], [0.6, 0.7], [0.1, 0.2])	([0.7, 0.8], [0.6, 0.7], [0.1, 0.2])	([0.2, 0.3], [0.4, 0.5], [0.1, 0.2])
A_6	([0.6, 0.7], [0.8, 0.9], [0.2, 0.3])	([0.1, 0.2], [0.3, 0.4], [0.2, 0.3])	([0.6, 0.7], [0.8, 0.9], [0.2, 0.3])	([0.2, 0.3], [0.4, 0.5], [0.1, 0.2])	([0.2, 0.3], [0.4, 0.5], [0.1, 0.2])	([0.1, 0.2], [0.3, 0.4], [0.2, 0.3])	([0.3, 0.4], [0.5, 0.6], [0.3, 0.4])	([0.6, 0.7], [0.8, 0.9], [0.2, 0.3])
A_7	([0.7, 0.8], [0.6, 0.7], [0.1, 0.2])	([0.2, 0.3], [0.4, 0.5], [0.1, 0.2])	([0.7, 0.8], [0.6, 0.7], [0.1, 0.2])	([0.6, 0.7], [0.8, 0.9], [0.2, 0.3])	([0.6, 0.7], [0.8, 0.9], [0.2, 0.3])	([0.2, 0.3], [0.4, 0.5], [0.1, 0.2])	([0.7, 0.8], [0.6, 0.7], [0.1, 0.2])	([0.7, 0.8], [0.6, 0.7], [0.1, 0.2])
A_8	([0.3, 0.4], [0.5, 0.6], [0.3, 0.4])	([0.6, 0.7], [0.8, 0.9], [0.2, 0.3])	([0.3, 0.4], [0.5, 0.6], [0.3, 0.4])	([0.1, 0.3], [0.4, 0.5], [0.1, 0.2])	([0.1, 0.3], [0.4, 0.5], [0.1, 0.2])	([0.6, 0.7], [0.8, 0.9], [0.2, 0.3])	([0.6, 0.7], [0.8, 0.9], [0.2, 0.3])	([0.3, 0.4], [0.5, 0.6], [0.3, 0.4])
A_9	([0.7, 0.8], [0.6, 0.7], [0.1, 0.2])	([0.7, 0.8], [0.6, 0.7], [0.1, 0.2])	([0.7, 0.8], [0.6, 0.7], [0.1, 0.2])	([0.7, 0.8], [0.6, 0.7], [0.1, 0.2])	([0.1, 0.2], [0.3, 0.4], [0.2, 0.3])	([0.7, 0.8], [0.6, 0.7], [0.1, 0.2])	([0.2, 0.3], [0.4, 0.5], [0.1, 0.2])	([0.7, 0.8], [0.6, 0.7], [0.1, 0.2])

Table 2. The second IVINNs.

	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8
A_1	([0.8, 0.9], [0.7, 0.8], [0.0, 0.1])	([0.6, 0.7], [0.8, 0.9], [0.2, 0.3])	([0.2, 0.3], [0.4, 0.5], [0.1, 0.2])	([0.1, 0.2], [0.3, 0.4], [0.2, 0.3])	([0.3, 0.4], [0.5, 0.6], [0.3, 0.4])	([0.7, 0.8], [0.6, 0.7], [0.1, 0.2])	([0.3, 0.4], [0.5, 0.6], [0.3, 0.4])	([0.8, 0.9], [0.7, 0.8], [0.0, 0.1])
A_2	([0.7, 0.8], [0.6, 0.7], [0.1, 0.2])	([0.3, 0.4], [0.5, 0.6], [0.3, 0.4])	([0.7, 0.8], [0.6, 0.7], [0.1, 0.2])	([0.6, 0.7], [0.8, 0.9], [0.2, 0.3])	([0.2, 0.3], [0.4, 0.5], [0.1, 0.2])	([0.1, 0.2], [0.3, 0.4], [0.2, 0.3])	([0.1, 0.2], [0.3, 0.4], [0.2, 0.3])	([0.8, 0.9], [0.7, 0.8], [0.0, 0.1])
A_3	([0.6, 0.7], [0.8, 0.9], [0.2, 0.3])	([0.3, 0.4], [0.5, 0.6], [0.3, 0.4])	([0.2, 0.3], [0.4, 0.5], [0.1, 0.2])	([0.1, 0.2], [0.3, 0.4], [0.2, 0.3])	([0.3, 0.4], [0.5, 0.6], [0.3, 0.4])	([0.8, 0.9], [0.7, 0.8], [0.0, 0.1])	([0.1, 0.2], [0.3, 0.4], [0.2, 0.3])	([0.1, 0.2], [0.3, 0.4], [0.2, 0.3])
A_4	([0.2, 0.3], [0.4, 0.5], [0.1, 0.2])	([0.7, 0.8], [0.6, 0.7], [0.1, 0.2])	([0.6, 0.7], [0.8, 0.9], [0.2, 0.3])	([0.1, 0.3], [0.4, 0.5], [0.1, 0.2])	([0.8, 0.9], [0.7, 0.8], [0.0, 0.1])	([0.1, 0.3], [0.4, 0.5], [0.1, 0.2])	([0.6, 0.7], [0.8, 0.9], [0.2, 0.3])	([0.2, 0.3], [0.4, 0.5], [0.1, 0.2])
A_5	([0.1, 0.2], [0.3, 0.4], [0.2, 0.3])	([0.3, 0.4], [0.5, 0.6], [0.3, 0.4])	([0.1, 0.2], [0.3, 0.4], [0.2, 0.3])	([0.8, 0.9], [0.7, 0.8], [0.0, 0.1])	([0.7, 0.8], [0.6, 0.7], [0.1, 0.2])	([0.6, 0.7], [0.8, 0.9], [0.2, 0.3])	([0.1, 0.3], [0.4, 0.5], [0.1, 0.2])	([0.6, 0.7], [0.8, 0.9], [0.2, 0.3])
A_6	([0.6, 0.7], [0.8, 0.9], [0.2, 0.3])	([0.1, 0.3], [0.4, 0.5], [0.1, 0.2])	([0.8, 0.9], [0.7, 0.8], [0.0, 0.1])	([0.7, 0.8], [0.6, 0.7], [0.1, 0.2])	([0.6, 0.7], [0.8, 0.9], [0.2, 0.3])	([0.2, 0.3], [0.4, 0.5], [0.1, 0.2])	([0.3, 0.4], [0.5, 0.6], [0.3, 0.4])	([0.7, 0.8], [0.6, 0.7], [0.1, 0.2])
A_7	([0.1, 0.3], [0.4, 0.5], [0.1, 0.2])	([0.6, 0.7], [0.8, 0.9], [0.2, 0.3])	([0.7, 0.8], [0.6, 0.7], [0.1, 0.2])	([0.6, 0.7], [0.8, 0.9], [0.2, 0.3])	([0.2, 0.3], [0.4, 0.5], [0.1, 0.2])	([0.1, 0.2], [0.3, 0.4], [0.2, 0.3])	([0.6, 0.7], [0.8, 0.9], [0.2, 0.3])	([0.8, 0.9], [0.7, 0.8], [0.0, 0.1])
A_8	([0.8, 0.9], [0.7, 0.8], [0.0, 0.1])	([0.2, 0.3], [0.4, 0.5], [0.1, 0.2])	([0.6, 0.7], [0.8, 0.9], [0.2, 0.3])	([0.2, 0.3], [0.4, 0.5], [0.1, 0.2])	([0.1, 0.2], [0.3, 0.4], [0.2, 0.3])	([0.2, 0.3], [0.4, 0.5], [0.1, 0.2])	([0.2, 0.3], [0.4, 0.5], [0.1, 0.2])	([0.7, 0.8], [0.6, 0.7], [0.1, 0.2])
A_9	([0.7, 0.8], [0.6, 0.7], [0.1, 0.2])	([0.1, 0.2], [0.3, 0.4], [0.2, 0.3])	([0.2, 0.3], [0.4, 0.5], [0.1, 0.2])	([0.1, 0.2], [0.3, 0.4], [0.2, 0.3])	([0.1, 0.2], [0.3, 0.4], [0.2, 0.3])	([0.1, 0.2], [0.3, 0.4], [0.2, 0.3])	([0.1, 0.2], [0.3, 0.4], [0.2, 0.3])	([0.3, 0.4], [0.5, 0.6], [0.3, 0.4])

Table 3. The third IVINNs.

	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8
A_1	([0.7, 0.8], [0.6, 0.7], [0.1, 0.2])	([0.6, 0.7], [0.8, 0.9], [0.2, 0.3])	([0.2, 0.3], [0.4, 0.5], [0.1, 0.2])	([0.1, 0.2], [0.3, 0.4], [0.2, 0.3])	([0.3, 0.4], [0.5, 0.6], [0.3, 0.4])	([0.1, 0.3], [0.4, 0.5], [0.1, 0.2])	([0.3, 0.4], [0.5, 0.6], [0.3, 0.4])	([0.3, 0.4], [0.5, 0.6], [0.3, 0.4])
A_2	([0.1, 0.3], [0.4, 0.5], [0.1, 0.2])	([0.8, 0.9], [0.7, 0.8], [0.0, 0.1])	([0.7, 0.8], [0.6, 0.7], [0.1, 0.2])	([0.6, 0.7], [0.8, 0.9], [0.2, 0.3])	([0.2, 0.3], [0.4, 0.5], [0.1, 0.2])	([0.1, 0.2], [0.3, 0.4], [0.2, 0.3])	([0.1, 0.2], [0.3, 0.4], [0.2, 0.3])	([0.7, 0.8], [0.6, 0.7], [0.1, 0.2])
A_3	([0.3, 0.4], [0.5, 0.6], [0.3, 0.4])	([0.8, 0.9], [0.7, 0.8], [0.0, 0.1])	([0.2, 0.3], [0.4, 0.5], [0.1, 0.2])	([0.1, 0.2], [0.3, 0.4], [0.2, 0.3])	([0.3, 0.4], [0.5, 0.6], [0.3, 0.4])	([0.3, 0.4], [0.5, 0.6], [0.3, 0.4])	([0.2, 0.3], [0.4, 0.5], [0.1, 0.2])	([0.8, 0.9], [0.7, 0.8], [0.0, 0.1])
A_4	([0.1, 0.2], [0.3, 0.4], [0.2, 0.3])	([0.7, 0.8], [0.6, 0.7], [0.1, 0.2])	([0.6, 0.7], [0.8, 0.9], [0.2, 0.3])	([0.7, 0.8], [0.6, 0.7], [0.1, 0.2])	([0.8, 0.9], [0.7, 0.8], [0.0, 0.1])	([0.7, 0.8], [0.6, 0.7], [0.1, 0.2])	([0.6, 0.7], [0.8, 0.9], [0.2, 0.3])	([0.1, 0.2], [0.3, 0.4], [0.2, 0.3])
A_5	([0.2, 0.3], [0.4, 0.5], [0.1, 0.2])	([0.8, 0.9], [0.7, 0.8], [0.0, 0.1])	([0.1, 0.2], [0.3, 0.4], [0.2, 0.3])	([0.3, 0.4], [0.5, 0.6], [0.3, 0.4])	([0.1, 0.3], [0.4, 0.5], [0.1, 0.2])	([0.3, 0.4], [0.5, 0.6], [0.3, 0.4])	([0.1, 0.3], [0.4, 0.5], [0.1, 0.2])	([0.2, 0.3], [0.4, 0.5], [0.1, 0.2])
A_6	([0.6, 0.7], [0.8, 0.9], [0.2, 0.3])	([0.1, 0.2], [0.3, 0.4], [0.2, 0.3])	([0.2, 0.3], [0.4, 0.5], [0.1, 0.2])	([0.7, 0.8], [0.6, 0.7], [0.1, 0.2])	([0.1, 0.3], [0.4, 0.5], [0.1, 0.2])	([0.7, 0.8], [0.6, 0.7], [0.1, 0.2])	([0.3, 0.4], [0.5, 0.6], [0.3, 0.4])	([0.6, 0.7], [0.8, 0.9], [0.2, 0.3])
A_7	([0.1, 0.3], [0.4, 0.5], [0.1, 0.2])	([0.2, 0.3], [0.4, 0.5], [0.1, 0.2])	([0.1, 0.2], [0.3, 0.4], [0.2, 0.3])	([0.8, 0.9], [0.7, 0.8], [0.0, 0.1])	([0.6, 0.7], [0.8, 0.9], [0.2, 0.3])	([0.6, 0.7], [0.8, 0.9], [0.2, 0.3])	([0.1, 0.3], [0.4, 0.5], [0.1, 0.2])	([0.1, 0.3], [0.4, 0.5], [0.1, 0.2])
A_8	([0.8, 0.9], [0.7, 0.8], [0.0, 0.1])	([0.6, 0.7], [0.8, 0.9], [0.2, 0.3])	([0.8, 0.9], [0.7, 0.8], [0.0, 0.1])	([0.1, 0.2], [0.3, 0.4], [0.2, 0.3])	([0.2, 0.3], [0.4, 0.5], [0.1, 0.2])	([0.2, 0.3], [0.4, 0.5], [0.1, 0.2])	([0.6, 0.7], [0.8, 0.9], [0.2, 0.3])	([0.8, 0.9], [0.7, 0.8], [0.0, 0.1])
A_9	([0.7, 0.8], [0.6, 0.7], [0.1, 0.2])	([0.1, 0.3], [0.4, 0.5], [0.1, 0.2])	([0.1, 0.3], [0.4, 0.5], [0.1, 0.2])	([0.1, 0.3], [0.4, 0.5], [0.1, 0.2])	([0.1, 0.2], [0.3, 0.4], [0.2, 0.3])	([0.1, 0.2], [0.3, 0.4], [0.2, 0.3])	([0.2, 0.3], [0.4, 0.5], [0.1, 0.2])	([0.7, 0.8], [0.6, 0.7], [0.1, 0.2])

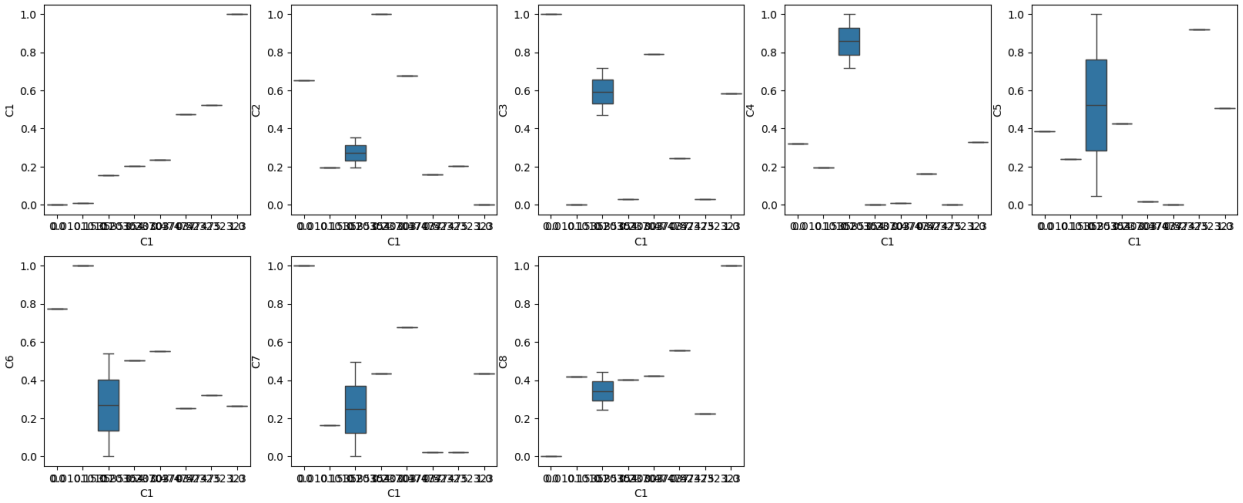

Fig 2. The normalization matrix.

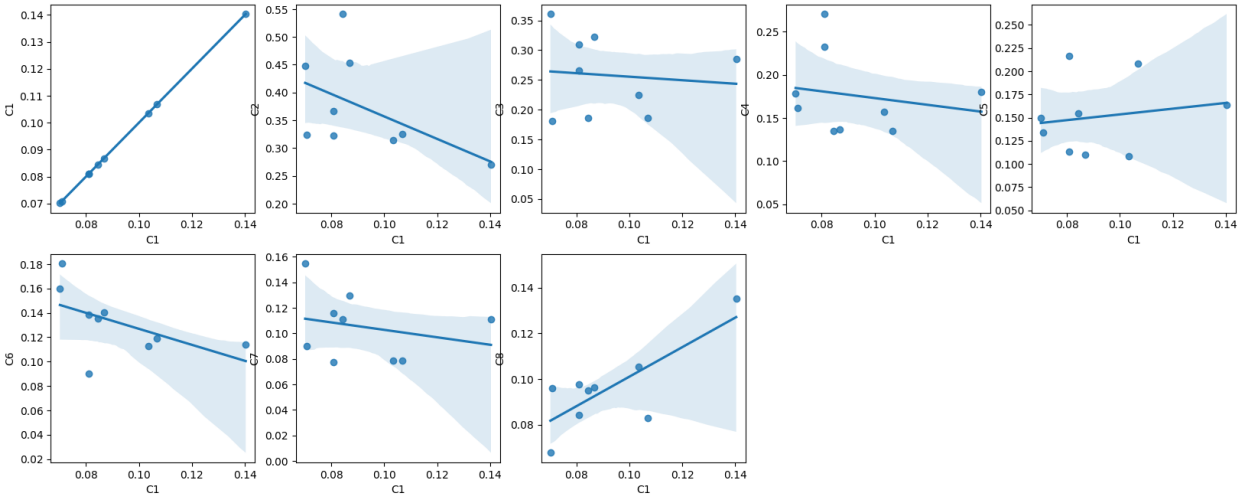

Fig 3. The weighted normalized matrix.

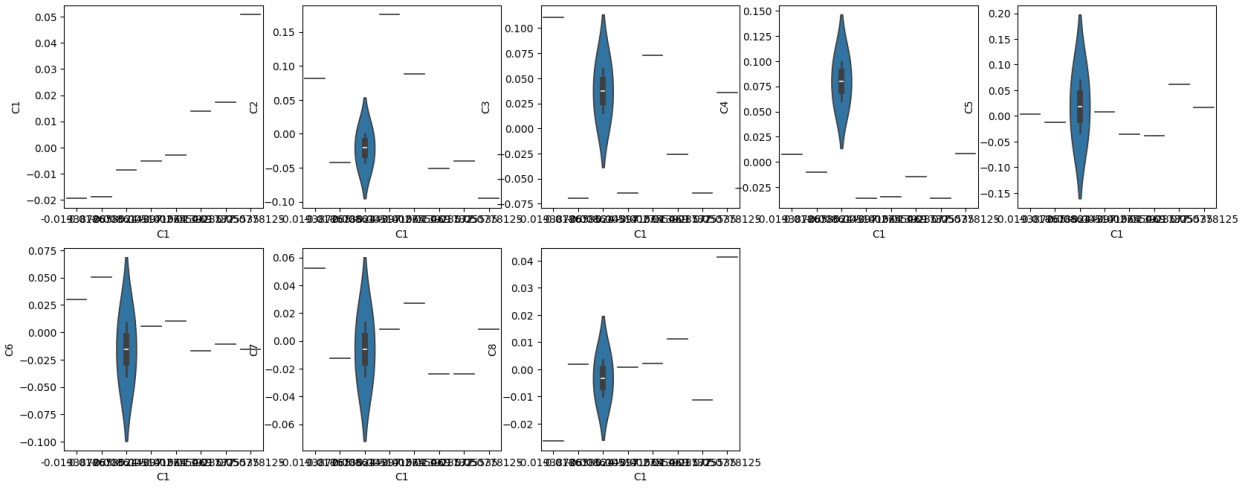

Fig 4. The total distances.

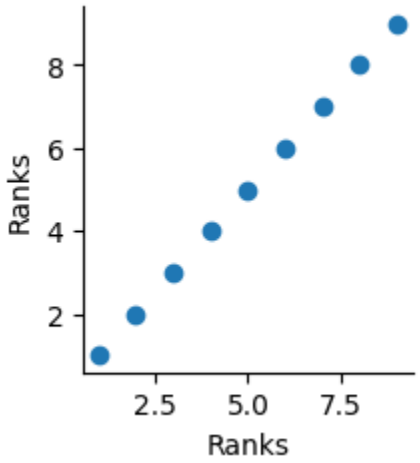

Fig 5. The ranks of alternatives.

Based on the Second set, we compute the normalization matrix as shown in Fig 6.

Then we compute the weighted normalized decision matrix as shown in Fig 7. Then we compute the border areas and total distance as shown in Fig 8.

Finally, we obtain the final ranks of alternatives as shown in Fig 9. We show the alternative 4 is the best and alternative 9 is the worst.

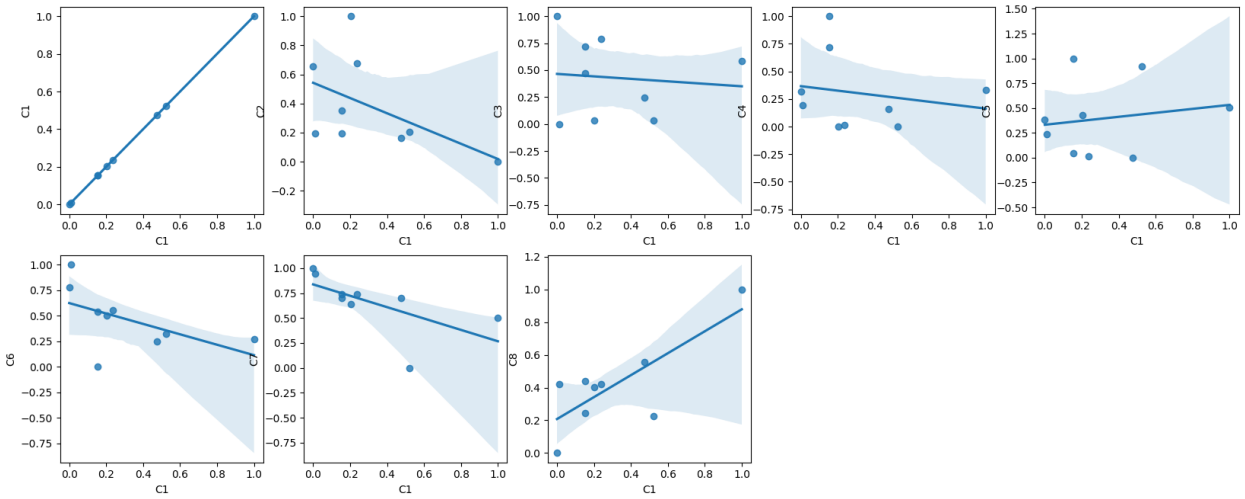

Fig 6. The normalization matrix.

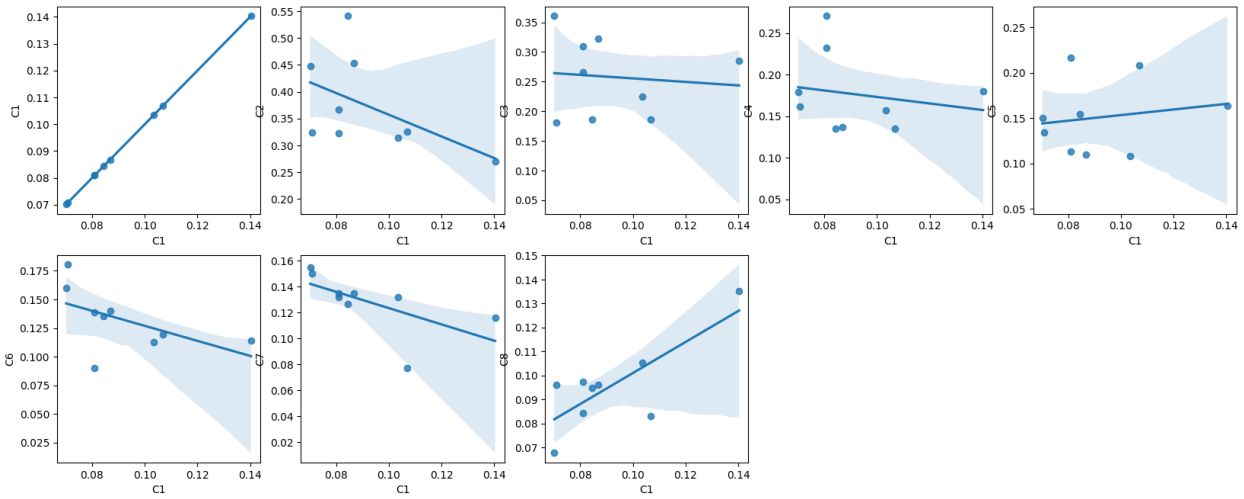

Fig 7. The weighted normalized matrix.

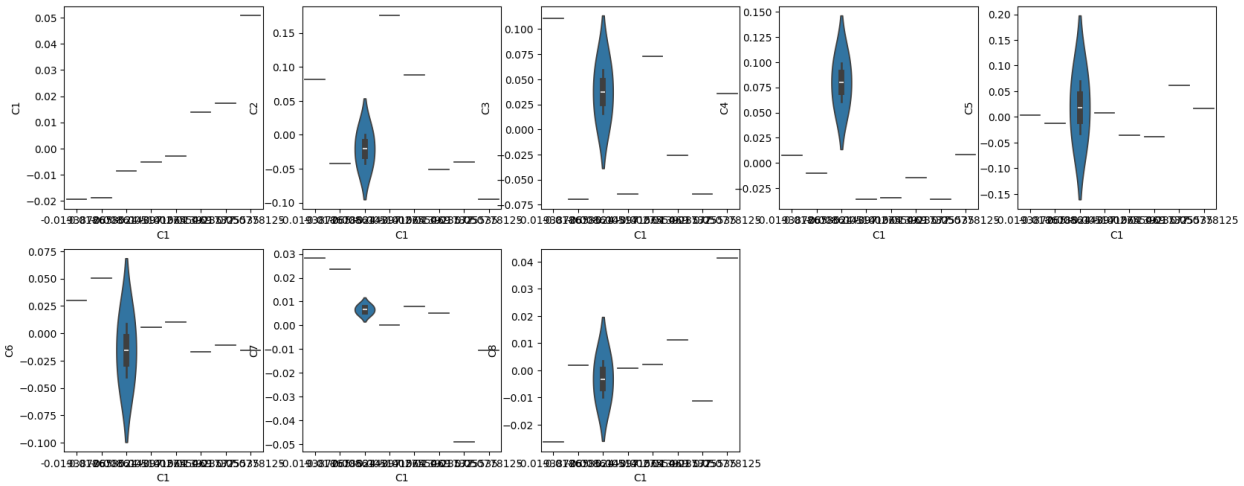

Fig 8. The total distances.

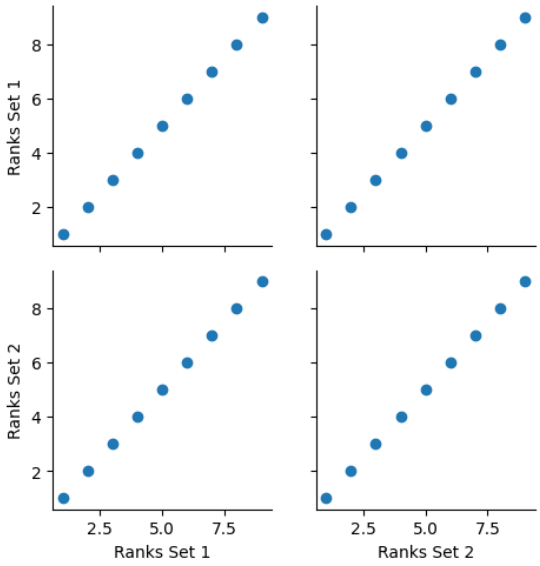

Fig 9. The final ranks of alternatives.

4.2. Sensitivity analysis

This part shows the sensitivity analysis to show the ranks of alternatives under different cases. We change the criteria weights by 9 cases to show different values of each alternative. We increase the weights of the criteria by 28% and other criteria have the same weights.

We applied the MABAC method under these cases, to show the ranks of the criteria. In the first case, we work on the same weights of all criteria. Then we increase the criteria weights by 28%. We obtain the total distance of each alternative with each case as shown in Fig 10. Then we rank the alternatives under different cases as shown in Fig 11. We show the rank of alternatives is stable under different cases. So, the MABAC method obtains stability in the ranks.

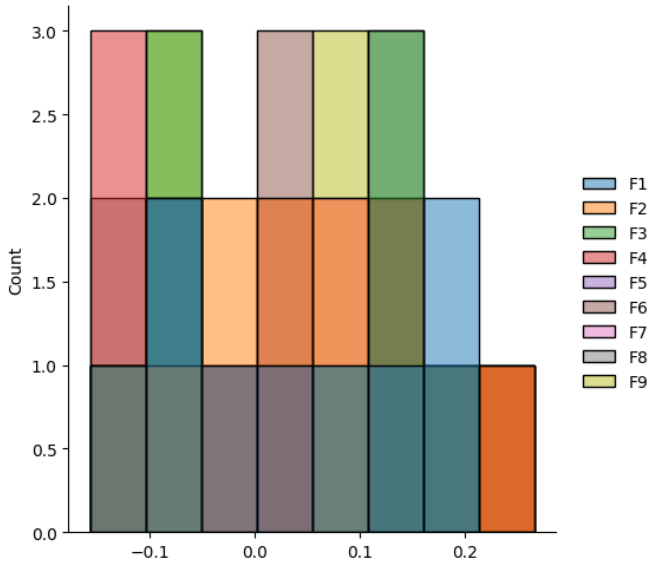

Fig 10. The total distance in sensitivity analysis.

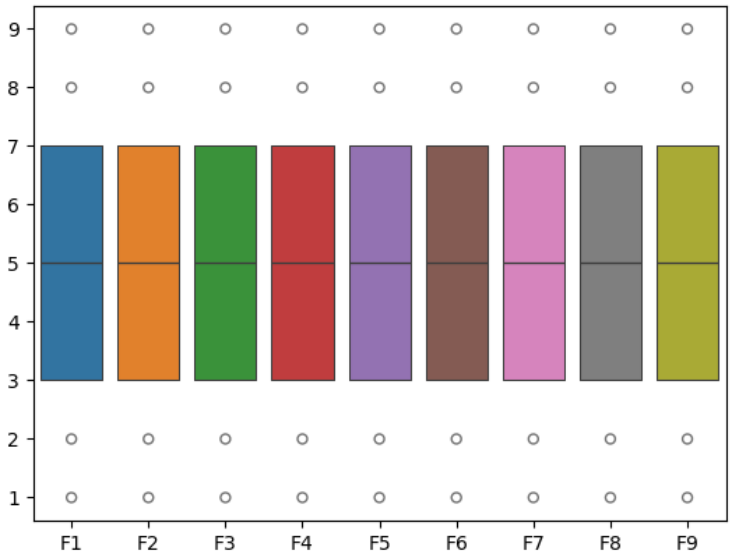

Fig 11. The ranks in sensitivity analysis.

4.3. Comparative analysis

This part shows the comparative analysis between the proposed approach and other methods to show the effectiveness of the proposed approach. We compared the proposed approach with the Proposed Model, TODIM Method, COPRAS Method, ELECTRE Method as shown in Fig 12. The proposed approach has the alternative 4 is the best and alternative 9 is the worst. The results show the proposed approach is effective compared to other methods.

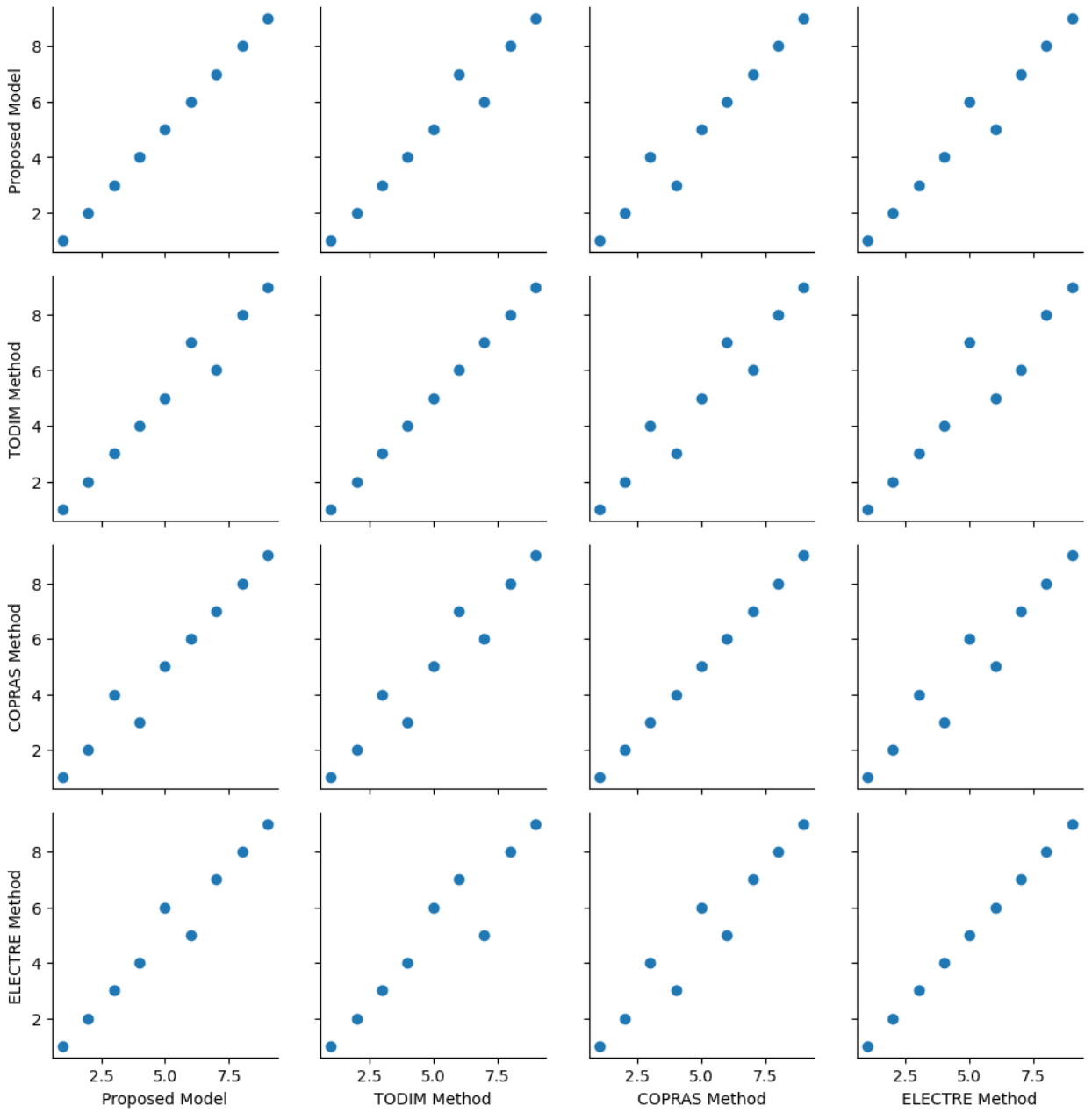

Fig 12. The comparative results

5. Conclusions

This study evaluated transportation efficiency by conducting a set of criteria and alternatives. Two methods are used in this study, such as BWM to compute the criteria weights and the MABAC method to rank the alternatives. These methods are used under the Interval Valued Intuitionistic Neutrosophic number (IVINN) to deal with vague information. SuperHyperSoft set is used with this model to treat various criteria and sub criteria. The results show alternative 4 is the best and alternative 9 is the worst. The sensitivity analysis is conducted by using nine cases.

The ranks of alternatives under these cases are stable. We compared the proposed approach with four methods. The results show the proposed approach is effective.

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New Energy Vehicle Industry Chain Evaluation by Developing Decision-Making Methodology with HyperSoft Set and Neutrosophic Sets

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Abstract: The New Energy Vehicle (NEV) industry has emerged as a crucial sector in the global transition toward sustainable transportation. This study proposed a decision-making methodology for NEV industry chain evaluation. We used two decision making methods, such as SWARA method to compute the criteria weights and the MAIRCA method to rank the alternatives. These methods are used under the neutrosophic sets to deal with uncertainty data. Then we used the HyperSoft Set to show the relationship between the criteria and their values. This study proves the validation of the proposed approach by applying the proposed approach in case study. Four experts are invited in this study to evaluate six criteria and seven alternatives. Comparative analysis between our method and other methods shows the effectiveness of the proposed approach. Sensitivity analysis is conducted to show the stability of the ranks of the proposed approach.

Keywords: New Energy Vehicle (NEV); HyperSoft Set; Neutrosophic HyperSoft Set; Interval Valued Pythagorean Neutrosophic Set (IVPNS).

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1. Introduction and Related Work

The New Energy Vehicle (NEV) industry has emerged as a crucial sector in the global transition toward sustainable transportation. With increasing concerns over climate change, energy security, and environmental degradation, governments and industries worldwide have invested heavily in electric vehicles (EVs), hybrid-electric vehicles (HEVs), and hydrogen fuel cell vehicles (HFCVs). The NEV industry chain consists of multiple interconnected segments, including battery production, vehicle manufacturing, charging infrastructure, supply chain logistics, policy regulation, and recycling technologies[1], [2]. Understanding and evaluating the performance and resilience of this industry chain is essential for sustainable development and long-term competitiveness.

Despite its rapid growth, the NEV industry faces several challenges, such as technological innovation constraints, supply chain vulnerabilities, high infrastructure costs, and regulatory uncertainties. These challenges necessitate a comprehensive evaluation framework that considers multiple criteria affecting the industry's overall efficiency and effectiveness[3], [4]. Multi-Criteria Decision-Making (MCDM) methodologies provide a structured approach for analyzing complex, multi-dimensional problems by integrating qualitative and quantitative factors. Among these methodologies, HyperSoft Set theory has gained attention due to its ability to handle uncertainties, overlapping attributes, and hierarchical decision structures.

To assess the NEV industry chain's performance, this study evaluates seven alternatives representing key stakeholders, including battery manufacturers, electric vehicle producers, charging infrastructure providers, supply chain firms, autonomous mobility companies, regulatory bodies, and sustainability-focused enterprises. The HyperSoft Set model allows for a nuanced evaluation by assigning lower-bound, middle-value, and upper-bound scores to each alternative under the selected criteria[5], [6].

The findings of this research will provide practical insights for policymakers, industry leaders, and investors, helping them identify strengths, weaknesses, and potential areas for improvement in the NEV industry chain. Additionally, the application of HyperSoft Set theory in this context will demonstrate its effectiveness in handling complex decision-making problems with uncertain and interrelated parameters. By offering a structured and data-driven evaluation framework, this study contributes to the sustainable growth and strategic development of the NEV industry, ensuring its resilience and competitiveness in the global market.

Due to their intrinsic ambiguity, the data used in many complex problems—including those in computer science, engineering, social and economic contexts, medical science, and more—may not always have clear, exact, and predictable characteristics. Most of these intricate issues have been approached using a range of ideas. To address these issues, Zadeh developed the notion of a fuzzy set, in which the concept of a membership function is represented by the symbol μ, providing the range [0,1] to signify the degree of belongingness to the set in question[7], [8].

Uncertainty and indeterminacy are crucial in a neutrosophic setting. First proposed by Smarandache in 1995, the idea of a neutrosophic set (NS) enables indeterminacy to be represented as three components: truth, indeterminacy, and false membership[9], [10].

To determine the shortest path in each network, the A* search technique is frequently used. On the other hand, the nodes in a conventional A* search algorithm are thought to have crisp values, or a single value. In many real-world situations when ambiguity or uncertainty are present, this assumption might not hold true. An interval-valued Neutrosophic Pythagorean (IVNP) environment can offer a more reliable and accurate representation in certain situations. Uncertain and imprecise data, which are common in executive issues, can be effectively modeled using interval-valued Neutrosophic Pythagorean sets (IVNPS). Because the values of the nodes

in the graph can fluctuate within specific intervals rather than having fixed values, these sets offer a more adaptable method of capturing uncertainty.

The interval representation is a strong tool in executive processes and can manage vague or insufficient information with ease. Raut et al. [11] suggested an enhanced A* search method that makes use of the IVPN environment. This method seeks to determine a graph's shortest path in the presence of ambiguity and uncertainty. To deal with the uncertainty in node values and edge weights, their model integrated the IVNPS theory into the A* search framework. To determine the heuristic function and choose the best node to grow next, it makes use of the neutrosophic Pythagorean distance idea.

While neutrosophic sets, a more general concept that includes fuzzy sets and intuitionistic fuzzy sets, are designed to represent real-world data that is uncertain, imprecise, inconsistent, and incomplete, the interval-valued Pythagorean fuzzy set (IVPFS) offers a fresh method for dealing with vagueness and uncertainty. Furthermore, rather than concentrating just on one value, the creation of Interval Value Neutrosophic Sets (IVNS) improves accuracy when solving issues involving a range of integers inside the actual unit interval. Notwithstanding these developments, there is still a lack of study on issues related to actual implementation, comparisons with current approaches, and the application of these ideas in many domains.

By putting forth a unique idea based on the Interval Valued Pythagorean Neutrosophic Set (IVPNS), a generalization of the IVPFS and INS, Razak et al. [12] aimed to close this research gap. The creation of IVPNS offers a more thorough framework for dealing with ambiguity, uncertainty, and partial information in a variety of domains, which promotes stronger decision-making, enhanced problem-solving skills, and better complex system management. Additionally, their study compares IVPFS and IVNS and presents algebraic operations for IVPNS, such as addition, multiplication, scalar multiplication, and exponentiation. To show these procedures in action, their paper includes illustrated numerical examples. Furthermore, the algebraic features of IVPNS are presented and carefully proved in their paper, with particular attention paid to their commutative and associative characteristics.

The rest of this study is organized as follows: Section 2 presents the HyperSoft set definition and with the neutrosophic sets. Section 3 shows the definition of the IVPNS. Section 4 shows the steps of the proposed approach to computing the criteria weights and ranking the alternatives. Section 5 shows the results and discussion. Section 6 shows comparative analysis with four MCDM methods to show the effectiveness of the proposed approach. Section 7 shows the sensitivity analysis to show the stability of the ranks. Section 8 shows the conclusion of this study.

2. HyperSoft Set (HSS)

Let U be the universal set and P(U) is the power set of U. Let $A^1, A^2, C^3, \dots A^n$ be a set of attributes for $n \geq 1$and the corresponding values are $R^1, R^2, R^3, \dots R^n$, $R^i \cap R^j = \emptyset \; for \; i \neq j$.

A pair $(F, R^1 \times R^2 \times R^3 \ldots . R^n)$ is a HSS over U where F is a mapping with $F: R^1 \times R^2 \times R^3 \ldots R^n \rightarrow P(U)$ [13].

(Neutrosophic HyperSoft Set (NHSS))

Let U be the universal set and P(U) is the power set of U. Let $A^1, A^2, A^3, \ldots A^n$ be a set of attributes for $n \geq 1$and the corresponding values are $R^1, R^2, R^3, \ldots R^n$, $R^i \cap R^j = \emptyset \; for \; i \neq j$ and the relation $R^1 \times R^2 \times R^3 \ldots . R^n = S$

A pair (F, S) is a NHSS over U where $(F, R^1 \times R^2 \times R^3 \ldots . R^n) = R^1 \times R^2 \times R^3 \ldots . L^n \rightarrow P(U)$ with $F(R^1 \times R^2 \times R^3 \ldots . R^n) = \{T(F(S)), I(F(S)), F(F(S)), x \in U\}$ where T,I, and F refer to truth, indeterminacy, and falsity functions[14].

3. Interval Valued Pythagorean Neutrosophic Set (IVPNS)[12]

This section shows some definitions of the IVPNS:

Let two IVPNNs such as $N_1 = \left([b_{N_1}^L, b_{N_1}^U], [I_{N_1}^L, I_{N_1}^U], [S_{N_1}^L, S_{N_1}^U],\right)$ and $N_2 = \left([b_{N_2}^L, b_{N_2}^U], [I_{N_2}^L, I_{N_2}^U], [S_{N_2}^L, S_{N_2}^U],\right)$

$$N_1 \oplus N_2 = \begin{pmatrix} [b_{N_1}^L + b_{N_2}^L - b_{N_1}^L b_{N_2}^L, b_{N_1}^U + b_{N_2}^U - b_{N_1}^u b_{N_2}^U], \\ [I_{N_1}^L I_{N_2}^L, I_{N_1}^U I_{N_2}^U], \\ [S_{N_1}^L S_{N_2}^L, S_{N_1}^U S_{N_2}^U] \end{pmatrix} \quad (1)$$

$$N_1 \otimes N_2 = \begin{pmatrix} [b_{N_1}^L b_{N_2}^L, b_{N_1}^U b_{N_2}^U], \\ [I_{N_1}^L + I_{N_2}^L - I_{N_1}^L I_{N_2}^L, I_{N_1}^U + I_{N_2}^U - I_{N_1}^u I_{N_2}^U], \\ [S_{N_1}^L + S_{N_2}^L - S_{N_1}^L S_{N_2}^L, S_{N_1}^U + S_{N_2}^U - S_{N_1}^u S_{N_2}^U] \end{pmatrix} \quad (2)$$

$$\varphi N_1 = \begin{pmatrix} [1 - (1 - b_{N_1}^L)^\varphi, 1 - (1 - b_{N_1}^U)^\varphi], \\ [(I_{N_1}^L)^\varphi, (I_{N_1}^Y)^\varphi], \\ [(S_{N_1}^L)^\varphi, (S_{N_1}^Y)^\varphi] \end{pmatrix} \quad (3)$$

$$N_1^\varphi = \begin{pmatrix} [(b_{N_1}^L)^\varphi, (b_{N_1}^Y)^\varphi], \\ [1 - (1 - I_{N_1}^L)^\varphi, 1 - (1 - I_{N_1}^U)^\varphi], \\ [1 - (1 - S_{N_1}^L)^\varphi, 1 - (1 - S_{N_1}^U)^\varphi] \end{pmatrix} \quad (4)$$

$$0 \leq b_{N_1}^U + S_{N_1}^U \leq 1 \quad (5)$$

$$0 \leq b_{N_1}^U + I_{N_1}^U + S_{N_1}^U \leq 2 \quad (6)$$

$$0 \leq (b_{N_1}^U)^2 + (I_{N_1}^U)^2 + (S_{N_1}^U)^2 \leq 2 \quad (7)$$

Let $\pi_A(x) = [\pi_A^U(x), \pi_A^L(x)]$ is called IVPNS, where

$$\pi_A^U(x) = \sqrt{1 - \left(b_A^U(x)\right)^2 - \left(I_A^U(x)\right)^2 - \left(S_A^U(x)\right)^2} \tag{8}$$

$$\pi_A^L(x) = \sqrt{1 - \left(b_A^L(x)\right)^2 - \left(I_A^L(x)\right)^2 - \left(S_A^L(x)\right)^2} \tag{9}$$

4. Multi-Criteria Decision-Making Methods

This section uses the different MCDM methods to compute the criteria weights and rank the alternatives. Two MCDM methods are used in this study such as: SWARA method to compute the criteria weights and the MAIRCA method to rank the alternatives. MCDM methods are used under the neutrosophic sets.

IVPN-SWARA[15], [16]

1) Ranking the criteria in terms of relative importance

2) Compute the coefficient

$$F_j = \begin{cases} 1 & if\ j = 1 \\ S_j + 1 & if\ j > 1 \end{cases} \tag{10}$$

Where S_j refers to the initial value by experts.

3) Compute the initial weights of criteria

$$h_j = \begin{cases} 1 & if\ j = 1 \\ \frac{h_j}{F-j} & if\ j > 1 \end{cases} \tag{11}$$

4) Compute the relative wights of criteria.

$$W_j = \frac{h_j}{\sum_{j=1}^{n} h_j} \tag{12}$$

IVPN- MAIRCA[17], [18]

1) Build the decision matrix.

2) Compute the elements $t_{p_{ij}}$ of the theoretical rating matrix

$$t_{p_{ij}} = x_{ij} * w_j \tag{13}$$

3) Compute the elements of real rating matrix

$$u_{ij} = t_{p_{ij}} \left(\frac{x_{ij} - \min x_i}{\max x_i - \min x_i}\right) \text{ for positive criteria} \tag{14}$$

$$u_{ij} = t_{p_{ij}} \left(\frac{x_{ij} - \max x_i}{\min x_i - \max x_i}\right) \text{ for cost criteria} \tag{15}$$

4) Compute the total gap matrix

$$k_{ij} = t_{p_{ij}} - u_{ij} \quad (16)$$

5) Compute the final values of the criterion function

$$D_i = \sum_{i=1}^{m} k_{ij} \quad (17)$$

6) Rank the alternatives.

5. Results and Discussions

This section shows the results and discussion of the proposed approach by applying it on a case study to show the validation of the proposed approach. This section shows the preferences of experts and decision makers to evaluate the criteria and alternatives. Four experts are invited in this study to evaluate 6 criteria and seven alternatives. The criteria of this study are:

Technological Innovation: Level of R&D, battery technology, AI integration, and smart mobility advancements.

Supply Chain Stability: Efficiency of sourcing raw materials, battery production, and semiconductor supply.

Environmental Impact: Sustainability of materials, carbon footprint, and recycling processes.

Market Competitiveness: Market share, brand positioning, cost-effectiveness, and customer adoption rate

Infrastructure Readiness: Availability of charging stations, battery swapping networks, and government incentives.

Government Policies: Subsidies, regulations, and support for NEV adoption.

The alternatives of this study are organized as follows: Electric Vehicle Manufacturers, Supply Chain and Material Providers, Battery Manufacturers, Government and Regulatory Bodies, Autonomous and Smart Mobility Solutions, Recycling and Sustainability Companie, Charging Infrastructure Providers

IVPN-SWARA

1) We ranked the criteria in terms of relative importance

2) We Compute the coefficient using Eq. (10).

3) We compute the initial weights of the criteria using Eq. (11) as shown in Fig 1.

4) We compute the relative wights of criteria using Eq. (12) as shown in Fig 2.

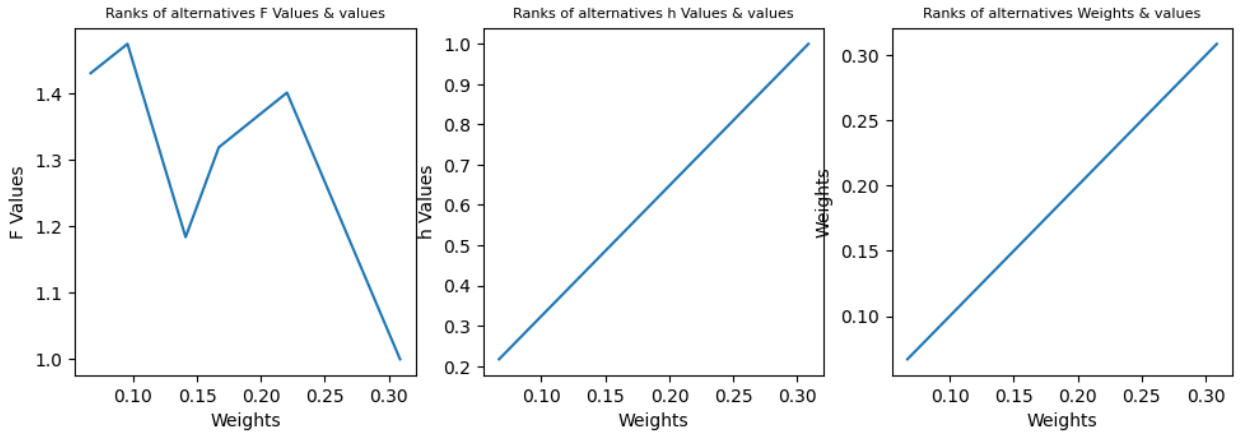

Fig 1. The SWARA results.

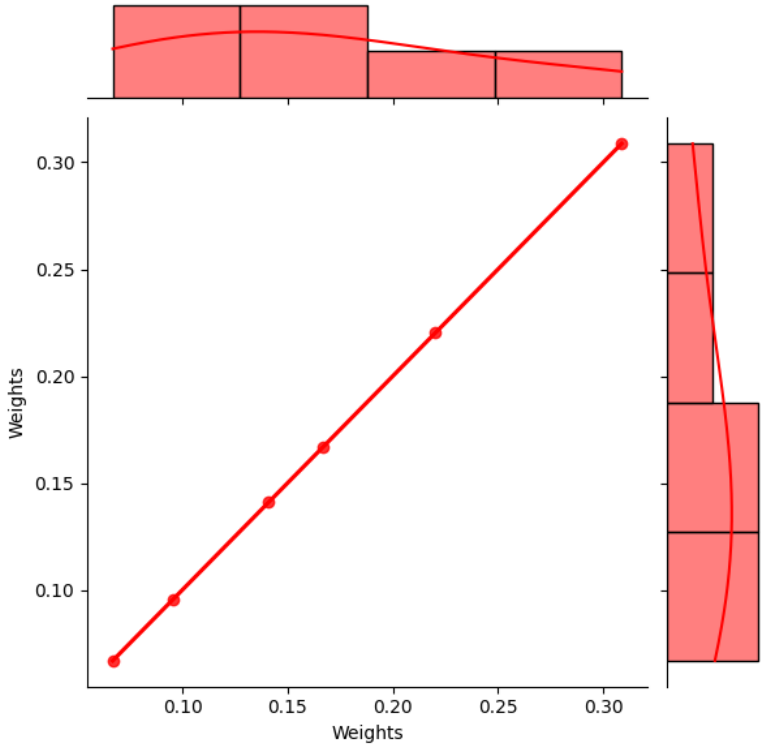

Fig 2. The relative wights of criteria.

IVPN- MAIRCA

After computing the criteria weights, we applied the NHSS to select the best values of each criterion. We select the best values such as: $R1-3\times R2-1\times R3-2\times R4-4\times R5-1\times R6-3$

1) We used the IVPNNs to build the decision matrix as shown in Table 1-4. Then we applied the score function to obtain crisp values. Then we combine the decision matrix.

2) Then we compute the elements $t_{p_{ij}}$ of the theoretical rating matrix using Eq. (13) as shown in Table 5.

3) We compute the elements of real rating matrix using Eq. (14) and Eq. (15) as shown in Table 6.

4) We compute the total gap matrix using Eq. (16) as shown in Table 7.

5) We compute the final values of the criterion function using Eq. (17).

6) Then we rank the alternatives as shown in Fig 3.

Table 1. The first IVPNNs

	$R_{1\text{-}3}$	$R_{2\text{-}1}$	$R_{3\text{-}2}$	$R_{4\text{-}4}$	$R_{5\text{-}1}$	$R_{6\text{-}3}$
A_1	([0.4,0.6],[0.3,0.5],[0.3,0.4])	([0.4,0.5],[0.3,0.4],[0.3,0.5])	([0.5,0.6],[0.4,0.5],[0.3,0.4])	([0.5,0.6],[0.3,0.5],[0.3,0.4])	([0.4,0.6],[0.3,0.5],[0.2,0.3])	([0.3,0.5],[0.4,0.5],[0.5,0.6])
A_2	([0.4,0.6],[0.3,0.5],[0.3,0.4])	([0.3,0.4],[0.2,0.4],[0.3,0.5])	([0.3,0.5],[0.4,0.5],[0.5,0.6])	([0.4,0.6],[0.3,0.5],[0.2,0.3])	([0.5,0.6],[0.3,0.5],[0.3,0.4])	([0.5,0.6],[0.4,0.5],[0.3,0.4])
A_3	([0.4,0.5],[0.3,0.4],[0.3,0.5])	([0.5,0.6],[0.4,0.5],[0.3,0.4])	([0.5,0.6],[0.3,0.5],[0.3,0.4])	([0.4,0.6],[0.3,0.5],[0.2,0.3])	([0.3,0.5],[0.4,0.5],[0.5,0.6])	([0.4,0.5],[0.3,0.4],[0.3,0.5])
A_4	([0.5,0.6],[0.3,0.5],[0.3,0.4])	([0.5,0.6],[0.4,0.5],[0.3,0.4])	([0.4,0.5],[0.3,0.4],[0.3,0.5])	([0.4,0.6],[0.3,0.5],[0.3,0.4])	([0.3,0.4],[0.2,0.4],[0.3,0.5])	([0.4,0.6],[0.3,0.5],[0.3,0.4])
A_5	([0.4,0.6],[0.3,0.5],[0.2,0.3])	([0.5,0.6],[0.3,0.5],[0.3,0.4])	([0.5,0.6],[0.4,0.5],[0.3,0.4])	([0.4,0.5],[0.3,0.4],[0.3,0.5])	([0.4,0.6],[0.3,0.5],[0.3,0.4])	([0.3,0.4],[0.2,0.4],[0.3,0.5])
A_6	([0.4,0.6],[0.3,0.5],[0.2,0.3])	([0.4,0.6],[0.3,0.5],[0.2,0.3])	([0.5,0.6],[0.3,0.5],[0.3,0.4])	([0.4,0.6],[0.3,0.5],[0.2,0.3])	([0.3,0.4],[0.2,0.4],[0.3,0.5])	([0.3,0.5],[0.4,0.5],[0.5,0.6])
A_7	([0.3,0.5],[0.4,0.5],[0.5,0.6])	([0.3,0.5],[0.4,0.5],[0.5,0.6])	([0.5,0.6],[0.4,0.5],[0.3,0.4])	([0.3,0.5],[0.4,0.5],[0.5,0.6])	([0.3,0.5],[0.4,0.5],[0.5,0.6])	([0.4,0.6],[0.3,0.5],[0.2,0.3])

Table 2. The second IVPNNs

	$R_{1\text{-}3}$	$R_{2\text{-}1}$	$R_{3\text{-}2}$	$R_{4\text{-}4}$	$R_{5\text{-}1}$	$R_{6\text{-}3}$
A_1	([0.5,0.6],[0.3,0.5],[0.3,0.4])	([0.4,0.5],[0.3,0.4],[0.3,0.5])	([0.5,0.6],[0.4,0.5],[0.3,0.4])	([0.5,0.6],[0.3,0.5],[0.3,0.4])	([0.4,0.6],[0.3,0.5],[0.2,0.3])	([0.3,0.5],[0.4,0.5],[0.5,0.6])
A_2	([0.5,0.6],[0.4,0.5],[0.3,0.4])	([0.3,0.4],[0.2,0.4],[0.3,0.5])	([0.3,0.5],[0.4,0.5],[0.5,0.6])	([0.5,0.6],[0.3,0.5],[0.3,0.4])	([0.5,0.6],[0.3,0.5],[0.3,0.4])	([0.5,0.6],[0.4,0.5],[0.3,0.4])
A_3	([0.4,0.5],[0.3,0.4],[0.3,0.5])	([0.5,0.6],[0.4,0.5],[0.3,0.4])	([0.5,0.6],[0.3,0.5],[0.3,0.4])	([0.5,0.6],[0.4,0.5],[0.3,0.4])	([0.5,0.6],[0.3,0.5],[0.3,0.4])	([0.4,0.5],[0.3,0.4],[0.3,0.5])
A_4	([0.4,0.6],[0.3,0.5],[0.3,0.4])	([0.5,0.6],[0.4,0.5],[0.3,0.4])	([0.4,0.5],[0.3,0.4],[0.3,0.5])	([0.4,0.5],[0.3,0.4],[0.3,0.5])	([0.5,0.6],[0.4,0.5],[0.3,0.4])	([0.4,0.6],[0.3,0.5],[0.3,0.4])
A_5	([0.3,0.4],[0.2,0.4],[0.3,0.5])	([0.5,0.6],[0.3,0.5],[0.3,0.4])	([0.5,0.6],[0.4,0.5],[0.3,0.4])	([0.4,0.6],[0.3,0.5],[0.3,0.4])	([0.4,0.5],[0.3,0.4],[0.3,0.5])	([0.3,0.4],[0.2,0.4],[0.3,0.5])
A_6	([0.5,0.6],[0.3,0.5],[0.3,0.4])	([0.5,0.6],[0.3,0.5],[0.3,0.4])	([0.5,0.6],[0.3,0.5],[0.3,0.4])	([0.3,0.4],[0.2,0.4],[0.3,0.5])	([0.4,0.6],[0.3,0.5],[0.3,0.4])	([0.5,0.6],[0.3,0.5],[0.3,0.4])
A_7	([0.5,0.6],[0.4,0.5],[0.3,0.4])	([0.5,0.6],[0.4,0.5],[0.3,0.4])	([0.5,0.6],[0.4,0.5],[0.3,0.4])	([0.3,0.5],[0.4,0.5],[0.5,0.6])	([0.5,0.6],[0.3,0.5],[0.3,0.4])	([0.5,0.6],[0.4,0.5],[0.3,0.4])

Table 3. The third IVPNNs

	$R_{1\text{-}3}$	$R_{2\text{-}1}$	$R_{3\text{-}2}$	$R_{4\text{-}4}$	$R_{5\text{-}1}$	$R_{6\text{-}3}$
A_1	([0.4,0.6],[0.3,0.5],[0.3,0.4])	([0.4,0.5],[0.3,0.4],[0.3,0.5])	([0.5,0.6],[0.4,0.5],[0.3,0.4])	([0.5,0.6],[0.3,0.5],[0.3,0.4])	([0.4,0.6],[0.3,0.5],[0.2,0.3])	([0.3,0.5],[0.4,0.5],[0.5,0.6])
A_2	([0.3,0.4],[0.2,0.4],[0.3,0.5])	([0.3,0.4],[0.2,0.4],[0.3,0.5])	([0.3,0.5],[0.4,0.5],[0.5,0.6])	([0.4,0.6],[0.3,0.5],[0.2,0.3])	([0.5,0.6],[0.3,0.5],[0.3,0.4])	([0.5,0.6],[0.4,0.5],[0.3,0.4])
A_3	([0.3,0.5],[0.4,0.5],[0.5,0.6])	([0.4,0.6],[0.3,0.5],[0.3,0.4])	([0.5,0.6],[0.3,0.5],[0.3,0.4])	([0.4,0.6],[0.3,0.5],[0.2,0.3])	([0.3,0.5],[0.4,0.5],[0.5,0.6])	([0.4,0.5],[0.3,0.4],[0.3,0.5])
A_4	([0.4,0.6],[0.3,0.5],[0.2,0.3])	([0.3,0.4],[0.2,0.4],[0.3,0.5])	([0.4,0.6],[0.3,0.5],[0.3,0.4])	([0.4,0.6],[0.3,0.5],[0.3,0.4])	([0.4,0.6],[0.3,0.5],[0.3,0.4])	([0.4,0.6],[0.3,0.5],[0.3,0.4])
A_5	([0.5,0.6],[0.3,0.5],[0.3,0.4])	([0.3,0.5],[0.4,0.5],[0.5,0.6])	([0.3,0.4],[0.2,0.4],[0.3,0.5])	([0.4,0.6],[0.3,0.5],[0.3,0.4])	([0.3,0.4],[0.2,0.4],[0.3,0.5])	([0.4,0.6],[0.3,0.5],[0.3,0.4])
A_6	([0.5,0.6],[0.4,0.5],[0.3,0.4])	([0.4,0.6],[0.3,0.5],[0.2,0.3])	([0.3,0.5],[0.4,0.5],[0.5,0.6])	([0.3,0.4],[0.2,0.4],[0.3,0.5])	([0.3,0.5],[0.4,0.5],[0.5,0.6])	([0.3,0.4],[0.2,0.4],[0.3,0.5])
A_7	([0.4,0.5],[0.3,0.4],[0.3,0.5])	([0.5,0.6],[0.3,0.5],[0.3,0.4])	([0.4,0.6],[0.3,0.5],[0.2,0.3])	([0.3,0.5],[0.4,0.5],[0.5,0.6])	([0.4,0.6],[0.3,0.5],[0.2,0.3])	([0.3,0.5],[0.4,0.5],[0.5,0.6])

Table 4. The fourth IVPNNs

	$R_{1\text{-}3}$	$R_{2\text{-}1}$	$R_{3\text{-}2}$	$R_{4\text{-}4}$	$R_{5\text{-}1}$	$R_{6\text{-}3}$
A_1	([0.3,0.5],[0.4,0.5],[0.5,0.6])	([0.3,0.4],[0.2,0.4],[0.3,0.5])	([0.4,0.5],[0.3,0.4],[0.3,0.5])	([0.3,0.4],[0.2,0.4],[0.3,0.5])	([0.4,0.6],[0.3,0.5],[0.2,0.3])	([0.5,0.6],[0.3,0.5],[0.3,0.4])
A_2	([0.3,0.4],[0.2,0.4],[0.3,0.5])	([0.4,0.6],[0.3,0.5],[0.3,0.4])	([0.4,0.6],[0.3,0.5],[0.3,0.4])	([0.4,0.6],[0.3,0.5],[0.3,0.4])	([0.5,0.6],[0.3,0.5],[0.3,0.4])	([0.5,0.6],[0.4,0.5],[0.3,0.4])
A_3	([0.3,0.4],[0.2,0.4],[0.3,0.5])	([0.4,0.5],[0.3,0.4],[0.3,0.5])	([0.3,0.4],[0.2,0.4],[0.3,0.5])	([0.4,0.5],[0.3,0.4],[0.3,0.5])	([0.5,0.6],[0.4,0.5],[0.3,0.4])	([0.4,0.5],[0.3,0.4],[0.3,0.5])
A_4	([0.4,0.5],[0.3,0.4],[0.3,0.5])	([0.4,0.5],[0.3,0.4],[0.3,0.5])	([0.4,0.5],[0.3,0.4],[0.3,0.5])	([0.4,0.6],[0.3,0.5],[0.2,0.3])	([0.5,0.6],[0.4,0.5],[0.3,0.4])	([0.4,0.5],[0.3,0.4],[0.3,0.5])
A_5	([0.4,0.6],[0.3,0.5],[0.3,0.4])	([0.4,0.6],[0.3,0.5],[0.3,0.4])	([0.4,0.6],[0.3,0.5],[0.3,0.4])	([0.4,0.6],[0.3,0.5],[0.3,0.4])	([0.4,0.5],[0.3,0.4],[0.3,0.5])	([0.4,0.6],[0.3,0.5],[0.3,0.4])
A_6	([0.3,0.4],[0.2,0.4],[0.3,0.5])	([0.3,0.4],[0.2,0.4],[0.3,0.5])	([0.3,0.4],[0.2,0.4],[0.3,0.5])	([0.4,0.5],[0.3,0.4],[0.3,0.5])	([0.4,0.6],[0.3,0.5],[0.3,0.4])	([0.3,0.4],[0.2,0.4],[0.3,0.5])
A_7	([0.4,0.6],[0.3,0.5],[0.3,0.4])	([0.5,0.6],[0.4,0.5],[0.3,0.4])	([0.5,0.6],[0.3,0.5],[0.3,0.4])	([0.4,0.6],[0.3,0.5],[0.2,0.3])	([0.5,0.6],[0.3,0.5],[0.3,0.4])	([0.4,0.6],[0.3,0.5],[0.2,0.3])

Table 5. The values of $t_{p_{ij}}$

	$R_{1\text{-}3}$	$R_{2\text{-}1}$	$R_{3\text{-}2}$	$R_{4\text{-}4}$	$R_{5\text{-}1}$	$R_{6\text{-}3}$
A_1	0.123824457	0.050161	0.087296	0.06184	0.057046	0.014069
A_2	0.098423312	0.042129	0.032018	0.078408	0.052872	0.040561
A_3	0.056737177	0.108065	0.073205	0.073027	0.032327	0.01818
A_4	0.146472851	0.087005	0.054005	0.064341	0.042735	0.028533
A_5	0.132886438	0.092476	0.07459	0.060178	0.026729	0.019192
A_6	0.142777216	0.101346	0.054153	0.037361	0.027466	0.01405

A_7	0.112054264	0.102498	0.098594	0.031211	0.042999	0.03169

Table 6. The values of real rating matrix.

	$R_{1\text{-}3}$	$R_{2\text{-}1}$	$R_{3\text{-}2}$	$R_{4\text{-}4}$	$R_{5\text{-}1}$	$R_{6\text{-}3}$
A_1	0.092572	0.006110116	0.014815	0.021708	0.057046	1.02E-05
A_2	0.045722	0	0.032018	0	0.045592	0.040561
A_3	0	0.108065404	0.027917	0.008326	0.005969	0.002832
A_4	0.146473	0.059215961	0.03617	0.019176	0.022563	0.015588
A_5	0.112767	0.070612309	0.026894	0.023244	0	0.003722
A_6	0.136897	0.091018769	0.036148	0.032492	0.000668	0
A_7	0.069075	0.093842464	0	0.031211	0.023077	0.021086

Table 7. The total gap matrix.

	$R_{1\text{-}3}$	$R_{2\text{-}1}$	$R_{3\text{-}2}$	$R_{4\text{-}4}$	$R_{5\text{-}1}$	$R_{6\text{-}3}$
A_1	0.031252064	0.044051	0.072481	0.040133	0	0.014059
A_2	0.05270139	0.042129	0	0.078408	0.00728	0
A_3	0.056737177	0	0.045288	0.0647	0.026358	0.015348
A_4	0	0.02779	0.017835	0.045165	0.020172	0.012945
A_5	0.020119647	0.021864	0.047696	0.036934	0.026729	0.015469
A_6	0.005880075	0.010328	0.018004	0.004869	0.026798	0.01405
A_7	0.042978999	0.008655	0.098594	0	0.019922	0.010604

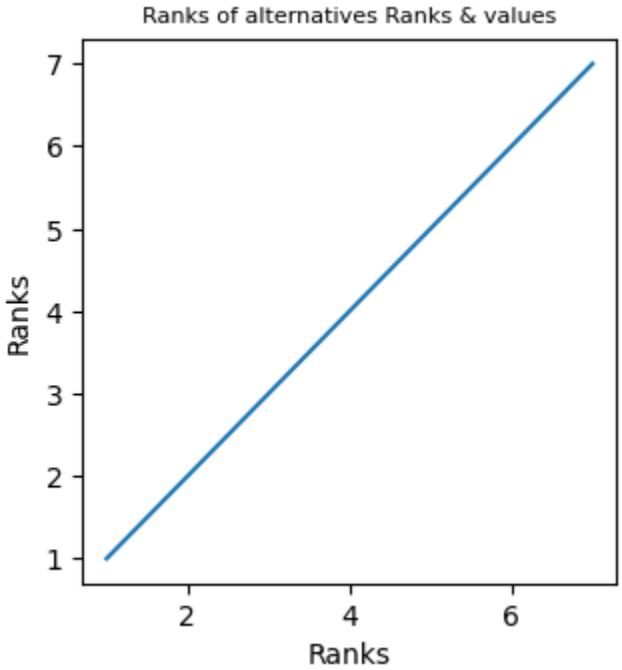

Fig 3. The rank of alternatives.

6. Comparative Analysis

This section shows the comparative analysis between the proposed approach and other MCDM methods. The aim of the comparative analysis is to show the effectiveness of the proposed approach compared to other MCDM methods. This section compares the proposed approach with four MCDM methods such as VIKOR, MOOSRA, MOORA, and MULTIMOORA methods. We used the same weights of the SWARA method in the comparison results.

Fig 4 shows the results of the comparative analysis. We show alternative 3 is the best and alternative 6 is the worst. We show our method is effective compared to other MCDM methods. Our method has relationships between other MCDM methods.

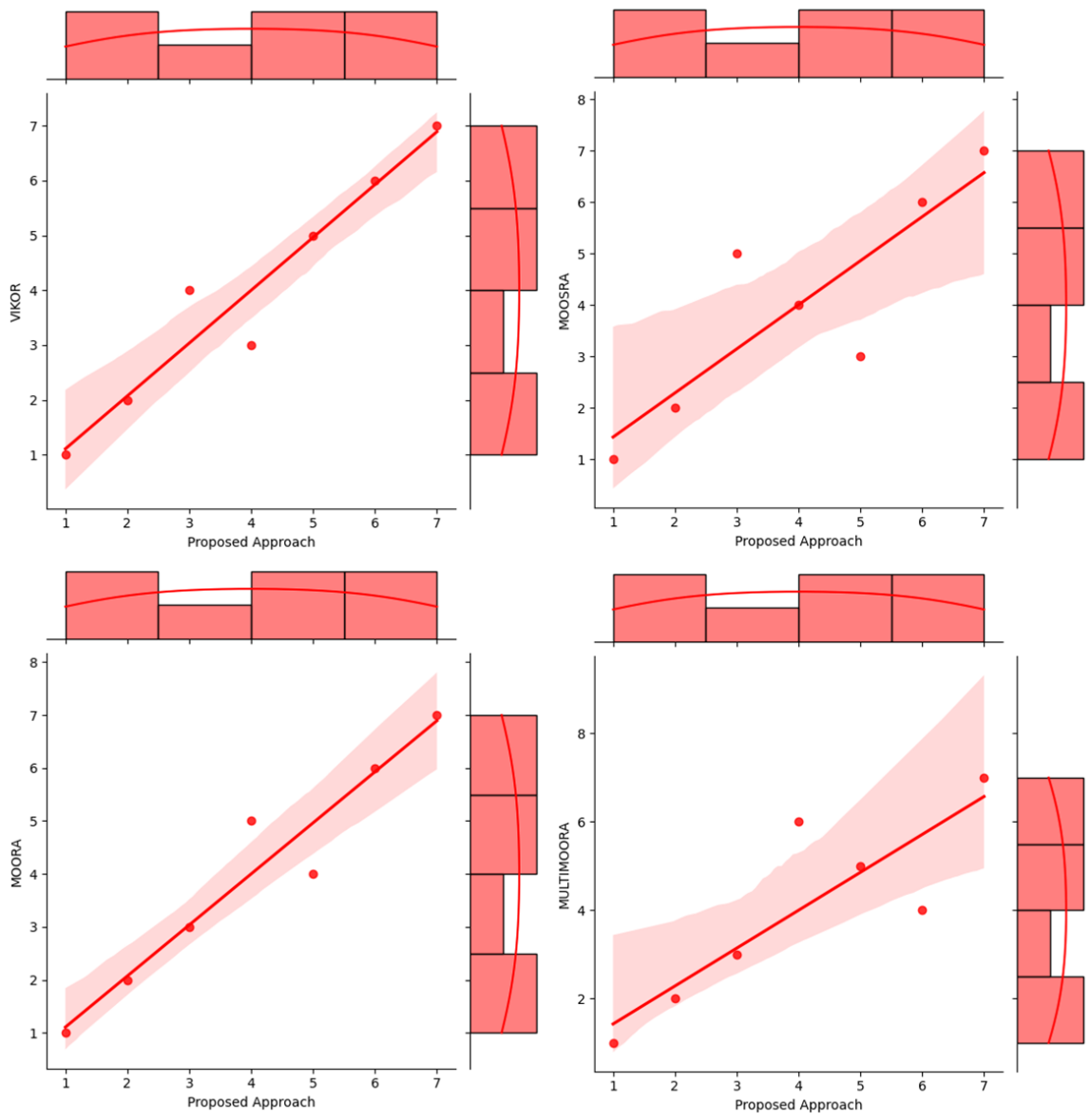

Fig 4. Comparative results.

7. Sensitivity Analysis

This section shows the sensitivity analysis results between the ranks of the alternatives. in this section we proposed seven cases in criteria weights to rank the alternatives under these cases to show the stability of the ranks. Fig 5 shows the different criteria weights. In the first case, we put all the criteria the same weights, then we rank the alternatives.

The results show alternative 3 is the best and alternative 6 is the worst. Fig 6 shows the different ranks of the alternatives. In the second case, we increase the first criterion by 25% weight and other criteria have the same weight. In this case, we show alternative 3 is the best and alternative 6 is the worst. In the third case, we increase the second criterion by 25% weight and other criteria have the same weight. In this case, we show alternative 3 is the best and alternative 6 is the worst. In the fourth case, we increase the third criterion by 25% weight and other criteria have the same weight. In this case, we show alternative 3 is the best and alternative 6 is the worst.

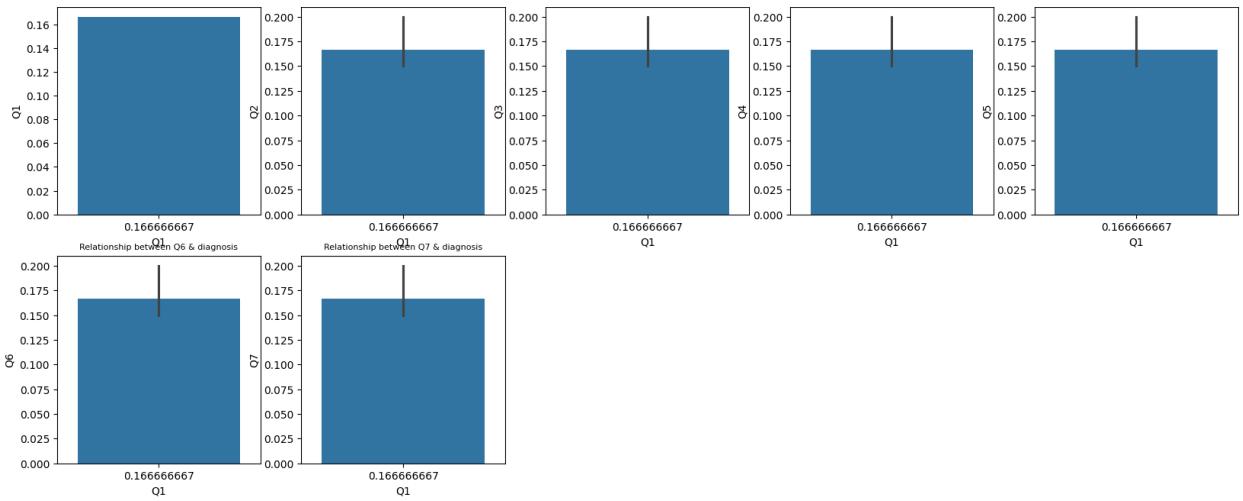

Fig 5. Seven criteria weights.

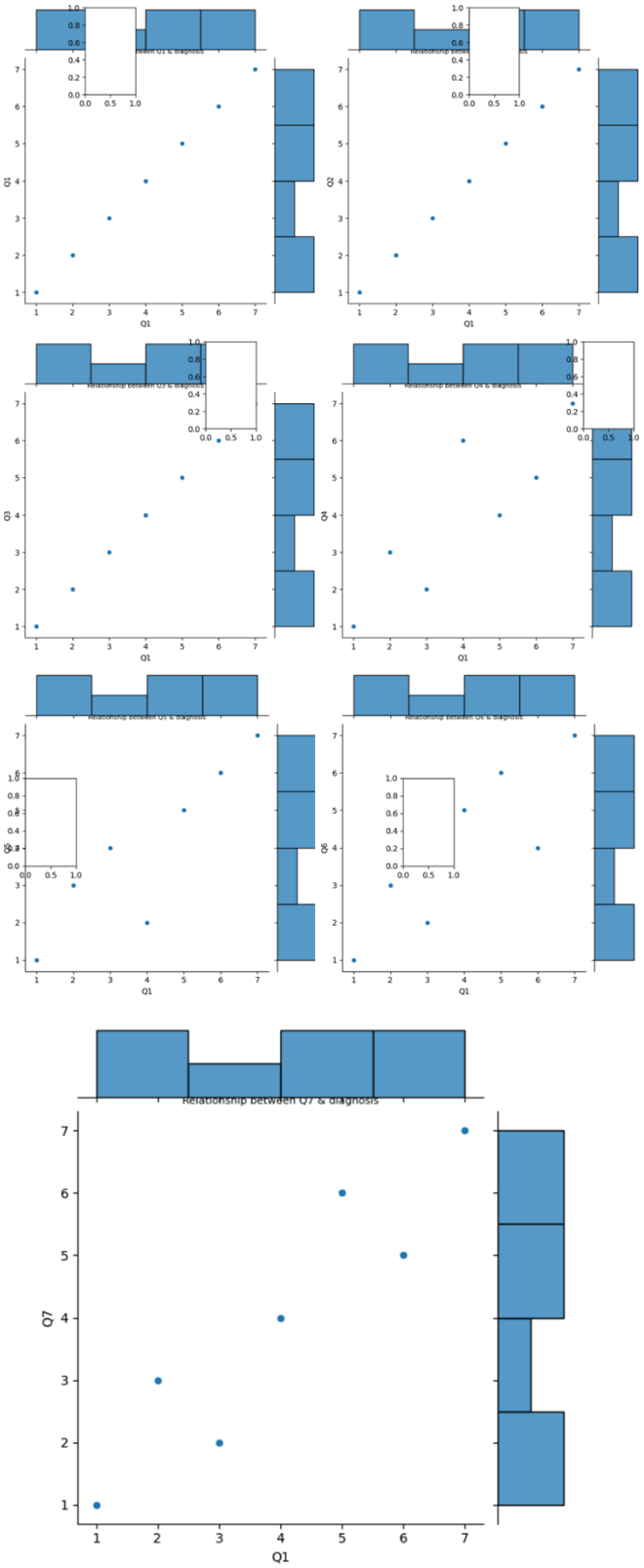

Fig 6. Analysis of ranks.

8. Conclusions and Future Works

This study proposed a MCDM approach for evaluation of New Energy Vehicle. We used two MCDM methods such as SWARA method to compute the criteria weights and MAIRCA method to rank the alternatives. These methods are used under the Interval Valued Pythagorean Neutrosophic Set (IVPNS) to deal with the vague data. The HyperSoft set is used to show the relationship between the criteria. We evaluated six criteria and seven alternatives in this study. The results show alternative 3 is the best and alternative 6 is the worst. We compared our model with the other four MCDM methods. The results show our method is effective compared to other methods. We applied the sensitivity analysis by seven cases. The results show our model is stable of ranks under different cases.

The future works, the Interval Valued Pythagorean Neutrosophic Set can be used in the MCDM problems to dela with uncertainty and vague information. The SWARA method can be used to compute the criteria weights. The MAIRCA can be used to rank the alternatives.

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Neutrosophic HyperSoft Set (NHSS) for Teaching Quality Assessment in University Interior Design Courses

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Abstract: Teaching quality evaluation in university interior design courses have different criteria and values. So, we used the multi-criteria decision making (MCDM) approach to deal with various criteria. Two MCDM methods are used such as Entropy to compute the criteria weights and RAFSI method to rank the alternatives. Two MCDM methods are used under the neutrosophic-Z number (NZN) to deal with uncertain and vague information. This study uses the neutrosophic HyperSoft set (NHSS) to find the relationship between different criteria. Seven criteria and eight alternatives are used in this study. An empirical application is applied to show the validation of the proposed approach. The sensitivity analysis is conducted to show the stability of the ranks of the alternatives.

Keywords: HyperSoft Set; Neutrosophic-Z-Number (NZN); University Interior Design Courses; Uncertainty Information; Evaluation Problem.

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1. Introduction and Literature Review

Interior design is a multidisciplinary field that blends art, architecture, technology, and human psychology to create functional and aesthetically pleasing indoor environments. University interior design courses aim to equip students with knowledge in design principles, materials, space planning, sustainability, lighting, digital modeling, and user experience. The quality of teaching in these courses directly impacts students' creativity, technical expertise, and industry readiness. Thus, a systematic evaluation of teaching quality is essential to maintain academic standards, improve curricula, and enhance learning outcomes[1], [2]. There are various aims of the Teaching Quality

- Ensures high educational standards and alignment with industry needs.
- Identifies strengths and weaknesses in teaching methods, course structure, and student engagement.
- Helps in faculty development and curriculum enhancement.

- Improves student satisfaction and learning experience.
- Guides policymakers, accreditation bodies, and university administrations in decision-making[3], [4].

However, teaching quality in interior design courses is difficult to measure due to its subjective, creative, and technical nature. A structured Multi-Criteria Decision-Making (MCDM) approach can provide a more objective, data-driven, and comprehensive evaluation system[5].

It is well recognized that Zadeh's fuzzy sets are crucial to modern scientific and technological applications. The notion of Z-numbers was further established by Zadeh in 2011 to characterize the reliability and constraint of the evaluation by an order pair of fuzzy numbers in uncertain scenarios. This concept is broader and closely tied to dependability than the traditional fuzzy number. As a result, the Z-number suggests a greater capacity to characterize human knowledge and judgments using an order pair of fuzzy numbers that correlate to reliability and limitation. It has received a lot of attention since then[6], [7].

The truth, falsity, and indeterminacy membership degrees independently characterize neutrosophic sets in an indeterminate and inconsistent environment, yet the Z-numbers discussed above are unable to represent them. Neutrosophic sets have since been used in a variety of fields. Nevertheless, the reliability measures associated with the truth, falsity, and indeterminacy membership degrees are absent from the neutrosophic set. The hybrid information mixing the truth, falsity, and indeterminacy degrees with their associated reliability degrees may be described by three order pairs of fuzzy numbers if the Z-number concept is extended to the neutrosophic set[8], [9]. Information expressions and decision-making techniques are essential study subjects in MCDM challenges.

The Z-number, which Zadeh presented as an extension of the conventional fuzzy number, has more capacity to represent human knowledge and judgments of dependability and restraint as an order pair of fuzzy numbers. The truth, falsity, and indeterminacy degrees define a neutrosophic set in an indeterminate and inconsistent environment, but they do not include reliability-related metrics. As a generalization of the Z-number and the neutrosophic set, Du et al.[10] first introduced the idea of a neutrosophic Z-number (NZN) set, which is a new framework of neutrosophic values combined with the neutrosophic measures of reliability, to describe the hybrid information of combining the truth, falsity, and indeterminacy degrees with their corresponding reliability degrees.

They proposed a scoring function for grading neutrosophic Z-numbers (NZNs) and their operations. To aggregate NZN information and examine its features, we then introduce NZN weighted arithmetic averaging. In the NZN environment, MCDM strategy is created by NZN operators and the score function. The relevance and efficacy of the established MCDM technique in the NZN scenario are finally illustrated with an example pertaining to the business partner selection problem.

The following are the article's primary contributions for the first study:

(a) In indeterminate and inconsistent scenarios, the suggested NZN set may resolve the information expression problem of the truth, falsity, and indeterminacy values together with their associated reliability measures by the three order pairs of fuzzy numbers.

(b) The score function of NZN is used to rank NZNs, which offer helpful mathematical tools for MCDM problems in NZN settings.

(c) In addition to improving MCDM dependability, the proposed MCDM strategy offers a fresh, practical solution for MCDM issues in the NZN context.

(d) Two MCDM methods are applied such as Entropy to compute the criteria weights and the RAFSI method to rank the alternatives. These methods are used under the hypersoft set to deal with various criteria values.

(f) The sensitivity analysis is conducted to show the stability of the ranks under different cases.

The following structures make up the study's organization. The concept of a NZN set, operations of NZNs, and a scoring function of NZN for comparing NZNs are introduced in section 2. The section 3 shows the steps of the proposed approach to compute the criteria weights and ranking the alternatives. Section 4 shows the results of the proposed approach. Section 5 shows the conclusion of this study.

2. Neutrosophic Z-number set (NZN)

An order pair of fuzzy numbers, Z = (V, R), associated with a real-valued uncertain variable X, was originally described by Zadeh in 2011. The first component, V, is a fuzzy constraint on the values that X can take, while the second component, R, is a measure of reliability for V[10].

Definition 1

Set X as a universe set, then the NZN can be defined as:

$$S_Z = \{(x, T(V,R)x, I(V,R)x, F(V,R)x) | x \in X\} \quad (1)$$

$$T(V,R)x = \left(T_V(x), T_R(x)\right), I(V,R)x = \left(I_V(x), I_R(x)\right), F(V,R)x = \left(F_V(x), F_R(x)\right) \quad (2)$$

$$0 \leq T_V(x) + I_V(x) + F_V(x) \leq 3 \quad (3)$$

$$0 \leq T_R(x) + I_R(x) + F_R(x) \leq 3 \quad (4)$$

Definition 2

Let $s_{Z_1} = \left(T_1(V,R), I_1(V,R), F_1(V,R)\right) = \left(\left(T_{V_1}(x), T_{R_1}(x)\right), \left(I_{V_1}(x), I_{R_1}(x)\right), \left(F_{V_1}(x), F_{R_1}(x)\right)\right)$

$s_{Z_2} = \left(T_2(V,R), I_2(V,R), F_2(V,R)\right) = \left(\left(T_{V_2}(x), T_{R_2}(x)\right), \left(I_{V_2}(x), I_{R_2}(x)\right), \left(F_{V_2}(x), F_{R_2}(x)\right)\right)$, two NZNs and their operations as:

$$s_{Z_1} \supseteq s_{Z_2} \Leftrightarrow T_{V_1}(x) \geq T_{V_2}(x), T_{R_1}(x) \geq T_{R_2}(x), I_{V_1}(x) \leq I_{V_2}(x), I_{R_1}(x) \leq I_{R_2}(x), F_{V_1}(x) \leq F_{V_2}(x), F_{R_1}(x) \leq F_{R_2}(x) \quad (5)$$

$$s_{Z_1} = s_{Z_2} \Leftrightarrow s_{Z_1} \supseteq s_{Z_2} \; and \; s_{Z_2} \supseteq s_{Z_1} \quad (6)$$

$$s_{Z_1} \cup s_{Z_2} = \begin{pmatrix} \left(T_{V_1}(x) \vee T_{V_2}(x), T_{R_1}(x) \vee T_{R_2}(x)\right), \\ \left(I_{V_1}(x) \wedge I_{V_2}(x), I_{R_1}(x) \wedge I_{R_2}(x)\right), \\ \left(F_{V_1}(x) \wedge F_{V_2}(x), F_{R_1}(x) \wedge F_{R_1}(x)\right) \end{pmatrix} \quad (7)$$

$$s_{Z_1} \cap s_{Z_2} = \begin{pmatrix} \left(T_{V_1}(x) \wedge T_{V_2}(x), T_{R_1}(x) \wedge T_{R_2}(x)\right), \\ \left(I_{V_1}(x) \vee I_{V_2}(x), I_{R_1}(x) \vee I_{R_2}(x)\right), \\ \left(F_{V_1}(x) \vee F_{V_2}(x), F_{R_1}(x) \vee F_{R_1}(x)\right) \end{pmatrix} \quad (8)$$

$$\left(s_{Z_1}\right)^C = \left(\left(F_{V_1}(x), F_{R_1}(x)\right), \left(1 - I_{V_1}(x), 1 - I_{R_1}(x)\right), \left(T_{V_1}(x), T_{R_1}(x)\right)\right) \quad (9)$$

$$s_{Z_1} \oplus s_{Z_2} = \begin{pmatrix} \left(T_{V_1}(x) + T_{V_2}(x) - T_{V_1}(x)T_{V_2}(x), T_{R_1}(x) + T_{R_2}(x) - T_{R_1}(x)T_{R_2}(x)\right), \\ \left(I_{V_1}(x)I_{V_2}(x), I_{R_1}(x)I_{R_2}(x)\right), \\ \left(F_{V_1}(x)F_{V_2}(x), F_{R_1}(x)F_{R_2}(x)\right) \end{pmatrix} \quad (10)$$

$$s_{Z_1} \otimes s_{Z_2} = \begin{pmatrix} \left(T_{V_1}(x)T_{V_2}(x), T_{R_1}(x)T_{R_2}(x)\right), \\ \left(I_{V_1}(x) + I_{V_2}(x) - I_{V_1}(x)I_{V_2}(x), I_{R_1}(x) + I_{R_2}(x) - I_{R_1}(x)I_{R_2}(x)\right), \\ \left(F_{V_1}(x) + F_{V_2}(x) - F_{V_1}(x)F_{V_2}(x), F_{R_1}(x) + F_{R_2}(x) - F_{R_1}(x)F_{R_2}(x)\right) \end{pmatrix} \quad (11)$$

$$\lambda s_{Z_1} = \begin{pmatrix} \left(1 - \left(1 - T_{V_1}(x)\right)^{\lambda}, 1 - \left(1 - T_{R_1}(x)\right)^{\lambda}\right), \\ \left(\left(I_{V_1}(x)\right)^{\lambda}, \left(I_{R_1}(x)\right)^{\lambda}\right), \\ \left(\left(F_{V_1}(x)\right)^{\lambda}, \left(F_{R_1}(x)\right)^{\lambda}\right) \end{pmatrix} \quad (12)$$

$$\left(s_{Z_1}\right)^{\lambda} = \begin{pmatrix} \left(\left(T_{V_1}(x)\right)^{\lambda}, \left(T_{R_1}(x)\right)^{\lambda}\right), \\ \left(1 - \left(1 - I_{V_1}(x)\right)^{\lambda}, 1 - \left(1 - I_{R_1}(x)\right)^{\lambda}\right), \\ \left(1 - \left(1 - I_{V_1}(x)\right)^{\lambda}, 1 - \left(1 - I_{R_1}(x)\right)^{\lambda}\right) \end{pmatrix} \quad (13)$$

Definition 3

The score function can be computed such as:

$$A\left(s_{Z_1}\right) = \frac{2+T_{V_1}(x)T_{R_1}(x)-I_{V_1}(x)I_{R_1}(x)-F_{V_1}(x)F_{R_1}(x)}{3} \quad (14)$$

Definition 4 (HyperSoft Set (HSS))[11]

Let U be the universal set and P(U) is the power set of U. Let $C^1, C^2, C^3, \dots C^n$ be a set of attributes for $n \geq 1$and the corresponding values are $L^1, L^2, L^3, \dots L^n$, $L^i \cap L^j = \emptyset \; for \; i \neq j$.

A pair $(F, L^1 \times L^2 \times L^3 \dots . L^n)$ is a HSS over U where F is a mapping with $F: L^1 \times L^2 \times L^3 \dots L^n \rightarrow P(U)$.

Definition 5 (Neutrosophic HyperSoft Set (NHSS))[12], [13]

Let U be the universal set and P(U) is the power set of U. Let $C^1, C^2, C^3, \dots C^n$ be a set of attributes for $n \geq 1$and the corresponding values are $L^1, L^2, L^3, \dots L^n$, $L^i \cap L^j = \emptyset \; for \; i \neq j$ and the relation $L^1 \times L^2 \times L^3 \dots . L^n = S$

A pair (F, S) is a NHSS over U where $(F, L^1 \times L^2 \times L^3 \dots . L^n) = L^1 \times L^2 \times L^3 \dots . L^n \rightarrow P(U)$ with $F(L^1 \times L^2 \times L^3 \dots . L^n) = \{T(F(S)), I(F(S)), F(F(S)), x \in U\}$ where T,I, and F refer to truth, indeterminacy, and falsity functions.

3. NZN-Entropy-RAFSI

This section shows the steps of the proposed approach. We used the Entropy method to compute the criteria weights. We used the RAFSI method to rank the alternatives. We used the NZN to deal with vague and uncertainty information.

A new decision-making technique called Ranking of Alternatives by Functional mapping of criteria sub-intervals into Single Intervals (RAFSI) was initially put out by Žižović et al. The aim of this approach is to solve the rank reversal problem. They identified three specific advantages of the suggested approach. First, it facilitates the resolution of challenging real-world decision-making issues. Second, it has a novel data normalization process. Thirdly, it can get rid of problems with rank reversal. The values of the initial decision-making matrix are mapped and transferred by the RAFSI algorithm using a criterion interval. Different formats of decision-making values must be standardized into a criterion interval that includes values between 0 and 1[14], [15]. An overview of the original RAFSI method's computing process is provided below.

NZN-Entropy Method

- Build the decision matrix.

We used the NZN to evaluate the criteria and alternatives.

- Normalize the decision matrix

$$r_{ij} = \frac{a_{ij}}{\sum_{i=1}^{m} a_{ij}}; i = 1, \dots, m; j = 1, \dots, n \quad (15)$$

a_{ij} refers to the value in the decision matrix.

- Compute entropy value

$$e_j = -h \sum_{i=1}^{m} r_{ij} \ln r_{ij} \quad (16)$$

$$h = \frac{1}{\ln(m)} \quad (17)$$

- Compute the criteria weights.

$$W_j = \frac{1-e_j}{\sum_{j=1}^{n}(1-e_j)} \quad (18)$$

NZN-RAFSI-Method

- Compute the ideal $(a_j)^I$ and non-ideal $(a_j)^N$ values.

Each criterion must have two values defined by the expert, with the requirement that the ideal value for the maximum criteria type be larger than the non-ideal value, and vice versa for the lowest criteria type.

- Create criterion intervals for each entry in the main decision-making matrix. The following is the design of the criterion intervals: There are two types of intervals: (i) for maximum criterion and (ii) for minimum criteria. Next, all the criteria in the decision matrix must be constructed equally using the RAFSI approach and converted into the essential interval. It is truly the case that the procedure creates a series of numbers with intervals and locations inside the gap in between. The approach separated two interval mappings since the criterion had two kinds. First, it converts the lowest value into the maximum criteria and the highest value into the minimum criteria. Second, the value is mapped into both the minimal criterion and the value. Additionally, it is advised to provide. $a_1 = 0.1, a_{2_c} = 0.9$

$$g_m(a_{ij}) = \frac{a_{2_c} - a_1}{(a_j)^I - (a_j)^N} a_{ij} + \frac{(a_j)^I a_1 - a_{2_c}(a_j)^N}{(a_j)^I - (a_j)^N} \quad (19)$$

- Compute the harmonic and arithmetic means for the minim and maximum values

$$H = \frac{2}{\frac{1}{a_1} + \frac{1}{a_{2_c}}} \quad (20)$$

$$H = \frac{a_1 + a_{2_c}}{2} \quad (21)$$

- Normalize the decision matrix

$$Y_{ij} = \frac{a_{ij}}{2A} \quad (22)$$

$$Y_{ij} = \frac{H}{2a_{ij}} \quad (23)$$

- Compute the criteria function

$$V(A_i) = \sum_{j=1}^{n} w_j Y_{ij} \quad (24)$$

4. On Construction of NHSS-Entropy- RAFSI for MCDM Problem using NZN

This section shows the results of the proposed approach to compute the criteria weights and ranking the alternatives. Three experts are evaluated the criteria and alternatives under the neutrosophic sets to deal with uncertainty and vague information. This study uses seven criteria and eight alternatives to be evaluated.

The criteria of this study are technology integration, assessment and feedback mechanism, curriculum relevance, creativity, project-based learning, faculty expertise, and industry collaboration.

The alternatives of this study are flipped classroom model, design thinking and problem-solving, competency-based learning, blended learning, traditional lecture-based teaching, technology-driven education, project-based learning, industry-engaged learning

NZN-Entropy Method

- We Build the decision matrix using the NZN between the criteria and alternatives as shown in Table 1. Then we applied the score function to obtain one value. Then we combined these matrixes.

- We normalize the decision matrix using Eq. (15).

- Then we compute entropy value using Eq. (16) and Eq. (17).

- Then we compute the criteria weights such as W1 = 0.13893117, W2 = 0.130143474, W2 = 0.187458982, W4 = 0.137317575, W5 = 0.142512891, W6 = 0.157670497, W7 = 0.105965411. We show the criterion 3 is the best and the criterion 7 is the worst.

Table 1. The NZN numbers.

	C_1	C_2	C_3	C_4	C_5	C_6	C_7
A_1	((0.8,0.7),(0.1,0.8),(0.2,0.6))	((0.6,0.6),(0.2,0.6),(0.1,0.7))	((0.7,0.8),(0.1,0.7),(0.1,0.7))	((0.8,0.8),(0.4,0.7),(0.2,0.8))	((0.7,0.6),(0.2,0.7),(0.3,0.8))	((0.6,0.7),(0.1,0.7),(0.2,0.7))	((0.7,0.6),(0.2,0.7),(0.3,0.8))
A_2	((0.6,0.7),(0.1,0.7),(0.2,0.7))	((0.6,0.8),(0.1,0.7),(0.2,0.8))	((0.8,0.7),(0.1,0.8),(0.2,0.6))	((0.6,0.6),(0.2,0.6),(0.1,0.7))	((0.7,0.8),(0.1,0.7),(0.1,0.7))	((0.8,0.8),(0.4,0.7),(0.2,0.8))	((0.8,0.8),(0.4,0.7),(0.2,0.8))
A_3	((0.7,0.6),(0.2,0.7),(0.3,0.8))	((0.6,0.8),(0.1,0.7),(0.2,0.8))	((0.7,0.8),(0.1,0.7),(0.1,0.7))	((0.8,0.8),(0.4,0.7),(0.2,0.8))	((0.7,0.6),(0.2,0.7),(0.3,0.8))	((0.7,0.6),(0.2,0.7),(0.3,0.8))	((0.7,0.8),(0.1,0.7),(0.1,0.7))
A_4	((0.8,0.8),(0.4,0.7),(0.2,0.8))	((0.8,0.7),(0.1,0.8),(0.2,0.6))	((0.6,0.6),(0.2,0.6),(0.1,0.7))	((0.8,0.7),(0.1,0.8),(0.2,0.6))	((0.6,0.8),(0.1,0.7),(0.2,0.8))	((0.8,0.7),(0.1,0.8),(0.2,0.6))	((0.6,0.6),(0.2,0.6),(0.1,0.7))
A_5	((0.7,0.8),(0.1,0.7),(0.1,0.7))	((0.6,0.8),(0.1,0.7),(0.2,0.8))	((0.8,0.8),(0.4,0.7),(0.2,0.8))	((0.7,0.6),(0.2,0.7),(0.3,0.8))	((0.6,0.7),(0.1,0.7),(0.2,0.7))	((0.7,0.6),(0.2,0.7),(0.3,0.8))	((0.6,0.7),(0.1,0.7),(0.2,0.7))
A_6	((0.6,0.6),(0.2,0.6),(0.1,0.7))	((0.8,0.8),(0.4,0.7),(0.2,0.8))	((0.7,0.8),(0.1,0.7),(0.1,0.7))	((0.8,0.7),(0.1,0.8),(0.2,0.6))	((0.6,0.7),(0.1,0.7),(0.2,0.7))	((0.8,0.7),(0.1,0.8),(0.2,0.6))	((0.7,0.6),(0.2,0.7),(0.3,0.8))
A_7	((0.6,0.7),(0.1,0.7),(0.2,0.7))	((0.7,0.8),(0.1,0.7),(0.1,0.7))	((0.8,0.8),(0.4,0.7),(0.2,0.8))	((0.6,0.8),(0.1,0.7),(0.2,0.8))	((0.6,0.6),(0.2,0.6),(0.1,0.7))	((0.6,0.6),(0.2,0.6),(0.1,0.7))	((0.6,0.7),(0.1,0.7),(0.2,0.7))
A_8	((0.6,0.8),(0.1,0.7),(0.2,0.8))	((0.6,0.6),(0.2,0.6),(0.1,0.7))	((0.6,0.8),(0.1,0.7),(0.2,0.8))	((0.8,0.8),(0.4,0.7),(0.2,0.8))	((0.7,0.8),(0.1,0.7),(0.1,0.7))	((0.7,0.8),(0.1,0.7),(0.1,0.7))	((0.6,0.6),(0.2,0.6),(0.1,0.7))
	C_1	C_2	C_3	C_4	C_5	C_6	C_7

A_1	((0.6,0.8),(0.1,0.7),(0.2,0.8))	((0.6,0.6),(0.2,0.6),(0.1,0.7))	((0.7,0.8),(0.1,0.7),(0.1,0.7))	((0.8,0.8),(0.4,0.7),(0.2,0.8))	((0.7,0.6),(0.2,0.7),(0.3,0.8))	((0.8,0.7),(0.1,0.8),(0.2,0.6))	((0.7,0.6),(0.2,0.7),(0.3,0.8))
A_2	((0.8,0.7),(0.1,0.8),(0.2,0.6))	((0.7,0.6),(0.2,0.7),(0.3,0.8))	((0.8,0.7),(0.1,0.8),(0.2,0.6))	((0.6,0.6),(0.2,0.6),(0.1,0.7))	((0.7,0.8),(0.1,0.7),(0.1,0.7))	((0.8,0.8),(0.4,0.7),(0.2,0.8))	((0.8,0.8),(0.4,0.7),(0.2,0.8))
A_3	((0.6,0.6),(0.2,0.6),(0.1,0.7))	((0.7,0.6),(0.2,0.7),(0.3,0.8))	((0.7,0.8),(0.1,0.7),(0.1,0.7))	((0.8,0.8),(0.4,0.7),(0.2,0.8))	((0.7,0.6),(0.2,0.7),(0.3,0.8))	((0.6,0.8),(0.1,0.7),(0.2,0.8))	((0.8,0.8),(0.4,0.7),(0.2,0.8))
A_4	((0.7,0.8),(0.1,0.7),(0.1,0.7))	((0.8,0.7),(0.1,0.8),(0.2,0.6))	((0.6,0.6),(0.2,0.6),(0.1,0.7))	((0.6,0.7),(0.1,0.7),(0.2,0.7))	((0.6,0.8),(0.1,0.7),(0.2,0.8))	((0.6,0.7),(0.1,0.7),(0.2,0.7))	((0.6,0.6),(0.2,0.6),(0.1,0.7))
A_5	((0.8,0.8),(0.4,0.7),(0.2,0.8))	((0.7,0.6),(0.2,0.7),(0.3,0.8))	((0.8,0.8),(0.4,0.7),(0.2,0.8))	((0.6,0.8),(0.1,0.7),(0.2,0.8))	((0.8,0.7),(0.1,0.8),(0.2,0.6))	((0.6,0.6),(0.2,0.6),(0.1,0.7))	((0.6,0.7),(0.1,0.7),(0.2,0.7))
A_6	((0.6,0.6),(0.2,0.6),(0.1,0.7))	((0.6,0.7),(0.1,0.7),(0.2,0.7))	((0.6,0.8),(0.1,0.7),(0.2,0.8))	((0.8,0.7),(0.1,0.8),(0.2,0.6))	((0.6,0.6),(0.2,0.6),(0.1,0.7))	((0.7,0.8),(0.1,0.7),(0.1,0.7))	((0.7,0.6),(0.2,0.7),(0.3,0.8))
A_7	((0.6,0.7),(0.1,0.7),(0.2,0.7))	((0.6,0.6),(0.2,0.6),(0.1,0.7))	((0.8,0.7),(0.1,0.8),(0.2,0.6))	((0.6,0.6),(0.2,0.6),(0.1,0.7))	((0.7,0.8),(0.1,0.7),(0.1,0.7))	((0.8,0.8),(0.4,0.7),(0.2,0.8))	((0.6,0.6),(0.2,0.6),(0.1,0.7))
A_8	((0.6,0.8),(0.1,0.7),(0.2,0.8))	((0.7,0.8),(0.1,0.7),(0.1,0.7))	((0.6,0.6),(0.2,0.6),(0.1,0.7))	((0.7,0.8),(0.1,0.7),(0.1,0.7))	((0.8,0.8),(0.4,0.7),(0.2,0.8))	((0.7,0.8),(0.1,0.7),(0.1,0.7))	((0.7,0.8),(0.1,0.7),(0.1,0.7))
	C_1	C_2	C_3	C_4	C_5	C_6	C_7
A_1	((0.6,0.7),(0.1,0.7),(0.2,0.7))	((0.6,0.6),(0.2,0.6),(0.1,0.7))	((0.7,0.8),(0.1,0.7),(0.1,0.7))	((0.8,0.8),(0.4,0.7),(0.2,0.8))	((0.8,0.8),(0.4,0.7),(0.2,0.8))	((0.8,0.7),(0.1,0.8),(0.2,0.6))	((0.7,0.6),(0.2,0.7),(0.3,0.8))
A_2	((0.6,0.7),(0.1,0.7),(0.2,0.7))	((0.8,0.8),(0.4,0.7),(0.2,0.8))	((0.8,0.7),(0.1,0.8),(0.2,0.6))	((0.6,0.6),(0.2,0.6),(0.1,0.7))	((0.7,0.8),(0.1,0.7),(0.1,0.7))	((0.8,0.8),(0.4,0.7),(0.2,0.8))	((0.8,0.8),(0.4,0.7),(0.2,0.8))
A_3	((0.6,0.8),(0.1,0.7),(0.2,0.8))	((0.7,0.8),(0.1,0.7),(0.1,0.7))	((0.7,0.8),(0.1,0.7),(0.1,0.7))	((0.8,0.8),(0.4,0.7),(0.2,0.8))	((0.6,0.6),(0.2,0.6),(0.1,0.7))	((0.7,0.8),(0.1,0.7),(0.1,0.7))	((0.7,0.8),(0.1,0.7),(0.1,0.7))
A_4	((0.8,0.8),(0.4,0.7),(0.2,0.8))	((0.6,0.6),(0.2,0.6),(0.1,0.7))	((0.8,0.8),(0.4,0.7),(0.2,0.8))	((0.8,0.7),(0.1,0.8),(0.2,0.6))	((0.8,0.7),(0.1,0.8),(0.2,0.6))	((0.6,0.6),(0.2,0.6),(0.1,0.7))	((0.6,0.6),(0.2,0.6),(0.1,0.7))
A_5	((0.7,0.8),(0.1,0.7),(0.1,0.7))	((0.6,0.7),(0.1,0.7),(0.2,0.7))	((0.7,0.8),(0.1,0.7),(0.1,0.7))	((0.8,0.8),(0.4,0.7),(0.2,0.8))	((0.8,0.8),(0.4,0.7),(0.2,0.8))	((0.8,0.7),(0.1,0.8),(0.2,0.6))	((0.8,0.7),(0.1,0.8),(0.2,0.6))
A_6	((0.6,0.6),(0.2,0.6),(0.1,0.7))	((0.8,0.8),(0.4,0.7),(0.2,0.8))	((0.6,0.6),(0.2,0.6),(0.1,0.7))	((0.7,0.8),(0.1,0.7),(0.1,0.7))	((0.7,0.8),(0.1,0.7),(0.1,0.7))	((0.8,0.8),(0.4,0.7),(0.2,0.8))	((0.7,0.6),(0.2,0.7),(0.3,0.8))
A_7	((0.8,0.7),(0.1,0.8),(0.2,0.6))	((0.7,0.8),(0.1,0.7),(0.1,0.7))	((0.8,0.7),(0.1,0.8),(0.2,0.6))	((0.6,0.6),(0.2,0.6),(0.1,0.7))	((0.6,0.6),(0.2,0.6),(0.1,0.7))	((0.7,0.8),(0.1,0.7),(0.1,0.7))	((0.8,0.7),(0.1,0.8),(0.2,0.6))
A_8	((0.7,0.6),(0.2,0.7),(0.3,0.8))	((0.6,0.6),(0.2,0.6),(0.1,0.7))	((0.7,0.6),(0.2,0.7),(0.3,0.8))	((0.6,0.7),(0.1,0.7),(0.2,0.7))	((0.6,0.7),(0.1,0.7),(0.2,0.7))	((0.6,0.6),(0.2,0.6),(0.1,0.7))	((0.6,0.6),(0.2,0.6),(0.1,0.7))

NZN-RAFSI-Method

- In the NHSS, experts put three values of each criterion such as lower bound, middle value, and upper bound. We chose the best value of each criterion such as: $C_{1-3}, C_{2-3}, C_{3-3}, C_{4-3}, C_{5-3}, C_{6-3}, C_{7-3}$.

- We Compute the ideal $(a_j)^I$ and non-ideal $(a_j)^N$ values.

- We create criterion intervals for each entry in the main decision-making matrix using Eq. (19).

- then we compute the harmonic and arithmetic means for the minim and maximum values using Eq. (21)

- Then we normalize the decision matrix using Eq. (22) as shown in Table 2 and the weighted normalized decision matrix in Table 3.

- Then we compute the criteria function using Eq. (24) as shown in Fig 1. Then we ranked the alternatives as shown in Fig 2.

Table 2. Normalized matrix using the RAFSI method.

	$C_{1\text{-}3}$	$C_{2\text{-}3}$	$C_{3\text{-}3}$	$C_{4\text{-}3}$	$C_{5\text{-}3}$	$C_{6\text{-}3}$	$C_{7\text{-}3}$
A_1	0.149138	0.032692	0.225	0.058846	0.030941	0.188636	0.025
A_2	0.135345	0.025	0.18	0.031154	0.225	0.038636	0.129348
A_3	0.025	0.109615	0.225	0.058846	0.025	0.088636	0.225
A_4	0.149138	0.178846	0.045	0.160385	0.145792	0.102273	0.109783
A_5	0.225	0.028846	0.115	0.025	0.12797	0.025	0.168478
A_6	0.042241	0.071154	0.12	0.225	0.133911	0.211364	0.025
A_7	0.135345	0.225	0.14	0.055769	0.12599	0.125	0.159783
A_8	0.052586	0.128846	0.025	0.129615	0.139851	0.225	0.16413

Table 3. The weighted normalize values.

	$C_{1\text{-}3}$	$C_{2\text{-}3}$	$C_{3\text{-}3}$	$C_{4\text{-}3}$	$C_{5\text{-}3}$	$C_{6\text{-}3}$	$C_{7\text{-}3}$
A_1	0.02072	0.004255	0.042178	0.008081	0.004409	0.029742	0.002649
A_2	0.018804	0.003254	0.033743	0.004278	0.032065	0.006092	0.013706
A_3	0.003473	0.014266	0.042178	0.008081	0.003563	0.013975	0.023842
A_4	0.02072	0.023276	0.008436	0.022024	0.020777	0.016125	0.011633
A_5	0.03126	0.003754	0.021558	0.003433	0.018237	0.003942	0.017853
A_6	0.005869	0.00926	0.022495	0.030896	0.019084	0.033326	0.002649
A_7	0.018804	0.029282	0.026244	0.007658	0.017955	0.019709	0.016931
A_8	0.007306	0.016768	0.004686	0.017798	0.019931	0.035476	0.017392

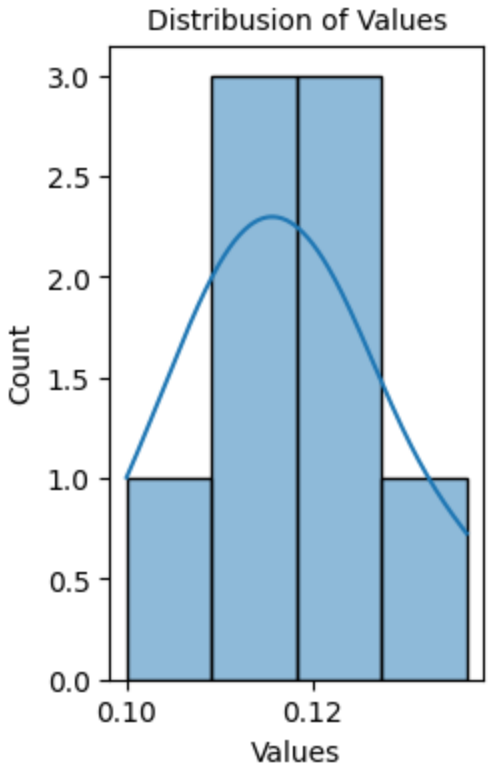

Fig 1. The values of criteria function.

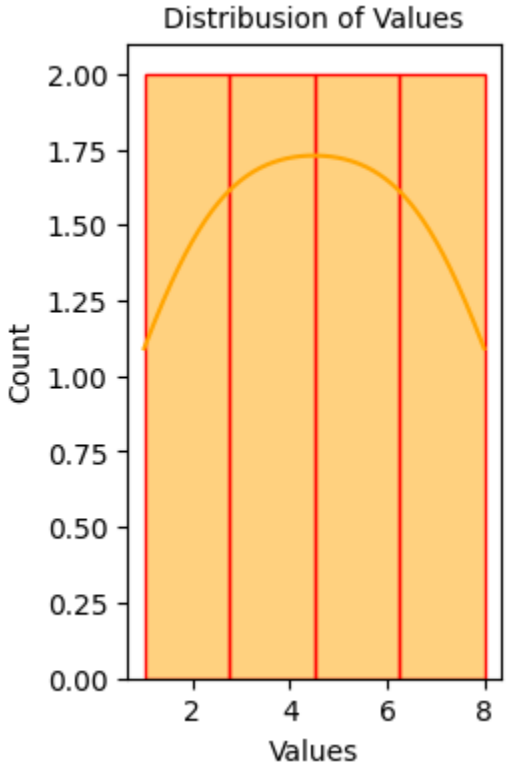

Fig 2. The rank of alternatives.

Sensitivity analysis

In this section, we change the proposed approach parameters then we ranked the alternatives under new parameters values to show different ranks. The aim of the sensitivity analysis is to show the stability of the ranks of the proposed approach. In this section, we use the criteria weights of the Entropy method to compute the criteria weights. We build the decision matrix using the NZN between the criteria and alternatives. Then we compute the entropy values of each criterion. Then we compute the criteria weights.

Then we ranked the alternatives under the Entropy method. We change value of the A parameters value between 0 and 1. Then we ranked the alternatives. In each case, we compute ethe value of criteria function under the ranking method as shown in Fig 3. Then we ranked the alternatives under the new values as shown in Fig 4. The results show the ranks of alternatives are stable under different values. The results show the alternative 6 is the best and alternative 5 is the worst in all cases.

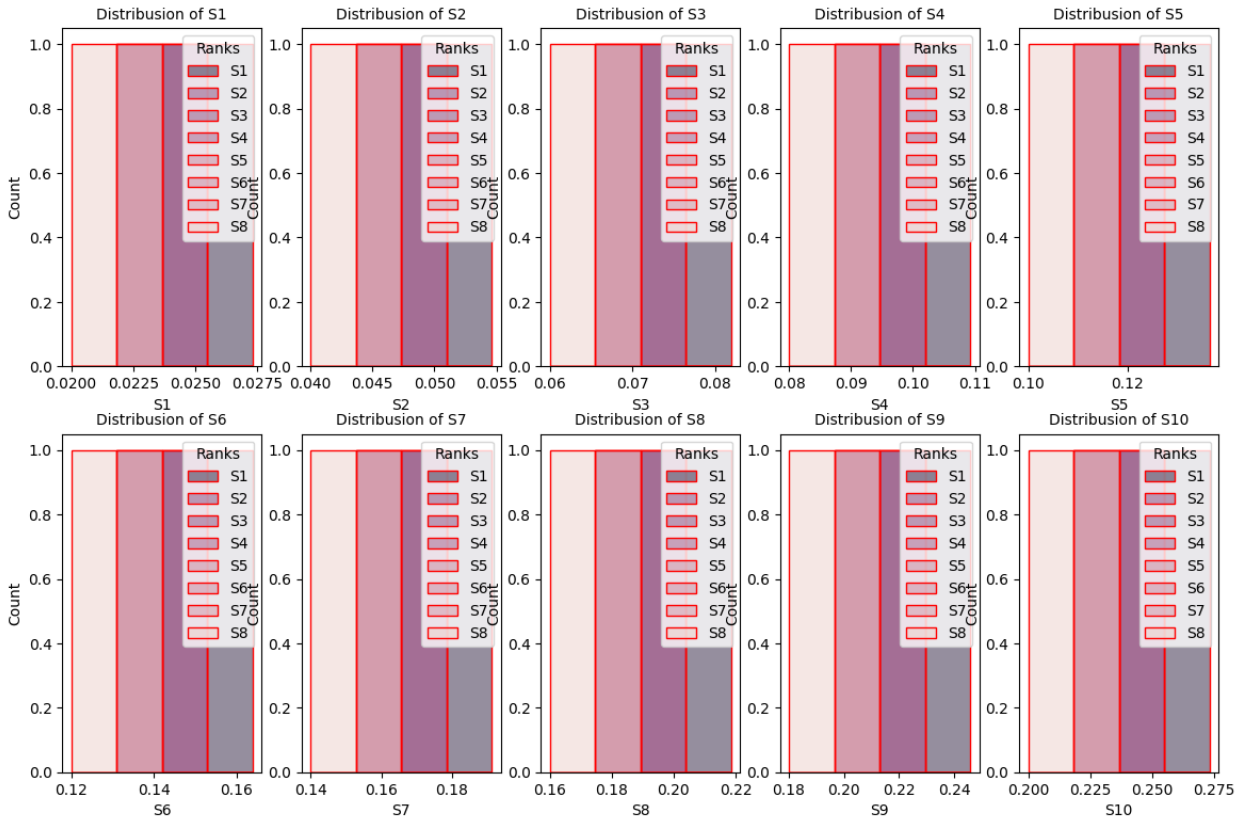

Fig 3. The different values of criteria function.

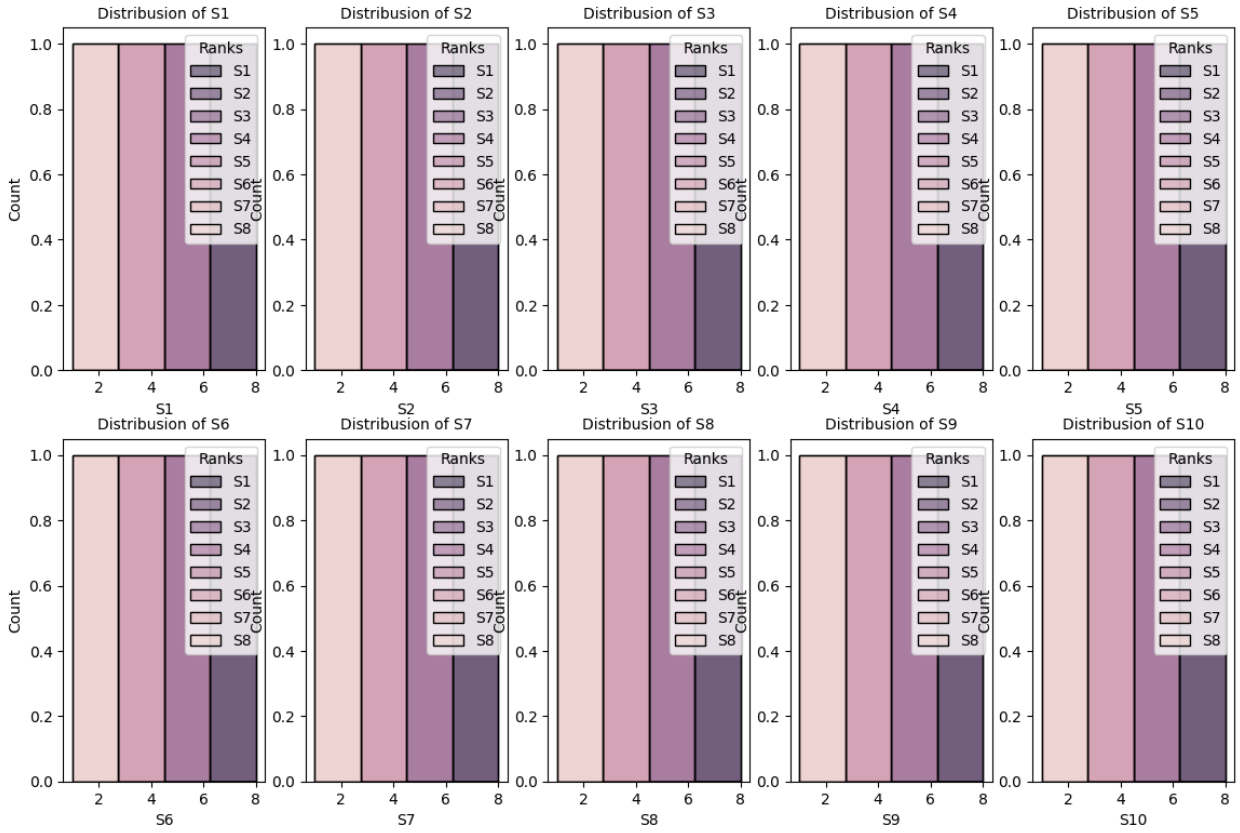

Fig 4. The different values of ranks of alternatives.

5. Conclusions and Future Works

Two MCDM methods are used in this study such as Entropy method to compute the criteria weights and RAFSI method to rank the alternatives. These methods are used under the neutrosophic-z number to deal with uncertainty data. We used the neutrosophic HyperSoft set to show the relationship between the criteria and values. Three experts are evaluated the criteria and alternatives. Seven criteria and eight alternatives are used in this study. We show alternative 6 and alternative 5. Sensitivity analysis results show the ranks of alternatives are stable under different cases.

In the future works, the proposed approach can be used in different MCDM problems to compute the criteria weights by the Entropy method and ranking the alternatives by the RAFSI method. These methods can be used under the uncertainty models to overcome the vague and uncertain information.

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The double integrals of 2- refined neutrosophic functions and its applications

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Abstract: The importance of this paper comes from the fact that it presented a study of the double integrals of 2- refined neutrosophic functions and its applications, where we presented several theories for the concept of the double integrals of 2- refined neutrosophic functions over a general region and over a rectangle, including the 2- refined neutrosophic Fubini's theorem. In addition to discussing applications of double integrals of 2- refined neutrosophic functions to calculate the area of region. Also, We took into account the definition of the positive 2- refined neutrosophic number, and the n-refined AH-Isometry.

Keywords: integrals, double, Fubini's theorem, neutrosophic functions, areas, 2- refined neutrosophic.

1. Introduction and Preliminaries

As an alternative to the existing logics, Smarandache proposed the neutrosophic logic to represent a mathematical model of uncertainty, vagueness, ambiguity, imprecision, undefined, unknown, incompleteness, inconsistency, redundancy, contradiction, where Smarandache made refined neutrosophic numbers available in the following form: $(a, b_1I_1, b_2I_2, \dots, b_nI_n)$ where $a, b_1, b_2, \dots, b_n \in R \text{ or } C$ [1]. Agboola introduced the concept of refined neutrosophic algebraic structures [2]. In addition, the refined neutrosophic rings I was studied in paper [3], where it assumed that I splits into two indeterminacies I_1 [contradiction (true (T) and false (F))] and I_2 [ignorance (true (T) or false (F))]. Abobala presented the papers on some special substructures of refined neutrosophic rings and a study of ah-substructures in n-refined neutrosophic vector spaces [6-7].

Alhasan.Y and Abdulfatah. R also presented a study on the division of refined neutrosophic number [8].

There are papers presented in n-Valued Refined Neutrosophic Logic and Its Applications in Physics, Neutrosophic Rings I [4-5] and studying the integral calculus according to the logic of neutrosophic by presenting a set of papers on that [9-10].

In addition, the AH-Isometry was extended to n-Refined AH-Isometry by Smarandache & Abobala in 2024 [11].

2. The double integrals of 2- refined neutrosophic functions

2.1 The double integrals of 2- refined neutrosophic functions over a rectangle

Theorem 1 (The 2- refined neutrosophic Fubini's theorem)

Let $f(x,y,I_1,I_2)$ integrable over the neutrosophic rectangle $R \cup I_1 \cup I_2 = \{(x,y,I_1,I_2): a + a_1I_1 + a_2I_2 \le x \le b + b_1I_1 + b_2I_2 \; and \; c + c_1I_1 + c_2I_2 \le y \le d + d_1I_1 + d_2I_2\}$, where: $a, a_1, a_2, b, b_1, b_2, c, c_1, c_2, d, d_1, d_2$ are real numbers, while $I_1, I_2 =$ indeterminacy.

Then the double integrals of 2- refined neutrosophic function $f(x,y,I_1,I_2)$ over a rectangle $R \cup I_1 \cup I_2$ given by the following formula:

$$\iint\limits_{R\cup I_1\cup I_2} f(x,y,I_1,I_2)dA = \int\limits_{a+a_1I_1+a_2I_2}^{b+b_1I_1+b_2I_2} \int\limits_{c+c_1I_1+c_2I_2}^{d+d_1I_1+d_2I_2} f(x,y,I_1,I_2)dydx$$

$$= \int\limits_{c+c_1I_1+c_2I_2}^{d+d_1I_1+d_2I_2} \int\limits_{a+a_1I_1+a_2I_2}^{b+b_1I_1+b_2I_2} f(x,y,I_1,I_2)dxdy$$

where:

- using the horizontal slice:

$$\int\limits_{c+c_1I_1+c_2I_2}^{d+d_1I_1+d_2I_2} \int\limits_{a+a_1I_1+a_2I_2}^{b+b_1I_1+b_2I_2} f(x,y,I_1,I_2)dxdy$$

- using the vertical slice:

$$\int\limits_{a+a_1I_1+a_2I_2}^{b+b_1I_1+b_2I_2} \int\limits_{c+c_1I_1+c_2I_2}^{d+d_1I_1+d_2I_2} f(x,y,I_1,I_2)dydx$$

Example 1

Let $R \cup I_1 \cup I_2 = \{(x,y,I_1,I_2): 0 \le x \le 2 + 2I_1 + 2I_2 \; and \; 1 + I_1 + I_2 \le y \le 4 + 4I_1 + 4I_2\}$, find:

$$\iint\limits_{R\cup I_1\cup I_2} (x + 4I_1xy + 2I_2y)\, dA$$

Solution:

- <u>using the vertical slice:</u>

$$\int\limits_0^{2+2I_1+2I_2} \int\limits_{1+I_1+I_2}^{4+4I_1+4I_2} (x + 4I_1xy + 2I_2y)dydx = \int\limits_0^{2+2I_1+2I_2} (xy + 2I_1xy^2 + I_2y^2)\Big|_{1+I_1+I_2}^{4+4I_1+4I_2} dx$$

$$= \int\limits_0^{2+2I_1+2I_2} [(4 + 4I_1 + 4I_2)x + 2I_1(4 + 4I_1 + 4I_2)^2x + I_2(4 + 4I_1 + 4I_2)^2]$$
$$- [(1 + I_1 + I_2)x + 2I_1(1 + I_1 + I_2)^2x + I_2(1 + I_1 + I_2)^2]dx$$

$$= \int_0^{2+2I_1+2I_2} [(4+4I_1+4I_2)x + 288I_1x + I_2(16+80I_1+48I_2)] - [(1+19I_1+I_2)x + I_2(1+I_1+I_2)^2]dx$$

$$= \int_0^{2+2I_1+2I_2} [(4+292I_1+4I_2)x + 80I_1+64I_2] - [(1+19I_1+I_2)x + 5I_1+4I_2]dx$$

$$= \int_0^{2+2I_1+2I_2} \big((3+273I_1+3I_2)x + 75I_1+60I_2\big)dx$$

$$= \left((3+273I_1+3I_2)\frac{x^2}{2} + (75I_1+60I_2)x\right)\Big|_0^{2+2I_1+2I_2}$$

$$= (3+273I_1+3I_2)\frac{(2+2I_1+2I_2)^2}{2} + (75I_1+60I_2)(2+2I_1+2I_2)$$

$$= (3+273I_1+3I_2)(2+10I_1+6I_2) + 570I_1+240I_2$$

$$= 6+4974I_1+42I_2+570I_1+240I_2$$

$$\boxed{= 6+5544I_1+282I_2}$$

- using the horizontal slice:

$$\int_{1+I_1+I_2}^{4+4I_1+4I_2}\int_0^{2+2I_1+2I_2} (x+4I_1xy+2I_2y)dxdy = \int_{1+I_1+I_2}^{4+4I_1+4I_2} \left(\frac{x^2}{2}+2I_1x^2y+2I_2xy\right)\Big|_0^{2+2I_1+2I_2} dy$$

$$= \int_{1+I_1+I_2}^{4+4I_1+4I_2} \left(\frac{(2+2I_1+2I_2)^2}{2} + 2I_1(2+2I_1+2I_2)^2y + 2I_2(2+2I_1+2I_2)y\right)dy$$

$$= \int_{1+I_1+I_2}^{4+4I_1+4I_2} (2+10I_1+6I_2+72I_1y+(4I_1+8I_2)y)dy$$

$$= \int_{1+I_1+I_2}^{4+4I_1+4I_2} [2+10I_1+6I_2+(76I_1+8I_2)y]dy$$

$$= \big((2+10I_1+6I_2)y + (38I_1+4I_2)y^2\big)\Big|_{1+I_1+I_2}^{4+4I_1+4I_2}$$

$$= [(2+10I_1+6I_2)(4+4I_1+4I_2) + (38I_1+4I_2)(4+4I_1+4I_2)^2] - [(2+10I_1+6I_2)(1+I_1+I_2) + (38I_1+4I_2)(1+I_1+I_2)^2]$$

$$= [(2+10I_1+6I_2)(4+4I_1+4I_2) + (38I_1+4I_2)(16+80I_1+48I_2)] - [(2+10I_1+6I_2)(1+I_1+I_2) + (38I_1+4I_2)(1+5I_1+3I_2)]$$

$$= [8+152I_1+56I_2+5792I_1+256I_2] - [2+38I_1+14I_2+362I_1+16I_2]$$

$$= 6 + 5544I_1 + 282I_2$$

Take note that the result was the same.

2.2 The double integrals of 2- refined neutrosophic functions over a general region

Theorem 2

Let $f(x, y, I_1, I_2)$ is continuous on the region $R \cup I_1 \cup I_2 = \{(x, y, I_1, I_2): a + a_1I_1 + a_2I_2 \leq x \leq b + b_1I_1 + b_2I_2 \text{ and } g_1(x, I_1, I_2) \leq y \leq g_2(x, I_1, I_2)\}$, where: a, a_1, a_2, b, b_1, b_2 are real numbers, while $I_1, I_2 =$ indeterminacy, and $g_1(x, I_1, I_2)$, $g_2(x, I_1, I_2)$ continuous 2- refined neutrosophic functions, where $g_1(x, I_1, I_2) \leq g_2(x, I_1, I_2)$, for all $x \in [a + a_1I_1 + a_2I_2\ , b + b_1I_1 + b_2I_2]$.

Then the double integrals of 2- refined neutrosophic function $f(x, y, I_1, I_2)$ over a region $R \cup I_1 \cup I_2$ given by the following formula:

$$\iint_{R \cup I_1 \cup I_2} f(x, y, I_1, I_2)\, dA = \int_{a+a_1I_1+a_2I_2}^{b+b_1I_1+b_2I_2} \int_{g_1(x,I_1,I_2)}^{g_2(x,I_1,I_2)} f(x, y, I_1, I_2) dy dx$$

Example 2

Let $R \cup I_1 \cup I_2 = \{(x, y, I): 0 \leq x \leq 3 + 3I_1 + 3I_2 \text{ and } 0 \leq y \leq (1 + I_1 + I_2)x\}$, then:

$$\iint_{R \cup I_1 \cup I_2} \left(2e^{x^2} - siny\right) dA = \int_0^{3+3I_1+3I_2} \int_0^{(1+I_1+I_2)x} \left(2I_1e^{x^2} - \cos y\right) dy dx$$

$$= \int_0^{3+3I_1+3I_2} \left(2I_1ye^{x^2} - \sin y\right)\Big|_0^{(1+I_1+I_2)x} dx$$

$$= \int_0^{3+3I_1+3I_2} \left(2I_1(1 + I_1 + I_2)xe^{x^2} - sin(1 + I_1 + I_2)x\right) dx$$

$$= \int_0^{3+3I_1+3I_2} \left(6I_1xe^{x^2} - sin(1 + I_1 + I_2)x\right) dx$$

$$= \left(3I_1e^{x^2} + \frac{1}{1 + I_1 + I_2} cos(1 + I_1 + I_2)x\right)\Big|_0^{3+3I_1+3I_2}$$

$$= \left[3I_1e^{9+45I_1+27I_2} + \frac{1}{1 + I_1 + I_2} cos\left((1 + I_1 + I_2)(3 + 3I_1 + 3I_2)\right)\right] - \left[3I_1 + \frac{1}{1 + I_1 + I_2}\right]$$

$$= \left[3I_1e^{9+45I_1+27I_2} + \frac{1}{1 + I_1 + I_2} cos(3 + 15I_1 + 9I_2)\right] - \left[3I_1 + \frac{1}{1 + I_1 + I_2}\right]$$

$$= 3I_1[e^9 + I_1(e^{81} - e^{36}) + I_2(e^{36} - e^9)] + \left(1 + \frac{1}{6}I_1 + \frac{1}{2}I_2\right)[\cos 3 + I_1(\cos 27 - \cos 9) + I_2(\cos 9 - \cos 3)] - 1 - \frac{19}{6}I_1 - \frac{1}{2}I_2$$

Example 3

Let $R \cup I_1 \cup I_2 = \{(x, y, I_1, I_2): 0 \leq x \leq 1 + I_1 + I_2 \; and \; 0 \leq y \leq x\}$, then:

$$\iint\limits_{R \cup I_1 \cup I_2} \frac{cos\left(\pi + \frac{\pi}{2} I_1 + \frac{\pi}{3} I_2\right) x}{x} \, dA = \int\limits_0^{1+I_1+I_2} \int\limits_0^{x} \frac{cos\left(\pi + \frac{\pi}{2} I_1 + \frac{\pi}{3} I_2\right) x}{x} dy dx$$

$$= \int\limits_0^{1+I_1+I_2} \frac{cos\left(\pi + \frac{\pi}{2} I_1 + \frac{\pi}{3} I_2\right) x}{x} \, y \Big|_0^x \, dx$$

$$= \int\limits_0^{1+I_1+I_2} \left(\left[\frac{cos\left(\pi + \frac{\pi}{2} I_1 + \frac{\pi}{3} I_2\right) x}{x} \, x \right] - \left[\frac{cos\left(\pi + \frac{\pi}{2} I_1 + \frac{\pi}{3} I_2\right) x}{x} \, (0) \right] \right) dx$$

$$= \int\limits_0^{1+I_1+I_2} cos\left(\pi + \frac{\pi}{2} I_1 + \frac{\pi}{3} I_2\right) x \, dx$$

$$= \frac{1}{\pi + \frac{\pi}{2} I_1 + \frac{\pi}{3} I_2} sin\left(\pi + \frac{\pi}{2} I_1 + \frac{\pi}{3} I_2\right) x \Big|_0^{1+I_1+I_2}$$

$$= \left[\frac{1}{\pi + \frac{\pi}{2} I_1 + \frac{\pi}{3} I_2} sin\left(\left(\pi + \frac{\pi}{2} I_1 + \frac{\pi}{3} I_2\right) (1 + I_1 + I_2) \right) \right] - [0]$$

$$= \frac{1}{\pi + \frac{\pi}{2} I_1 + \frac{\pi}{3} I_2} sin\left(\left(\pi + \frac{\pi}{2} I_1 + \frac{\pi}{3} I_2\right) (1 + I_1 + I_2) \right)$$

$$= \left(\frac{1}{\pi} - \frac{9}{44\pi} I_1 - \frac{1}{4\pi} I_2\right) sin\left(\pi + \frac{11\pi}{6} I_1 + \frac{2\pi}{3} I_2\right)$$

$$= \left(\frac{1}{\pi} - \frac{9}{44\pi} I_1 - \frac{1}{4\pi} I_2\right) \left[sin(\pi) + I_1 \left[sin\left(\frac{21\pi}{6}\right) - sin\left(\frac{5\pi}{3}\right) \right] + I_2 \left[sin\left(\frac{5\pi}{3}\right) - sin(\pi) \right] \right]$$

$$= \left(\frac{1}{\pi} - \frac{9}{44\pi} I_1 - \frac{1}{4\pi} I_2\right) \left(\left(-1 + \frac{\sqrt{3}}{2}\right) I_1 + \frac{\sqrt{3}}{2} I_2 \right)$$

$$= -\frac{1}{\pi} I_1 + \frac{\sqrt{3}}{2\pi} I_1 + \frac{\sqrt{3}}{2\pi} I_2 + \frac{9}{44\pi} I_1 - \frac{9\sqrt{3}}{44\pi} I_1 + \frac{1}{4\pi} I_1 - \frac{\sqrt{3}}{8\pi} I_1 - \frac{\sqrt{3}}{8\pi} I_2$$

$$= \left(-\frac{6}{11\pi} + \frac{15\sqrt{3}}{44\pi}\right) I_1 + \frac{3\sqrt{3}}{8\pi} I_2$$

Example 4

$$\int_0^{\frac{\pi}{3}+\pi I_1-\frac{\pi}{3}I_2}\int_0^{x} 3I_1 x\sin y\,dydx = \int_0^{\frac{\pi}{3}+\pi I_1-\frac{\pi}{3}I_2} -3I_1 x\cos y|_0^x dx$$

$$= \int_0^{\frac{\pi}{3}+\pi I_1-\frac{\pi}{3}I_2} -3I_1 x\cos x\,dx$$

derivation	integration
$(+)\ -3I_1 x$	$\cos x$
$(-)\ -3I_1$	$\sin x$
0	$-\cos x$

$$= [-3I_1 x\sin x - 3I_1\cos x]|_0^{\frac{\pi}{3}+\pi I_1-\frac{\pi}{3}I_2}$$

$$= \left[-3I_1\left(\frac{\pi}{3}+\pi I_1-\frac{\pi}{3}I_2\right)\sin\left(\frac{\pi}{3}+\pi I_1-\frac{\pi}{3}I_2\right) - 3I_1\cos\left(\frac{\pi}{3}+\pi I_1-\frac{\pi}{3}I_2\right)\right] - [-3I_1]$$

$$= \left[-3\pi I_1\sin\left(\frac{\pi}{3}+\pi I_1-\frac{\pi}{3}I_2\right) - 3I_1\cos\left(\frac{\pi}{3}+\pi I_1-\frac{\pi}{3}I_2\right)\right] + 3I_1$$

$$= \left[-3\pi I_1\left(\sin\left(\frac{\pi}{3}\right)+I_1[\sin(\pi)-\sin(0)]+I_2\left[\sin(0)-\sin\left(\frac{\pi}{3}\right)\right]\right)\right.$$
$$\left. -3I_1\left(\cos\left(\frac{\pi}{3}\right)+I_1[\cos(\pi)-\cos(0)]+I_2\left[\cos(0)-\cos\left(\frac{\pi}{3}\right)\right]\right)\right] + 3I_1$$

$$= \left[-3\pi I_1\left(\frac{\sqrt{3}}{2}-\frac{\sqrt{3}}{2}I_2\right) - 3I_1\left(\frac{1}{2}-2I_1+\frac{1}{2}I_2\right)\right] + 3I_1 = 6I_1$$

Theorem 3

Let $f(x,y,I_1,I_2)$ is continuous on the region

$R\cup I_1\cup I_2 = \{(x,y,I_1,I_2)\colon c+c_1I_1+c_2I_2 \le y \le d+d_1I_1+d_2I_2 \text{ and } h_1(y,I_1,I_2) \le x \le h_2(y,I_1,I_2)\}$, where: c, c_1, c_2, d, d_1, d_2 are real numbers, while $I_1, I_2 =$ indeterminacy, and $h_1(y,I_1,I_2)$, $h_2(y,I_1,I_2)$ continuous 2- refined neutrosophic functions, where $h_1(y,I_1,I_2) \le h_2(y,I_1,I_2)$, for all $y \in [c+c_1I_1+c_2I_2\ , d+d_1I_1+d_2I_2]$.

Then the double integrals of 2- refined neutrosophic function $f(x,y,I_1,I_2)$ over a region $R\cup I_1\cup I_2$ given by the following formula:

$$\iint_{R\cup I_1\cup I_2} f(x,y,I_1,I_2)\,dA = \int_{c+c_1I_1+c_2I_2}^{d+d_1I_1+d_2I_2}\int_{h_1(y,I_1,I_2)}^{h_2(y,I_1,I_2)} f(x,y,I_1,I_2)\,dxdy$$

Example 5

Let $R\cup I_1\cup I_2 = \{(x,y,I_1,I_2)\colon 1+I_1+I_2 \le y \le 2+2I_1+2I_2 \text{ and } (1+I_1+I_2)y \le x \le 3I_1y^2\}$, then:

$$\iint_{R\cup I_1\cup I_2} 2\,dA = \int_{1+I_1+I_2}^{2+2I_1+2I_2}\int_{(1+I_1+I_2)y}^{3I_1y^2} 2\,dxdy = \int_{1+I_1+I_2}^{2+2I_1+2I_2} 2x|_{(1+I_1+I_2)y}^{3I_1y^2}\,dy$$

$$= 2 \int_{1+I_1+I_2}^{2+2I_1+2I_2} (3I_1y^2 - (1 + I_1 + I_2)y)\, dy$$

$$= (2I_1y^3 - (1 + I_1 + I_2)y^2)|_{1+I_1+I_2}^{2+2I_1+2I_2}$$

$$= [2I_1(2 + 2I_1 + 2I_2)^3 - (1 + I_1 + I_2)(2 + 2I_1 + 2I_2)^2] - [2I_1(1 + I_1 + I_2)^3 - (1 + I_1 + I_2)(1 + I_1 + I_2)^2]$$

$$= (1 + I_1 + I_2)^3[14I_1 - 3] = (1 + 19I_1 + 7I_2)[14I_1 - 3]$$

$$= -3 + 321I_1 - 21I_2$$

Theorem 4

Let $f(x, y, I_1, I_2)$ and $h(x, y, I_1, I_2)$ be integrable over region $R \cup I_1 \cup I_2$, and let $c + c_1I_1 + c_2I_2$ any constant, then:

(1) $$\iint_{R \cup I_1 \cup I_2} (c + c_1I_1 + c_2I_2) f(x, y, I_1, I_2)\, dA = (c + c_1I_1 + c_2I_2) \iint_{R \cup I_1 \cup I_2} f(x, y, I_1, I_2)\, dA$$

(2) $$\iint_{R \cup I_1 \cup I_2} [f(x, y, I_1, I_2) + h(x, y, I_1, I_2)]\, dA = \iint_{R \cup I_1 \cup I_2} f(x, y, I_1, I_2)\, dA + \iint_{R \cup I_1 \cup I_2} h(x, y, I_1, I_2)\, dA$$

(3) If $R \cup I_1 \cup I_2 = R_1 \cup R_2 \cup I_1 \cup I_2$, where R_1, R_2 are not overlap regions, then:

$$\iint_{R \cup I_1 \cup I_2} f(x, y, I_1, I_2)\, dA = \iint_{R_1 \cup I_1 \cup I_2} f(x, y, I_1, I_2)\, dA + \iint_{R_2 \cup I_1 \cup I_2} f(x, y, I_1, I_2)\, dA$$

3.2 Applications of double integrals of 2- refined neutrosophic functions

We can use double integrals of 2- refined neutrosophic functions to calculate the area of region:

$$A = \iint_{R \cup I_1 \cup I_2} dxdy = \iint_{R \cup I_1 \cup I_2} dydx$$

Example 6

Let's find the area of the plane region bounded by the curve of $y = x^2$ and $y = (1 + I_1 + I_2)x$ by using the double integrals of 2- refined neutrosophic functions.

intersection points:

$$x^2 = (1 + I_1 + I_2)x$$

$$x(x - 1 + I_1 + I_2) = 0$$

$$\Rightarrow \begin{cases} x = 0 \\ x = 1 + I_1 + I_2 \end{cases}$$

then:

$$\begin{cases} y = 0 \\ y = 1 + 5I_1 + 3I_2 \end{cases}$$

so, the intersection points are: $(0, 0)$ and $\left(1 - \frac{1}{4}I_1, 1 + 5I_1 + 3I_2\right)$

- using the horizontal slice:

$$\int_0^{1+I_1+I_2}\int_{x^2}^{(1+I_1+I_2)x} dydx = \int_0^{1-\frac{1}{4}I_1} y|_{x^2}^{(1+I_1+I_2)x}dx$$

$$= \int_0^{1+I_1+I_2} \left((1+I_1+I_2)x - x^2\right)dx$$

$$= \left((1+I_1+I_2)\frac{x^2}{2} - \frac{x^3}{3}\right)\Big|_0^{1+I_1+I_2}$$

$$= (1+I_1+I_2)\frac{(1+I_1+I_2)^2}{2} - \frac{(1+I_1+I_2)^3}{3}$$

$$= \frac{(1+I_1+I_2)^3}{2} - \frac{(1+I_1+I_2)^3}{3}$$

$$= \frac{1}{6} + \frac{19}{6}I_1 + \frac{7}{6}I_2$$

- using the vertical slice:

$$\int_0^{1+5I_1+3I_2}\int_{\left(1-\frac{1}{6}I_1-\frac{1}{2}I_2\right)y}^{\sqrt{y}} dxdy = \int_0^{1+5I_1+3I_2} x|_{\left(1-\frac{1}{6}I_1-\frac{1}{2}I_2\right)y}^{\sqrt{y}}dy$$

$$= \int_0^{1+5I_1+3I_2} \left(\sqrt{y} - \left(1-\frac{1}{6}I_1-\frac{1}{2}I_2\right)y\right)dy$$

$$= \left(\frac{2}{3}\sqrt{y^3} - \left(1-\frac{1}{6}I_1-\frac{1}{2}I_2\right)\frac{y^2}{2}\right)\Big|_0^{1+5I_1+3I_2}$$

$$= \left[\frac{2}{3}\sqrt{(1+5I_1+3I_2)^3} - \left(1-\frac{1}{6}I_1-\frac{1}{2}I_2\right)\frac{(1+5I_1+3I_2)^2}{2}\right] - [0]$$

$$= \frac{2}{3}\sqrt{1+665I_1+63I_2} - \left(\frac{1}{2}+\frac{19}{2}I_1+\frac{7}{2}I_2\right)$$

$$= \frac{2}{3}(1+19I_1+7I_2) - \left(\frac{1}{2}+\frac{19}{2}I_1+\frac{7}{2}I_2\right)$$

$$= \frac{1}{6} + \frac{19}{6}I_1 + \frac{7}{6}I_2$$

Take note that the result was the same.

Definition 1

Let $R(I_1, I_2) = \{a + bI_1 + cI_2;\ a, b, c \in R\}$ be the real neutrosophic field, then $a + bI_1 + cI_2 \geq 0$ if and only if $a \geq 0, a + c \geq 0, a + b + c \geq 0$

Example 7

Let's find the area of the plane region bounded by the curve of $y = 1 + 5I_1 + 3I_2 - x^2$ and $x - axis$ by using the double integrals of 2- refined neutrosophic functions.

$$1 + 5I_1 + 3I_2 - x^2 = 0$$
$$x^2 = 1 + 5I_1 + 3I_2$$

Let's find: $\sqrt{1 + 5I_1 + 3I_2}$

$$\sqrt{1 + 5I_1 + 3I_2} = x_1 + x_2I_1 + x_3I_2$$

$$1 + 5I_1 + 3I_2 = (x_1 + x_2I_1 + x_3I_2)^2$$

$$1 + 5I_1 + 3I_2 = (x_1 + x_2I_1)^2 + 2(x_1 + x_2I_1)(x_3I_2) + (x_3I_2)^2$$

$$1 + 5I_1 + 3I_2 = {x_1}^2 + 2x_1x_2I_1 + (x_2I_1)^2 + 2(x_1 + x_2I_1)(x_3I_2) + (x_3I_2)^2$$

$$1 + 5I_1 + 3I_2 = {x_1}^2 + 2x_1x_2I_1 + {x_2}^2I_1 + 2xx_3I_2 + 2x_2x_3I_1 + {x_3}^2I_2$$

$$1 + 5I_1 + 3I_2 = {x_1}^2 + ({x_2}^2 + 2x_1x_2 + 2x_2x_3)I_1 + ({x_3}^2 + 2x_1x_3)I_2$$

Whence:

$$\begin{cases} {x_1}^2 = 1 \\ {x_2}^2 + 2x_1x_2 + 2x_2x_3 = 5 \\ {x_3}^2 + 2x_1x_3 = 3 \end{cases}$$

$$\Rightarrow \begin{cases} x_1 = \pm 1 \\ {x_2}^2 + 2x_1x_2 + 2x_2x_3 = 5 \ (1) \\ {x_3}^2 + 2x_1x_3 = 3 \ (2) \end{cases}$$

Case1: $x_1 = 1$ by substitution in (2)

$${x_3}^2 + 2x_3 - 3 = 0$$
$$(x_3 - 1)(x_3 + 3) = 0$$

Then:

$$\begin{cases} x_3 = 1 \\ x_3 = -3 \end{cases}$$

- For $x_3 = 1$ we substitute in (1)

$${x_2}^2 + 4x_2 - 5 = 0$$
$$(x_2 + 5)(x_2 - 1) = 0$$

Then:

$$\begin{cases} x_2 = 1 \\ x_2 = -5 \end{cases}$$

Hence:

$$\sqrt{1 + 5I_1 + 3I_2} = 1 + I_1 + I_2$$

Or:

$$= 1 - 5I_1 + I_2$$

- For $x_3 = -3$ we substitute in (1)

$${x_2}^2 - 4x_2 - 5 = 0$$
$$(x_2 - 5)(x_2 + 1) = 0$$

Then:

$$\begin{cases} x_2 = 5 \\ x_2 = -1 \end{cases}$$

Hence:

$$\sqrt{1+5I_1+3I_2} = 1+5I_1-3I_2$$

Or:

$$= 1-I_1-3I_2$$

Case2: $x_1 = -1$ by substitution in (2)

$$x_3{}^2 - 2x_3 - 3 = 0$$
$$(x_3-3)(x_3+1) = 0$$

Then:

$$\begin{cases} x_3 = 3 \\ x_3 = -1 \end{cases}$$

- For $x_3 = 3$ we substitute in (1)

$$x_2{}^2 + 4x_2 - 5 = 0$$

$$(x_2-1)(x_2+5) = 0$$

Then:

$$\begin{cases} x_2 = 1 \\ x_2 = -5 \end{cases}$$

Hence:

$$\sqrt{1+5I_1+3I_2} = -1+I_1+3I_2$$

Or:

$$= -1-5I_1+3I_2$$

- For $x_3 = -1$ we substitute in (1)

$$x_2{}^2 - 4x_2 - 5 = 0$$
$$(x_2-5)(x_2+1) = 0$$

$$\begin{cases} x_2 = 5 \\ x_2 = -1 \end{cases}$$

Hence:

$$\sqrt{1+5I_1+3I_2} = -1+5I_1-I_2$$

Or:

$$= -1-I_1-I_2$$

hence intersection points are:

$$(1+I_1+I_2, 0), (-1-I_1-I_2, 0)$$

Or:

$$(1+5I_1-3I_2, 0), (-1-5I_1+3I_2, 0)$$

Or:

$$(1-I_1-3I_2, 0), (-1+I_1+3I_2, 0)$$

Or:

$$(1-5I_1+I_2, 0), (-1+5I_1-I_2, 0)$$

For: $(1+I_1+I_2, 0), (-1-I_1-I_2, 0)$

$$A=\int_{-1-I_1-I_2}^{1+I_1+I_2}\int_{0}^{1+5I_1+3I_2-x^2} dydx=\int_{-1-I_1-I_2}^{1+I_1+I_2} y|_0^{1+5I_1+3I_2-x^2}dx$$

$$=\int_{-1-I_1-I_2}^{1+I_1+I_2}(1+5I_1+3I_2-x^2)dx$$

$$=\left((1+5I_1+3I_2)x-\frac{x^3}{3}\right)\Bigg|_{-1-I_1-I_2}^{1+I_1+I_2}$$

$$=\left[(1+5I_1+3I_2)(1+I_1+I_2)-\frac{(1+I_1+I_2)^3}{3}\right]-\left[(1+5I_1+3I_2)(-1-I_1-I_2)-\frac{(-1-I_1-I_2)^3}{3}\right]$$

$$=2\left[(1+5I_1+3I_2)(1+I_1+I_2)-\frac{(1+I_1+I_2)^3}{3}\right]$$

$$=2\left[1+19I_1+7I_2-\left(\frac{1+19I_1+7I_2}{3}\right)\right]$$

$$=\frac{4}{3}+\frac{76}{3}I_1+\frac{28}{3}I_2>0$$

$becous$: $\frac{4}{3}>0, \frac{4}{3}+\frac{28}{3}>0$ and $\frac{4}{3}+\frac{76}{3}+\frac{28}{3}, see\ definition\ 1$

For: $(1+5I_1-3I_2,0),(-1-5I_1+3I_2,0)$

$$A=\int_{-1-5I_1+3I_2}^{1+5I_1-3I_2}\int_{0}^{1+5I_1+3I_2-x^2} dydx=\int_{-1-5I_1+3I_2}^{1+5I_1-3I_2} y|_0^{1+5I_1+3I_2-x^2}dx$$

$$=\int_{-1-5I_1+3I_2}^{1+5I_1-3I_2}(1+5I_1+3I_2-x^2)dx$$

$$=\left((1+5I_1+3I_2)x-\frac{x^3}{3}\right)\Bigg|_{-1-5I_1+3I_2}^{1+5I_1-3I_2}$$

$$=\left[(1+5I_1+3I_2)(1+5I_1-3I_2)-\frac{(1+5I_1-3I_2)^3}{3}\right]$$
$$-\left[(1+5I_1+3I_2)(-1-5I_1+3I_2)-\frac{(-1-5I_1+3I_2)^3}{3}\right]$$

$$=2\left[(1+5I_1+3I_2)(1+5I_1-3I_2)-\frac{(1+5I_1-3I_2)^3}{3}\right]$$

$$=2\left[1+35I_1-9I_2-\left(\frac{1+35I_1-9I_2}{3}\right)\right]$$

$$= \frac{4}{3} + \frac{140}{3} I_1 - 12 I_2 \ (rejected)$$

For: $(1 - I_1 - 3I_2, 0), (-1 + I_1 + 3I_2, 0)$

$$A = \int_{-1+I_1+3I_2}^{1-I_1-3I_2} \int_{0}^{1+5I_1+3I_2-x^2} dydx = \int_{-1+I_1+3I_2}^{1-I_1-3I_2} y|_0^{1+5I_1+3I_2-x^2} dx$$

$$= \int_{-1+I_1+3I_2}^{1-I_1-3I_2} (1 + 5I_1 + 3I_2 - x^2) dx$$

$$= \left((1 + 5I_1 + 3I_2)x - \frac{x^3}{3} \right) \Bigg|_{-1+I_1+3I_2}^{1-I_1-3I_2}$$

$$= \left[(1 + 5I_1 + 3I_2)(1 - I_1 - 3I_2) - \frac{(1 - I_1 - 3I_2)^3}{3} \right] - \left[(1 + 5I_1 + 3I_2)(-1 + I_1 + 3I_2) - \frac{(-1 + I_1 + 3I_2)^3}{3} \right]$$

$$= 2 \left[(1 + 5I_1 + 3I_2)(1 - I_1 - 3I_2) - \frac{(1 - I_1 - 3I_2)^3}{3} \right]$$

$$= 2 \left[1 - 19I_1 - 9I_2 - \left(\frac{1 - 19I_1 - 9I_2}{3} \right) \right]$$

$$= \frac{4}{3} - \frac{76}{3} I_1 - 12 I_2 \ (rejected)$$

For: $(1 - 5I_1 + I_2, 0), (-1 + 5I_1 - I_2, 0)$

$$A = \int_{-1+5I_1-I_2}^{1-5I_1+I_2} \int_{0}^{1+5I_1+3I_2-x^2} dydx = \int_{-1+5I_1-I_2}^{1-5I_1+I_2} y|_0^{1+5I_1+3I_2-x^2} dx$$

$$= \int_{-1+5I_1-I_2}^{1-5I_1+I_2} (1 + 5I_1 + 3I_2 - x^2) dx$$

$$= \left((1 + 5I_1 + 3I_2)x - \frac{x^3}{3} \right) \Bigg|_{-1+5I_1-I_2}^{1-5I_1+I_2}$$

$$= \left[(1 + 5I_1 + 3I_2)(1 - 5I_1 + I_2) - \frac{(1 - 5I_1 + I_2)^3}{3} \right] - \left[(1 + 5I_1 + 3I_2)(-1 + 5I_1 - I_2) - \frac{(-1 + 5I_1 - I_2)^3}{3} \right]$$

$$= 2 \left[(1 + 5I_1 + 3I_2)(1 - 5I_1 + I_2) - \frac{(1 - 5I_1 + I_2)^3}{3} \right]$$

$$= 2 \left[1 - 35I_1 + 7I_2 - \left(\frac{1 - 35I_1 + 7I_2}{3} \right) \right]$$

$$= \frac{4}{3} - \frac{140}{3} I_1 + \frac{28}{3} I_2 \ (rejected)$$

3. Conclusions

The paper presented an important study on the double integrals of 2- refined neutrosophic functions and its applications, where it presented a set of theorems in the double integrals of 2-refined neutrosophic functions, discussed several methods to calculate these integrals using the vertical slice or the horizontal slice, where we obtained the same results in both methods. Finally, we presented applications of the double integrals of 2- refined neutrosophic functions to calculate the area of region, and concluded that we take positive values by using the definition of the positive 2-refined neutrosophic number.

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Quality Review of Computer Digital Media Art Design using Neutrosophic Logic: A Case Study

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Abstract: Evaluating the quality of Computer Digital Media Art Design is complex because it involves both subjective artistic elements and objective technical criteria. This study uses the multi-criteria decision making (MCDM) method to deal with different criteria. Two MCDM methods are used in this study such as Preference Selection Index (PSI) Method to compute the criteria weights and the RATGOS method to rank the alternatives. These methods are used neutrosophic sets to dela with vague and uncertainty information. Eight criteria and seven alternatives are used to select the best alternatives. Four experts have evaluated these criteria and alternatives. An example is conducted to show the validation of the proposed approach. The sensitivity analysis and comparative analysis are applied in this study to show the stability of the ranks and the effectiveness of the proposed approach.

Keywords: Trapezoidal interval valued neutrosophic set (TIVNS); Computer Digital Media; Art Design; Neutrosophic Logic.

1. Introduction and Related Works

Computer Digital Media Art Design is a multidisciplinary field that blends artistic creativity with digital technology to create visually compelling and interactive media content. It encompasses graphic design, animation, visual effects (VFX), digital painting, 3D modeling, augmented reality (AR), virtual reality (VR), and interactive media. This field is widely applied in advertising, film, gaming, web design, user experience (UX) design, and immersive technologies[1], [2].

With the rise of digital transformation, AI-assisted design tools, real-time rendering engines, and interactive platforms have further expanded the capabilities of digital media artists. These technological advancements have made digital media art design more dynamic, immersive, and accessible across multiple devices and platforms.

Evaluating the quality of Computer Digital Media Art Design is complex because it involves both subjective artistic elements and objective technical criteria. Traditional evaluation methods may rely on human judgment, which can be biased and inconsistent[3], [4]. Therefore, a structured approach such as Multi-Criteria Decision-Making (MCDM) is beneficial for quality assessment.

Key challenges include:

- Creativity & Subjectivity: Art and aesthetics are difficult to quantify.
- Technical Complexity: Requires knowledge of digital tools and techniques.
- Cross-Platform Adaptability: Digital art must perform well across different screens, resolutions, and interfaces.
- Industry Relevance: Trends in digital media change rapidly, requiring continuous updates.
- User Experience & Interaction: Evaluating how well the design engages users.

For evaluating Computer Digital Media Art Design, MCDM methods can systematically analyze both qualitative and quantitative factors, ensuring a more objective and reliable decision-making process. In 1965, the term fuzzy set (FS) was used to describe a set whose components have degrees of membership. It primarily addresses a variety of real-world scenarios where the data has some degree of ambiguity[5], [6].

Only the components' membership value—not their non-membership value—is addressed by the FS notion. Atannasov's 1975 introduction of intuitionistic FS (IFS), which permits both the membership function (MF) and non-membership function, resolved this problem. Data indeterminacy may be present in real-world scenarios, which is why FS and IFS are unable to handle it. Smarandache's introduction of the neutrosophic set (NS), a generalization of FS and IFS, solved this issue.

All the components in NS have varying degrees of membership, indeterminacy, and non-membership, and the sum of these MFs must be less than or equal to three[7], [8]. Each of the three MFs operates independently of the others. Wang et al. created single valued NSs since the NSs are hard to use in practical situations. Fuzzy numbers and intuitionistic fuzzy numbers can be used to represent elemental uncertainty. Neutroposophic numbers are also highly helpful in expressing the components' indeterminacy and uncertainty. As a result, the NS's enhancement of the real number domain to neutrosophic numbers is a specific case.

Numerous approaches, including FS, IFS, interval valued IFS, triangular IFS, trapezoidal IFS, and NS, have been suggested by researchers to address inconsistency, uncertainty, ambiguity, impreciseness, and indeterminacy. In each of these environments, the information can be represented as a triangle or a trapezoid. Additionally, the membership values fall within the range of [0, 1] for real units. Therefore, in real-world situations when the information is

ambiguous and indeterminate within a few ranges of permitted behavior, the trapezoidal interval valued neutrosophic number (TIVNN) is helpful. TIVNN is therefore the key to extracting the MFs of truth, indeterminacy, and falsehood, whose values rely on the intervals as well as the trapezoidal neutrosophic numbers.

Decision-makers find it difficult to articulate their opinions and judgments with a precise number for a single valued membership degree in a neutrophilic setting because of the intricacy of the decision-making process. Despite being a unique case of the neutrosophic, the interval number did not provide a definitive solution to the MCDM issue. The score function and accuracy function of trapezoidal interval valued neutrosophic numbers, together with their illustrative qualities, have thus been described by Broumi et al. [9]. These characteristics provide the trapezoidal interval valued neutrosophic number a crucial theoretical foundation. To address the neutrosophic MCDM issue in network analysis, they also suggested a clever technique known as the trapezoidal interval valued neutrosophic variant of Bellman's algorithm. Additionally, a comparison with the current algorithm has been conducted.

It has been shown that a single valued neutrosophic set may encompass the indeterminacy in real-world decision-making problems. It is an expansion of the neutrosophic set with interval values. Most real-world issues contain some degree of uncertainty, and determining the network's MCDM issue is one of the most well-known examples. In study of Broumi et al [10], interval valued neutrosophic numbers are used to solve MCDM issue and a new scoring function for these numbers is provided. By considering interval-valued neutrosophic numbers, trapezoidal, and triangular interval-valued neutrosophic numbers with an illustrated example, new methods are also presented to determine the neutrosophic MCDM. A comparison between the suggested algorithm and the current approach has been conducted.

The interval-valued intuitionistic fuzzy number (IVIFN), the trapezoidal fuzzy number (TrFN), the neutrosophic set and its operational laws, and the trapezoidal neutrosophic set (TrNS) and its operational laws are all covered in Kiran Khatter [11]. A suggested interval valued trapezoidal neutrosophic set (IVTrNS) and its operating rules are based on the combination of IVIFN and TrNS. The accuracy and scoring functions for the suggested interval valued trapezoidal neutrosophic number (IVTrNN) are also included. Then, to aggregate the neutrosophic trapezoidal information in the unit interval of real numbers, an interval valued trapezoidal neutrosophic operator is introduced. Following a numerical example of NFRs prioritizing a technique is finally devised to address the issues in the MCDM environment.

2. Trapezoidal interval valued neutrosophic set (TIVNS)

The neutrosophic set can be defined as $A = \left(x: T_A(x), I_A(x), F_A(x)\right)$, where T, I, F refer to the truth, indeterminacy, and falsity membership functions where satisfy the condition:[9]

$$-0 \leq T_A(x) + T_A(x) + T_A(x) \leq 3 + \quad (1)$$

TIVNN with truth, indeterminacy, and falsity

$$T_x(Z) = \begin{cases} \frac{(Z-a)t_x}{(b-a)}, & a \le z < b \\ t_x, & b \le z < c \\ \frac{(d-z)t_x}{(d-c)}, & c \le z < d \\ 0, & otherwise \end{cases} \quad (2)$$

$$i_x(Z) = \begin{cases} \frac{(b-z)+(z-a)i_x}{(b-a)}, & a \le z < b \\ i_x, & b \le z < c \\ \frac{(z-c)+(d-z)i_x}{(d-c)}, & c \le z < d \\ 0, & otherwise \end{cases} \quad (3)$$

$$f_x(Z) = \begin{cases} \frac{(b-z)+(z-a)f_x}{(b-a)}, & a \le z < b \\ f_x, & b \le z < c \\ \frac{(z-c)+(d-z)i_x}{(d-c)}, & c \le z < d \\ 0, & otherwise \end{cases} \quad (4)$$

The score function can be defined as:

$$S(x) = \frac{1}{16}(a+b+c+d)(t_- - t^- - i_- - i^- - f_- - f^-) \quad (5)$$

Where $S(x) \in [0,1]$ and $0 \le a \le b \le c \le d \le 1, t_x, i_x, f_x$ are $subset$ of $[0,1]$

$where$ $t_x = [t_-, t^-], i_x = [i_-, i^-], f_x = [f_-, f^-]$

Ranking Method

Let two TIVNNs such as a and b, the ranking of the two numbers by the score and accuracy function such as:

$$if\ s(r^N) < s(s^N)\ then\ r^N < s^N \quad (6)$$

$$if\ s(r^N) = s(s^N)\ and\ if \quad (7)$$

$$a(r^N) < a(s^N)\ then\ r^N < s^N \quad (8)$$

$$a(r^N) > a(s^N)\ then\ r^N > s^N \quad (9)$$

$$a(r^N) = a(s^N)\ then\ r^N = s^N \quad (10)$$

3. Materials and Methods

This section shows the steps of the proposed approach under the neutrosophic sets. We used two MCDM methods such as PSI to compute the criteria weights and RATGOS method to rank the alternatives.

TIVNNs- Preference Selection Index (PSI)

We use the PSI method to compute the criteria weights[12], [13].

a) Preparing the decision matrix.

b) Normalizing the decision matrix.

The normalized decision matrix is computed for the positive and cost criteria.

$$B_{ij} = \frac{x_{ij}}{\max x_{ij}} \tag{11}$$

$$B_{ij} = \frac{\min x_{ij}}{x_{ij}} \tag{12}$$

c) Compute the mean value of each criterion.

$$E = \frac{1}{N}\sum_{i=1}^{m} B_{ij} \tag{13}$$

d) Compute the preference variation value

$$Q_j = \sum_{i=1}^{m}\left[B_{ij} - E\right]^2 \tag{14}$$

e) Compute the deviation of preference value

$$U_j - 1 - Q_j \tag{15}$$

f) Compute the criteria weights.

$$w_j = \frac{U_j}{\sum_{j=1}^{n} U_j} \tag{16}$$

TIVNNs-RATGOS

This part shows the rank of the alternatives under the TIVNNs to deal with uncertainty and vague information[14], [15].

a) Compute the optimal value

The optimal value of each criterion is computed for positive and cost criteria such as:

$$Optimal = \max x_j \tag{17}$$

$$Optimal = \max 1/x_j \tag{18}$$

b) Compute the normalization values for positive and cost value

$$A = \frac{x_{ij}}{optimal} \tag{19}$$

$$A = \frac{optimal}{x_{ij}} \tag{20}$$

c) Compute the weighted normalized decision matrix.

$$R_{ij} = W_j A \tag{21}$$

d) Compute the geometric mean.

f) Rank the alternatives.

4. Results and Discussion

This section shows the results of the proposed approach to obtaining the criteria weights and ranking the alternatives. This study are evaluated using four experts and decision makers. We collected eight criteria and seven alternatives in this study. The criteria of this study are Content Relevance and Message Clarity, Aesthetic Appeal, Creativity and Originality, Market and Industry Relevance, User Experience and Interactivity, Adaptability and Cross-Platform Compatibility, Efficiency of Workflow and Production Process, Technical Proficiency. The alternatives of this study are: Interactive Media, 3D Modeling, Motion Graphics, UX-Focused Design, Generative Art, AI-Integrated Digital Design, Cross-Platform.

TIVNNs- Preference Selection Index (PSI)

a) We build the decision matrix using the TIVNNs as shown in Tables 1-4. Then we apply the score function. Then we combined the decision matrix.

b) We used Eq. (11) to normalize the decision matrix as shown in Table 5.

c) Then we compute the mean value of each criterion using Eq. (13).

d) Then we compute the preference variation value using Eq. (14).

e) Then we compute the deviation of preference value using Eq. (15).

f) Then we compute the criteria weights using Eq. (16) as shown in Fig 1.

Table 1. The first decision matrix.

	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8
A_1	((0.1,0.2,0.3,0.4);[0.1,0.2],[0.2,0.3],[0.4,0.5])	((0.2,0.5,0.7,0.8);[0.2,0.4],[0.3,0.5],[0.1,0.2])	((0.3,0.7,0.8,0.9);[0.3,0.4],[0.1,0.2],[0.3,0.5])	((0.1,0.5,0.7,0.9);[0.1,0.3],[0.3,0.4],[0.2,0.3])	((0.2,0.4,0.8,0.9);[0.2,0.3],[0.2,0.5],[0.4,0.5])	((0.3,0.4,0.5,1);[0.3,0.6],[0.1,0.2],[0.1,0.4])	((0.7,0.8,0.9,1);[0.4,0.6],[0.2,0.4],[0.1,0.3])	((0.3,0.7,0.8,0.9);[0.3,0.4],[0.1,0.2],[0.3,0.5])
A_2	((0.1,0.2,0.3,0.4);[0.1,0.2],[0.2,0.3],[0.4,0.5])	((0.7,0.8,0.9,1);[0.4,0.6],[0.2,0.4],[0.1,0.3])	((0.3,0.4,0.5,1);[0.3,0.6],[0.1,0.2],[0.1,0.4])	((0.2,0.4,0.8,0.9);[0.2,0.3],[0.2,0.5],[0.4,0.5])	((0.1,0.5,0.7,0.9);[0.1,0.3],[0.3,0.4],[0.2,0.3])	((0.3,0.7,0.8,0.9);[0.3,0.4],[0.1,0.2],[0.3,0.5])	((0.1,0.2,0.3,0.4);[0.1,0.2],[0.2,0.3],[0.4,0.5])	((0.1,0.5,0.7,0.9);[0.1,0.3],[0.3,0.4],[0.2,0.3])
A_3	((0.2,0.5,0.7,0.8);[0.2,0.4],[0.3,0.5],[0.1,0.2])	((0.3,0.7,0.8,0.9);[0.3,0.4],[0.1,0.2],[0.3,0.5])	((0.1,0.5,0.7,0.9);[0.1,0.3],[0.3,0.4],[0.2,0.3])	((0.2,0.4,0.8,0.9);[0.2,0.3],[0.2,0.5],[0.4,0.5])	((0.3,0.4,0.5,1);[0.3,0.6],[0.1,0.2],[0.1,0.4])	((0.2,0.5,0.7,0.8);[0.2,0.4],[0.3,0.5],[0.1,0.2])	((0.2,0.5,0.7,0.8);[0.2,0.4],[0.3,0.5],[0.1,0.2])	((0.2,0.4,0.8,0.9);[0.2,0.3],[0.2,0.5],[0.4,0.5])
A_4	((0.1,0.5,0.7,0.9);[0.1,0.3],[0.3,0.4],[0.2,0.3])	((0.3,0.7,0.8,0.9);[0.3,0.4],[0.1,0.2],[0.3,0.5])	((0.2,0.5,0.7,0.8);[0.2,0.4],[0.3,0.5],[0.1,0.2])	((0.1,0.2,0.3,0.4);[0.1,0.2],[0.2,0.3],[0.4,0.5])	((0.7,0.8,0.9,1);[0.4,0.6],[0.2,0.4],[0.1,0.3])	((0.1,0.2,0.3,0.4);[0.1,0.2],[0.2,0.3],[0.4,0.5])	((0.3,0.7,0.8,0.9);[0.3,0.4],[0.1,0.2],[0.3,0.5])	((0.3,0.4,0.5,1);[0.3,0.6],[0.1,0.2],[0.1,0.4])
A_5	((0.2,0.4,0.8,0.9);[0.2,0.3],[0.2,0.5],[0.4,0.5])	((0.1,0.5,0.7,0.9);[0.1,0.3],[0.3,0.4],[0.2,0.3])	((0.3,0.7,0.8,0.9);[0.3,0.4],[0.1,0.2],[0.3,0.5])	((0.2,0.5,0.7,0.8);[0.2,0.4],[0.3,0.5],[0.1,0.2])	((0.1,0.2,0.3,0.4);[0.1,0.2],[0.2,0.3],[0.4,0.5])	((0.7,0.8,0.9,1);[0.4,0.6],[0.2,0.4],[0.1,0.3])	((0.1,0.5,0.7,0.9);[0.1,0.3],[0.3,0.4],[0.2,0.3])	((0.7,0.8,0.9,1);[0.4,0.6],[0.2,0.4],[0.1,0.3])
A_6	((0.2,0.4,0.8,0.9);[0.2,0.3],[0.2,0.5],[0.4,0.5])	((0.2,0.4,0.8,0.9);[0.2,0.3],[0.2,0.5],[0.4,0.5])	((0.1,0.5,0.7,0.9);[0.1,0.3],[0.3,0.4],[0.2,0.3])	((0.2,0.4,0.8,0.9);[0.2,0.3],[0.2,0.5],[0.4,0.5])	((0.7,0.8,0.9,1);[0.4,0.6],[0.2,0.4],[0.1,0.3])	((0.3,0.4,0.5,1);[0.3,0.6],[0.1,0.2],[0.1,0.4])	((0.2,0.4,0.8,0.9);[0.2,0.3],[0.2,0.5],[0.4,0.5])	((0.3,0.4,0.5,1);[0.3,0.6],[0.1,0.2],[0.1,0.4])
A_7	((0.3,0.4,0.5,1);[0.3,0.6],[0.1,0.2],[0.1,0.4])	((0.3,0.4,0.5,1);[0.3,0.6],[0.1,0.2],[0.1,0.4])	((0.3,0.7,0.8,0.9);[0.3,0.4],[0.1,0.2],[0.3,0.5])	((0.3,0.4,0.5,1);[0.3,0.6],[0.1,0.2],[0.1,0.4])	((0.3,0.4,0.5,1);[0.3,0.6],[0.1,0.2],[0.1,0.4])	((0.2,0.4,0.8,0.9);[0.2,0.3],[0.2,0.5],[0.4,0.5])	((0.3,0.4,0.5,1);[0.3,0.6],[0.1,0.2],[0.1,0.4])	((0.2,0.4,0.8,0.9);[0.2,0.3],[0.2,0.5],[0.4,0.5])

Table 2. The second decision matrix.

	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8

A_1	((0.1,0.5,0.7,0.9);[0.1,0.3],[0.3,0.4],[0.2,0.3])	((0.2,0.5,0.7,0.8);[0.2,0.4],[0.3,0.5],[0.1,0.2])	((0.3,0.7,0.8,0.9);[0.3,0.4],[0.1,0.2],[0.3,0.5])	((0.1,0.5,0.7,0.9);[0.1,0.3],[0.3,0.4],[0.2,0.3])	((0.2,0.4,0.8,0.9);[0.2,0.3],[0.2,0.5],[0.4,0.5])	((0.3,0.4,0.5,1);[0.3,0.6],[0.1,0.2],[0.1,0.4])	((0.7,0.8,0.9,1);[0.4,0.6],[0.2,0.4],[0.1,0.3])	((0.3,0.7,0.8,0.9);[0.3,0.4],[0.1,0.2],[0.3,0.5])
A_2	((0.3,0.7,0.8,0.9);[0.3,0.4],[0.1,0.2],[0.3,0.5])	((0.7,0.8,0.9,1);[0.4,0.6],[0.2,0.4],[0.1,0.3])	((0.3,0.4,0.5,1);[0.3,0.6],[0.1,0.2],[0.1,0.4])	((0.1,0.5,0.7,0.9);[0.1,0.3],[0.3,0.4],[0.2,0.3])	((0.1,0.5,0.7,0.9);[0.1,0.3],[0.3,0.4],[0.2,0.3])	((0.3,0.7,0.8,0.9);[0.3,0.4],[0.1,0.2],[0.3,0.5])	((0.1,0.5,0.7,0.9);[0.1,0.3],[0.3,0.4],[0.2,0.3])	((0.1,0.5,0.7,0.9);[0.1,0.3],[0.3,0.4],[0.2,0.3])
A_3	((0.2,0.5,0.7,0.8);[0.2,0.4],[0.3,0.5],[0.1,0.2])	((0.3,0.7,0.8,0.9);[0.3,0.4],[0.1,0.2],[0.3,0.5])	((0.1,0.5,0.7,0.9);[0.1,0.3],[0.3,0.4],[0.2,0.3])	((0.3,0.7,0.8,0.9);[0.3,0.4],[0.1,0.2],[0.3,0.5])	((0.1,0.5,0.7,0.9);[0.1,0.3],[0.3,0.4],[0.2,0.3])	((0.2,0.5,0.7,0.8);[0.2,0.4],[0.3,0.5],[0.1,0.2])	((0.3,0.7,0.8,0.9);[0.3,0.4],[0.1,0.2],[0.3,0.5])	((0.2,0.4,0.8,0.9);[0.2,0.3],[0.2,0.5],[0.4,0.5])
A_4	((0.1,0.2,0.3,0.4);[0.1,0.2],[0.2,0.3],[0.4,0.5])	((0.3,0.7,0.8,0.9);[0.3,0.4],[0.1,0.2],[0.3,0.5])	((0.2,0.5,0.7,0.8);[0.2,0.4],[0.3,0.5],[0.1,0.2])	((0.2,0.5,0.7,0.8);[0.2,0.4],[0.3,0.5],[0.1,0.2])	((0.3,0.7,0.8,0.9);[0.3,0.4],[0.1,0.2],[0.3,0.5])	((0.1,0.2,0.3,0.4);[0.1,0.2],[0.2,0.3],[0.4,0.5])	((0.2,0.5,0.7,0.8);[0.2,0.4],[0.3,0.5],[0.1,0.2])	((0.3,0.4,0.5,1);[0.3,0.6],[0.1,0.2],[0.1,0.4])
A_5	((0.7,0.8,0.9,1);[0.4,0.6],[0.2,0.4],[0.1,0.3])	((0.1,0.5,0.7,0.9);[0.1,0.3],[0.3,0.4],[0.2,0.3])	((0.3,0.7,0.8,0.9);[0.3,0.4],[0.1,0.2],[0.3,0.5])	((0.1,0.2,0.3,0.4);[0.1,0.2],[0.2,0.3],[0.4,0.5])	((0.2,0.5,0.7,0.8);[0.2,0.4],[0.3,0.5],[0.1,0.2])	((0.7,0.8,0.9,1);[0.4,0.6],[0.2,0.4],[0.1,0.3])	((0.1,0.2,0.3,0.4);[0.1,0.2],[0.2,0.3],[0.4,0.5])	((0.2,0.4,0.8,0.9);[0.2,0.3],[0.2,0.5],[0.4,0.5])
A_6	((0.1,0.5,0.7,0.9);[0.1,0.3],[0.3,0.4],[0.2,0.3])	((0.1,0.5,0.7,0.9);[0.1,0.3],[0.3,0.4],[0.2,0.3])	((0.1,0.5,0.7,0.9);[0.1,0.3],[0.3,0.4],[0.2,0.3])	((0.7,0.8,0.9,1);[0.4,0.6],[0.2,0.4],[0.1,0.3])	((0.1,0.2,0.3,0.4);[0.1,0.2],[0.2,0.3],[0.4,0.5])	((0.1,0.5,0.7,0.9);[0.1,0.3],[0.3,0.4],[0.2,0.3])	((0.7,0.8,0.9,1);[0.4,0.6],[0.2,0.4],[0.1,0.3])	((0.3,0.4,0.5,1);[0.3,0.6],[0.1,0.2],[0.1,0.4])
A_7	((0.3,0.7,0.8,0.9);[0.3,0.4],[0.1,0.2],[0.3,0.5])	((0.3,0.7,0.8,0.9);[0.3,0.4],[0.1,0.2],[0.3,0.5])	((0.3,0.7,0.8,0.9);[0.3,0.4],[0.1,0.2],[0.3,0.5])	((0.3,0.4,0.5,1);[0.3,0.6],[0.1,0.2],[0.1,0.4])	((0.1,0.5,0.7,0.9);[0.1,0.3],[0.3,0.4],[0.2,0.3])	((0.3,0.7,0.8,0.9);[0.3,0.4],[0.1,0.2],[0.3,0.5])	((0.3,0.4,0.5,1);[0.3,0.6],[0.1,0.2],[0.1,0.4])	((0.2,0.4,0.8,0.9);[0.2,0.3],[0.2,0.5],[0.4,0.5])

Table 3. The third decision matrix.

	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8
A_1	((0.1,0.2,0.3,0.4);[0.1,0.2],[0.2,0.3],[0.4,0.5])	((0.2,0.5,0.7,0.8);[0.2,0.4],[0.3,0.5],[0.1,0.2])	((0.3,0.7,0.8,0.9);[0.3,0.4],[0.1,0.2],[0.3,0.5])	((0.1,0.5,0.7,0.9);[0.1,0.3],[0.3,0.4],[0.2,0.3])	((0.2,0.4,0.8,0.9);[0.2,0.3],[0.2,0.5],[0.4,0.5])	((0.3,0.4,0.5,1);[0.3,0.6],[0.1,0.2],[0.1,0.4])	((0.7,0.8,0.9,1);[0.4,0.6],[0.2,0.4],[0.1,0.3])	((0.3,0.7,0.8,0.9);[0.3,0.4],[0.1,0.2],[0.3,0.5])
A_2	((0.7,0.8,0.9,1);[0.4,0.6],[0.2,0.4],[0.1,0.3])	((0.7,0.8,0.9,1);[0.4,0.6],[0.2,0.4],[0.1,0.3])	((0.3,0.4,0.5,1);[0.3,0.6],[0.1,0.2],[0.1,0.4])	((0.2,0.4,0.8,0.9);[0.2,0.3],[0.2,0.5],[0.4,0.5])	((0.1,0.5,0.7,0.9);[0.1,0.3],[0.3,0.4],[0.2,0.3])	((0.3,0.7,0.8,0.9);[0.3,0.4],[0.1,0.2],[0.3,0.5])	((0.1,0.2,0.3,0.4);[0.1,0.2],[0.2,0.3],[0.4,0.5])	((0.1,0.5,0.7,0.9);[0.1,0.3],[0.3,0.4],[0.2,0.3])
A_3	((0.3,0.4,0.5,1);[0.3,0.6],[0.1,0.2],[0.1,0.4])	((0.1,0.2,0.3,0.4);[0.1,0.2],[0.2,0.3],[0.4,0.5])	((0.1,0.5,0.7,0.9);[0.1,0.3],[0.3,0.4],[0.2,0.3])	((0.2,0.4,0.8,0.9);[0.2,0.3],[0.2,0.5],[0.4,0.5])	((0.3,0.4,0.5,1);[0.3,0.6],[0.1,0.2],[0.1,0.4])	((0.2,0.5,0.7,0.8);[0.2,0.4],[0.3,0.5],[0.1,0.2])	((0.7,0.8,0.9,1);[0.4,0.6],[0.2,0.4],[0.1,0.3])	((0.2,0.4,0.8,0.9);[0.2,0.3],[0.2,0.5],[0.4,0.5])
A_4	((0.2,0.4,0.8,0.9);[0.2,0.3],[0.2,0.5],[0.4,0.5])	((0.7,0.8,0.9,1);[0.4,0.6],[0.2,0.4],[0.1,0.3])	((0.1,0.2,0.3,0.4);[0.1,0.2],[0.2,0.3],[0.4,0.5])	((0.1,0.2,0.3,0.4);[0.1,0.2],[0.2,0.3],[0.4,0.5])	((0.1,0.2,0.3,0.4);[0.1,0.2],[0.2,0.3],[0.4,0.5])	((0.1,0.2,0.3,0.4);[0.1,0.2],[0.2,0.3],[0.4,0.5])	((0.3,0.4,0.5,1);[0.3,0.6],[0.1,0.2],[0.1,0.4])	((0.3,0.4,0.5,1);[0.3,0.6],[0.1,0.2],[0.1,0.4])
A_5	((0.1,0.5,0.7,0.9);[0.1,0.3],[0.3,0.4],[0.2,0.3])	((0.3,0.4,0.5,1);[0.3,0.6],[0.1,0.2],[0.1,0.4])	((0.7,0.8,0.9,1);[0.4,0.6],[0.2,0.4],[0.1,0.3])	((0.1,0.2,0.3,0.4);[0.1,0.2],[0.2,0.3],[0.4,0.5])	((0.7,0.8,0.9,1);[0.4,0.6],[0.2,0.4],[0.1,0.3])	((0.1,0.2,0.3,0.4);[0.1,0.2],[0.2,0.3],[0.4,0.5])	((0.2,0.4,0.8,0.9);[0.2,0.3],[0.2,0.5],[0.4,0.5])	((0.7,0.8,0.9,1);[0.4,0.6],[0.2,0.4],[0.1,0.3])
A_6	((0.3,0.7,0.8,0.9);[0.3,0.4],[0.1,0.2],[0.3,0.5])	((0.2,0.4,0.8,0.9);[0.2,0.3],[0.2,0.5],[0.4,0.5])	((0.3,0.4,0.5,1);[0.3,0.6],[0.1,0.2],[0.1,0.4])	((0.7,0.8,0.9,1);[0.4,0.6],[0.2,0.4],[0.1,0.3])	((0.3,0.4,0.5,1);[0.3,0.6],[0.1,0.2],[0.1,0.4])	((0.7,0.8,0.9,1);[0.4,0.6],[0.2,0.4],[0.1,0.3])	((0.1,0.5,0.7,0.9);[0.1,0.3],[0.3,0.4],[0.2,0.3])	((0.3,0.4,0.5,1);[0.3,0.6],[0.1,0.2],[0.1,0.4])
A_7	((0.2,0.5,0.7,0.8);[0.2,0.4],[0.3,0.5],[0.1,0.2])	((0.1,0.5,0.7,0.9);[0.1,0.3],[0.3,0.4],[0.2,0.3])	((0.2,0.4,0.8,0.9);[0.2,0.3],[0.2,0.5],[0.4,0.5])	((0.3,0.4,0.5,1);[0.3,0.6],[0.1,0.2],[0.1,0.4])	((0.2,0.4,0.8,0.9);[0.2,0.3],[0.2,0.5],[0.4,0.5])	((0.3,0.4,0.5,1);[0.3,0.6],[0.1,0.2],[0.1,0.4])	((0.3,0.7,0.8,0.9);[0.3,0.4],[0.1,0.2],[0.3,0.5])	((0.2,0.4,0.8,0.9);[0.2,0.3],[0.2,0.5],[0.4,0.5])

Table 4. The fourth decision matrix.

	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8
A_1	((0.3,0.4,0.5,1);[0.3,0.6],[0.1,0.2],[0.1,0.4])	((0.7,0.8,0.9,1);[0.4,0.6],[0.2,0.4],[0.1,0.3])	((0.2,0.5,0.7,0.8);[0.2,0.4],[0.3,0.5],[0.1,0.2])	((0.7,0.8,0.9,1);[0.4,0.6],[0.2,0.4],[0.1,0.3])	((0.2,0.4,0.8,0.9);[0.2,0.3],[0.2,0.5],[0.4,0.5])	((0.1,0.5,0.7,0.9);[0.1,0.3],[0.3,0.4],[0.2,0.3])	((0.7,0.8,0.9,1);[0.4,0.6],[0.2,0.4],[0.1,0.3])	((0.2,0.5,0.7,0.8);[0.2,0.4],[0.3,0.5],[0.1,0.2])
A_2	((0.7,0.8,0.9,1);[0.4,0.6],[0.2,0.4],[0.1,0.3])	((0.1,0.2,0.3,0.4);[0.1,0.2],[0.2,0.3],[0.4,0.5])	((0.1,0.2,0.3,0.4);[0.1,0.2],[0.2,0.3],[0.4,0.5])	((0.1,0.2,0.3,0.4);[0.1,0.2],[0.2,0.3],[0.4,0.5])	((0.1,0.5,0.7,0.9);[0.1,0.3],[0.3,0.4],[0.2,0.3])	((0.3,0.7,0.8,0.9);[0.3,0.4],[0.1,0.2],[0.3,0.5])	((0.1,0.2,0.3,0.4);[0.1,0.2],[0.2,0.3],[0.4,0.5])	((0.7,0.8,0.9,1);[0.4,0.6],[0.2,0.4],[0.1,0.3])
A_3	((0.7,0.8,0.9,1);[0.4,0.6],[0.2,0.4],[0.1,0.3])	((0.2,0.5,0.7,0.8);[0.2,0.4],[0.3,0.5],[0.1,0.2])	((0.7,0.8,0.9,1);[0.4,0.6],[0.2,0.4],[0.1,0.3])	((0.2,0.5,0.7,0.8);[0.2,0.4],[0.3,0.5],[0.1,0.2])	((0.3,0.7,0.8,0.9);[0.3,0.4],[0.1,0.2],[0.3,0.5])	((0.2,0.5,0.7,0.8);[0.2,0.4],[0.3,0.5],[0.1,0.2])	((0.2,0.5,0.7,0.8);[0.2,0.4],[0.3,0.5],[0.1,0.2])	((0.3,0.7,0.8,0.9);[0.3,0.4],[0.1,0.2],[0.3,0.5])
A_4	((0.2,0.5,0.7,0.8);[0.2,0.4],[0.3,0.5],[0.1,0.2])	((0.2,0.5,0.7,0.8);[0.2,0.4],[0.3,0.5],[0.1,0.2])	((0.2,0.5,0.7,0.8);[0.2,0.4],[0.3,0.5],[0.1,0.2])	((0.2,0.4,0.8,0.9);[0.2,0.3],[0.2,0.5],[0.4,0.5])	((0.3,0.7,0.8,0.9);[0.3,0.4],[0.1,0.2],[0.3,0.5])	((0.2,0.5,0.7,0.8);[0.2,0.4],[0.3,0.5],[0.1,0.2])	((0.2,0.4,0.8,0.9);[0.2,0.3],[0.2,0.5],[0.4,0.5])	((0.2,0.4,0.8,0.9);[0.2,0.3],[0.2,0.5],[0.4,0.5])
A_5	((0.1,0.2,0.3,0.4);[0.1,0.2],[0.2,0.3],[0.4,0.5])	((0.1,0.2,0.3,0.4);[0.1,0.2],[0.2,0.3],[0.4,0.5])	((0.1,0.2,0.3,0.4);[0.1,0.2],[0.2,0.3],[0.4,0.5])	((0.1,0.2,0.3,0.4);[0.1,0.2],[0.2,0.3],[0.4,0.5])	((0.2,0.5,0.7,0.8);[0.2,0.4],[0.3,0.5],[0.1,0.2])	((0.1,0.2,0.3,0.4);[0.1,0.2],[0.2,0.3],[0.4,0.5])	((0.2,0.5,0.7,0.8);[0.2,0.4],[0.3,0.5],[0.1,0.2])	((0.1,0.2,0.3,0.4);[0.1,0.2],[0.2,0.3],[0.4,0.5])
A_6	((0.7,0.8,0.9,1);[0.4,0.6],[0.2,0.4],[0.1,0.3])	((0.7,0.8,0.9,1);[0.4,0.6],[0.2,0.4],[0.1,0.3])	((0.7,0.8,0.9,1);[0.4,0.6],[0.2,0.4],[0.1,0.3])	((0.2,0.5,0.7,0.8);[0.2,0.4],[0.3,0.5],[0.1,0.2])	((0.1,0.2,0.3,0.4);[0.1,0.2],[0.2,0.3],[0.4,0.5])	((0.7,0.8,0.9,1);[0.4,0.6],[0.2,0.4],[0.1,0.3])	((0.1,0.2,0.3,0.4);[0.1,0.2],[0.2,0.3],[0.4,0.5])	((0.2,0.5,0.7,0.8);[0.2,0.4],[0.3,0.5],[0.1,0.2])
A_7	((0.1,0.2,0.3,0.4);[0.1,0.2],[0.2,0.3],[0.4,0.5])	((0.3,0.7,0.8,0.9);[0.3,0.4],[0.1,0.2],[0.3,0.5])	((0.1,0.5,0.7,0.9);[0.1,0.3],[0.3,0.4],[0.2,0.3])	((0.2,0.4,0.8,0.9);[0.2,0.3],[0.2,0.5],[0.4,0.5])	((0.1,0.5,0.7,0.9);[0.1,0.3],[0.3,0.4],[0.2,0.3])	((0.2,0.4,0.8,0.9);[0.2,0.3],[0.2,0.5],[0.4,0.5])	((0.2,0.5,0.7,0.8);[0.2,0.4],[0.3,0.5],[0.1,0.2])	((0.7,0.8,0.9,1);[0.4,0.6],[0.2,0.4],[0.1,0.3])

Table 5. The normalized decision matrix.

	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8
A_1	0.481403	0.784038	0.973653	0.775962	0.506732	0.790988	1	0.947552
A_2	1	1	0.883832	0.40485	0.646267	0.82838	0.196324	0.857809
A_3	0.957492	0.602817	0.881437	0.619926	0.991432	0.63279	0.651471	0.613636
A_4	0.473433	0.879812	0.646707	0.377965	1	0.287632	0.526103	0.928322
A_5	0.659405	0.507042	0.978443	0.316289	0.875153	0.738255	0.327574	0.965618
A_6	0.841126	0.637559	1	1	0.809058	1	0.45625	1
A_7	0.698193	0.746479	0.799401	0.839747	0.732558	0.627037	0.619853	0.758159

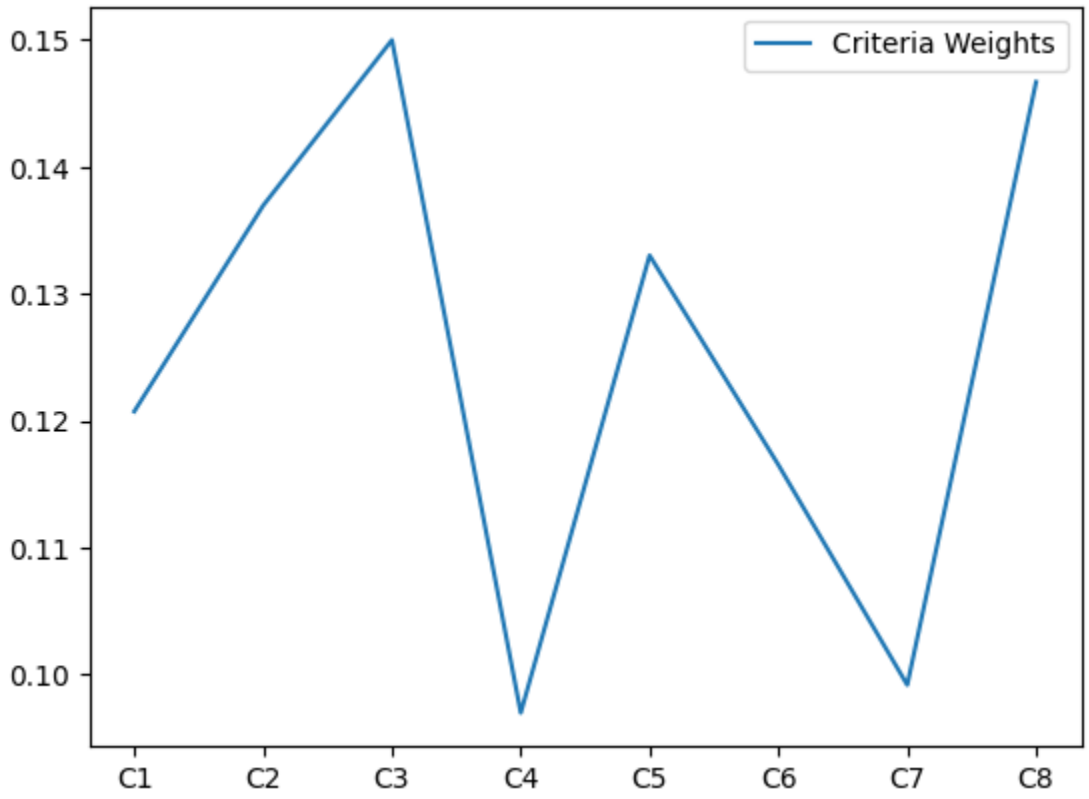

Fig 1. The criteria weights.

TIVNNs-RATGOS

a) We compute the optimal value using Eq. (17).

b) We compute the normalization values for positive and cost value using Eq. (19).

c) Eq. (21) is used to compute the weighted normalized decision matrix as shown in Table 6.

d) Then we compute the geometric mean.

f) Then we rank the alternatives as shown in Fig 2.

Table 6. The weighted normalized decision matrix.

	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8
A_1								
A_2	0.05811643	0.107367	0.146039	0.075247	0.067404	0.092154	0.099169	0.138989
A_3	0.120723091	0.136941	0.132566	0.039259	0.085964	0.09651	0.019469	0.125825
A_4	0.115591397	0.08255	0.132207	0.060116	0.131877	0.073723	0.064606	0.090009
A_5	0.057154237	0.120482	0.097	0.036652	0.133017	0.033511	0.052173	0.136168
A_6	0.079605396	0.069435	0.146757	0.030671	0.11641	0.086011	0.032485	0.141639
A_7	0.101543386	0.087308	0.14999	0.096972	0.107618	0.116505	0.045246	0.146682

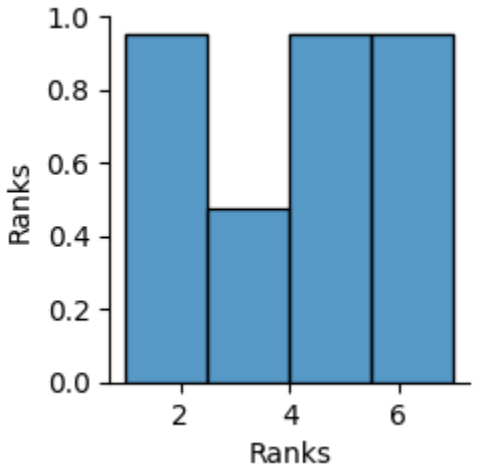

Fig 2. The ranks of alternatives.

Comparative analysis

This study compared the proposed approach with different MCDM methods to show the effectiveness of the proposed approach. The proposed approach is compared with the same weight of the CRITIC method. Then we compared the proposed approach with different MCDM methods with TOPSIS, COPRAS, MARCOS, and MABAC. Fig 3 shows the different ranks of alternatives under comparative analysis.

The results show the proposed approach is effectiveness compared other MCDM methods and the proposed approach has high correlation between other MCDM methods.

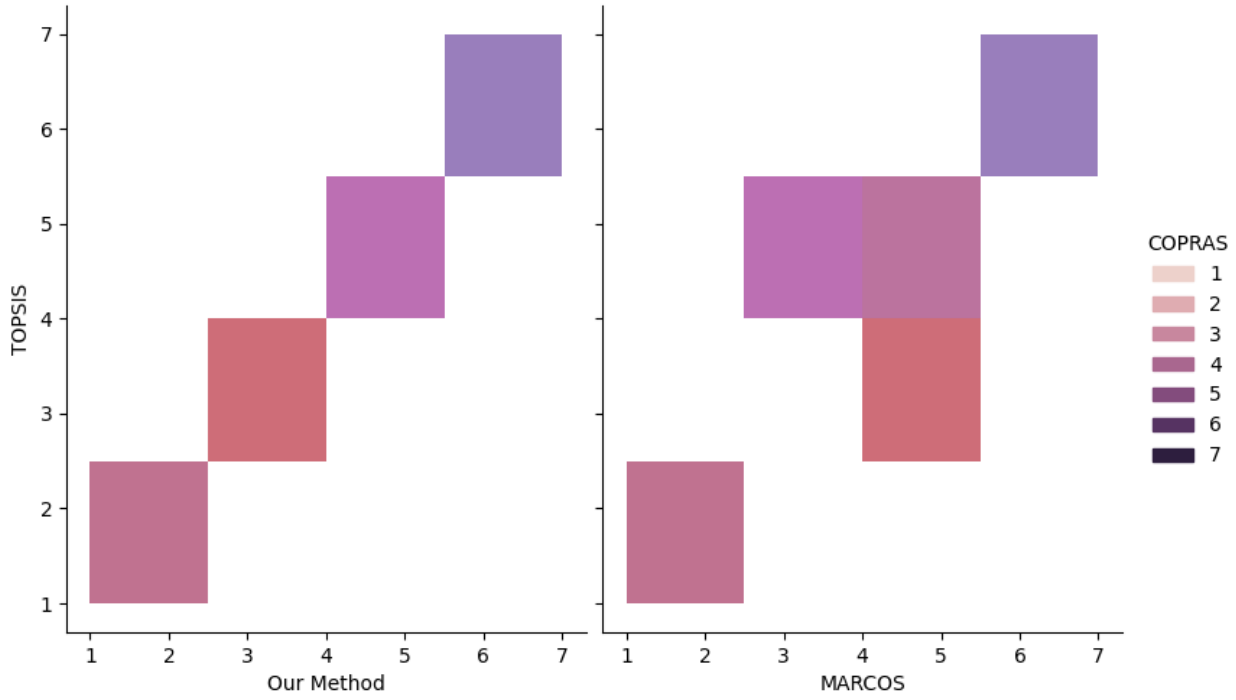

Fig 3. Comparative analysis results.

Sensitivity Analysis

We conducted sensitivity analysis by nine cases in criteria weights. We change the criteria weights by nine cases, then we ranked the alternatives under different criteria weights. Fig 4 shows the different criteria for weights. The first case, we put all criteria with the same weights. Then in the second case, we change the first criterion with 16% weights and all other criteria have ethe same weights. Then in the third case, we change the second criterion with 16% weights and other criteria have the same weights. Fig 5 and 6 shows the nine ranks of alternatives.

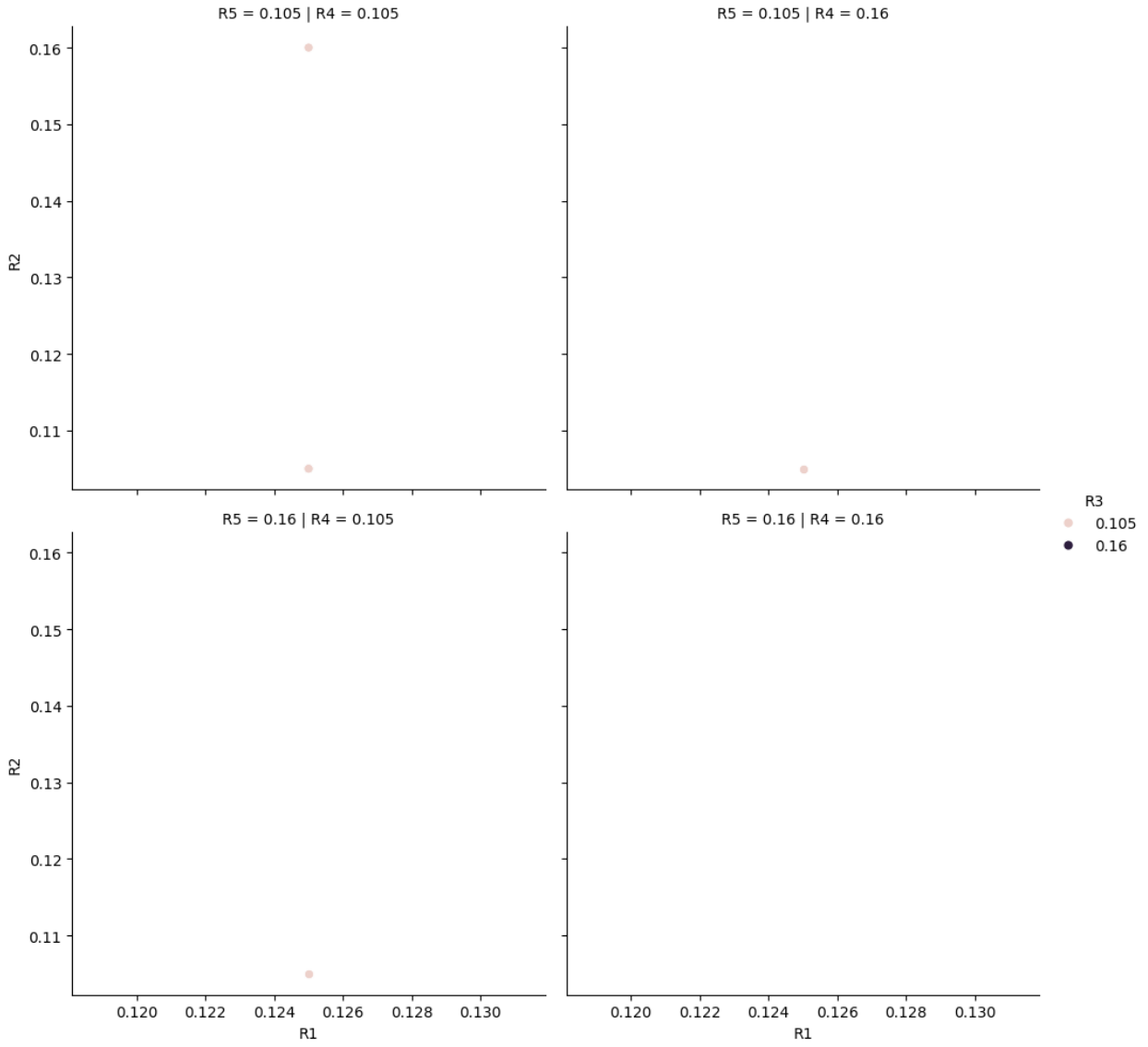

Fig 4-a. The nine criteria weights

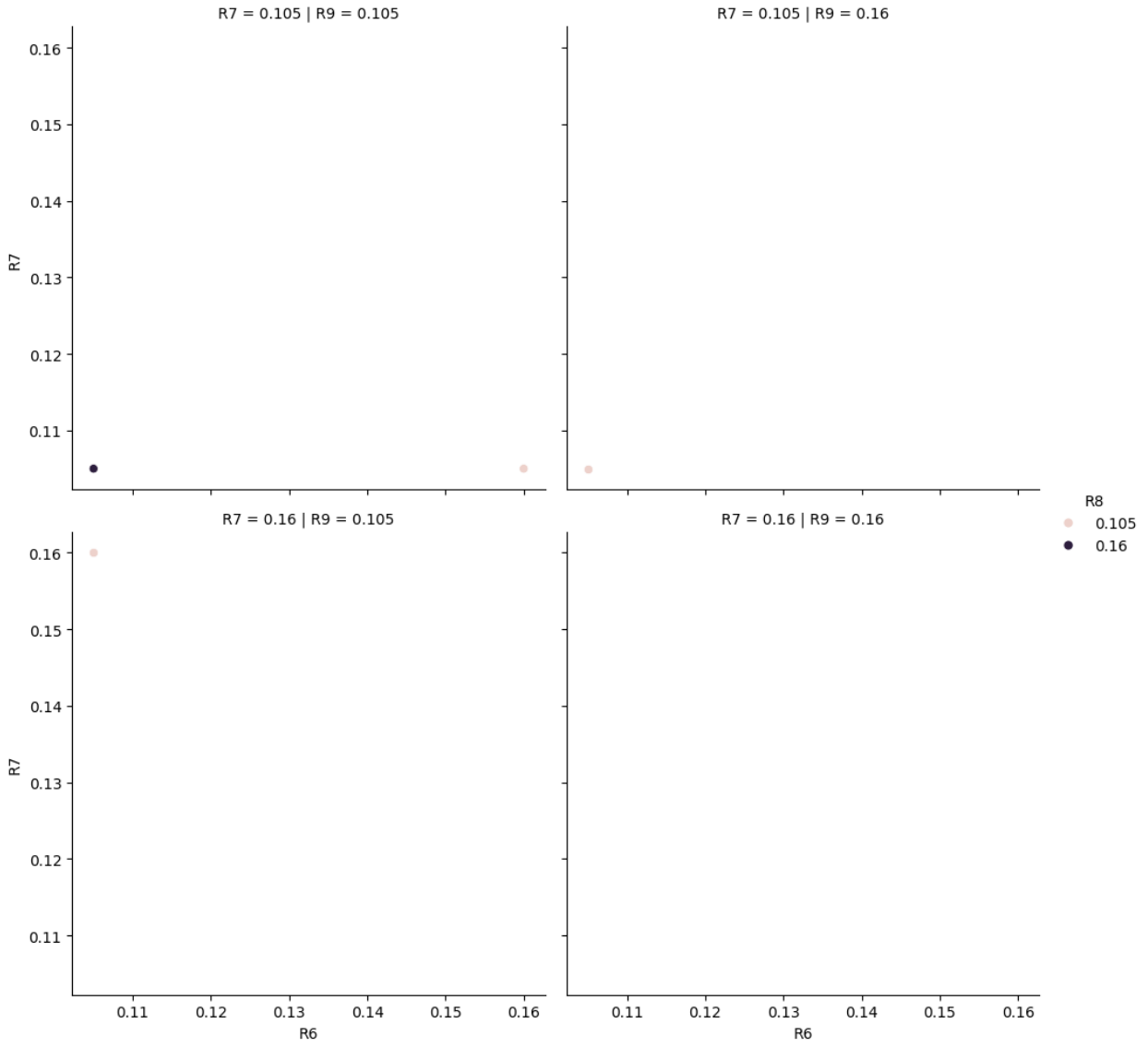

Fig 4-b. The nine criteria weights.

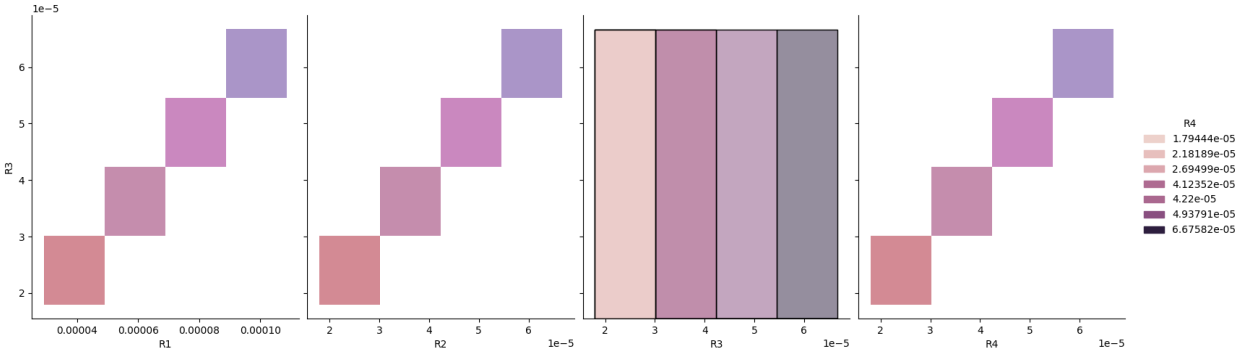

Fig 5-a. The ranking results under different cases.

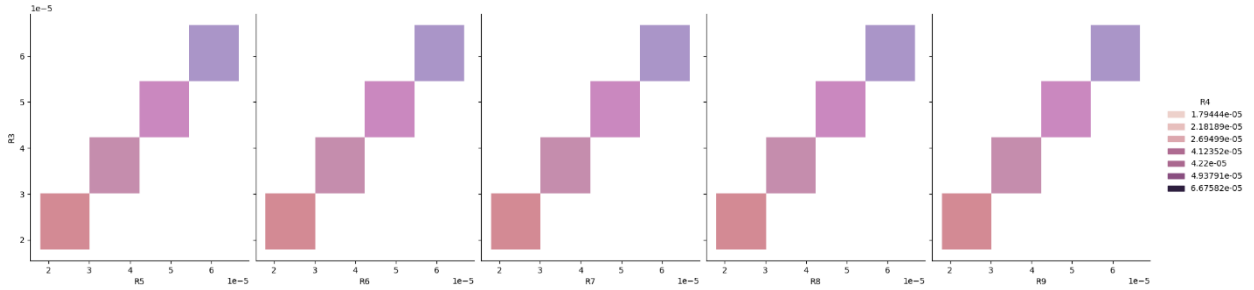

Fig 5-b. The ranking results under different cases.

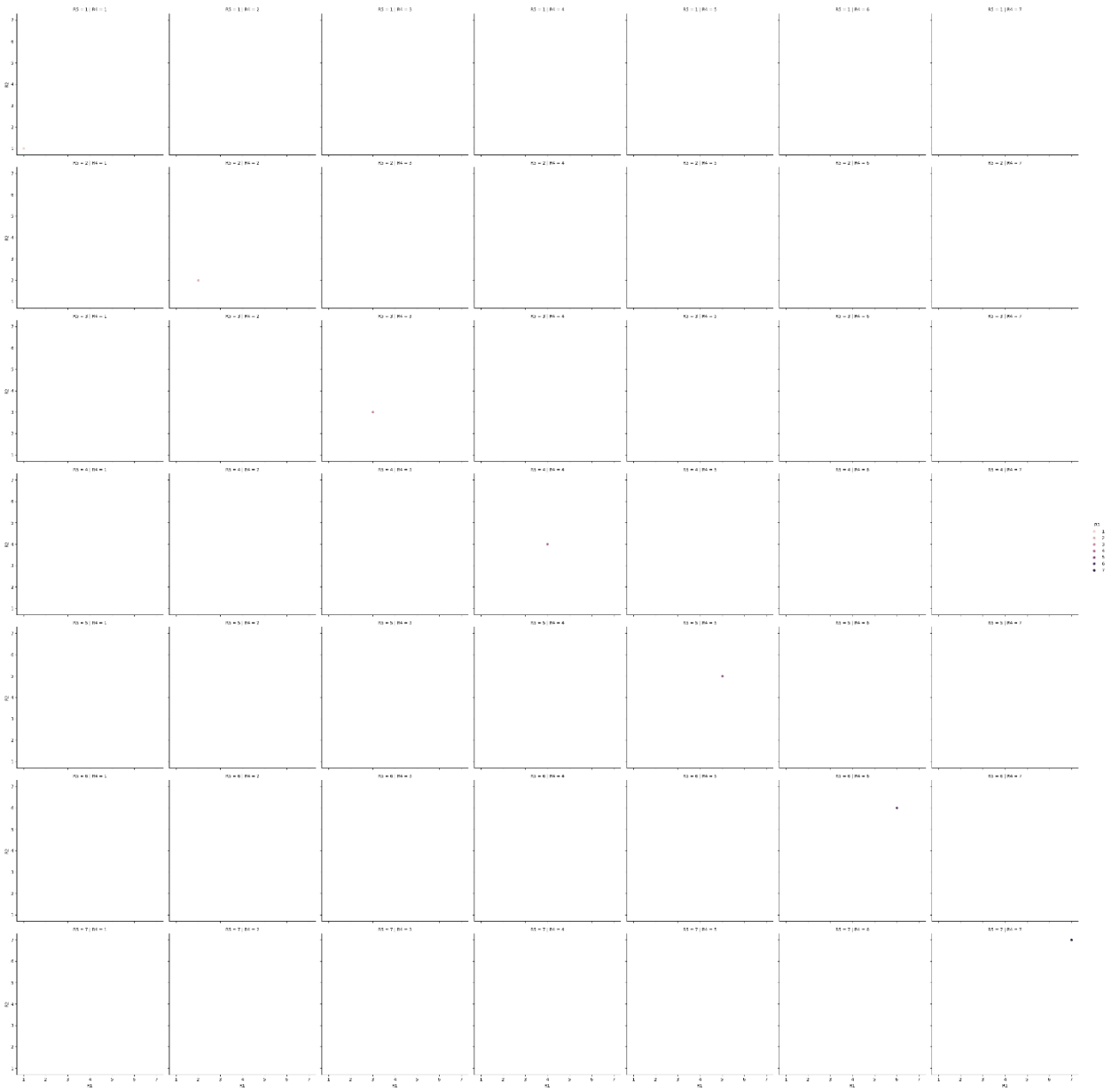

Fig 6. The nine ranks of alternatives.

The results show alternative 4 is the best and alternative 6 is the worst. The results show the ranks of alternatives are stable in different cases.

5. Conclusions and Future Directions

This study proposed MCDM approach to compute the criteria weights and rank the alternatives. The PSI method is used to compute the criteria weights. The RATGOS method is used to rank alternatives. The Trapezoidal interval valued neutrosophic set is used in this study to deal with vague and uncertainty information. Four experts are evaluated the criteria and alternatives. We compared the proposed approach under different MCDM methods. The results show the proposed approach is effective compared with MCDM methods. We conducted sensitivity analysis to show different ranks of alternatives. The results show the ranks of alternatives are stable in different cases.

In the future study, the criteria weights can be applied in different MCDM methods to compute the criteria weights and the ranking method to rank the alternatives. The proposed approach with the neutrosophic method can be extended with different criteria and alternatives to show different dimensions of the MCDM issues.

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Complex Fermatean Neutrosophic Sets and their Applications in Decision Making

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Abstract: Complex Neutrosophic Sets have emerged as a groundbreaking paradigm for handling uncertainty in complex-valued variables. By defining truth, indeterminacy, and falsity membership functions within the unit circle of the complex plane, this innovative framework provides a robust foundation for uncertainty management. This paper introduces a novel hesitance distance measure for Complex Fermatean Neutrosophic Sets, enabling the quantification of uncertainty in complex decision-making scenarios. The proposed distance measure is applied to solve a Multi-Criteria Decision Making Problem, demonstrating its effectiveness in handling uncertainty and ambiguity. This research has far-reaching implications for various fields, including decision theory, artificial intelligence, and operations research. The proposed hesitance distance measure provides a valuable tool for decision-makers to quantify uncertainty and make informed decisions in complex environments.

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Keywords: Neutrosophic Sets, Complex Neutrosophic Sets, Compex Fermatean Neutrosophic Sets

1. Introduction

Fuzzy Sets introduced by Zadeh[1] paved way to deal with problems with uncertainty. Atanassov[2] added a non-membership function to Fuzzy set to coin Intuitionistic fuzzy sets, which helped to solve problems in real life. The concept of neutrosophic sets (NS) and neutrosophic logic was introduced by Smarandache [3] as a generalization of fuzzy sets [1] and intuitionistic fuzzy sets [2].

The concept of Complex Fuzzy Sets was introduced by Ramot et al[4]. This was extended to Complex Intuitionistic Fuzzy Set[5], Complex Pythagorean Fuzzy Set[6] and Complex Neutrosophic Sets[7]. The idea of Fermatean Neutrosophic Fuzzy Sets was proposed by Antony and Jansi[8]. Later, Broumi et al[6], defined Complex Fermatean Neutrosophic sets.

Thangaraj Beaula and Vijaya[9-11] have adopted different fuzzy numbers and ranking methods to solve critical path problems. Vijaya et al[13,14] have used Pythagorean Fuzzy numbers and Neutrosophic Fuzzy numbers to find the solution of Decision making problem and critical path problems. Madad Khan[12] proposed a new distance measure in Complex Decision making problems. Ali, Smarandache, Khan & Ye[15-17] have discussed the applications of Complex Neutrosophic sets. Ranulfo Paiva Barbosa (Sobrinho), & Smarandache, F.[18] introduced Pura Vida Neutrosophic Algebra. Jayasudha and Raghavi[19] studied the properties of Neutrosophic Hypersoft Matrices.

Complex Fermatean Neutrosophic Sets(CFNS) provides a robust framework for handling uncertainty, ambiguity, and imprecision in decision-making problems. CFNS is particularly useful in complex decision scenarios where multiple criteria, constraints, and stakeholders are involved. CFNS can capture non-linear relationships between variables, which is essential in many real-world decision-making problems. CFNS can incorporate human judgment and expertise into the decision-making process, making it more realistic and effective.

CFNS lacks a comprehensive axiomatization, which is essential for establishing a solid theoretical foundation. The properties of CFNS, such as commutativity, associativity, and distributivity, require further investigation. Research on operators, such as union, intersection, and complement, is limited, and more work is needed to develop a comprehensive operator theory. Efficient algorithms for computing CFNS operations, such as membership functions and weighted aggregation, are needed. Research on distance measures for CFNS is limited, and more work is needed to develop effective distance measures. CFNS has limited applications in real-world problems, such as decision-making, optimization, and risk assessment. CFNS-based models, such as decision-making models and optimization models, require further development.

In this paper, we have used Hesitance degree of Neutrosophic Sets to coin the distance measure of Complex Neutrosphic Fermatean Sets. Using the distance measure and Score function, a Multicriteria Decision Making Problem is solved whose parameters are Complex Neutrosophic Fermatean Sets.

2. Preliminaries

Definition 2.1[1]

Let X be a Universal Set. Then a Fuzzy Set $\tilde{A}$ is usually characterized by the membership function $\mu_{\tilde{A}}(x)$, where $\mu_{\tilde{A}}(x): X \to [0,1]$ and it is denoted by $\tilde{A} = \{x, \mu_{\tilde{A}}(x); x \in X\}$

Definition 2.2[3]

Let X be a space of points (objects) with generic elements in X denoted by x; then the neutrosophic set A is an object having the form

$$A=\{< x : T_A(x), I_A(x), F_A(x) > x \in X\}$$

where the functions $T, I, F: X \to]0^-, 1^+[$ define respectively the truth-membership function, an indeterminacy-membership function, and a falsity-membership function of the element $x \in X$ to the set A with, condition $^-0 \le T_A(x) + I_A(x) + F_A(x) \le 3^+$ The functions $T_A(x), I_A(x)$ & $F_A(x)$ are real standard or nonstandard subsets of $]0^-, 1^+[$

Definition 2.3[4]

Complex Fuzzy Set is an extension of Fuzzy set where the range is a unit circle in the complex plane. The membership function is complex valued which is given by $\mu_{\tilde{A}}(x) = r(x)e^{i\theta(x)}$, where $r(x) \in [0,1], \theta(x) \in [0, 2\pi]$

Definition 2.4[7]

A Neutrosophic set can be extended to Complex Neutrosophic Fuzzy Set where the Truth membership, Indeterminacy membership and Falsity membership functions are all complex valued functions and is given by $A = \{x, T_A(x), I_A(x), F_A(x), x \in X\}$ where $T_A(x) = r_A(x)e^{i\theta_1(u)}$, $I_A(x) = s_A(x)e^{i\theta_2(u)}$, $F_A(x) = t_A(x)e^{i\theta_3(u)}$, $0^- \le r_A + s_A + t_A \le 3^+$, $0 \le \theta_1 + \theta_2 + \theta_3 \le 2\pi$

Definition 2.5[8]

A Fermatean Neutrosophic Set A defined on X is represented as

$$A = \{x, T_A(x), I_A(x), F_A(x), x \in X\}$$

with the condition $0^- \le T_A^3 + I_A^3 + F_A^3 \le 2^+$ and $0 \le T_A^3 + F_A^3 \le 1, 0 \le I_A^3 \le 1$

3. Complex Fermatean Neutrosophic Sets(CFNS)

Definition 3.1[6]

A Complex Fermatean Neutrosophic Set A on X is denoted by $A = \{x, T_A(x), I_A(x), F_A(x), x \in X\}$, where $T_A(x) = r_A(x)e^{2\pi i\theta_1(u)}$, $I_A(x) = s_A(x)e^{2\pi i\theta_2(u)}$, $F_A(x) = t_A(x)e^{2\pi i\theta_3(u)}$, represents the truth, indeterminacy and falsity membership functions with the condition $0^- \le r_A^3 + s_A^3 + t_A^3 \le 3^+$, $0^- \le \theta_1^3 + \theta_2^3 + \theta_3^3 \le 3^+$. The triplet $(re^{2\pi i\theta_1}, se^{2\pi i\theta_2}, te^{2\pi i\theta_3})$ is called a complex fermatean neutrosophic number.

3.2 Operations on Complex Fermatean Neutrosophic Sets

If $A = (x, r_1e^{2\pi i\theta_1}, s_1e^{2\pi i\theta_2}, t_1e^{2\pi i\theta_3})$ & $B = (x, r_2e^{2\pi i\phi_1}, s_2e^{2\pi i\phi_2}, t_2e^{2\pi i\phi_3})$ are any two complex fermatean neutrosophic sets, then their union, intersection and complement are defined as follows

Union: $A \cup B = \{x, \max(r_1, r_2)e^{2\pi i \max(\theta_1, \phi_1)}, \max(s_1, s_2)e^{2\pi i \max(\theta_2, \phi_2)}, \max(t_1, t_2)e^{2\pi i \max(\theta_3, \phi_3)})\}$

Intersection: $A \cap B = \{x, \min(r_1, r_2)e^{2\pi i \min(\theta_1, \phi_1)}, \min(s_1, s_2)e^{2\pi i \min(\theta_2, \phi_2)}, \min(t_1, t_2)e^{2\pi i \min(\theta_3, \phi_3)})\}$

Complement: $A^C = \{x, (1-r_1)e^{2\pi i(1-\theta_1)}, (1-s_1)e^{2\pi i(1-\theta_2)}, (1-t_1)e^{2\pi i(1-\theta_3)})\}$

Theorem 3.3

Operations on Complex Fermatean Neutrtosophic Sets satisfies Commutative, Associative, Distributive and Demorgan's properties

Proof:

Let $A = (x, r_1e^{2\pi i\theta_1}, s_1e^{2\pi i\theta_2}, t_1e^{2\pi i\theta_3})$, $B = (x, r_2e^{2\pi i\phi_1}, s_2e^{2\pi i\phi_2}, t_2e^{2\pi i\phi_3})$ &

$C = (x, r_3e^{2\pi i\omega_1}, s_3e^{2\pi i\omega_2}, t_3e^{2\pi i\omega_3})$ be complex fermatean neutrosophic sets.

Commutative property:

i. $A \cup B = B \cup A$

ii. $A \cap B = B \cap A$

$$A \cup B = \{x, \max(r_1, r_2)e^{2\pi i \max(\theta_1, \phi_1)}, \max(s_1, s_2)e^{2\pi i \max(\theta_2, \phi_2)}, \max(t_1, t_2)e^{2\pi i \max(\theta_3, \phi_3)})\}$$

$$= \{x, \max(r_2, r_1)e^{2\pi i \max(\phi_1, \theta_1)}, \max(s_2, s_1)e^{2\pi i \max(\phi_2, \theta_2)}, \max(t_2, t_1)e^{2\pi i \max(\phi_3, \theta_3)})\}$$

$$= B \cup A$$

$$A \cap B = \{x, \min(r_1, r_2)e^{2\pi i \min(\theta_1, \phi_1)}, \min(s_1, s_2)e^{2\pi i \min(\theta_2, \phi_2)}, \min(t_1, t_2)e^{2\pi i \min(\theta_3, \phi_3)})\}$$

$$= \{x, \min(r_2, r_1)e^{2\pi i \min(\phi_1, \theta_1)}, \min(s_2, s_1)e^{2\pi i \min(\phi_2, \theta_2)}, \min(t_2, t_1)e^{2\pi i \min(\phi_3, \theta_3)})\}$$

$$= B \cap A$$

Associative property

i. $A \cup (B \cup C) = (A \cup B) \cup C$

ii. $A \cap (B \cap C) = (A \cap B) \cap C$

$$A \cup (B \cup C) = \{x, \max(r_1, \max(r_2, r_3)e^{2\pi i \max(\theta_1, \max(\phi_1, \omega_1)}, \max(s_1, \max(s_2, s_3)e^{2\pi i \max(\theta_2, \max(\phi_2, \omega_3)},$$

$$\max(t_1, \max(t_2, t_3)e^{2\pi i \max(\theta_3, \max(\phi_3, \omega_3)})\}$$

$$= \{x, \max(r_1, r_2, r_3)e^{2\pi i \max(\theta_1, \phi_1, \omega_1)}, \max(s_1, s_2, s_3)e^{2\pi i \max(\theta_2, \phi_2, \omega_3)}, \max(t_1, t_2, t_3)e^{2\pi i \max(\theta_3, \phi_3, \omega_3)})\}$$

$$= (A \cup B) \cup C$$

Similarly $A \cap (B \cap C) = (A \cap B) \cap C$ can be proved.

Distributive property:

i. $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

ii. $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

$$A \cup B = \{x, \max(r_1, r_2)e^{2\pi i \max(\theta_1, \phi_1)}, \max(s_1, s_2)e^{2\pi i \max(\theta_2, \phi_2)}, \max(t_1, t_2)e^{2\pi i \max(\theta_3, \phi_3)})\}$$

$$A \cup C = \{x, \max(r_1, r_3)e^{2\pi i \max(\theta_1, \omega_1)}, \max(s_1, s_3)e^{2\pi i \max(\theta_2, \omega_2)}, \max(t_1, t_3)e^{2\pi i \max(\theta_3, \omega_3)})\}$$

$$(A \cup B) \cap (A \cup C) = \{x, \min(\max(r_1, r_2), \max(r_1, r_3))e^{2\pi i \min(\max(\theta_1, \phi_1), \max(\theta_1, \omega_1))},$$

$$\min(\max(s_1, s_2), \max(s_1, s_3))e^{2\pi i \min(\max(\theta_2, \phi_2), \max(\theta_2, \omega_2))},$$

$$\min(\max(t_1, t_2), \max(t_1, t_3))e^{2\pi i \min(\max(\theta_3, \phi_3), \max(\theta_3, \omega_3))}\}$$

$$= \{x, \max(r_1, \min(r_2, r_3))e^{2\pi i \max(\theta_1, \min(\phi_1, \omega_1))}, \max(s_1, \min(s_2, s_3))e^{2\pi i \max(\theta_2, \min(\phi_2, \omega_{12}))},$$

$$\max(t_1, \min(t_2, t_3))e^{2\pi i \max(\theta_3, \min(\phi_3, \omega_3))}\}$$

$$= A \cup (B \cap C)$$

Similarly $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ can be proved.

Demorgan's property

i. $(A \cup B)^c = A^c \cap B^c$

ii. $(A \cap B)^c = A^c \cup B^c$

$$A \cup B = \{x, \max(r_1, r_2)e^{2\pi i \max(\theta_1, \phi_1)}, \max(s_1, s_2)e^{2\pi i \max(\theta_2, \phi_2)}, \max(t_1, t_2)e^{2\pi i \max(\theta_3, \phi_3)})\}$$

$$(A \cup B)^c = \{x, 1 - \max(r_1, r_2)e^{2\pi i(1 - \max(\theta_1, \phi_1))}, 1 - \max(s_1, s_2)e^{2\pi i(1 - \max(\theta_2, \phi_2))}, 1 - \max(t_1, t_2)e^{2\pi i(1 - \max(\theta_3, \phi_3))})\}$$

$$A^C = \{x, (1 - r_1)e^{2\pi i(1 - \theta_1)}, (1 - s_1)e^{2\pi i(1 - \theta_2)}, (1 - t_1)e^{2\pi i(1 - \theta_3)})\}$$

$$B^C = \{x, (1 - r_2)e^{2\pi i(1 - \phi_1)}, (1 - s_2)e^{2\pi i(1 - \phi_2)}, (1 - t_2)e^{2\pi i(1 - \phi_3)})\}$$

$$A^c \cap B^c = \{x, \min(1 - r_1, 1 - r_2)e^{2\pi i \min(1 - \theta_1, 1 - \phi_1)}, \min(1 - s_1, 1 - s_2)e^{2\pi i \min(1 - \theta_2, 1 - \phi_2)},$$

$$\min(1 - t_1, 1 - t_2)e^{2\pi i \min(1 - \theta_3, 1 - \phi_3)}\}$$

=

$$\{x, 1 - \max(r_1, r_2)e^{2\pi i(1 - \max(\theta_1, \phi_1))}, 1 - \max(s_1, s_2)e^{2\pi i(1 - \max(\theta_2, \phi_2))}, 1 - \max(t_1, t_2)e^{2\pi i(1 - \max(\theta_3, \phi_3))})\}$$

$$= (A \cup B)^c$$

Similarly $(A \cap B)^c = A^c \cup B^c$ can be proved.

4. Distance measures of Complex Fermatean Neutrosophic set

Definition 4.1

Let $A = (x, r_1 e^{2\pi i\theta_1}, s_1 e^{2\pi i\theta_2}, t_1 e^{2\pi i\theta_3})$, $B = (x, r_2 e^{2\pi i\phi_1}, s_2 e^{2\pi i\phi_2}, t_2 e^{2\pi i\phi_3})$ & be any two complex fermatean neutrosophic sets. Then the Hamming distance between A and B is defined as

$$d_H(A,B) = \frac{1}{3}\left[\left|T_A - T_B\right| + \left|I_A - I_B\right| + \left|F_A - F_B\right|\right]$$

Definition 4.2

Let $A = (x, r_1 e^{2\pi i\theta_1}, s_1 e^{2\pi i\theta_2}, t_1 e^{2\pi i\theta_3})$, $B = (x, r_2 e^{2\pi i\phi_1}, s_2 e^{2\pi i\phi_2}, t_2 e^{2\pi i\phi_3})$ & be any two complex fermatean neutrosophic sets. Then the Normalised Hamming distance between A and B is defined as

$$d_{NH}(A,B) = \frac{1}{3n}\sum_{1}^{n}\left|T_{A_i} - T_{B_i}\right| + \left|I_{A_i} - I_{B_i}\right| + \left|F_{A_i} - F_{B_i}\right|$$

Definition 4.3

If $A = (re^{2\pi i\theta_1}, se^{2\pi i\theta_2}, te^{2\pi i\theta_3})$ is any complex Fermatean Neutrosophic number, then the hesitance degree of A is given by

$$H_A = \frac{s}{r+s+t}$$

Definition 4.4

Let $A = (x, r_1 e^{2\pi i\theta_1}, s_1 e^{2\pi i\theta_2}, t_1 e^{2\pi i\theta_3})$, $B = (x, r_2 e^{2\pi i\phi_1}, s_2 e^{2\pi i\phi_2}, t_2 e^{2\pi i\phi_3})$ & be any two complex fermatean neutrosophic sets. The distance measure incorporating hesitance degree is defined as

$$D_{HD}(A,B) = \frac{1}{3n}\sum_{i=1}^{n}\left|T_{A_i} - T_{B_i}\right| + \left|I_{A_i} - I_{B_i}\right| + \left|F_{A_i} - F_{B_i}\right| + \left|\theta_{A_i} - \phi_{B_i}\right| + \left|H_{A_i} - H_{B_i}\right|$$

Theorem 4.5

The function defined above is a metric function

Proof:

i. It is obvious that

$$D_{HD}(A,B) \geq 0$$

ii. $$D_{HD}(A,B) = \frac{1}{3n}\sum_{i=1}^{n}\left|T_{A_i} - T_{B_i}\right| + \left|I_{A_i} - I_{B_i}\right| + \left|F_{A_i} - F_{B_i}\right| + \left|\theta_{A_i} - \phi_{B_i}\right| + \left|H_{A_i} - H_{B_i}\right|$$

$$= \frac{1}{3n}\sum_{i=1}^{n}\left|T_{B_i} - T_{A_i}\right| + \left|I_{B_i} - I_{A_i}\right| + \left|F_{B_i} - F_{A_i}\right| + \left|\theta_{B_i} - \phi_{A_i}\right| + \left|H_{B_i} - H_{A_i}\right|$$

$$= D_{HD}(B, A)$$

iii. $$D_{HD}(A,C) = \frac{1}{3n}\sum_{i=1}^{n}\left|T_{A_i} - T_{C_i}\right| + \left|I_{A_i} - I_{C_i}\right| + \left|F_{A_i} - F_{C_i}\right| + \left|\theta_{A_i} - \omega_{C_i}\right| + \left|H_{A_i} - H_{C_i}\right|$$

$$= \frac{1}{3n}\sum_{i=1}^{n}\left|T_{A_i} + T_{B_i} - T_{B_i} - T_{C_i}\right| + \left|I_{A_i} + I_{B_i} - I_{B_i} - I_{C_i}\right| + \left|F_{A_i} + F_{B_i} - F_{B_i} - F_{C_i}\right|$$

$$+ \left|\theta_{A_i} + \phi_{B_i} - \phi_{B_i} - \omega_{C_i}\right| + \left|H_{A_i} + H_{B_i} - H_{B_i} - H_{C_i}\right|$$

$$\leq \frac{1}{3n}\sum_{i=1}^{n}\left|T_{A_i} - T_{B_i}\right| + \left|T_{B_i} - T_{C_i}\right| + \left|I_{A_i} - I_{B_i}\right| + \left|I_{B_i} - I_{C_i}\right| + \left|F_{A_i} - F_{B_i}\right| + \left|F_{B_i} - F_{C_i}\right|$$

$$+ \left|\theta_{A_i} - \phi_{B_i}\right| + \left|\phi_{B_i} - \omega_{C_i}\right| + \left|H_{A_i} - H_{B_i}\right| + \left|H_{B_i} - H_{C_i}\right|$$

$$= D_{HD}(A,B) + D_{HD}(B,C)$$

Hence $D_{HD}(A,B) \leq D_{HD}(A,B) + D_{HD}(B,C)$

From i, ii and iii, we can see that $D_{HD}(A,B)$ is a metric function.

5. Complex Fermatean Neutrosophic Sets in Multi criteria decision making problems

Different algorithms for making decisions are made for particular situations and issues. A novel approach to decision-making based on CFNSs and distance measure is suggested in this section. By adding complex fermatean neutrosophic numbers, CNFS theory expands on the potential of conventional fuzzy set theory and provides a more sophisticated way to represent and manipulate uncertainty in decision-making scenarios. It provides decision-makers with a strong foundation for handling challenging and unpredictable situations and coming to wise decisions. CFNSs can be used to more accurately model conflicting criteria, preferences, and imprecise information in decision-making situations. Decision-makers are able to take into account a wider range of options and make better decisions.

Consider a Multi Criteria Decision Making(MCDM) problem with i alternatives $a_1, a_2, ...a_i$, j attributes $b_1, b_2, ...b_j$ and k experts $E_1, E_2, ...E_k$

Defintion 5.5

The Score function of an alternative $a_i; i = 1,2,....n$ is defined as

$$S(a_i) = \frac{\sum_{i=1}^{n} D_{HD}(a_i^{E_j}, a_i^{E_k})}{n}$$

5.6 Algorithm to solve a Multi Criteria Decision Making Problem based on Complex Fermatean Neutrosphic Sets:

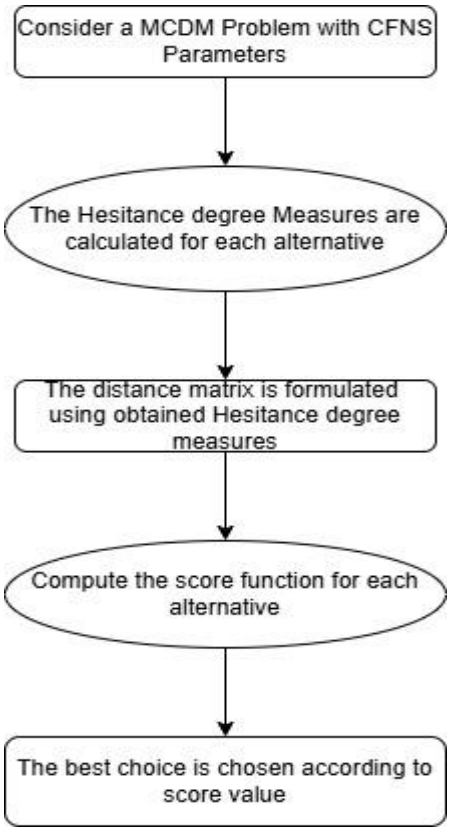
Step 1: Construct the decision making table where the entries are CFNS.

Step 2: Calculate the Hesitance distance measure between the values assessed by various experts for each alternative.

Step 3: Formulate the distance matrix utilizing the values of hesitance distance measure

Step 4: Compute the Score function for each alternative

Step 5: The Best Option can be chosen by ranking the alternatives using the Score function.


Consider a MCDM Problem with CFNS Parameters
The Hesitance degree Measures are calculated for each alternative
The distance matrix is formulated using obtained Hesitance degree measures
Compute the score function for each alternative
The best choice is chosen according to score value

Structured Outline of the Proposed algorithm

6 Illustrative example

Consider the following example of multi criteria decision making problem.

A university wants to select the best candidate for a faculty position. There are three candidates (A,B,C), five attributes Research Experience(RE), Teaching Experience(TE), Publication Record(PR), Communication Skills(CS) and Teamwork Ability(TA). Three experts Department Head(E1), Senior Faculty Member(E2), and External Reviewer(E3) have to analyse and submit the reports. The evaluated values of the candidates are given in Table I. Choose the best candidate.

Experts	Candidates	Attributes

		RE	TE	PR	CS	TA
E1	A	$(0.8e^{i2\pi(0.7)}, 0.7e^{i2\pi(0.5)}, 0.7e^{i2\pi(0.2)})$	$(0.6e^{i2\pi(0.4)}, 0.7e^{i2\pi(0.8)}, 0.7e^{i2\pi(0.5)})$	$(0.7e^{i2\pi(0.4)}, 0.8e^{i2\pi(0.6)}, 0.6e^{i2\pi(0.7)})$	$(0.9e^{i2\pi(0.6)}, 0.4e^{i2\pi(0.8)}, 0.5e^{i2\pi(0.2)})$	$(0.4e^{i2\pi(0.6)}, 0.5e^{i2\pi(0.7)}, 0.9e^{i2\pi(0.5)})$
	B	$(0.4e^{i2\pi(0.6)}, 0.6e^{i2\pi(0.7)}, 0.8e^{i2\pi(0.5)})$	$(0.8e^{i2\pi(0.6)}, 0.7e^{i2\pi(0.6)}, 0.6e^{i2\pi(0.2)})$	$(0.6e^{i2\pi(0.5)}, 0.8e^{i2\pi(0.7)}, 0.7e^{i2\pi(0.4)})$	$(0.7e^{i2\pi(0.6)}, 0.5e^{i2\pi(0.6)}, 0.9e^{i2\pi(0.5)})$	$(0.5e^{i2\pi(0.7)}, 0.7e^{i2\pi(0.5)}, 0.8e^{i2\pi(0.4)})$
	C	$(0.2e^{i2\pi(0.7)}, 0.8e^{i2\pi(0.5)}, 0.4e^{i2\pi(0.7)})$	$(0.4e^{i2\pi(0.7)}, 0.5e^{i2\pi(0.5)}, 0.6e^{i2\pi(0.4)})$	$(0.6e^{i2\pi(0.5)}, 0.4e^{i2\pi(0.8)}, 0.5e^{i2\pi(0.3)})$	$(0.5e^{i2\pi(0.8)}, 0.6e^{i2\pi(0.4)}, 0.7e^{i2\pi(0.2)})$	$(0.3e^{i2\pi(0.6)}, 0.7e^{i2\pi(0..7)}, 0.9e^{i2\pi(0.5)})$
E2	A	$(0.9e^{i2\pi(0.5)}, 0.6e^{i2\pi(0.4)}, 0.7e^{i2\pi(0.6)})$	$(0.8e^{i2\pi(0.7)}, 0.5e^{i2\pi(0.5)}, 0.6e^{i2\pi(0.3)})$	$(0.6e^{i2\pi(0.4)}, 0.9e^{i2\pi(0.6)}, 0.8e^{i2\pi(0.7)})$	$(0.7e^{i2\pi(0.8)}, 0.9e^{i2\pi(0.7)}, 0.5e^{i2\pi(0.2)})$	$(0.5e^{i2\pi(0.8)}, 0.6e^{i2\pi(0.2)}, 0.4e^{i2\pi(0.9)})$
	B	$(0.5e^{i2\pi(0.7)}, 0.7e^{i2\pi(0.8)}, 0.6e^{i2\pi(0.4)})$	$(0.7e^{i2\pi(0.7)}, 0.6e^{i2\pi(0.5)}, 0.4e^{i2\pi(0.3)})$	$(0.4e^{i2\pi(0.8)}, 0.5e^{i2\pi(0.6)}, 0.8e^{i2\pi(0.7)})$	$(0.8e^{i2\pi(0.7)}, 0.6e^{i2\pi(0.5)}, 0.9e^{i2\pi(0.7)})$	$(0.3e^{i2\pi(0.9)}, 0.4e^{i2\pi(0..7)}, 0.5e^{i2\pi(0.2)})$
	C	$(0.4e^{i2\pi(0.8)}, 0.5e^{i2\pi(0.5)}, 0.7e^{i2\pi(0.9)})$	$(0.6e^{i2\pi(0.6)}, 0.7e^{i2\pi(0.4)}, 0.8e^{i2\pi(0.2)})$	$(0.7e^{i2\pi(0.7)}, 0.5e^{i2\pi(0.6)}, 0.9e^{i2\pi(0.7)})$	$(0.8e^{i2\pi(0.5)}, 0.6e^{i2\pi(0.7)}, 0.4e^{i2\pi(0.3)})$	$(0.5e^{i2\pi(0.9)}, 0.4e^{i2\pi(0.3)}, 0.6e^{i2\pi(0.4)})$
E3	A	$(0.7e^{i2\pi(0.8)}, 0.6e^{i2\pi(0.4)}, 0.8e^{i2\pi(0.5)})$	$(0.5e^{i2\pi(0.7)}, 0.9e^{i2\pi(0.5)}, 0.4e^{i2\pi(0.6)})$	$(0.9e^{i2\pi(0.4)}, 0.7e^{i2\pi(0.6)}, 0.6e^{i2\pi(0.7)})$	$(0.8e^{i2\pi(0.3)}, 0.5e^{i2\pi(0.8)}, 0.7e^{i2\pi(0.4)})$	$(0.6e^{i2\pi(0.5)}, 0.8e^{i2\pi(0.7)}, 0.9e^{i2\pi(0.2)})$
	B	$(0.6e^{i2\pi(0.7)}, 0.6e^{i2\pi(0.8)}, 0.8e^{i2\pi(0.2)})$	$(0.9e^{i2\pi(0.6)}, 0.7e^{i2\pi(0.5)}, 0.6e^{i2\pi(0.6)})$	$(0.8e^{i2\pi(0.7)}, 0.9e^{i2\pi(0.4)}, 0.7e^{i2\pi(0.5)})$	$(0.7e^{i2\pi(0.9)}, 0.5e^{i2\pi(0.7)}, 0.4e^{i2\pi(0.8)})$	$(0.5e^{i2\pi(0.8)}, 0.4e^{i2\pi(0.6)}, 0.9e^{i2\pi(0.3)})$
	C	$(0.3e^{i2\pi(0.7)}, 0.4e^{i2\pi(0.5)}, 0.5e^{i2\pi(0.6)})$	$(0.5e^{i2\pi(0.8)}, 0.7e^{i2\pi(0.4)}, 0.6e^{i2\pi(0.3}$	$(0.4e^{i2\pi(0.6)}, 0.5e^{i2\pi(0.3)}, 0.7e^{i2\pi(0.4)})$	$(0.6e^{i2\pi(0.4)}, 0.8e^{i2\pi(0.6)}, 0.3e^{i2\pi(0.2)})$	$(0.8e^{i2\pi(0.5)}, 0.6e^{i2\pi(0.7)}, 0.4e^{i2\pi(0.5)})$

Table 1- Evaluated Values of Alternatives

The hesitance degree is calculated and represented in the following table

Experts	Candidates	Attributes				
		RE	TE	PR	CS	TA

E1	A	0.32	0.35	0.38	0.22	0.28
	B	0.33	0.33	0.38	0.24	0.35
	C	0.57	0.33	0.27	0.33	0.37
E2	A	0.27	0.26	0.39	0.43	0.4
	B	0.39	0.35	0.29	0.26	0.33
	C	0.31	0.33	0.24	0.33	0.27
E3	A	0.29	0.5	0.32	0.25	0.35
	B	0.3	0.32	0.38	0.31	0.22
	C	0.33	0.39	0.31	0.47	0.33

Table 2- Hesitance degree value

The Hesitance distance measure of each candidate corresponding to the evaluation of experts is calculated and represented in a matrix

$$D_{HD}(A^{E1}, A^{E2}) = 0.40$$

$$D_{HD}(A^{E1}, A^{E3}) = 0.30$$

$$D_{HD}(A^{E2}, A^{E3}) = 0.42$$

$$D_{HD}(B^{E1}, B^{E2}) = 0.334$$

$$D_{HD}(B^{E1}, B^{E3}) = 0.283$$

$$D_{HD}(B^{E2}, B^{E3}) = 0.356$$

$$D_{HD}(C^{E1}, C^{E2}) = 0.459$$

$$D_{HD}(C^{E1}, C^{E3}) = 0.368$$

$$D_{HD}(C^{E2}, C^{E3}) = 0.363$$

Distance measure matrix is given by

$$M = \begin{matrix} A \\ B \\ C \end{matrix} \begin{pmatrix} 0.40 & 0.30 & 0.42 \\ 0.334 & 0.283 & 0.356 \\ 0.459 & 0.368 & 0.363 \end{pmatrix}$$

Now the Score function is calculated for each candidate

$$S(A) = \frac{0.40 + 0.30 + 0.42}{3} = 0.37$$

$$S(B) = \frac{0.334 + 0.283 + 0.356}{3} = 0.324$$

$$S(C) = \frac{0.459 + 0.368 + 0.363}{3} = 0.396$$

We have $S(C) > S(A) > S(B)$

Hence C can be chosen the best among the three candidates.

6.1 Sensitivity Analysis

The study of Complex Fermatean Neutrosophic Sets is analysed for various values of truth, indeterminacy and falsity membership functions. Different truth, indeterminacy and falsity degree values are generated and each is used in the calculation of hesitance degree and the score function. It is found that, if the indeterminacy is of degree 1, then there will be no impact on final result. On the other hand if either truth or falsity is of degree 1, then the decision will be favourable in case of truth value 1 and unfavourable in case of falsity degree.

If hesitance degree increases then the corresponding attribute will also increase considerably thereby resulting in an increase in the distance measure. This impact on the score function and the corresponding choice will be chosen by the decision maker.

7 Discussion:

7.1 Need for study of CFNS

CFNS can handle complex uncertainty, ambiguity, and imprecision in real-world problems. CFNS can capture non-linear relationships between variables, which is common in real-world problems. CFNS can help assess risks in complex systems. CFNS provides opportunities for collaboration with other disciplines, such as mathematics, computer science, and engineering.

7.2 Impacts of CFNS in Decision Making

CFNS can lead to improved decision quality by incorporating uncertainty, ambiguity, and imprecision into the decision-making process. CFNS provides a transparent and auditable decision-making process, which can increase stakeholder trust and confidence. CFNS can help identify and manage risks more effectively by capturing uncertainty and ambiguity. CFNS can lead to innovative solutions by incorporating human judgment and expertise into the decision-making process.

7.3 Limitations of CFNS in Decision Making

CFNS requires complex computations, which can be time-consuming and challenging, especially for large-scale problems. The results obtained from CFNS can be difficult to interpret, especially for non-technical stakeholders.Currently, there is limited software support for CFNS, making it challenging to implement and apply.

Overall, CFNS has the potential to revolutionize decision-making by providing a robust framework for handling uncertainty, ambiguity, and imprecision. However, its limitations, such as computational complexity and data requirements, need to be addressed to fully realize its potential.

7.4 Future Scope of CFNS

CFNS-based models can be developed for solving real-world problems, such as medical diagnosis, financial forecasting, and environmental modeling. The application of CFNS in big data analytics can be investigated, such as data mining, data warehousing, and business intelligence.The application of CFNS in cognitive computing can be studied, such as natural language processing, machine learning, and expert systems. The use of Complex Fermatean Neutrosophic Sets can be investigated in image and signal processing, such as in image segmentation, denoising, and feature

extraction. Efficient computational algorithms and tools can be developed to support the widespread adoption of Complex Fermatean Neutrosophic Sets in MCDM problems.

8 Conclusion:

The integration of Complex Fermatean Neutrosophic Sets with Multi-Criteria Decision Making (MCDM) problems has far-reaching implications. By leveraging the hesitance degree distance measure, decision-makers can navigate complex decision-making landscapes with unprecedented ease. The proposed approach reduces computational complexity, enabling faster and more efficient decision-making. Complex Fermatean Neutrosophic Sets effectively model uncertainty and ambiguity, providing a more comprehensive understanding of decision-making scenarios. By capturing the nuances of uncertainty and ambiguity, decision-makers can make more informed choices, leading to improved outcomes. The proposed approach has the potential to revolutionize decision-making in complex environments, enabling organizations to make more informed, efficient, and effective decisions.

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Prioritization of Higher Education Management Strategies Based on the Neutrosophic Uncertainty Model for Sustainability

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Abstract: This study proposes the multi-criteria decision-making methodology (MCDM) to rank the strategies of higher education management. We used the neutrosophic sets to deal with uncertain information in the MCDM solutions. We used two MCDM methods in this study such as CRiteria Importance Through Intercriteria Correlation (CRITIC) and alternative ranking order method accounting for two-step normalization (AROMAN). The CRITIC method is used to compute the criteria weights and the AROMAN method is used to rank the alternatives. We use the Trapezoidal Neutrosophic Z-Number (TNZN) which is an extension of the neutrosophic set to deal with vague data. This study uses nine criteria and ten alternatives to rank the strategies of higher education management. An empirical application is conducted to show the effectiveness of the proposed approach. We conducted the sensitivity analysis with two cases to show the different ranks of the alternatives. The results show the rank of alternatives is stable under different cases.

Keywords: Trapezoidal Neutrosophic Z-Number (TNZN); Higher Education; Management Strategies; Uncertainty Model.

1. Introduction

Numerous applications of novel fuzzy decision-making techniques have been made in bipolar, simplified neutrosophic, hesitant fuzzy, and neutrosophic settings for fuzzy decision-making challenges. Since continuous fuzzy numbers of membership functions can represent genuine decision-making difficulties instead of fuzzy values, triangular and trapezoidal fuzzy numbers are typically employed[1], [2]. To solve multicriteria decision-making (MCDM) problems[3] with triangular intuitionistic fuzzy information, some researchers introduced triangular intuitionistic fuzzy sets (TIFSs), where the values of the membership and non-membership functions are triangular fuzzy numbers. They also extended triangular fuzzy numbers to intuitionistic fuzzy sets (IFSs). However, NZN is represented by exact/discrete truth, falsity, and indeterminacy Z-numbers instead of continuous Z-numbers, and TNN is characterized just by the trapezoidal fuzzy numbers of its truth, falsity, and indeterminacy membership functions without considering

their reliability measures[3], [4]. As a result, the shortcomings of TNN, NZN, and their MCDM techniques in their information representations and applications are exposed[5], [6]. To address these shortcomings of current information expressions and MCDM methods in the environments of TNNs and NZNs, this study must propose an MCDM method based on trapezoidal neutrosophic Z-numbers (TNZNs) to express both the reliability measures and the continuous Z-numbers of truth, falsity, and indeterminacy membership functions in MCDM problems[7].

In fuzzy decision theory and methodology, the modeling of decision-making and the presentation of information are crucial issues. However, the current multicriteria decision-making (MCDM) approaches for trapezoidal neutrosophic numbers (TNNs) and neutrosophic Z-numbers (NZNs) show their shortcomings, such as failing to take dependability measures into account in TNN and continuous Z-numbers into account in NZN. Trapezoidal neutrosophic Z-numbers (TNZNs), their aggregation processes, and an MCDM approach for addressing MCDM issues using TNZN information must be proposed to solve the shortcomings. As a result, Young et al. [8] suggested a TNZN set, a few fundamental TNZN operations, as well as the accuracy and score functions of TNZN and associated ranking rules.

Next, based on the operations of TNZNs, the TNZN operators are introduced. After that, an MCDM method is developed to solve MCDM problems in the TNZN environment by utilizing the suggested aggregation operators. Finally, an MCDM example of problem selection and ranking is used to demonstrate the efficiency and rationale of the established MCDM strategy in the context of TNZNs. Their study's primary benefit is that the created methodology improves decision-making efficiency and logic in the context of TNZNs while also making evaluation data continuous and dependable.

Z-numbers, the trapezium cloud model, and interval-valued linguistic neutrosophic sets (IVLNSs) are effective methods for representing randomness and uncertainty. Combining these approaches is the goal of Chen et al.[9] They start by going over pertinent ideas and operators before presenting a brand-new IVLNS and Z-number combo that creates a new expression format. To reduce information loss and distortion during quantification, they then suggest the Z-interval-valued linguistic neutrosophic set-trapezium–trapezium cloud (Z-IVLNS-TTC) model. Next, multi-objective programming (MOP) is used to build a new way to compute the objective weight vector. They provided a novel technique for determining the distance between Z-IVLNS-TTCs based on the p-norm, which is inspired by the TOPSIS method. To show how the suggested approach may be used in practice, a group decision-making problem is finally solved.

Recently, linear systems of equations with Z-numbers have garnered some attention. Some methods for resolving these systems have been devised. There is no analytical solution for these sorts of systems since they are filled with ambiguities and uncertainties. As a result, numerical techniques are typically employed to approximate their solution. A computational method for resolving linear systems with trapezoidal Z-numbers is introduced by Moosavi et al[10]. Their suggested method is set up so that the Z-number coefficients are first transformed to the

equivalent fuzzy numbers, and then the fuzzy coefficients are converted to real coefficients using a ranking function. The initial Z-numbers system in this trend transforms into a fuzzy linear system after two phases.

The aims of this study are presented as follows: (1) we define some definitions of TNZN with their operations to show the multiplication and summation forms. (2) we use the concept of the MCDM approach to deal with different criteria. (3) Two MCDM methods are used in this study such as the CRITIC method to compute the criteria weights and the AROMAN method to rank the alternatives. (4) Two MCDM methods are used under the TNZN to deal with uncertain information in the evaluation form. (5) We apply the sensitivity analysis to show the stability of the ranks of the alternatives.

The following sections make up the remainder of the article. As a prelude to this study, Section 2 presents some fundamental ideas about TNZNs. A TNZN set is proposed in Section 3, along with the fundamental operations of the decision-making methods. The effectiveness of the established MCDM strategy in the context of TNZNs is demonstrated in Section 4 by applying it to an MCDM problem selection example. Ultimately, Section 5 contains conclusions and recommendations for additional research.

2. Trapezoidal Neutrosophic Z-Number (TNZN) Sets

This section gives the following definitions of TNZN such as[8]:

Definition 1

Set U as universe set, the TNZN can be defined as:

$$Z = \left\{u, \left(\left(T_{Z_V}(u), T_{Z_R}(u)\right), \left(I_{Z_V}(u), I_{Z_R}(u)\right), \left(F_{Z_V}(u), F_{Z_R}(u)\right)\right) | u \in U\right\} \tag{1}$$

Where the previous function shows the truth, indeterminacy, and falsity membership function for the TNZN, and their values can be defined such as:

$$\left(T_{Z_V}(u), T_{Z_R}(u)\right) = \begin{pmatrix} \left(T_{Z_{V_1}}(u), T_{Z_{V_2}}(u), T_{Z_{V_3}}(u), T_{Z_{V_4}}(u)\right), \\ \left(T_{Z_{R_1}}(u), T_{Z_{R_2}}(u), T_{Z_{R_3}}(u), T_{Z_{R_4}}(u)\right) \end{pmatrix}; U \to [0,1] \times [0,1] \tag{2}$$

$$\left(I_{Z_V}(u), I_{Z_R}(u)\right) = \begin{pmatrix} \left(I_{Z_{V_1}}(u), I_{Z_{V_2}}(u), I_{Z_{V_3}}(u), I_{Z_{V_4}}(u)\right), \\ \left(I_{Z_{R_1}}(u), I_{Z_{R_2}}(u), I_{Z_{R_3}}(u), I_{Z_{R_4}}(u)\right) \end{pmatrix}; U \to [0,1] \times [0,1] \tag{3}$$

$$\left(F_{Z_V}(u), F_{Z_R}(u)\right) = \begin{pmatrix} \left(F_{Z_{V_1}}(u), F_{Z_{V_2}}(u), F_{Z_{V_3}}(u), F_{Z_{V_4}}(u)\right), \\ \left(F_{Z_{R_1}}(u), F_{Z_{R_2}}(u), F_{Z_{R_3}}(u), F_{Z_{R_4}}(u)\right) \end{pmatrix}; U \to [0,1] \times [0,1] \tag{4}$$

$$0 \le T_{Z_{V_1}}(u) + I_{Z_{V_1}}(u) + F_{Z_{V_1}}(u) \le 3 \tag{5}$$

$$0 \le T_{Z_{R_1}}(u) + I_{Z_{R_1}}(u) + F_{Z_{R_1}}(u) \le 3 \tag{6}$$

Definition 2

$$\text{Set } Z_1 = \begin{pmatrix} \left(\left(T_{Z_{V_{11}}}(u), T_{Z_{V_{12}}}(u), T_{Z_{V_{13}}}(u), T_{Z_{V_{14}}}(u)\right), \left(T_{Z_{R_{11}}}(u), T_{Z_{R_{12}}}(u), T_{Z_{R_{13}}}(u), T_{Z_{R_{14}}}(u)\right)\right), \\ \left(\left(I_{Z_{V_{11}}}(u), I_{Z_{V_{12}}}(u), I_{Z_{V_{13}}}(u), I_{Z_{V_{14}}}(u)\right), \left(I_{Z_{R_{11}}}(u), I_{Z_{R_{12}}}(u), I_{Z_{R_{13}}}(u), I_{Z_{R_{14}}}(u)\right)\right), \\ \left(\left(F_{Z_{V_{11}}}(u), F_{Z_{V_{12}}}(u), F_{Z_{V_{13}}}(u), F_{Z_{V_{14}}}(u)\right), \left(F_{Z_{R_{11}}}(u), F_{Z_{R_{12}}}(u), F_{Z_{R_{13}}}(u), F_{Z_{R_{14}}}(u)\right)\right) \end{pmatrix}$$

$$\text{and } Z_2 = \begin{pmatrix} \left(\left(T_{Z_{V_{21}}}(u), T_{Z_{V_{22}}}(u), T_{Z_{V_{23}}}(u), T_{Z_{V_{24}}}(u)\right), \left(T_{Z_{R_{21}}}(u), T_{Z_{R_{22}}}(u), T_{Z_{R_{23}}}(u), T_{Z_{R_{24}}}(u)\right)\right), \\ \left(\left(I_{Z_{V_{21}}}(u), I_{Z_{V_{22}}}(u), I_{Z_{V_{23}}}(u), I_{Z_{V_{24}}}(u)\right), \left(I_{Z_{R_{21}}}(u), I_{Z_{R_{22}}}(u), I_{Z_{R_{23}}}(u), I_{Z_{R_{24}}}(u)\right)\right), \\ \left(\left(F_{Z_{V_{21}}}(u), F_{Z_{V_{22}}}(u), F_{Z_{V_{23}}}(u), F_{Z_{V_{24}}}(u)\right), \left(F_{Z_{R_{21}}}(u), F_{Z_{R_{22}}}(u), F_{Z_{R_{23}}}(u), F_{Z_{R_{24}}}(u)\right)\right) \end{pmatrix} \text{ as}$$

two TNZNs and basic operations can be defined as:

$$Z_1 \oplus Z_2 = \begin{pmatrix} \left(\begin{pmatrix} T_{Z_{V_{11}}}(u) + T_{Z_{V_{21}}}(u) - T_{Z_{V_{11}}}(u)T_{Z_{V_{21}}}(u), \\ T_{Z_{V_{12}}}(u) + T_{Z_{V_{22}}}(u) - T_{Z_{V_{12}}}(u)T_{Z_{V_{22}}}(u), \\ T_{Z_{V_{13}}}(u) + T_{Z_{V_{23}}}(u) - T_{Z_{V_{13}}}(u)T_{Z_{V_{23}}}(u), \\ T_{Z_{V_{14}}}(u) + T_{Z_{V_{24}}}(u) - T_{Z_{V_{14}}}(u)T_{Z_{V_{24}}}(u) \end{pmatrix}, \begin{pmatrix} T_{Z_{R_{11}}}(u) + T_{Z_{R_{21}}}(u) - T_{Z_{R_{11}}}(u)T_{Z_{R_{21}}}(u), \\ T_{Z_{R_{12}}}(u) + T_{Z_{R_{22}}}(u) - T_{Z_{R_{12}}}(u)T_{Z_{R_{22}}}(u), \\ T_{Z_{R_{13}}}(u) + T_{Z_{R_{23}}}(u) - T_{Z_{R_{13}}}(u)T_{Z_{R_{23}}}(u), \\ T_{Z_{R_{14}}}(u) + T_{Z_{R_{24}}}(u) - T_{Z_{R_{14}}}(u)T_{Z_{R_{24}}}(u) \end{pmatrix}\right), \\ \left(\begin{pmatrix} I_{Z_{V_{11}}}(u)I_{Z_{V_{21}}}(u), I_{Z_{V_{12}}}(u)I_{Z_{V_{22}}}(u), \\ I_{Z_{V_{13}}}(u)I_{Z_{V_{23}}}(u), I_{Z_{V_{14}}}(u)I_{Z_{V_{24}}}(u) \end{pmatrix}, \begin{pmatrix} I_{Z_{R_{11}}}(u)I_{Z_{R_{21}}}(u), I_{Z_{R_{12}}}(u)I_{Z_{R_{22}}}(u), \\ I_{Z_{R_{13}}}(u)I_{Z_{R_{23}}}(u), I_{Z_{R_{14}}}(u)I_{Z_{R_{24}}}(u) \end{pmatrix}\right), \\ \left(\begin{pmatrix} F_{Z_{V_{11}}}(u)I_{Z_{V_{21}}}(u), F_{Z_{V_{12}}}(u)F_{Z_{V_{22}}}(u), \\ F_{Z_{V_{13}}}(u)F_{Z_{V_{23}}}(u), F_{Z_{V_{14}}}(u)F_{Z_{V_{24}}}(u) \end{pmatrix}, \begin{pmatrix} F_{Z_{R_{11}}}(u)F_{Z_{R_{21}}}(u), F_{Z_{R_{12}}}(u)F_{Z_{R_{22}}}(u), \\ F_{Z_{R_{13}}}(u)F_{Z_{R_{23}}}(u), F_{Z_{R_{14}}}(u)F_{Z_{R_{24}}}(u) \end{pmatrix}\right) \end{pmatrix} \tag{7}$$

$$Z_1 \otimes Z_2 = \left(\begin{array}{c} \left(\begin{pmatrix} T_{Z_{V_{11}}}(u)T_{Z_{V_{21}}}(u), T_{Z_{V_{12}}}(u)T_{Z_{V_{22}}}(u), \\ T_{Z_{V_{13}}}(u)T_{Z_{V_{23}}}(u), T_{Z_{V_{14}}}(u)T_{Z_{V_{24}}}(u) \end{pmatrix}, \right. \\ \left. \begin{pmatrix} T_{Z_{R_{11}}}(u)T_{Z_{R_{21}}}(u), T_{Z_{R_{12}}}(u)T_{Z_{R_{22}}}(u), \\ T_{Z_{R_{13}}}(u)T_{Z_{R_{23}}}(u), T_{Z_{R_{14}}}(u)T_{Z_{R_{24}}}(u) \end{pmatrix} \right), \\ \left(\begin{pmatrix} I_{Z_{V_{11}}}(u) + I_{Z_{V_{21}}}(u) - I_{Z_{V_{11}}}(u)I_{Z_{V_{21}}}(u), \\ I_{Z_{V_{12}}}(u) + I_{Z_{V_{22}}}(u) - I_{Z_{V_{12}}}(u)I_{Z_{V_{22}}}(u), \\ I_{Z_{V_{13}}}(u) + I_{Z_{V_{23}}}(u) - I_{Z_{V_{13}}}(u)I_{Z_{V_{23}}}(u), \\ I_{Z_{V_{14}}}(u) + I_{Z_{V_{24}}}(u) - I_{Z_{V_{14}}}(u)I_{Z_{V_{24}}}(u) \end{pmatrix}, \right. \\ \left. \begin{pmatrix} I_{Z_{R_{11}}}(u) + I_{Z_{R_{21}}}(u) - I_{Z_{R_{11}}}(u)I_{Z_{R_{21}}}(u), \\ I_{Z_{R_{12}}}(u) + I_{Z_{R_{22}}}(u) - I_{Z_{R_{12}}}(u)I_{Z_{R_{22}}}(u), \\ I_{Z_{R_{13}}}(u) + I_{Z_{R_{23}}}(u) - I_{Z_{R_{13}}}(u)I_{Z_{R_{23}}}(u) \\ , I_{Z_{R_{14}}}(u) + I_{Z_{R_{24}}}(u) - I_{Z_{R_{14}}}(u)I_{Z_{R_{24}}}(u) \end{pmatrix} \right), \\ \left(\begin{pmatrix} F_{Z_{V_{11}}}(u) + F_{Z_{V_{21}}}(u) - F_{Z_{V_{11}}}(u)F_{Z_{V_{21}}}(u), \\ F_{Z_{V_{12}}}(u) + F_{Z_{V_{22}}}(u) - F_{Z_{V_{12}}}(u)F_{Z_{V_{22}}}(u), \\ F_{Z_{V_{13}}}(u) + F_{Z_{V_{23}}}(u) - F_{Z_{V_{13}}}(u)F_{Z_{V_{23}}}(u), \\ F_{Z_{V_{14}}}(u) + F_{Z_{V_{24}}}(u) - F_{Z_{V_{14}}}(u)F_{Z_{V_{24}}}(u) \end{pmatrix}, \right. \\ \left. \begin{pmatrix} F_{Z_{R_{11}}}(u) + F_{Z_{R_{21}}}(u) - F_{Z_{R_{11}}}(u)F_{Z_{R_{21}}}(u), \\ F_{Z_{R_{12}}}(u) + F_{Z_{R_{22}}}(u) - F_{Z_{R_{12}}}(u)F_{Z_{R_{22}}}(u), \\ F_{Z_{R_{13}}}(u) + F_{Z_{R_{23}}}(u) - F_{Z_{R_{13}}}(u)F_{Z_{R_{23}}}(u), \\ F_{Z_{R_{14}}}(u) + F_{Z_{R_{24}}}(u) - F_{Z_{R_{14}}}(u)F_{Z_{R_{24}}}(u) \end{pmatrix} \right) \end{array} \right) \tag{8}$$

$$\gamma Z_1 = \left(\begin{array}{c} \left(\begin{pmatrix} 1 - \left(1 - T_{Z_{V_{11}}}(u)\right)^{\gamma}, 1 - \left(1 - T_{Z_{V_{12}}}(u)\right)^{\gamma}, \\ 1 - \left(1 - T_{Z_{V_{13}}}(u)\right)^{\gamma}, 1 - \left(1 - T_{Z_{V_{14}}}(u)\right)^{\gamma} \end{pmatrix}, \right. \\ \left. \begin{pmatrix} 1 - \left(1 - T_{Z_{R_{11}}}(u)\right)^{\gamma}, 1 - \left(1 - T_{Z_{R_{12}}}(u)\right)^{\gamma}, \\ 1 - \left(1 - T_{Z_{R_{13}}}(u)\right)^{\gamma}, 1 - \left(1 - T_{Z_{R_{14}}}(u)\right)^{\gamma} \end{pmatrix} \right), \\ \left(\left(\left(I_{Z_{V_{11}}}(u)\right)^{\gamma}, \left(I_{Z_{V_{12}}}(u)\right)^{\gamma}, \left(I_{Z_{V_{13}}}(u)\right)^{\gamma}, \left(I_{Z_{V_{14}}}(u)\right)^{\gamma} \right), \right. \\ \left. \left(\left(I_{Z_{R_{11}}}(u)\right)^{\gamma}, \left(I_{Z_{R_{12}}}(u)\right)^{\gamma}, \left(I_{Z_{R_{13}}}(u)\right)^{\gamma}, \left(I_{Z_{R_{14}}}(u)\right)^{\gamma} \right) \right), \\ \left(\left(\left(F_{Z_{V_{11}}}(u)\right)^{\gamma}, \left(F_{Z_{V_{12}}}(u)\right)^{\gamma}, \left(F_{Z_{V_{13}}}(u)\right)^{\gamma}, \left(F_{Z_{V_{14}}}(u)\right)^{\gamma} \right), \right. \\ \left. \left(\left(F_{Z_{R_{11}}}(u)\right)^{\gamma}, \left(F_{Z_{R_{12}}}(u)\right)^{\gamma}, \left(F_{Z_{R_{13}}}(u)\right)^{\gamma}, \left(F_{Z_{R_{14}}}(u)\right)^{\gamma} \right) \right) \end{array} \right) \tag{9}$$

$$Z_1^{\gamma}=\left(\begin{array}{c}\left(\left(\begin{array}{c}\left(\left(T_{Z_{V_{11}}}(u)\right)^{\gamma},\left(T_{Z_{V_{12}}}(u)\right)^{\gamma},\left(T_{Z_{V_{13}}}(u)\right)^{\gamma},\left(T_{Z_{V_{14}}}(u)\right)^{\gamma}\right),\\ \left(\left(T_{Z_{R_{11}}}(u)\right)^{\gamma},\left(T_{Z_{R_{12}}}(u)\right)^{\gamma},\left(T_{Z_{R_{13}}}(u)\right)^{\gamma},\left(T_{Z_{R_{14}}}(u)\right)^{\gamma}\right)\end{array}\right),\\ \left(\begin{array}{c}\left(\begin{pmatrix}1-\left(1-I_{Z_{V_{11}}}(u)\right)^{\gamma},1-\left(1-I_{Z_{V_{12}}}(u)\right)^{\gamma},\\ 1-\left(1-I_{Z_{V_{13}}}(u)\right)^{\gamma},1-\left(1-I_{Z_{V_{14}}}(u)\right)^{\gamma}\end{pmatrix},\\ \left(\begin{pmatrix}1-\left(1-I_{Z_{R_{11}}}(u)\right)^{\gamma},1-\left(1-I_{Z_{R_{12}}}(u)\right)^{\gamma},\\ 1-\left(1-I_{Z_{R_{13}}}(u)\right)^{\gamma},1-\left(1-I_{Z_{R_{14}}}(u)\right)^{\gamma}\end{pmatrix}\end{array}\right),\\ \left(\begin{array}{c}\left(\begin{pmatrix}1-\left(1-F_{Z_{V_{11}}}(u)\right)^{\gamma},1-\left(1-F_{Z_{V_{12}}}(u)\right)^{\gamma},\\ 1-\left(1-F_{Z_{V_{13}}}(u)\right)^{\gamma},1-\left(1-F_{Z_{V_{14}}}(u)\right)^{\gamma}\end{pmatrix},\\ \left(\begin{pmatrix}1-\left(1-F_{Z_{R_{11}}}(u)\right)^{\gamma},1-\left(1-F_{Z_{R_{12}}}(u)\right)^{\gamma},\\ 1-\left(1-F_{Z_{R_{13}}}(u)\right)^{\gamma},1-\left(1-F_{Z_{R_{14}}}(u)\right)^{\gamma}\end{pmatrix}\end{array}\right)\end{array}\right) \tag{10}$$

3. TNZN-CRITIC-AROMAN Framework

This section shows the steps of the proposed approach. We used the CRITIC methodology to compute the criteria weights. The AROMAN methodology is used to rank the alternatives.

3.1 Problem Definition

A. Build the decision matrix.

We build the decision matrix using the TNZN between the criteria and the alternatives.

$$X=\begin{bmatrix}x_{11} & \cdots & x_{1n}\\ \vdots & \ddots & \vdots\\ x_{m1} & \cdots & x_{mn}\end{bmatrix}_{m\times n};i=1,\dots m;j=1,\dots,n \tag{11}$$

B. Apply the score function.

We apply the score function to obtain crisp values in the decision matrix[8].

C. Combine the decision matrix.

3.2 CRITIC Methodology

This part is used to compute the criteria weights[11], [12].

A. Normalize the combined decision matrix.

$$n_{ij}=\frac{x_{ij}-x_i^-}{x_i^+-x_i^-} \tag{12}$$

$$n_{ij} = \frac{x_i^- - x_{ij}}{x_i^- - x_i^+} \quad (13)$$

Where $x_i^- = \min x_{ij}$ and $x_i^+ = \max x_{ij}$

B. Compute the correlation coefficient for criteria p_{jk}.

C. Compute the Index C

$$C_j = \rho_j \sum_{k=1}^{n} (1 - p_{jk}) \quad (14)$$

Where ρ refers to the standard deviation.

D. Compute the criteria weights.

$$W_j = \frac{C_j}{\sum_{j=1}^{n} C_j} \quad (15)$$

3.3 AROMAN Methodology

This part is used to rank the alternatives[13], [14].

A. Compute the linear normalization

$$Y_{ij} = \frac{x_{ij} - \min x_{ij}}{\max x_{ij} - \min x_{ij}} \quad (16)$$

B. Compute the vector normalization

$$A_{ij} = \frac{x_{ij}}{\sqrt{\sum_{i=1}^{m} x_{ij}^2}} \quad (17)$$

C. Combine the normalization matrix

$$D_{ij} = \frac{\beta Y_{ij} + (1-\beta) A_{ij}}{2} \quad (18)$$

The values of β between 0 and 1.

D. Compute the weighted normalized decision matrix.

$$E_{ij} = W_j D_{ij} \quad (19)$$

E. Compute the maximized beneficial criteria and minimized non-beneficial criteria.

$$F_i = \sum_{j=1}^{n} (E_{ij})^{max} \quad (20)$$

$$Q_i = \sum_{j=1}^{n} (E_{ij})^{min} \quad (21)$$

F. Compute the value of R

$$R_i = Q_i^{\varphi} + F_i^{(1-\varphi)} \quad (22)$$

4. Case Study

In this section, the proposed method for ranking the strategies in higher education management is applied through a case study. The case study has various criteria and alternatives. It involves three parts. In the first part, we define the problem dimension and the decision matrix. In the second part, we define the weights of criteria by the CRITIC method. In the third part, we rank the alternatives using the ranking method.

4.1 Problem Definition

We invited three experts to evaluate the criteria and alternatives. These experts have more experience in higher education management. Three experts have evaluated nine criteria and ten alternatives.

The nine criteria are: research output, student support services, administrative efficiency, academic quality, faculty competence, financial management, resource allocation, infrastructure, curriculum design

The ten alternatives are: industry-academia collaboration, comprehensive student support services, outcome-based education model, policy implementation, faculty development programs, student-centered learning approaches, international accreditation, data-driven decision-making, sustainable financial management, smart campus & digital transformation

A. We used TNZN to build the decision matrix using Eq. (11) as shown in Table 1.

B. We applied the score function to obtain crisp values.

C. We combine the decision matrix into one matrix.

Table 1. The decision matrix.

	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8	C_9
A_1	((0.3, 0.4, 0.5, 0.5), (0.5, 0.6, 0.7, 0.8)), ((0.1, 0.2, 0.3, 0.4), (0.4, 0.5, 0.6, 0.7)), ((0.0, 0.1, 0.1, 0.1), (0.5, 0.6, 0.7, 0.8))	((0.1, 0.1, 0.1, 0.1), (0.5, 0.6, 0.7, 0.8)), ((0.1, 0.1, 0.1, 0.1), (0.6, 0.7, 0.8, 0.9)), ((0.6, 0.7, 0.8, 0.9), (0.5, 0.6, 0.7, 0.8))	((0.7, 0.7, 0.7, 0.7), (0.4, 0.5, 0.6, 0.7)), ((0.0, 0.1, 0.2, 0.3), (0.4, 0.5, 0.6, 0.7)), ((0.1, 0.1, 0.1, 0.1), (0.3, 0.4, 0.5, 0.6))	((0.0, 0.1, 0.2, 0.2), (0.4, 0.5, 0.6, 0.7)), ((0.1, 0.1, 0.1, 0.1), (0.3, 0.4, 0.5, 0.6)), ((0.5, 0.6, 0.7, 0.8), (0.5, 0.6, 0.7, 0.8))	((0.2, 0.3, 0.4, 0.5), (0.6, 0.7, 0.8, 0.9)), ((0.0, 0.1, 0.2, 0.3), (0.5, 0.6, 0.7, 0.8)), ((0.0, 0.1, 0.2, 0.3), (0.4, 0.5, 0.6, 0.7))	((0.6, 0.7, 0.7, 0.8), (0.5, 0.5, 0.5, 0.5)), ((0.1, 0.1, 0.1, 0.1), (0.5, 0.6, 0.7, 0.8)), ((0.0, 0.1, 0.1, 0.2), (0.4, 0.5, 0.5, 0.6))	((0.2, 0.3, 0.4, 0.5), (0.6, 0.7, 0.8, 0.9)), ((0.0, 0.1, 0.2, 0.3), (0.5, 0.6, 0.7, 0.8)), ((0.0, 0.1, 0.2, 0.3), (0.4, 0.5, 0.6, 0.7))	((0.2, 0.3, 0.4, 0.5), (0.6, 0.7, 0.8, 0.9)), ((0.0, 0.1, 0.2, 0.3), (0.5, 0.6, 0.7, 0.8)), ((0.0, 0.1, 0.2, 0.3), (0.4, 0.5, 0.6, 0.7))	((0.3, 0.4, 0.5, 0.5), (0.5, 0.6, 0.7, 0.8)), ((0.1, 0.2, 0.3, 0.4), (0.4, 0.5, 0.6, 0.7)), ((0.0, 0.1, 0.1, 0.1), (0.5, 0.6, 0.7, 0.8))
A_2	((0.6, 0.7, 0.7, 0.8), (0.5, 0.5, 0.5, 0.5)), ((0.1, 0.1, 0.1, 0.1), (0.5, 0.6, 0.7, 0.8)), ((0.0, 0.1, 0.1, 0.2), (0.4, 0.5, 0.5, 0.6))	((0.4, 0.5, 0.6, 0.7), (0.4, 0.5, 0.6, 0.7)), ((0.0, 0.1, 0.2, 0.3), (0.3, 0.4, 0.5, 0.6)), ((0.1, 0.1, 0.1, 0.1), (0.3, 0.4, 0.5, 0.6))	((0.3, 0.4, 0.5, 0.5), (0.5, 0.6, 0.7, 0.8)), ((0.1, 0.2, 0.3, 0.4), (0.4, 0.5, 0.6, 0.7)), ((0.0, 0.1, 0.1, 0.1), (0.5, 0.6, 0.7, 0.8))	((0.1, 0.1, 0.1, 0.1), (0.5, 0.6, 0.7, 0.8)), ((0.1, 0.1, 0.1, 0.1), (0.6, 0.7, 0.8, 0.9)), ((0.6, 0.7, 0.8, 0.9), (0.5, 0.6, 0.7, 0.8))	((0.7, 0.7, 0.7, 0.7), (0.4, 0.5, 0.6, 0.7)), ((0.0, 0.1, 0.2, 0.3), (0.4, 0.5, 0.6, 0.7)), ((0.1, 0.1, 0.1, 0.1), (0.6, 0.7, 0.8, 0.9))	((0.0, 0.1, 0.2, 0.2), (0.4, 0.5, 0.6, 0.7)), ((0.1, 0.1, 0.1, 0.1), (0.3, 0.4, 0.5, 0.6)), ((0.5, 0.6, 0.7, 0.8), (0.5, 0.6, 0.7, 0.8))	((0.0, 0.1, 0.2, 0.2), (0.4, 0.5, 0.6, 0.7)), ((0.1, 0.1, 0.1, 0.1), (0.3, 0.4, 0.5, 0.6)), ((0.5, 0.6, 0.7, 0.8), (0.5, 0.6, 0.7, 0.8))	((0.3, 0.4, 0.5, 0.5), (0.5, 0.6, 0.7, 0.8)), ((0.1, 0.2, 0.3, 0.4), (0.4, 0.5, 0.6, 0.7)), ((0.0, 0.1, 0.1, 0.1), (0.5, 0.6, 0.7, 0.8))	((0.1, 0.1, 0.1, 0.1), (0.5, 0.6, 0.7, 0.8)), ((0.1, 0.1, 0.1, 0.1), (0.6, 0.7, 0.8, 0.9)), ((0.6, 0.7, 0.8, 0.9), (0.5, 0.6, 0.7, 0.8))
A_3	((0.2, 0.3, 0.4, 0.5), (0.6, 0.7, 0.8, 0.9)), ((0.0, 0.1, 0.2, 0.3), (0.5, 0.6, 0.7, 0.8)), ((0.0, 0.1, 0.2, 0.3), (0.4, 0.5, 0.6, 0.7))	((0.4, 0.5, 0.6, 0.7), (0.4, 0.5, 0.6, 0.7)), ((0.0, 0.1, 0.2, 0.3), (0.3, 0.4, 0.5, 0.6)), ((0.1, 0.1, 0.1, 0.1), (0.3, 0.4, 0.5, 0.6))	((0.7, 0.7, 0.7, 0.7), (0.4, 0.5, 0.6, 0.7)), ((0.0, 0.1, 0.2, 0.3), (0.4, 0.5, 0.6, 0.7)), ((0.1, 0.1, 0.1, 0.1), (0.3, 0.4, 0.5, 0.6))	((0.0, 0.1, 0.2, 0.2), (0.4, 0.5, 0.6, 0.7)), ((0.1, 0.1, 0.1, 0.1), (0.3, 0.4, 0.5, 0.6)), ((0.5, 0.6, 0.7, 0.8), (0.5, 0.6, 0.7, 0.8))	((0.2, 0.3, 0.4, 0.5), (0.6, 0.7, 0.8, 0.9)), ((0.0, 0.1, 0.2, 0.3), (0.5, 0.6, 0.7, 0.8)), ((0.0, 0.1, 0.2, 0.3), (0.4, 0.5, 0.6, 0.7))	((0.2, 0.3, 0.4, 0.5), (0.6, 0.7, 0.8, 0.9)), ((0.0, 0.1, 0.2, 0.3), (0.5, 0.6, 0.7, 0.8)), ((0.0, 0.1, 0.2, 0.3), (0.4, 0.5, 0.6, 0.7))	((0.7, 0.7, 0.7, 0.7), (0.4, 0.5, 0.6, 0.7)), ((0.0, 0.1, 0.2, 0.3), (0.4, 0.5, 0.6, 0.7)), ((0.1, 0.1, 0.1, 0.1), (0.3, 0.4, 0.5, 0.6))	((0.4, 0.5, 0.6, 0.7), (0.4, 0.5, 0.6, 0.7)), ((0.0, 0.1, 0.2, 0.3), (0.3, 0.4, 0.5, 0.6)), ((0.1, 0.1, 0.1, 0.1), (0.3, 0.4, 0.5, 0.6))	((0.7, 0.7, 0.7, 0.7), (0.4, 0.5, 0.6, 0.7)), ((0.0, 0.1, 0.2, 0.3), (0.4, 0.5, 0.6, 0.7)), ((0.1, 0.1, 0.1, 0.1), (0.3, 0.4, 0.5, 0.6))
A_4	((0.0, 0.1, 0.2, 0.2), (0.4, 0.5, 0.6, 0.7)), ((0.1, 0.1, 0.1, 0.1), (0.3, 0.4, 0.5, 0.6)), ((0.5, 0.6, 0.7, 0.8), (0.5, 0.6, 0.7, 0.8))	((0.3, 0.4, 0.5, 0.5), (0.5, 0.6, 0.7, 0.8)), ((0.1, 0.2, 0.3, 0.4), (0.4, 0.5, 0.6, 0.7)), ((0.0, 0.1, 0.1, 0.1), (0.5, 0.6, 0.7, 0.8))	((0.1, 0.1, 0.1, 0.1), (0.5, 0.6, 0.7, 0.8)), ((0.1, 0.1, 0.1, 0.1), (0.6, 0.7, 0.8, 0.9)), ((0.6, 0.7, 0.8, 0.9), (0.5, 0.6, 0.7, 0.8))	((0.3, 0.4, 0.5, 0.5), (0.5, 0.6, 0.7, 0.8)), ((0.1, 0.1, 0.1, 0.1), (0.4, 0.5, 0.6, 0.7)), ((0.0, 0.1, 0.2, 0.3), (0.3, 0.4, 0.5, 0.6))	((0.4, 0.5, 0.6, 0.7), (0.4, 0.5, 0.6, 0.7)), ((0.0, 0.1, 0.2, 0.3), (0.3, 0.4, 0.5, 0.6)), ((0.1, 0.1, 0.1, 0.1), (0.4, 0.5, 0.6, 0.7))	((0.3, 0.4, 0.5, 0.5), (0.5, 0.6, 0.7, 0.8)), ((0.1, 0.2, 0.3, 0.4), (0.4, 0.5, 0.6, 0.7)), ((0.0, 0.1, 0.1, 0.1), (0.5, 0.6, 0.7, 0.8))	((0.1, 0.1, 0.1, 0.1), (0.5, 0.6, 0.7, 0.8)), ((0.1, 0.1, 0.1, 0.1), (0.6, 0.7, 0.8, 0.9)), ((0.6, 0.7, 0.8, 0.9), (0.5, 0.6, 0.7, 0.8))	((0.0, 0.1, 0.2, 0.2), (0.4, 0.5, 0.6, 0.7)), ((0.1, 0.1, 0.1, 0.1), (0.3, 0.4, 0.5, 0.6)), ((0.5, 0.6, 0.7, 0.8), (0.5, 0.6, 0.7, 0.8))	((0.0, 0.1, 0.2, 0.2), (0.4, 0.5, 0.6, 0.7)), ((0.1, 0.1, 0.1, 0.1), (0.3, 0.4, 0.5, 0.6)), ((0.5, 0.6, 0.7, 0.8), (0.5, 0.6, 0.7, 0.8))
A_5	((0.7, 0.7, 0.7, 0.7), (0.4, 0.5, 0.6, 0.7)), ((0.0, 0.1, 0.2, 0.3), (0.4, 0.5, 0.6, 0.7)), ((0.1, 0.1, 0.1, 0.1), (0.3, 0.4, 0.5, 0.6))	((0.4, 0.5, 0.6, 0.7), (0.4, 0.5, 0.6, 0.7)), ((0.0, 0.1, 0.2, 0.3), (0.3, 0.4, 0.5, 0.6)), ((0.1, 0.1, 0.1, 0.1), (0.3, 0.4, 0.5, 0.6))	((0.0, 0.1, 0.2, 0.2), (0.4, 0.5, 0.6, 0.7)), ((0.1, 0.1, 0.1, 0.1), (0.3, 0.4, 0.5, 0.6)), ((0.5, 0.6, 0.7, 0.8), (0.5, 0.6, 0.7, 0.8))	((0.2, 0.3, 0.4, 0.5), (0.6, 0.7, 0.8, 0.9)), ((0.0, 0.1, 0.2, 0.3), (0.5, 0.6, 0.7, 0.8)), ((0.0, 0.1, 0.2, 0.3), (0.4, 0.5, 0.6, 0.7))	((0.6, 0.7, 0.7, 0.8), (0.5, 0.5, 0.5, 0.5)), ((0.1, 0.1, 0.1, 0.1), (0.5, 0.6, 0.7, 0.8)), ((0.0, 0.1, 0.1, 0.2), (0.4, 0.5, 0.5, 0.6))	((0.2, 0.3, 0.4, 0.5), (0.6, 0.7, 0.8, 0.9)), ((0.0, 0.1, 0.2, 0.3), (0.5, 0.6, 0.7, 0.8)), ((0.0, 0.1, 0.2, 0.3), (0.4, 0.5, 0.6, 0.7))	((0.6, 0.7, 0.7, 0.8), (0.5, 0.5, 0.5, 0.5)), ((0.1, 0.1, 0.1, 0.1), (0.5, 0.6, 0.7, 0.8)), ((0.0, 0.1, 0.2, 0.3), (0.4, 0.5, 0.5, 0.6))	((0.7, 0.7, 0.7, 0.7), (0.4, 0.5, 0.6, 0.7)), ((0.0, 0.1, 0.2, 0.3), (0.4, 0.5, 0.6, 0.7)), ((0.1, 0.1, 0.1, 0.1), (0.3, 0.4, 0.5, 0.6))	((0.4, 0.5, 0.6, 0.7), (0.4, 0.5, 0.6, 0.7)), ((0.0, 0.1, 0.2, 0.3), (0.3, 0.4, 0.5, 0.6)), ((0.1, 0.1, 0.1, 0.1), (0.3, 0.4, 0.5, 0.6))
A_6	((0.1, 0.1, 0.1, 0.1), (0.5, 0.6, 0.7, 0.8)), ((0.1, 0.1, 0.1, 0.1), (0.6, 0.7, 0.8, 0.9)), ((0.6, 0.7, 0.8, 0.9), (0.5, 0.6, 0.7, 0.8))	((0.0, 0.1, 0.2, 0.2), (0.4, 0.5, 0.6, 0.7)), ((0.1, 0.1, 0.1, 0.1), (0.3, 0.4, 0.5, 0.6)), ((0.5, 0.6, 0.7, 0.8), (0.5, 0.6, 0.7, 0.8))	((0.7, 0.7, 0.7, 0.7), (0.4, 0.5, 0.6, 0.7)), ((0.0, 0.1, 0.2, 0.3), (0.4, 0.5, 0.6, 0.7)), ((0.1, 0.1, 0.1, 0.1), (0.3, 0.4, 0.5, 0.6))	((0.3, 0.4, 0.5, 0.5), (0.5, 0.6, 0.7, 0.8)), ((0.1, 0.2, 0.3, 0.4), (0.4, 0.5, 0.6, 0.7)), ((0.0, 0.1, 0.1, 0.1), (0.5, 0.6, 0.7, 0.8))	((0.6, 0.7, 0.7, 0.8), (0.5, 0.5, 0.5, 0.5)), ((0.1, 0.1, 0.1, 0.1), (0.5, 0.6, 0.7, 0.8)), ((0.0, 0.1, 0.1, 0.2), (0.4, 0.5, 0.5, 0.6))	((0.2, 0.3, 0.4, 0.5), (0.6, 0.7, 0.8, 0.9)), ((0.0, 0.1, 0.2, 0.3), (0.5, 0.6, 0.7, 0.8)), ((0.0, 0.1, 0.1, 0.1), (0.4, 0.5, 0.6, 0.7))	((0.1, 0.1, 0.1, 0.1), (0.5, 0.6, 0.7, 0.8)), ((0.0, 0.1, 0.2, 0.3), (0.6, 0.7, 0.8, 0.9)), ((0.0, 0.1, 0.2, 0.3), (0.4, 0.5, 0.6, 0.7))	((0.3, 0.4, 0.5, 0.5), (0.5, 0.6, 0.7, 0.8)), ((0.1, 0.1, 0.1, 0.1), (0.6, 0.7, 0.8, 0.9)), ((0.6, 0.7, 0.8, 0.9), (0.5, 0.6, 0.7, 0.8))

A_7	((0.6, 0.7, 0.7, 0.8), (0.5, 0.5, 0.5, 0.5)), ((0.1, 0.1, 0.1, 0.1), (0.5, 0.6, 0.7, 0.8)), ((0.0, 0.1, 0.1, 0.2), (0.4, 0.5, 0.5, 0.6))	((0.7, 0.7, 0.7, 0.7), (0.4, 0.5, 0.6, 0.7)), ((0.0, 0.1, 0.2, 0.3), (0.4, 0.5, 0.6, 0.7)), ((0.1, 0.1, 0.1, 0.1), (0.3, 0.4, 0.5, 0.6))	((0.0, 0.1, 0.2, 0.2), (0.4, 0.5, 0.6, 0.7)), ((0.1, 0.1, 0.1, 0.1), (0.3, 0.4, 0.5, 0.6)), ((0.5, 0.6, 0.7, 0.8), (0.5, 0.6, 0.7, 0.8))	((0.4, 0.5, 0.6, 0.7), (0.4, 0.5, 0.6, 0.7)), ((0.0, 0.1, 0.2, 0.3), (0.3, 0.4, 0.5, 0.6)), ((0.1, 0.1, 0.1, 0.1), (0.3, 0.4, 0.5, 0.6))	((0.1, 0.1, 0.1, 0.1), (0.5, 0.6, 0.7, 0.8)), ((0.1, 0.1, 0.1, 0.1), (0.6, 0.7, 0.8, 0.9)), ((0.6, 0.7, 0.8, 0.9), (0.5, 0.6, 0.7, 0.8))	((0.1, 0.1, 0.1, 0.1), (0.5, 0.6, 0.7, 0.8)), ((0.1, 0.1, 0.1, 0.1), (0.6, 0.7, 0.8, 0.9)), ((0.6, 0.7, 0.8, 0.9), (0.5, 0.6, 0.7, 0.8))	((0.6, 0.7, 0.7, 0.8), (0.5, 0.5, 0.5, 0.5)), ((0.1, 0.1, 0.1, 0.1), (0.5, 0.6, 0.7, 0.8)), ((0.0, 0.1, 0.1, 0.2), (0.4, 0.5, 0.5, 0.6))	((0.6, 0.7, 0.7, 0.8), (0.5, 0.5, 0.5, 0.5)), ((0.1, 0.1, 0.1, 0.1), (0.5, 0.6, 0.7, 0.8)), ((0.0, 0.1, 0.1, 0.2), (0.4, 0.5, 0.5, 0.6))	((0.4, 0.5, 0.6, 0.7), (0.4, 0.5, 0.6, 0.7)), ((0.0, 0.1, 0.2, 0.3), (0.3, 0.4, 0.5, 0.6)), ((0.1, 0.1, 0.1, 0.1), (0.3, 0.4, 0.5, 0.6))
A_8	((0.4, 0.5, 0.6, 0.7), (0.4, 0.5, 0.6, 0.7)), ((0.0, 0.1, 0.2, 0.3), (0.3, 0.4, 0.5, 0.6)), ((0.1, 0.1, 0.1, 0.1), (0.3, 0.4, 0.5, 0.6))	((0.1, 0.1, 0.1, 0.1), (0.5, 0.6, 0.7, 0.8)), ((0.1, 0.1, 0.1, 0.1), (0.6, 0.7, 0.8, 0.9)), ((0.6, 0.7, 0.8, 0.9), (0.5, 0.6, 0.7, 0.8))	((0.4, 0.5, 0.6, 0.7), (0.4, 0.5, 0.6, 0.7)), ((0.0, 0.1, 0.2, 0.3), (0.3, 0.4, 0.5, 0.6)), ((0.1, 0.1, 0.1, 0.1), (0.5, 0.6, 0.7, 0.8))	((0.0, 0.1, 0.2, 0.2), (0.4, 0.5, 0.6, 0.7)), ((0.1, 0.1, 0.1, 0.1), (0.3, 0.4, 0.5, 0.6)), ((0.5, 0.6, 0.7, 0.8), (0.5, 0.6, 0.7, 0.8))	((0.7, 0.7, 0.7, 0.7), (0.4, 0.5, 0.6, 0.7)), ((0.0, 0.1, 0.2, 0.3), (0.4, 0.5, 0.6, 0.7)), ((0.1, 0.1, 0.1, 0.1), (0.3, 0.4, 0.5, 0.6))	((0.7, 0.7, 0.7, 0.7), (0.4, 0.5, 0.6, 0.7)), ((0.0, 0.1, 0.2, 0.3), (0.4, 0.5, 0.6, 0.7)), ((0.1, 0.1, 0.1, 0.1), (0.3, 0.4, 0.5, 0.6))	((0.1, 0.1, 0.1, 0.1), (0.5, 0.6, 0.7, 0.8)), ((0.1, 0.1, 0.1, 0.1), (0.6, 0.7, 0.8, 0.9)), ((0.6, 0.7, 0.8, 0.9), (0.5, 0.6, 0.7, 0.8))	((0.4, 0.5, 0.6, 0.7), (0.4, 0.5, 0.6, 0.7)), ((0.0, 0.1, 0.2, 0.3), (0.3, 0.4, 0.5, 0.6)), ((0.1, 0.1, 0.1, 0.1), (0.5, 0.6, 0.7, 0.8))	((0.3, 0.4, 0.5, 0.5), (0.5, 0.6, 0.7, 0.8)), ((0.1, 0.2, 0.3, 0.4), (0.4, 0.5, 0.6, 0.7)), ((0.0, 0.1, 0.1, 0.1), (0.5, 0.6, 0.7, 0.8))
A_9	((0.3, 0.4, 0.5, 0.5), (0.5, 0.6, 0.7, 0.8)), ((0.1, 0.2, 0.3, 0.4), (0.4, 0.5, 0.6, 0.7)), ((0.0, 0.1, 0.1, 0.1), (0.5, 0.6, 0.7, 0.8))	((0.6, 0.7, 0.7, 0.8), (0.5, 0.5, 0.5, 0.5)), ((0.1, 0.1, 0.1, 0.1), (0.5, 0.6, 0.7, 0.8)), ((0.0, 0.1, 0.1, 0.2), (0.4, 0.5, 0.5, 0.6

	((0.1, 0.1, 0.1, 0.1), (0.5, 0.6, 0.7, 0.8)), ((0.0, 0.1, 0.1, 0.2), (0.4, 0.5, 0.5, 0.6))	((0.1, 0.1, 0.1, 0.1), (0.3, 0.4, 0.5, 0.6)), ((0.5, 0.6, 0.7, 0.8), (0.5, 0.6, 0.7, 0.8))	((0.1, 0.2, 0.3, 0.4), (0.4, 0.5, 0.6, 0.7)), ((0.0, 0.1, 0.1, 0.1), (0.5, 0.6, 0.7, 0.8))	((0.1, 0.1, 0.1, 0.1), (0.6, 0.7, 0.8, 0.9)), ((0.6, 0.7, 0.8, 0.9), (0.5, 0.6, 0.7, 0.8))	((0.0, 0.1, 0.2, 0.3), (0.4, 0.5, 0.6, 0.7)), ((0.1, 0.1, 0.1, 0.1), (0.3, 0.4, 0.5, 0.6))	((0.1, 0.1, 0.1, 0.1), (0.3, 0.4, 0.5, 0.6)), ((0.5, 0.6, 0.7, 0.8), (0.5, 0.6, 0.7, 0.8))	((0.1, 0.1, 0.1, 0.1), (0.3, 0.4, 0.5, 0.6)), ((0.5, 0.6, 0.7, 0.8), (0.5, 0.6, 0.7, 0.8))	((0.1, 0.2, 0.3, 0.4), (0.4, 0.5, 0.6, 0.7)), ((0.0, 0.1, 0.1, 0.1), (0.5, 0.6, 0.7, 0.8))	((0.1, 0.1, 0.1, 0.1), (0.6, 0.7, 0.8, 0.9)), ((0.6, 0.7, 0.8, 0.9), (0.5, 0.6, 0.7, 0.8))
A_3	((0.4, 0.5, 0.6, 0.7), (0.4, 0.5, 0.6, 0.7)), ((0.0, 0.1, 0.2, 0.3), (0.3, 0.4, 0.5, 0.6)), ((0.1, 0.1, 0.1, 0.1), (0.3, 0.4, 0.5, 0.6))	((0.7, 0.7, 0.7, 0.7), (0.4, 0.5, 0.6, 0.7)), ((0.0, 0.1, 0.2, 0.3), (0.4, 0.5, 0.6, 0.7)), ((0.1, 0.1, 0.1, 0.1), (0.3, 0.4, 0.5, 0.6))	((0.7, 0.7, 0.7, 0.7), (0.4, 0.5, 0.6, 0.7)), ((0.0, 0.1, 0.2, 0.3), (0.4, 0.5, 0.6, 0.7)), ((0.1, 0.1, 0.1, 0.1), (0.3, 0.4, 0.5, 0.6))	((0.0, 0.1, 0.2, 0.2), (0.4, 0.5, 0.6, 0.7)), ((0.1, 0.1, 0.1, 0.1), (0.3, 0.4, 0.5, 0.6)), ((0.5, 0.6, 0.7, 0.8), (0.5, 0.6, 0.7, 0.8))	((0.1, 0.1, 0.1, 0.1), (0.5, 0.6, 0.7, 0.8)), ((0.1, 0.1, 0.1, 0.1), (0.6, 0.7, 0.8, 0.9)), ((0.6, 0.7, 0.8, 0.9), (0.5, 0.6, 0.7, 0.8))	((0.7, 0.7, 0.7, 0.7), (0.4, 0.5, 0.6, 0.7)), ((0.0, 0.1, 0.2, 0.3), (0.4, 0.5, 0.6, 0.7)), ((0.1, 0.1, 0.1, 0.1), (0.3, 0.4, 0.5, 0.6))	((0.7, 0.7, 0.7, 0.7), (0.4, 0.5, 0.6, 0.7)), ((0.0, 0.1, 0.2, 0.3), (0.4, 0.5, 0.6, 0.7)), ((0.1, 0.1, 0.1, 0.1), (0.3, 0.4, 0.5, 0.6))	((0.2, 0.3, 0.4, 0.5), (0.6, 0.7, 0.8, 0.9)), ((0.0, 0.1, 0.2, 0.3), (0.5, 0.6, 0.7, 0.8)), ((0.0, 0.1, 0.2, 0.3), (0.4, 0.5, 0.6, 0.7)), ((0.0, 0.1, 0.2, 0.3), (0.5, 0.6, 0.7, 0.8))	((0.7, 0.7, 0.7, 0.7), (0.4, 0.5, 0.6, 0.7)), ((0.0, 0.1, 0.2, 0.3), (0.4, 0.5, 0.6, 0.7)), ((0.1, 0.1, 0.1, 0.1), (0.3, 0.4, 0.5, 0.6)), ((0.0, 0.1, 0.2, 0.3), (0.5, 0.6, 0.7, 0.8))
A_4	((0.0, 0.1, 0.2, 0.2), (0.4, 0.5, 0.6, 0.7)), ((0.1, 0.1, 0.1, 0.1), (0.3, 0.4, 0.5, 0.6)), ((0.5, 0.6, 0.7, 0.8), (0.5, 0.6, 0.7, 0.8))	((0.1, 0.1, 0.1, 0.1), (0.5, 0.6, 0.7, 0.8)), ((0.1, 0.1, 0.1, 0.1), (0.6, 0.7, 0.8, 0.9)), ((0.6, 0.7, 0.8, 0.9), (0.5, 0.6, 0.7, 0.8))	((0.0, 0.1, 0.2, 0.2), (0.4, 0.5, 0.6, 0.7)), ((0.1, 0.1, 0.1, 0.1), (0.3, 0.4, 0.5, 0.6)), ((0.5, 0.6, 0.7, 0.8), (0.5, 0.6, 0.7, 0.8))	((0.3, 0.4, 0.5, 0.5), (0.5, 0.6, 0.7, 0.8)), ((0.1, 0.2, 0.3, 0.4), (0.4, 0.5, 0.6, 0.7)), ((0.0, 0.1, 0.1, 0.1), (0.5, 0.6, 0.7, 0.8))	((0.3, 0.4, 0.5, 0.5), (0.5, 0.6, 0.7, 0.8)), ((0.1, 0.2, 0.3, 0.4), (0.4, 0.5, 0.6, 0.7)), ((0.0, 0.1, 0.1, 0.1), (0.5, 0.6, 0.7, 0.8))	((0.1, 0.1, 0.1, 0.1), (0.5, 0.6, 0.7, 0.8)), ((0.1, 0.1, 0.1, 0.1), (0.6, 0.7, 0.8, 0.9)), ((0.6, 0.7, 0.8, 0.9), (0.5, 0.6, 0.7, 0.8))	((0.1, 0.1, 0.1, 0.1), (0.5, 0.6, 0.7, 0.8)), ((0.1, 0.1, 0.1, 0.1), (0.6, 0.7, 0.8, 0.9)), ((0.6, 0.7, 0.8, 0.9), (0.5, 0.6, 0.7, 0.8))	((0.0, 0.1, 0.2, 0.2), (0.4, 0.5, 0.6, 0.7)), ((0.0, 0.1, 0.1, 0.1), (0.3, 0.4, 0.5, 0.6)), ((0.5, 0.6, 0.7, 0.8), (0.5, 0.6, 0.7, 0.8))	((0.0, 0.1, 0.2, 0.2), (0.4, 0.5, 0.6, 0.7)), ((0.1, 0.1, 0.1, 0.1), (0.3, 0.4, 0.5, 0.6)), ((0.5, 0.6, 0.7, 0.8), (0.5, 0.6, 0.7, 0.8))
A_5	((0.7, 0.7, 0.7, 0.7), (0.4, 0.5, 0.6, 0.7)), ((0.0, 0.1, 0.2, 0.3), (0.4, 0.5, 0.6, 0.7)), ((0.1, 0.1, 0.1, 0.1), (0.3, 0.4, 0.5, 0.6))	((0.6, 0.7, 0.7, 0.8), (0.5, 0.5, 0.5, 0.5)), ((0.1, 0.1, 0.1, 0.1), (0.5, 0.6, 0.7, 0.8)), ((0.1, 0.1, 0.1, 0.2), (0.4, 0.5, 0.5, 0.6))	((0.7, 0.7, 0.7, 0.7), (0.4, 0.5, 0.6, 0.7)), ((0.0, 0.1, 0.2, 0.3), (0.4, 0.5, 0.6, 0.7)), ((0.1, 0.1, 0.1, 0.1), (0.3, 0.4, 0.5, 0.6))	((0.0, 0.1, 0.2, 0.2), (0.4, 0.5, 0.6, 0.7)), ((0.1, 0.1, 0.1, 0.1), (0.3, 0.4, 0.5, 0.6)), ((0.5, 0.6, 0.7, 0.8), (0.5, 0.6, 0.7, 0.8))	((0.0, 0.1, 0.2, 0.2), (0.4, 0.5, 0.6, 0.7)), ((0.1, 0.1, 0.1, 0.1), (0.3, 0.4, 0.5, 0.6)), ((0.5, 0.6, 0.7, 0.8), (0.5, 0.6, 0.7, 0.8))	((0.3, 0.4, 0.5, 0.5), (0.5, 0.6, 0.7, 0.8)), ((0.1, 0.2, 0.3, 0.4), (0.4, 0.5, 0.6, 0.7)), ((0.0, 0.1, 0.1, 0.1), (0.5, 0.6, 0.7, 0.8))	((0.3, 0.4, 0.5, 0.5), (0.5, 0.6, 0.7, 0.8)), ((0.1, 0.2, 0.3, 0.4), (0.4, 0.5, 0.6, 0.7)), ((0.0, 0.1, 0.1, 0.1), (0.5, 0.6, 0.7, 0.8))	((0.7, 0.7, 0.7, 0.7), (0.4, 0.5, 0.6, 0.7)), ((0.0, 0.1, 0.2, 0.3), (0.4, 0.5, 0.6, 0.7)), ((0.1, 0.1, 0.1, 0.1), (0.3, 0.4, 0.5, 0.6))	((0.2, 0.3, 0.4, 0.5), (0.6, 0.7, 0.8, 0.9)), ((0.0, 0.1, 0.2, 0.3), (0.5, 0.6, 0.7, 0.8)), ((0.0, 0.1, 0.2, 0.3), (0.4, 0.5, 0.6, 0.7))
A_6	((0.1, 0.1, 0.1, 0.1), (0.5, 0.6, 0.7, 0.8)), ((0.1, 0.1, 0.1, 0.1), (0.6, 0.7, 0.8, 0.9)), ((0.6, 0.7, 0.8, 0.9), (0.5, 0.6, 0.7, 0.8))	((0.0, 0.1, 0.2, 0.2), (0.4, 0.5, 0.6, 0.7)), ((0.1, 0.1, 0.1, 0.1), (0.3, 0.4, 0.5, 0.6)), ((0.5, 0.6, 0.7, 0.8), (0.5, 0.6, 0.7, 0.8))	((0.1, 0.1, 0.1, 0.1), (0.5, 0.6, 0.7, 0.8)), ((0.1, 0.1, 0.1, 0.1), (0.6, 0.7, 0.8, 0.9)), ((0.6, 0.7, 0.8, 0.9), (0.5, 0.6, 0.7, 0.8))	((0.7, 0.7, 0.7, 0.7), (0.4, 0.5, 0.6, 0.7)), ((0.0, 0.1, 0.2, 0.3), (0.4, 0.5, 0.6, 0.7)), ((0.1, 0.1, 0.1, 0.2), (0.4, 0.5, 0.6, 0.7))	((0.7, 0.7, 0.7, 0.7), (0.4, 0.5, 0.6, 0.7)), ((0.0, 0.1, 0.2, 0.3), (0.4, 0.5, 0.6, 0.7)), ((0.1, 0.1, 0.1, 0.1), (0.3, 0.4, 0.5, 0.6)), ((0.5, 0.6, 0.7, 0.8), (0.5, 0.6, 0.7, 0.8))	((0.0, 0.1, 0.2, 0.2), (0.4, 0.5, 0.6, 0.7)), ((0.1, 0.1, 0.1, 0.1), (0.5, 0.6, 0.7, 0.8)), ((0.0, 0.1, 0.1, 0.1), (0.5, 0.6, 0.7, 0.8))	((0.2, 0.3, 0.4, 0.5), (0.6, 0.7, 0.8, 0.9)), ((0.0, 0.1, 0.2, 0.3), (0.5, 0.6, 0.7, 0.8)), ((0.0, 0.1, 0.2, 0.3), (0.4, 0.5, 0.6, 0.7))	((0.1, 0.1, 0.1, 0.1), (0.5, 0.6, 0.7, 0.8)), ((0.1, 0.1, 0.1, 0.1), (0.6, 0.7, 0.8, 0.9)), ((0.0, 0.1, 0.2, 0.3), (0.5, 0.6, 0.7, 0.8)), ((0.0, 0.1, 0.2, 0.3), (0.4, 0.5, 0.6, 0.7))	((0.2, 0.3, 0.4, 0.5), (0.6, 0.7, 0.8, 0.9)), ((0.0, 0.1, 0.2, 0.3), (0.5, 0.6, 0.7, 0.8)), ((0.0, 0.1, 0.2, 0.3), (0.4, 0.5, 0.6, 0.7))
A_7	((0.3, 0.4, 0.5, 0.5), (0.5, 0.6, 0.7, 0.8)), ((0.1, 0.2, 0.3, 0.4), (0.4, 0.5, 0.6, 0.7)), ((0.0, 0.1, 0.1, 0.1), (0.5, 0.6, 0.7, 0.8))	((0.7, 0.7, 0.7, 0.7), (0.4, 0.5, 0.6, 0.7)), ((0.0, 0.1, 0.2, 0.3), (0.4, 0.5, 0.6, 0.7)), ((0.1, 0.1, 0.1, 0.1), (0.3, 0.4, 0.5, 0.6))	((0.3, 0.4, 0.5, 0.5), (0.5, 0.6, 0.7, 0.8)), ((0.1, 0.2, 0.3, 0.4), (0.4, 0.5, 0.6, 0.7)), ((0.0, 0.1, 0.2, 0.3), (0.5, 0.6, 0.7, 0.8))	((0.1, 0.1, 0.1, 0.1), (0.5, 0.6, 0.7, 0.8)), ((0.1, 0.1, 0.1, 0.1), (0.6, 0.7, 0.8, 0.9)), ((0.6, 0.7, 0.8, 0.9), (0.5, 0.6, 0.7, 0.8))	((0.1, 0.1, 0.1, 0.1), (0.5, 0.6, 0.7, 0.8)), ((0.1, 0.1, 0.1, 0.1), (0.6, 0.7, 0.8, 0.9)), ((0.6, 0.7, 0.8, 0.9), (0.5, 0.6, 0.7, 0.8))	((0.7, 0.7, 0.7, 0.7), (0.4, 0.5, 0.6, 0.7)), ((0.0, 0.1, 0.2, 0.3), (0.4, 0.5, 0.6, 0.7)), ((0.1, 0.1, 0.1, 0.1), (0.3, 0.4, 0.5, 0.6)), ((0.5, 0.6, 0.7, 0.8), (0.5, 0.6, 0.7, 0.8))	((0.3, 0.4, 0.5, 0.5), (0.5, 0.6, 0.7, 0.8)), ((0.1, 0.2, 0.3, 0.4), (0.4, 0.5, 0.6, 0.7)), ((0.0, 0.1, 0.1, 0.1), (0.5, 0.6, 0.7, 0.8))	((0.3, 0.4, 0.5, 0.5), (0.5, 0.6, 0.7, 0.8)), ((0.1, 0.2, 0.3, 0.4), (0.4, 0.5, 0.6, 0.7)), ((0.0, 0.1, 0.1, 0.1), (0.5, 0.6, 0.7, 0.8))	((0.2, 0.3, 0.4, 0.5), (0.6, 0.7, 0.8, 0.9)), ((0.0, 0.1, 0.2, 0.3), (0.5, 0.6, 0.7, 0.8)), ((0.0, 0.1, 0.2, 0.3), (0.4, 0.5, 0.6, 0.7))
A_8	((0.2, 0.3, 0.4, 0.5), (0.6, 0.7, 0.8, 0.9)), ((0.0, 0.1, 0.2, 0.3), (0.5, 0.6, 0.7, 0.8)), ((0.0, 0.1, 0.2, 0.3), (0.4, 0.5, 0.6, 0.7))	((0.1, 0.1, 0.1, 0.1), (0.5, 0.6, 0.7, 0.8)), ((0.1, 0.1, 0.1, 0.1), (0.6, 0.7, 0.8, 0.9)), ((0.6, 0.7, 0.8, 0.9), (0.5, 0.6, 0.7, 0.8))	((0.2, 0.3, 0.4, 0.5), (0.6, 0.7, 0.8, 0.9)), ((0.0, 0.1, 0.2, 0.3), (0.5, 0.6, 0.7, 0.8)), ((0.0, 0.1, 0.2, 0.3), (0.4, 0.5, 0.6, 0.7))	((0.6, 0.7, 0.7, 0.8), (0.5, 0.5, 0.5, 0.5)), ((0.1, 0.1, 0.1, 0.1), (0.5, 0.6, 0.7, 0.8)), ((0.0, 0.1, 0.1, 0.2), (0.4, 0.5, 0.5, 0.6))	((0.6, 0.7, 0.7, 0.8), (0.5, 0.5, 0.5, 0.5)), ((0.1, 0.1, 0.1, 0.1), (0.5, 0.6, 0.7, 0.8)), ((0.0, 0.1, 0.1, 0.2), (0.4, 0.5, 0.5, 0.6))	((0.1, 0.1, 0.1, 0.1), (0.5, 0.6, 0.7, 0.8)), ((0.1, 0.1, 0.1, 0.1), (0.6, 0.7, 0.8, 0.9)), ((0.6, 0.7, 0.8, 0.9), (0.5, 0.6, 0.7, 0.8))	((0.1, 0.1, 0.1, 0.1), (0.5, 0.6, 0.7, 0.8)), ((0.1, 0.2, 0.3, 0.4), (0.4, 0.5, 0.6, 0.7)), ((0.0, 0.1, 0.1, 0.1), (0.5, 0.6, 0.7, 0.8))	((0.2, 0.3, 0.4, 0.5), (0.6, 0.7, 0.8, 0.9)), ((0.0, 0.1, 0.2, 0.3), (0.5, 0.6, 0.7, 0.8)), ((0.0, 0.1, 0.2, 0.3), (0.4, 0.5, 0.6, 0.7))	((0.6, 0.7, 0.7, 0.8), (0.5, 0.5, 0.5, 0.5)), ((0.1, 0.1, 0.1, 0.1), (0.5, 0.6, 0.7, 0.8)), ((0.0, 0.1, 0.1, 0.2), (0.4, 0.5, 0.5, 0.6))
A_9	((0.3, 0.4, 0.5, 0.5), (0.5, 0.6, 0.7, 0.8)), ((0.1, 0.2, 0.3, 0.4), (0.4, 0.5, 0.6, 0.7)), ((0.0, 0.1, 0.1, 0.1), (0.5, 0.6, 0.7, 0.8))	((0.3, 0.4, 0.5, 0.5), (0.5, 0.6, 0.7, 0.8)), ((0.1, 0.2, 0.3, 0.4), (0.4, 0.5, 0.6, 0.7)), ((0.0, 0.1, 0.1, 0.1), (0.5, 0.6, 0.7, 0.8))	((0.3, 0.4, 0.5, 0.5), (0.5, 0.6, 0.7, 0.8)), ((0.1, 0.2, 0.3, 0.4), (0.4, 0.5, 0.6, 0.7)), ((0.0, 0.1, 0.1, 0.1), (0.5, 0.6, 0.7, 0.8))	((0.0, 0.1, 0.2, 0.2), (0.4, 0.5, 0.6, 0.7)), ((0.1, 0.1, 0.1, 0.1), (0.3, 0.4, 0.5, 0.6)), ((0.5, 0.6, 0.7, 0.8), (0.5, 0.6, 0.7, 0.8))	((0.3, 0.4, 0.5, 0.5), (0.5, 0.6, 0.7, 0.8)), ((0.1, 0.2, 0.3, 0.4), (0.4, 0.5, 0.6, 0.7)), ((0.0, 0.1, 0.1, 0.1), (0.5, 0.6, 0.7, 0.8))	((0.7, 0.7, 0.7, 0.7), (0.4, 0.5, 0.6, 0.7)), ((0.0, 0.1, 0.2, 0.3), (0.4, 0.5, 0.6, 0.7)), ((0.1, 0.1, 0.1, 0.1), (0.3, 0.4, 0.5, 0.6))	((0.3, 0.4, 0.5, 0.5), (0.5, 0.6, 0.7, 0.8)), ((0.1, 0.2, 0.3, 0.4), (0.4, 0.5, 0.6, 0.7)), ((0.0, 0.1, 0.1, 0.1), (0.5, 0.6, 0.7, 0.8))	((0.1, 0.1, 0.1, 0.1), (0.5, 0.6, 0.7, 0.8)), ((0.5, 0.6, 0.7, 0.8), (0.5, 0.6, 0.7, 0.8)), ((0.6, 0.7, 0.8, 0.9), (0.5, 0.6, 0.7, 0.8))
A_{10}	((0.4, 0.5, 0.6, 0.7), (0.4, 0.5, 0.6, 0.7)), ((0.0, 0.1, 0.2, 0.3), (0.3, 0.4, 0.5, 0.6)), ((0.1, 0.1, 0.1, 0.1), (0.3, 0.4, 0.5, 0.6))	((0.0, 0.1, 0.2, 0.2), (0.4, 0.5, 0.6, 0.7)), ((0.1, 0.1, 0.1, 0.1), (0.3, 0.4, 0.5, 0.6)), ((0.5, 0.6, 0.7, 0.8), (0.5, 0.6, 0.7, 0.8))	((0.7, 0.7, 0.7, 0.7), (0.4, 0.5, 0.6, 0.7)), ((0.0, 0.1, 0.2, 0.3), (0.4, 0.5, 0.6, 0.7)), ((0.1, 0.1, 0.1, 0.2), (0.4, 0.5, 0.6, 0.7))	((0.1, 0.1, 0.1, 0.1), (0.5, 0.6, 0.7, 0.8)), ((0.1, 0.1, 0.1, 0.1), (0.6, 0.7, 0.8, 0.9)), ((0.6, 0.7, 0.8, 0.9), (0.5, 0.6, 0.7, 0.8))	((0.3, 0.4, 0.5, 0.5), (0.5, 0.6, 0.7, 0.8)), ((0.1, 0.2, 0.3, 0.4), (0.4, 0.5, 0.6, 0.7)), ((0.0, 0.1, 0.1, 0.1), (0.5, 0.6, 0.7, 0.8))	((0.2, 0.3, 0.4, 0.5), (0.6, 0.7, 0.8, 0.9)), ((0.0, 0.1, 0.2, 0.3), (0.5, 0.6, 0.7, 0.8)), ((0.0, 0.1, 0.2, 0.3), (0.4, 0.5, 0.6, 0.7))	((0.0, 0.1, 0.2, 0.2), (0.4, 0.5, 0.6, 0.7)), ((0.1, 0.1, 0.1, 0.1), (0.3, 0.4, 0.5, 0.6)), ((0.5, 0.6, 0.7, 0.8), (0.5, 0.6, 0.7, 0.8))	((0.2, 0.3, 0.4, 0.5), (0.6, 0.7, 0.8, 0.9)), ((0.0, 0.1, 0.2, 0.3), (0.5, 0.6, 0.7, 0.8)), ((0.0, 0.1, 0.2, 0.3), (0.4, 0.5, 0.6, 0.7))	((0.7, 0.7, 0.7, 0.7), (0.4, 0.5, 0.6, 0.7)), ((0.0, 0.1, 0.2, 0.3), (0.4, 0.5, 0.6, 0.7)), ((0.1, 0.1, 0.1, 0.1), (0.3, 0.4, 0.5, 0.6))

4.2 CRITIC Methodology

A. We normalize the combined decision matrix using Eq. (12).

B. Then we compute the correlation coefficient for the criteria p_{jk} as shown in Fig 1.

C. Then we compute the Index C using Eq. (14).

D. Then we compute the criteria weights using Eq. (15). The criteria weights are: C1= 0.1019, C2 = 0.113, C3 = 0.093, C4 = 0.157, C5 = 0.117, C6 = 0.098, C7 = 0.097, C8 = 0.101, C9 = 0.118.

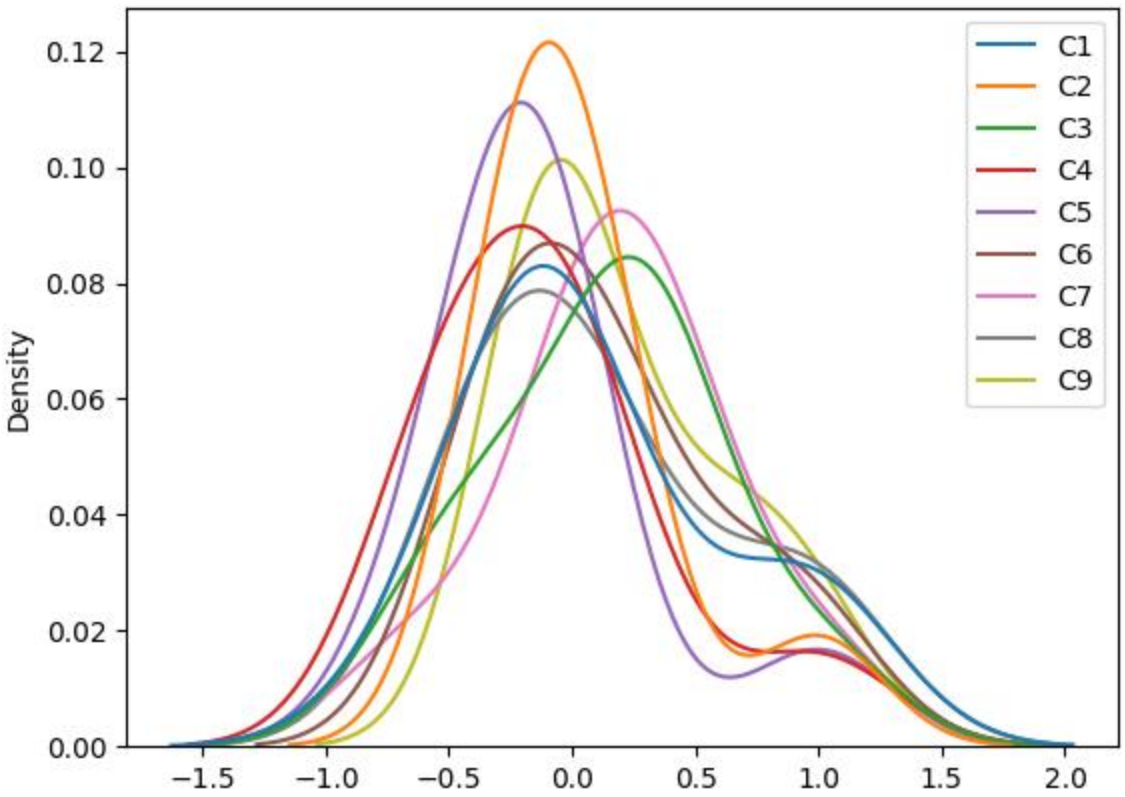

Fig 1. The correlation between the criteria.

4.3 AROMAN Methodology

A. We compute the linear normalization using Eq. (16) as shown in Table 2.

Table 2. The linear normalization.

	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8	C_9
A_1	0.978076	0	1	0.153497	0.488889	0.933382	0.847747	0.887324	0.842163
A_2	1	0.675525	0.768032	0	1	0	0.144336	0.897887	0
A_3	0.61754	1	1	0.153497	0.43873	1	0.7838	0.548415	1
A_4	0.464068	0.580964	0	0.988342	0.846349	0.593049	0	0.257042	0.130795
A_5	0.7838	0.988875	0.394114	0.707902	0.570159	0.504707	1	0.649648	0.863135
A_6	0	0.459827	0.620312	1	0.605079	0.663287	0.847747	0	0.774834
A_7	1	0.746601	0.542412	0.356218	0.233016	0.322954	0.643118	1	0.863135
A_8	0.952497	0.3733	0.539527	0.821891	0.655238	0.703114	0.367844	0.892606	0.842163
A_9	0.858709	0.701483	0.912291	0.735104	0	0.283128	0.7838	0.822183	0
A_{10}	0.900122	0.146477	0.697057	0	0.818413	0.83635	0.378806	0.472711	1

B. We compute the vector normalization using Eq. (17) as shown in Table 3.

Table 3. The vector normalization.

	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8	C_9
A_1	0.338304	0.250508	0.354494	0.275912	0.306813	0.347689	0.344493	0.333625	0.337868
A_2	0.340641	0.326438	0.328192	0.258896	0.360354	0.26036	0.264883	0.334412	0.237403

A_3	0.299878	0.36291	0.354494	0.275912	0.301559	0.353922	0.337255	0.30836	0.356697
A_4	0.283521	0.315809	0.241106	0.368457	0.344259	0.315847	0.248547	0.286639	0.253006
A_5	0.317598	0.361659	0.285794	0.337369	0.315327	0.307581	0.361724	0.315906	0.340369
A_6	0.234061	0.302193	0.311442	0.369749	0.318985	0.322418	0.344493	0.267477	0.329836
A_7	0.340641	0.334427	0.302609	0.298384	0.28001	0.290576	0.321333	0.342024	0.340369
A_8	0.335578	0.292467	0.302282	0.350005	0.324239	0.326145	0.290179	0.334018	0.337868
A_9	0.325582	0.329356	0.344549	0.340385	0.2556	0.28685	0.337255	0.328768	0.237403
A_{10}	0.329996	0.266972	0.320144	0.258896	0.341332	0.338611	0.291419	0.302716	0.356697

C. Then we combine the normalization matrix using Eq. (18) as shown in Table 4. We put the values of β with 0.5.

Table 4. The combined normalization.

	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8	C_9
A_1	0.329095	0.062627	0.338624	0.107352	0.198926	0.320268	0.29806	0.305237	0.295008
A_2	0.33516	0.250491	0.274056	0.064724	0.340089	0.06509	0.102305	0.308075	0.059351
A_3	0.229354	0.340727	0.338624	0.107352	0.185072	0.33848	0.280264	0.214194	0.339174
A_4	0.186897	0.224193	0.060276	0.3392	0.297652	0.227224	0.062137	0.13592	0.09595
A_5	0.27535	0.337634	0.169977	0.261318	0.221371	0.203072	0.340431	0.241389	0.300876
A_6	0.058515	0.190505	0.232938	0.342437	0.231016	0.246426	0.29806	0.066869	0.276168
A_7	0.33516	0.270257	0.211255	0.16365	0.128256	0.153383	0.241113	0.335506	0.300876
A_8	0.322019	0.166442	0.210452	0.292974	0.244869	0.257315	0.164506	0.306656	0.295008
A_9	0.296073	0.25771	0.31421	0.268872	0.0639	0.142495	0.280264	0.287738	0.059351
A_{10}	0.307529	0.103362	0.2543	0.064724	0.289936	0.29374	0.167556	0.193857	0.339174

D. Then we compute the weighted normalized decision matrix using Eq. (19) as shown in Table 5.

Table 5. The weighted normalization.

	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8	C_9
A_1	0.03355	0.007116	0.031659	0.01696	0.023414	0.031406	0.029149	0.030944	0.034814
A_2	0.034169	0.028463	0.025622	0.010225	0.040029	0.006383	0.010005	0.031232	0.007004
A_3	0.023382	0.038717	0.031659	0.01696	0.021784	0.033192	0.027408	0.021715	0.040026
A_4	0.019054	0.025475	0.005635	0.053587	0.035035	0.022282	0.006077	0.013779	0.011323
A_5	0.028071	0.038365	0.015892	0.041283	0.026056	0.019914	0.033292	0.024472	0.035506
A_6	0.005965	0.021647	0.021778	0.054099	0.027191	0.024165	0.029149	0.006779	0.032591
A_7	0.034169	0.030709	0.019751	0.025854	0.015096	0.015041	0.02358	0.034013	0.035506
A_8	0.032829	0.018913	0.019676	0.046284	0.028822	0.025233	0.016088	0.031088	0.034814
A_9	0.030184	0.029283	0.029377	0.042477	0.007521	0.013973	0.027408	0.02917	0.007004
A_{10}	0.031352	0.011745	0.023775	0.010225	0.034126	0.028805	0.016386	0.019653	0.040026

E. Then we compute the maximized beneficial criteria and minimized non-beneficial criteria using Eq. (20 and 21).

F. Then we compute the value of R using Eq. (22). We put the value of φ with 0.5. Then we rank the alternatives as shown in Fig 2.

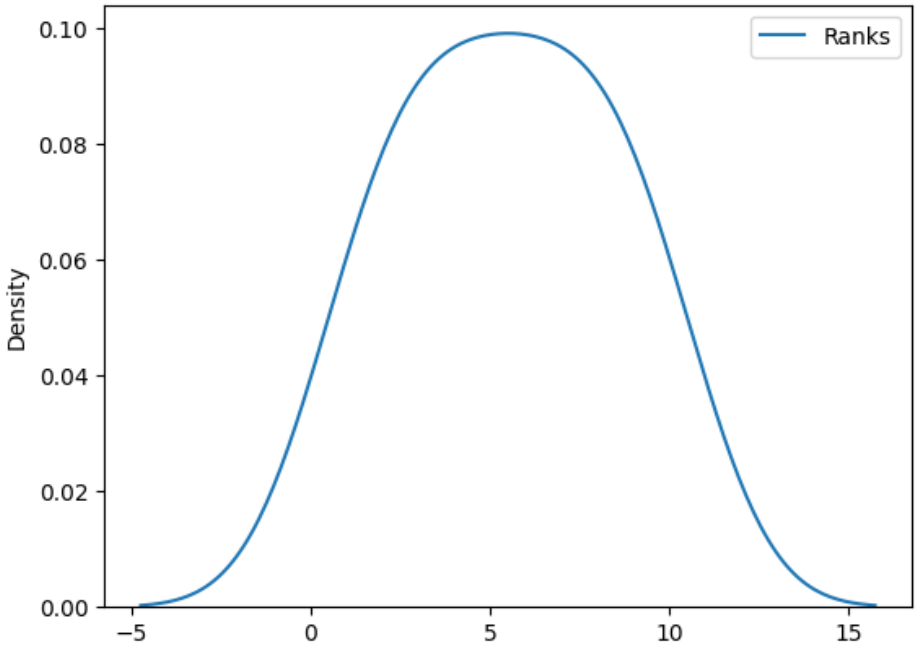

Fig 2. The rank of alternatives.

Sensitivity analysis

In this section, we conducted a sensitivity analysis with two parts. We show the sensitivity analysis to show the different ranks of alternatives. We show that the ranks of alternatives are stable under different cases.

In the first case, we change the value of β between 0 and 1. We conducted ten cases between the ranks of alternatives. Fig 3 shows the different ranks of alternatives under different cases. We show alternative 3 is the best and alternative 2 is the worst.

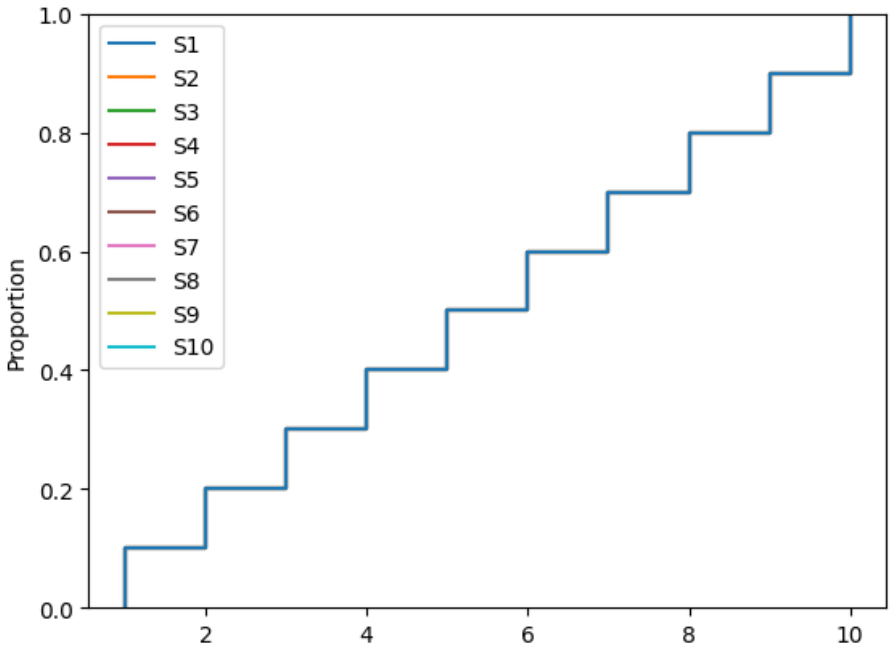

Fig 3. The ranks of the first case.

In the second case, we change the value of φ between 0 and 1. We conducted ten cases between the ranks of alternatives. Fig 4 shows the different ranks of alternatives under different cases. We show alternative 3 is the best and alternative 2 is the worst.

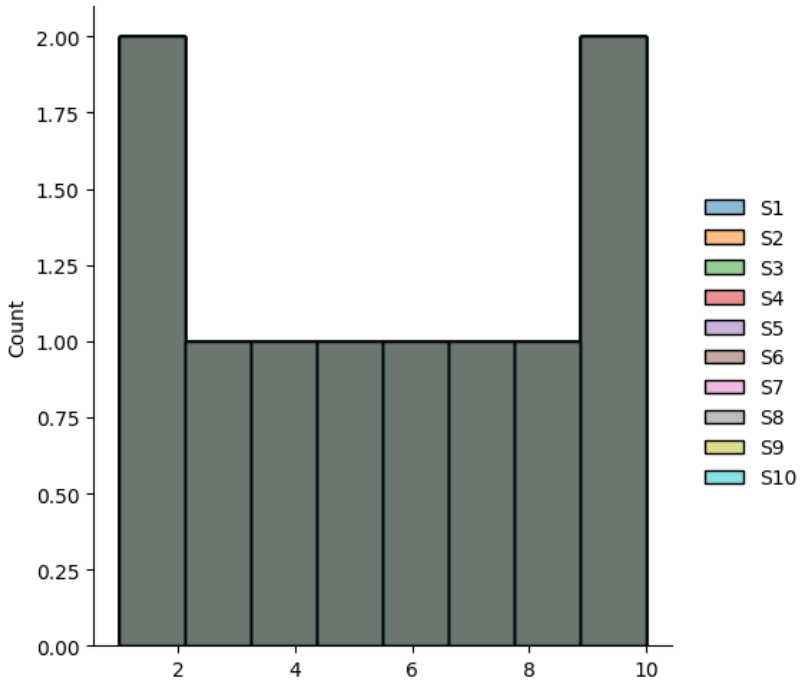

Fig 4. The ranks of the second case.

5. Conclusions and Future Works

Two MCDMs are used in this study to evaluate and rank the higher education management strategies. Two MCDM methods are used under the Trapezoidal Neutrosophic Z-Number (TNZN) to deal with uncertainty and vague information. We used the CRITIC method to compute the criteria weight and the AROMAN method to rank the alternatives. Three experts have assessed the criteria and alternatives. We used nine criteria and ten alternatives. The results show alternative 3 is the best and alternative 2 is the worst. The results of the sensitivity analysis show the ranks of the alternative are stable under different cases.

In future works, the proposed approach can be applied to different MCDM problems to compute the criteria weights and rank the alternatives. The proposed approach can be applied under different uncertainty models to deal with uncertainty data.

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